Empirical Geoid Modelling Using Classical Gravimetric Method.

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ABSTRACT:
This paper presents Gravimetric geoid modelling using the remove-restore-compute approach for Lagos State. The direct method of computation of the stokes integral was used in computing the geoidal undulation from archived gravity anormaly values obtained from BGI. An empirical geoid solution for the study area was thereafter developed from the resulting gravimetric geoid model using least squares technique. The obtained gravimetric geoid had a RMSE of 2.37cm when compared with GPS/Levelling geoid of the same area, while the empirical geoid model had a RMSE of 6.6mm when compared with the Gravimetric Geoid Model.

SUMMARY
Gravimetric geoid modelling is a well known method of regional geoid computation that has been utilised in many countries. Conventionally, the gravimetric geoid is computed by evaluation of the stokes integral in the Remove-Restore-Compute (RRC) procedure. Owing to the computational rigours and limitation involved in evaluating the stokes integral, several modifications and techniques have been developed some of which include the modified stokes kernel and Fast Fourier Transform (FFT) methods. In this paper, the direct integration of the stokes integral has been used to compute a regional geoid model for Lagos state using the RRC technique.

Based on the result obtained from the gravimetric geoid model, the ordinary least squares was thereafter used to develop an empirical geoid model of the study area.

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1.0 INTRODUCTION:

The geoid is an equi-potential surface of the earth's gravity field which best fits in the least squares sense, the mean sea level (Deakin, 1996). It may also be conceptualised as a surface which would coincide with that surface to which the oceans would conform over the entire earth, if free to adjust to the combined effect of the earth's mass attraction (gravitation) and centrifugal forces of the earth's rotation (Olaleye et al, 2013)

Being an equipotential surface means that gravitational potential all over its surface is equal thus to efficiently model it (i.e. geoid) require accurate and precise measurement of several quantities which include but are not limited to: astronomic co-ordinates, deflections of the verticals, gravity field measurement, azimuths, tidal observation e.t.c

Of these field quantities, the most important requirement that single-handedly describes the spatial-variability of the earth's shape both on land and sea with respect to its density (invariably mass deficit or surplus) is the gravity measurement.

Gravimetric methods of geoid determination models have been published for many regions based on the Astro-gravimetric, Remove-Restore-Compute (RRC), Helmert's Scheme and ellipsoidal Bruns formula approach; many of which the comparison of their results with GPS/Levelling data yield some acceptable measure of accuracy to centimetre level (Featherstone et al, 2003; Featherstone, 2000)

Furthermore, recent advancements in Satellite Geodesy has provided the possibility of deriving gravity related parameters from satellites (Gravity Space Missions) like the CHAMP and GRACE which provide Global Geopotential Models (GGM) (Kirby et al. 1998). Although, the error estimates for GGMs are too optimistic and are presented as global averages and so are not necessarily representative of the performance of the GGM in a particular region (Kirby et al. 1998), the resulting Global Geopotential models provide the long-wavelength part of the geoid which simply could be integrated with the gravimetric geoid computation from gravity anormaly values (short wave-length part of the geoid) in the RRC technique.

Nowadays, with the ability of GNSS systems to provide ellipsoidal heights of earth surface points, the determination and availability of a high-resolution and accurate regional geoid model is increasingly becoming a necessity in several geosciences and geodetic applications.

Consequently, with the continuous rise in MSL as a consequence of global warming, all urban development and engineering activities must be properly and accurately controlled altimetrically by referencing them to the Mean Sea Level (MSL) and not the ellipsoid or other arbitrary datum if appropriate measures are to be taken to forestall the possibility of mass inundation in the nearest future especially in flood prone cities like Lagos State. However, spirit levelling which is the

dominant technique for providing elevation above MSL is labour intensive and prone to human errors over long distances thus **precise geoid definition** is the **only solution** so as to efficiently convert ellipsoidal height obtained via GPS to their orthometric equivalent.
This paper thus provides an empirical geoid model for Lagos State based on gravimetric method of geoid computation.

2.0 STUDY AREA:

Lagos State has a land area of about 3500 sq. Km and is a low-lying coastal state that is bounded in the South by the Atlantic Ocean and the Lagoon. Several other tributaries from the Lagoon extend into the state some of which include the five cowries, the Iddo Port, Apapa port amongs others.

Though, it is the state where the Apapa Datum (which serves as the height reference for the Nation) is, its large longitudinal extent makes spirit leveling across the state a difficult task.

Besides, its proximity to the Atlantic Ocean and accessibility of the rain-bearing winds makes it experience several cases of flooding especially during the rainy season between May and October.

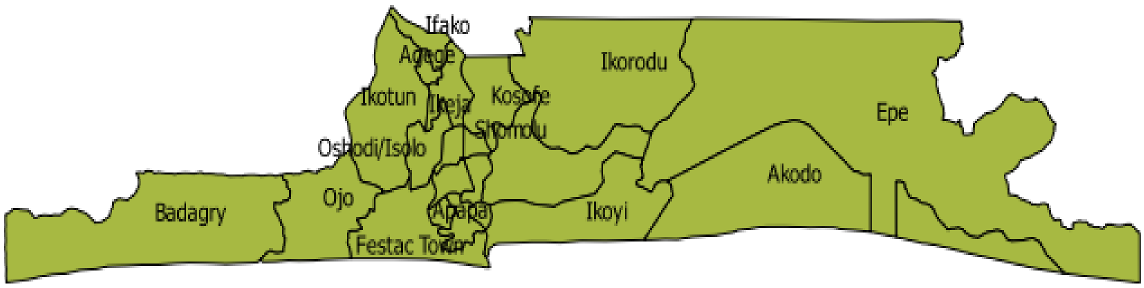

Figure 1: Administrative Map of Lagos State.

3.0 METHODOLOGY:

The paper aims to develop an empirical geoid model for Lagos State based on the gravimetric approach. This will enhance easy transformation of GNSS measured ellipsoidal heights into their Orthometric equivalent with minimal computational rigour.

The Stokes integral is a basic mathematical formulation upon which gravimetric geoid modelling is based. Point gravity anormaly at regular block intervals are obtained throughout the study area and evaluated using the Direct Stokes computation method. However, because the longer wavelengths are more prominent at the geoid, computation of geoidal undulation is best achieved in the RRC technique where the gravity anormaly observed on the earth surface (short wavelength) are combined with the Earth's Gravitational Model Derived gravity Anormaly (Long and Medium Wavelength)

Mathematically:

$$N_{GGM} = \frac{GM}{\gamma} \sum_{n=0}^{\infty} [\frac{a}{r}]^n \sum_{m=2}^{n} [S_{nm} \sin m\lambda + C_{nm} \cos m\lambda] P_{nm} \cos\theta \qquad \text{Equ. 1}$$

$$\Delta g_{ggm} = \frac{GM1}{R^2} \sum_{n=2}^{m} \left(\left(\frac{a}{r}\right)^n (n-1) \right) \sum_{m=0}^{n} (C_{nm} Cos\ (m\lambda) + S_{nm} \sin(m\lambda)) Pnm(Cos\theta) \qquad \text{Equ. 2}$$

Where:

N_{GGM} = EGM Derived Geoidal Undulation

GM = Gravity – Mass constant of GEM 2008

γ = Normal Gravity

a = Equitorial Scale Factor of GEM 2008

r, θ, λ = geocentric radius, spherical co-latitude and longitude of computation Point.

P_{nm} = fully – normalized legendre function

C_{nm}, S_{nm} = Fully – Normalised co-efficients of GEM 2008 to degree and order 2159.

Somiglaniana in 1929 developed a rigorous formula for normal gravity (Heskanem& Moritz (1967)

$$\gamma = \frac{a\gamma_a \ cos^2 \phi \ + \ b\gamma_b \ sin^2 \phi}{a^2 cos^2 \phi \ + \ b^2 sin^2 \phi} \qquad \text{Equ. 3(a)}$$

However, the International gravity formulae of 1980 given in Equ 3(b) below was used:

$$\gamma = 978032.67715(1 + 0.0053024 sin^2 \phi - 0.0000058 sin^2 \phi) mgal \qquad \text{Equ. 3(b)}$$

Where

γ_a = Theoretical Gravity at the Equator

γ_b = Theoretical Gravity at the Poles

ϕ = Geodetic Latitude

Equations 1 and 2 models the Long wavelength part of the geoid and long wavlenght contribution to the gravity anormaly respectively. The fully normalised legendre polynomials and the other EGM based parameters are obtained from http://earth-info.nga.mil/GandG/wgs84/gravitymod/egm2008/.

Thereafter, the measured terrain gravity is then reduced to the geoid. The various corrections applied in the reduction process are given by (Torge, 1980, Telford et al, 1990 and Regnolds, 1998):

1. Instrumental Drift Correction (dgD)
2. Earth tide correction ($dgET$)
3. Eotrors correction (when taken on a moving platform)
4. Free Air correction: 0.3086h meters (dgf)
5. Bouguer correction: 0.1119h meters (dgB)
6. Terrain correction: ($dgTc$)
7. Local latitude correction: 8.108 sin $2\theta.$ (dgL)
8. Isostatic correction (dgl)

Therefore, (Idowu, 2011) provides an all encompassing model for gravity reduction.
Where g_r= corrected gravity
g_{obs}= observed gravity
$g_r = g_{obs} - dg_D \pm dg_{ET} \pm dg_{EC} + dg_F - dg_B + dg_L \pm dgl$ Equ. 4
For this research, the Bouguer and Free-Air correction were applied to smoothen the gravity dataas shown below:

$$g_B = g - 0.1119H + 0.3086H \quad \text{Equ. 5(a)}$$

$$g_B = g + 0.1967H \quad \text{Equ. 5(b)}$$

Using the somiglanian formulae for normal gravity computation then gravity anormaly was computed for each observation point within the study area.

$$\Delta g = g_B - \gamma \quad \text{Equ. 6(a)}$$

$$\Delta g_{res} = \Delta g - \Delta g_{GGM} \quad \text{Equ. 6(b)}$$

Evaluating the stokes integral:

$$N_{Remove} = \frac{R}{4\pi G} \iint \Delta g_{res}\, s(\psi) d\sigma \quad \text{Equ. 7}$$

$$N_{Restore} = N_{GGM} + N_{Remove} \quad \text{Equ. 8}$$

Where $s(\psi) = \frac{1}{\sin\left(\frac{\psi}{2}\right)} - 6\sin\left(\frac{\psi}{2}\right) + 1 - 5\cos(\psi) - 3\cos(\psi)\ In(\sin\left(\frac{\psi}{2}\right) + sin^2(\frac{\psi}{2}))$

3.1 EVALUATING THE STOKES INTEGRAL:

In this research, the direct integral method of computation of the sto

kes integral was used instead of the Least Squares Collocation or Fast Fourier approach. The direct computation method is as illustrated in the next few sections:

$$N_{Remove}=\frac{R}{4\pi\gamma}\Delta g_{res}\iint_{\sigma}\frac{1}{\sin\left(\frac{\psi}{2}\right)}-6\sin\left(\frac{\psi}{2}\right)+\ 1-5\cos(\psi)-\ 3\cos(\psi)\ \ In(\sin\left(\frac{\psi}{2}\right)+\ sin^2(\frac{\psi}{2}))\ d\sigma \quad \text{Equ8.1}$$

Note that:

$$\iint_{\sigma}s(\psi)\ d_{\sigma}=\ \iint_{\propto,\psi}\sin(\psi)\mathrm{d}_{\psi}\ \mathrm{d}_{\alpha}$$

$$for\ 0\leq\psi\leq\pi\ ;0\leq\ \alpha\leq 2\pi$$

Therefore equation 3.7(c) becomes:

$$N_{Remove}=\frac{R}{4\pi\gamma}\Delta g\iint_{\alpha=0,\psi=0}^{2\pi,\pi}\frac{1}{\sin\left(\frac{\psi}{2}\right)}-6\sin\left(\frac{\psi}{2}\right)+\ 1-5\cos(\psi)-\ 3\cos(\psi)\ \ In(\sin\left(\frac{\psi}{2}\right)+\ sin^2(\frac{\psi}{2}))\ d\alpha d\psi \text{Equ 8.2}$$

Where:

$(\psi)=\ cos^{-1}([\sin(x)\sin(x_0)]+[\cos(\mathrm{x})\cos(x_0)]*\{\cos[(\mathrm{y}_0)-(\mathrm{y})]\})$ Inversine Rule

$x_o\ ,y_o\ =$ Dummy Point.

$x_p\ ,y_p\ =$ Computation Point

R = 6388137m = 57296 rad

But the first integral is:

$$\int_{\alpha=0}^{2\pi}d_{\alpha}=2\pi$$

Let:

$$c=\frac{R\Delta g_{res}}{4\pi\gamma}$$

Therefore, the solution to the first integral

$$c=\frac{R\Delta g_{res}}{2\gamma} \quad \text{Equ8.3}$$

Equations 8.4 – 8.6 below are as given by Featherstone and Oliver (1997):

$$N_{Remove}=c\int_0^{\psi}\left[2\cos\left(\frac{\psi}{2}\right)-12sin^2\left(\frac{\psi}{2}\right)\cos\left(\frac{\psi}{2}\right)+\sin\psi-\frac{5}{2}\sin 2\psi+\frac{3}{2}\sin 2\psi|_n\left\{\sin\left(\frac{\psi}{2}\right)+sin^2\left(\frac{\psi}{2}\right)\right\}\right]d\psi \text{Equ8.4}$$

Equation 8.4 above is divided into 5 definite integrals.

$$N_1.=c\int_0^{\psi}[2\cos\left(\frac{\psi}{2}\right)d\psi=[4c\ \left(\frac{\psi}{2}\right)\Big]_0^{\psi} \quad \text{Equ 8.5(a)}$$

$N_{2.} = c\int_0^{\psi}[-12sin^2\left(\frac{\psi}{2}\right)\cos\left(\frac{\psi}{2}\right)d\psi = \left[-8c\ sin^3\left(\frac{\psi}{2}\right)\right]_0^{\psi}$ Equ8.5(b)

$N_{3.} = c\int_0^{\psi}\sin\psi d\psi = [-\cos 4(\psi)]_0^{\psi}$ Equ8.5(c)

$N_{4.} = c\int_0^{\psi} - {}^5/_2 \sin 2(\psi)d\psi = \left[\frac{5c}{4}\cos 2(\psi)\right]_0^{\psi}$ Equ8.5(d)

$N_{5.} = c\int_0^{\psi}[-{}^3/_2\sin(\psi)\ ln\left\{\sin\left(\frac{\psi}{2}\right) + sin^2\left(\frac{\psi}{2}\right)\right\}]\ d\psi.$ Equ8.5(e)

$= c\left[-\frac{3}{2}sin^2\psi|n\left\{\sin\left(\frac{\psi}{2}\right) + sin^2\left(\frac{\psi}{2}\right)\right\} - 3sin^4\left(\frac{\psi}{2}\right) + 2sin^3\left(\frac{\psi}{2}\right) + 3sin^2\left(\frac{\psi}{2}\right)\right]_0^{\psi}$ Equ 8.5(e)

$N_{Remove} = \sum_{i=1}^{5} N_i$

$\boldsymbol{N_{Remove} = c\ [-\frac{3}{2}sin^2\psi\left|n\left\{\sin\left(\frac{\psi}{2}\right) + sin^2\left(\frac{\psi}{2}\right)\right\} + 4\sin\left(\frac{\psi}{2}\right) + 3sin^2\left(\frac{\psi}{2}\right) - 6sin^3\left(\frac{\psi}{2}\right) - 3sin^4\left(\frac{\psi}{2}\right) - \cos\psi + \frac{5}{4}\cos(\psi)\right]_0^{\psi}}$ **Equ8.6**

Equation 8.6 was evaluated using matLab functions named SphericalDistance2.m to compute the inversine or spherical distance from the earth's centre to the computation point while the second is the named stokes.m to finally evaluate the stokes integral.

3.2 DEVELOPMENT OF EMPIRICAL MODEL FOR THE "LAGOS – STATE GRAVIMETRIC GEOID" USING FIRST DEGREE ALGEBRAIC EQUATION.

A series of least squares observation equations were formulated as illustrated in equation 9 to fit in the obtained gravimetric geoid into an algebraic equation:

$N = (h - H) = N_0 + N_1 e + N_2 n + \delta_N$ Equ. 9

The solution to Equ. 9 can be further mathematically expressed as:

$(h - H) = N_0 + N_1 e + N_2 n$

$$(h - H) = (1 \quad e \quad n) \cdot \begin{matrix} N_0 \\ N_1 \\ N_2 \end{matrix}$$

Now, X = the required parameters = $\begin{matrix} N_0 \\ N_1 \\ N_2 \end{matrix}$

A matlab Code named ModelEquation.m was used to run this and the results yielded as follows:

$N_0 = 28.08584$
$N_1 = -0.00000842$
$N_2 = 0.0000000000639972$

Therefore, the empirical solution model Equation yields:

$$(h - H) = 28.08584 - 0.00000842e + 0.0000000000639972n \qquad \text{Equ. 10(a)}$$

Re-arranging Equ. 22(a)

$$\mathrm{H} = -28.08584 + 0.00000842e - 0.0000000000639972n + h \qquad \text{Equ. 10(b)}$$

4.0 DATA USED:

A summary of all the data used is as shown in the table 1:

Table 1: Showing Summary of the Source of Data Used.

S/N	DATA	SOURCE	ACCURACY
1.	Ellipsoidal Height	Office of the Surveyor General of Lagos State – Interspatial Surveys. (2nd Order State-wide Controls Network)	2nd – Order Accuracy
2.	Orthometric Heights	Office of the Surveyor General of Lagos State – Interspatial Surveys. (2nd Order State-wide Controls Network)	2nd – Order Accuracy
3.	Terrain-Gravity Anomaly	Bureau Gravimetric Internationale (BGI)	Archived Data. Accuracy not given.
4.	Satellite-Gravity Anomaly	http://earth-info.nga.mil/GandG/wgs84/gravitymod/egm2008/	

5.0 ANALYSIS OF RESULTS:

5.1COMPARISON OF GRAVIMETRIC GEOID WITH GPS/LEVELLING DATA:

The result obtained from gravimetric geoid computation was validated using the GPS/Levelling data. Table 2 and Fig 2 below provides an extract of the obtained result:

Table 2: Showing Comparison between the gravimetric geoid and the GPS/Levelling geoid.

CONTROL NO	N_GPS	N_GRAV	Diff.
ZTT14-1A	23.086	23.143	0.057
ZTT30-04A	23.149	23.272	0.123
ZTT2-57A	22.274	22.288	0.014
ZTT30-08A	23.109	23.202	0.093
ZTT30-09A	23.096	23.181	0.085
ZTT30-1	23.181	23.328	0.147
ZTT30-10	23.088	23.167	0.079
ZTT30-11A	23.081	23.154	0.073
ZTT30-13A	23.063	23.119	0.056
ZTT30-14A	23.05	23.095	0.045
ZTT30-15A	23.04	23.076	0.036

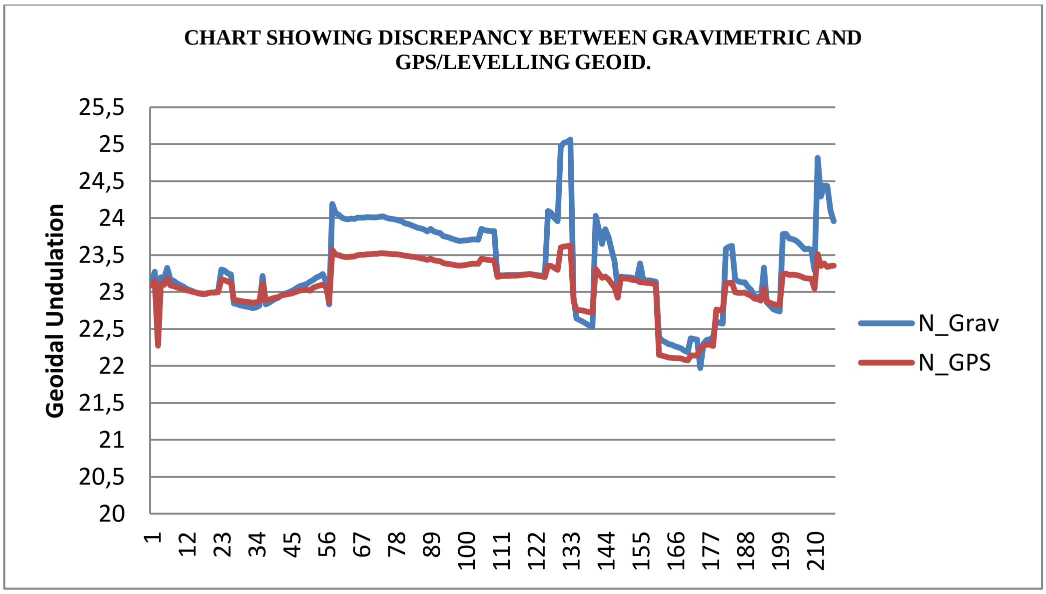

Figure 2: Discrepancy between Gravimetric and GPS/Levelling Geoid.

The Root Mean Square Error (RMSE) of the discrepancy 2.37cm.

5.1 COMPARISON OF GRAVIMETRIC GEOID WITH EMPIRICAL - MODEL:

The empirical model was also tested and the result obtained compared with the gravimetric geoid. An extract of the results and the residuals is as shown in table 3.

Table 3: Showing Comparison between the gravimetric geoid and the empirical geoid.

CONTROL NAME	N_Algebraic	N_Grav	Diff
ZTT14-1A	23.006	23.1431	0.137
ZTT30-04A	23.328	23.272	-0.056
ZTT2-57A	23.151	22.2878	-0.864
ZTT30-08A	23.328	23.2016	-0.127
ZTT30-09A	23.329	23.1811	-0.148
ZTT30-1	23.331	23.328	-0.003
ZTT30-10	23.329	23.1675	-0.162
ZTT30-11A	23.330	23.1537	-0.176
ZTT30-13A	23.328	23.1191	-0.209
ZTT30-14A	23.327	23.0954	-0.231
ZTT30-15A	23.326	23.0761	-0.250

The Root Mean Square Error (RMSE) of the discrepancy is 6.6mm.

6.0 GRAPHICAL PLOTS OF THE LAGOS STATE GEOID

The results obtained were plotted to show the spatial variability of the Lagos State geoid across the study area. The plots are as shown below:

6.1 GPS / LEVELLING GEOID PLOT:

Figures 3 – 5 show a graphical plot of the GPS/Levelling Geoid, gravimetric geoid and empirical geoid model respectively. While there seems to a major discrepancy between the GPS/Levelling and the gravimetric geoid, the empirical model appears more consistent with the GPS/Levelling data. Thus the empirical model parameters are essential for smoothing the gravimetric geoid.

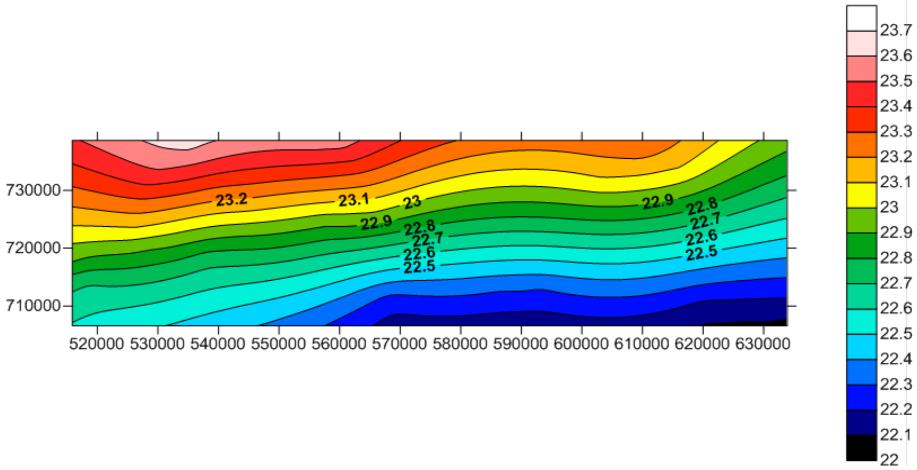

Figure 3: Local Geoid of Lagos State plotted from GPS/Levelling Geoid

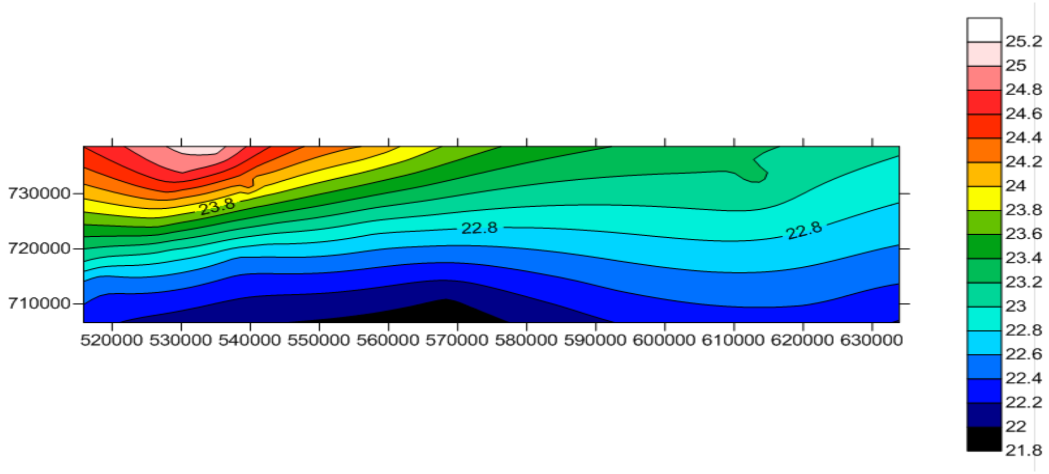

Figure 4: Local Geoid of Lagos State plotted from Gravimetric Method

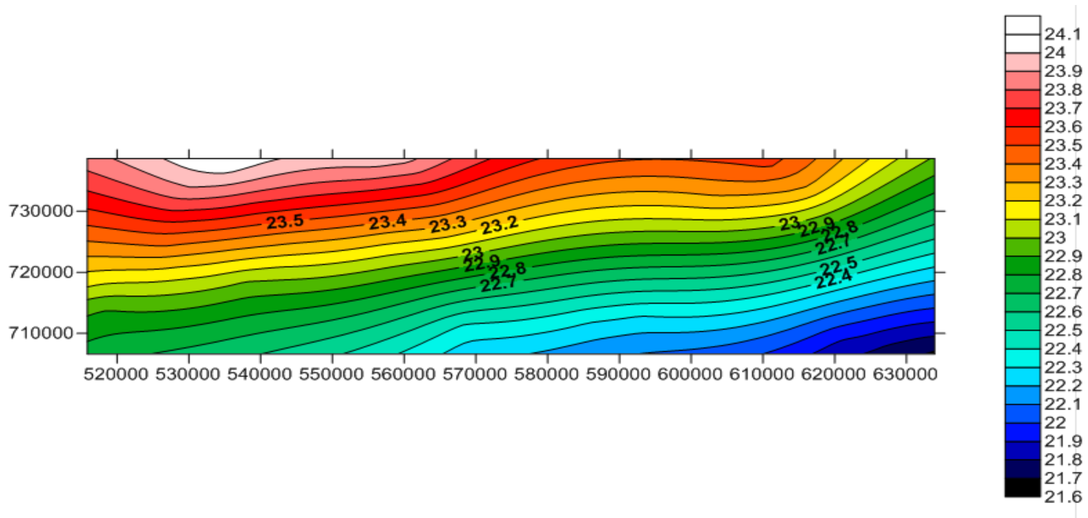

Figure 5: Local Geoid of Lagos State plotted from Empirical Model.

7.0 CONCLUSION:

The stokes integral has proven to be a veritable tool for gravimetric geoid computation of Lagos State. The remove-restore-compute technique was utilized in the solution algorithm after which empirical solutions were proposed for the study area. The empirical solution has thus further improved the accuracy of the gravimetric geoid. It thus can be concluded that:

1. This empirical geoid model gives the numerous GPS users across Lagos State the privilege of directly transforming ellipsoidal heights obtained with the GPS into their Orthometric Equivalent.
2. This method is relatively cheap and inexpensive to use.
3. It completely eliminates the stress of spirit levelling.

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