BLOCK IMPLICIT HYBRID LINEAR MULTISTEP METHODS FOR SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

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Abstract
The idea of multistep collocation is employed to reformulate the Hybrid Backward Differentiation Formula (HBDF) for cases k = 4 and 6 into continuous forms which were evaluated at some interpolation and collocation points. This gives rise to discrete schemes that are combined to form the efficient block method for the solution of Ordinary Differential Equations. The requirement of starting values and the overlap of solution model which are associated with conventional Linear Multistep Methods are eliminated by this approach. A convergence analysis of the derived hybrid schemes is presented to establish their effectiveness and reliability. Numerical examples carried out further substantiates their convergence.

Keywords: Linear Multistep method (LMM), Backward Differentiation Formula (BDF), Block Solution, Implicit, Hybrid

Introduction
Consider the Initial Value Problem of the form

$$y' = f(x, y) \qquad y(x_0) = y_0 \qquad (1)$$

Where the solution y is assumed to be a differentiable function on an interval $[x_0, b], b < \infty$. Many methods for solving (1) exist, one particular me thod is the Linear Multistep Method. The Method require less evaluation of the derivative function f than one step methods in the range of integral $[x_0, b]$. For this reasons they have been very popular and important for solving (1) num erically. But these methods have certain limitations such as the overlap of solution models and the requirements of a starting value. Other limitations include they yield the discretely solution values $y_1, ..., y_N$ hence uneconomical for producing dense o utput. A continuous formulation is desirable in this respect. The collocation method is probably the most important numerical procedure for the construction of continuous methods (Lie and Norsett, 1989; Awoyemi, Jator and sirisena, 1994; Onumanyi, sirisena and jator, 1999).

In this research paper, an attempt is made to reformulate the hybrid form of some BDF schemes, specifically $(k = 4 \, and \, 6)$ into continuous forms by the idea of multistep collocation.

The Continous Multistep Collocation (CMM) Method
Lambert (1973; 1991) adopted the continous finite difference (CFD) approximation approach by the idea of interpolation and collocation. Later, Lie and Norsett (1989), Onumanyi (1994; 1999) referred to it as Multistep Collocation (MC). The method is presented below

$$\underline{a} = (a_0, a_1, \dots\dots, a_{(t+m-1)})^T, \varphi(x) = (\varphi_0(x), \varphi_1(x), \dots\dots, \varphi_{(t+m-1)})^T \qquad (2)$$

where $a_r, r = 0, \ldots\ldots, t+m-1$ are undetermined constants, $\varphi_r(x)$ are specified basis functions, T denotes transpose of, t denotes the number of interpolation points and m denotes the number of distinct collocation points. We consid er a continuous approximation (interpolant) Y(x) to y(x) in the form

$$y(x) = \sum_{r=0}^{t+m-1} a_r\varphi_r(x) = \underline{a}^T\varphi(x) \qquad (3)$$

which is valid in the sub -intervals $x_n \le x \le x_{n+k}$, where n = 0,k......,N − k. The quantities $x_0 = a, x_N = b, k, m, n, t$ $and\ \varphi_r(x), r = 0,1\ldots, t+m-1$ are specified values. The constant co -efficient a_r of (3) can be determined using the conditions

$$y(x_{n+j}) = y_{n+j},\ j = 0,1,\ldots, t-1 \qquad (4)$$

$$y'(\bar{x}_j) = f_{n+j}\ \ j = 0,1,\ldots, m-1 \qquad (5)$$

Where

$$f_{n+j} = f(x_{n+j}, y_{n+j}) \qquad (6)$$

The distinct collocation points $x_0, \ldots\ldots\ldots x_{m-1}$, can be chosen freely from the set $[x_n, x_{n+k}]$. Equation (4),(5) and (6) are denoted by a single set of algebraic equations of the form

$$D\underline{a} = \underline{F} \qquad (7)$$

Where $\underline{F} = (y_n, y_{n+1}\ldots\ldots y_{n+t-1}, f_n, f_{n+1}, f_{n+m-1})^T$ (8)

$$\underline{a} = D^{-1}\underline{F} \qquad (9)$$

where $\underline{\underline{D}}$ is the non-singular matrix of dimension $(t+m)$ below

$$\underline{D} = \begin{pmatrix} \varphi_0(x_n) & \cdots & \varphi_{t+m-1}(x_n) \\ \vdots & \vdots & \vdots \\ \varphi_0(x_{n+t-1}) & \cdots & \varphi_{t+m-1}(x_{n+t-1}) \\ \varphi_0'(\bar{x}_0) & \cdots & \varphi_{t+m-1}'(\bar{x}_0) \\ \vdots & \vdots & \vdots \\ \varphi_0'(\bar{x}_{m-1}) & \cdots & \varphi_{t+m-1}'(\bar{x}_{m-1}) \end{pmatrix} \qquad (10)$$

By substituting (9) int o (3), we obtain the MC formula

$$y(x) = F^T \underline{\underline{C}}^T \varphi(x), \quad x_n \le x \le x_{n+k}\ \ n = 0, k, \ldots\ldots, N-k \qquad (11)$$

where

$$\underline{\underline{C}} \equiv \underline{\underline{D}}^{-1} = (c_{ij}), i,j = 1, \ldots, t+m-1$$

$$\underline{C} = \begin{pmatrix} c_{11} & \cdots & c_{1t} & c_{1,t+1} & \cdots & c_{1,t+m} \\ c_{21} & \cdots & c_{2t} & c_{2,t+1} & \cdots & c_{2,t+m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{t+m1} & \cdots & c_{t+m,t} & c_{t+m,t+1} & \cdots & c_{t+m,t+m} \end{pmatrix}$$

with the numerical elements denoted by $c_{ij}, i,j = 1, \ldots, k+m$. By expanding $\underline{C}^T\varphi(x)$ in (11) yields the following

$$y(x) = (F)^T \begin{pmatrix} \sum_{r=0}^{t+m-1} C_{r+1.1}\varphi_r(x) \\ \vdots \\ \sum_{r=0}^{t+m-1} C_{r+1.k+m}\varphi_r(x) \end{pmatrix} \qquad (12)$$

$$y(x) = \sum_{j=0}^{t-1}(\sum_{r=0}^{t+m-1} C_{r+1,j+1}\varphi_r(x) \quad + \sum_{j=0}^{m-1} h\left(\sum_{r=0}^{k+m-1}(\frac{C_{r+1,j+1}}{h}\varphi_r(x)\right) f_{n+j} \quad (13)$$

$$y(x) = \sum_{j=0}^{t-1} \alpha_j(x) y_{n+j} + h\sum_{j=0}^{m-1} \beta_j(x) f_{n+j} \qquad (14)$$

where we construct $\alpha_j(x) and\, \beta_j(x)$ explicitly by

$$\alpha_j(x) = \sum_{r=0}^{t+m-1} C_{r+1,j+1}\varphi_r(x)\ j = 0,1 \dots\dots t-1 \qquad (15)$$

$$\beta_j(x) = \sum_{r=0}^{k+m-1}(\frac{C_{r+1,j+1}}{h}\varphi_r(x), j = 0,1 \dots \dots, m-1 \qquad (16)$$

a_r can be determined as follows:

$$y(x) = \left\{\sum_{r=0}^{t-1} \alpha_{j,r+1} y_{n+j} + h\sum_{j=0}^{m-1} \beta_{j,r+1} f_{n+j}\right\}\varphi_r(x) \qquad (17)$$

For $K = 4$, the general form of the method upon addition of one off grid point is expressed as;

$$(x) = {}_1(x)y_n + \alpha_2(x)y_{n+1} + \alpha_3(x)y_{n+2} + \alpha_4(x)y_{n+3} + \alpha_5(x)y_{n+\frac{1}{2}} + \beta_0(x)f_{n+4}$$

(18)

The matrix D of the proposed method is expressed as:

$$\begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 \\ 1 & x_n+h & (x_n+h)^2 & (x_n+h)^3 & (x_n+h)^4 & (x_n+h)^5 \\ 1 & x_n+2h & (x_n+2h)^2 & (x_n+2h)^3 & (x_n+2h)^4 & (x_n+2h)^5 \\ 1 & x_n+3h & (x_n+3h)^2 & (x_n+3h)^3 & (x_n+3h)^4 & (x_n+3h)^5 \\ 1 & x_n+\frac{1}{2}h & (x_n+\frac{1}{2}h)^2 & (x_n+\frac{1}{2}h)^3 & (x_n+\frac{1}{2}h)^4 & (x_n+\frac{1}{2}h)^5 \\ 0 & 1 & 2x_n+8h & 3(x_n+4h)^2 & 4(x_n+4h)^3 & 5(x_n+4h)^4 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \beta_0 \end{bmatrix} = \begin{bmatrix} y_n \\ y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+1/2} \\ f_{n+4} \end{bmatrix} \qquad (19)$$

i:e $Da = F$

The matrix D in equation (19) which when solved by matrix inversion technique or Gaussian Elimination method will yield the continuous coefficients substituted in (18) to obtain continous form of the four step block hybrid BDF with one off step interpolation point.

$$\Big(\frac{1}{2388}\frac{1}{h^5}(2388h^5 + 9688h^4 + 13191x_n^2h^3 + 7666x_n^3h^2 + 1953x_n^4h + 178x_n^5) - \frac{1}{1194}\frac{1}{h^5}(4844h^4 + 13191x_nh^3 + 11499x_n^2h^2 + 3906x_n^3h + 445x_n^4)x + \frac{1}{2388}\frac{13191h^3+22998x_nh^2+11718x_n^2h+1780x_n^3}{h^5}x^2 - \frac{1}{1194}\frac{3833h^2+3906x_nh+890x_n^2}{h^5}x^3 + \frac{1}{2388}\frac{1953h+890x_n}{h^5}x^4 - \frac{89}{1194h^5}x^5\Big)y_n + \Big(\frac{1}{398}\frac{1}{h^5}(x_n(1534h^4 + 4694x_nh^3 + 3785x_n^2h^2 + 1139x_n^3h + 114x_n^4)) - \frac{1}{398}\frac{1}{h^5}(x_n(1536h^4 + 4694x_nh^3 + 3785x_n^2h^2 + 1139x_n^3h + 114x_n^4))x + \frac{1}{398}\frac{4694h^3+11355x_nh^2+6834x_n^2h+1140x_n^3}{h^5}x^2 - \frac{1}{398}\frac{3785h^2+4556x_nh+1140x_n^2}{h^5}x^3 + \frac{1}{398}\frac{1139h+570x_n}{h^5}x^4 - \frac{57}{199h^5}x^5\Big)y_{n+1} + \Big(-\frac{1}{2388}\frac{1}{h^5}(x_n(7591x_nh^3 + 7978x_n^2h^2 + 2837x_n^3h + 314x_n^4 + 2136h^4)) + \frac{1}{1194}\frac{1}{h^5}(7591x_nh^3 + 11967x_n^2h^2 + 5674x_n^3h + 785x_n^4 + 1068h^4)x - \frac{1}{2388}\frac{7591h^3+23934x_nh^2+17022x_n^2h+3140x_n^3}{h^5}x^2 + \frac{1}{1194}\frac{3989h^2+5674x_nh+1570x_n^2}{h^5}x^3 - \frac{1}{2388}\frac{2837h+1570x_n}{h^5}x^4 + \frac{157}{1194h^5}x^5\Big)y_{n+2} + \Big(\frac{1}{5970}\frac{1}{h^5}(x_n(4613x_n^2h^2 + 1893x_n^3h + 230x_n^4 + 4038x_nh^3 + 1088h^4)) - \frac{1}{5970}\frac{1}{h^5}(13839x_n^2h^2 + 7572x_n^3h + 1150x_n^4 + 785x_n^4 + 8076x_nh^3 + 1088h^4)x + \frac{1}{5970}\frac{13839x_nh^2+11358x_n^2h+2300x_n^3+4038h^3}{h^5}x^2 - \frac{1}{5970}\frac{4613h^2+7572x_nh+2300x_n^2}{h^5}x^3 +$$

$$\frac{1}{5970}\frac{1893h+1150x_n}{h^5}x^4+-\frac{23}{597h^5}x^5\Big)y_{n+3}+\Big(-\frac{32}{2985}\frac{1}{h^5}\big(x_n(25x_n^4+262x_n^3h+947x_n^2h^2+1382x_nh^3+672h^4)\big)+\frac{32}{2985}\frac{1}{h^5}(125x_n^4+1048x_n^3h+2841x_n^2h^2+2764x_nh^3+672h^4)x-\frac{32}{2985}\frac{250x_n^3+1572x_n^2h+2841x_nh^2+1382h^3}{h^5}x^2+\frac{32}{2985}\frac{250x_n^2+1048x_nh+947h^2}{h^5}x^3-\frac{32}{2985}\frac{125x_n+262h}{h^5}x^4+\frac{160}{597h^5}x^5\Big)y_{n+1/2}+$$
$$\Big(-\frac{1}{398}\frac{x_n(2x_n^4+13x_n^3h+28x_n^2h^2+23x_nh^3+6h^4)}{h^4}+\frac{1}{199}\frac{5x_n^4+26x_n^3h+42x_n^2h^2+23x_nh^3+3h^4}{h^4}x-\frac{1}{398}\frac{20x_n^3+78x_n^2h+84x_nh^2+23h^3}{h^4}x^2+\frac{2}{199}\frac{5x_n^2+13x_nh+7h^2}{h^4}x^3-\frac{1}{398}\frac{10x_n+13h}{h^4}x^4+\frac{1}{199h^4}x^5\Big)f_{n+4}\Big\}$$

(20)

Evaluating (20) at points $x=x_{n+4}$ and its derivative at $x=x_{n+3}, x=x_{n+1}, , x=x_{n+2}, x=x_{n+1/2}$ yields the followin g five discrete hybrid schemes which are used as a block integrator;

$$-\frac{784}{199}y_{n+1}+\frac{588}{199}y_{n+2}-\frac{2352}{995}y_{n+3}+y_{n+4}+\frac{3072}{995}y_{n+1/2}=\frac{147}{199}y_n+\frac{84}{199}hf_{n+4}$$
$$-y_{n+1}-\frac{925}{2415}y_{n+2}+\frac{143}{2415}y_{n+3}+\frac{3712}{2415}y_{n+1/2}=\frac{515}{2415}y_n-\frac{1990}{2415}hf_{n+1}+\frac{10}{2415}hf_{n+4}$$
$$\frac{6390}{2305}y_{n+1}-y_{n+2}-\frac{942}{2305}y_{n+3}-\frac{3968}{2305}y_{n+1/2}=-\frac{825}{2305}y_n-\frac{2985}{2305}hf_{n+2}-\frac{45}{2305}hf_{n+4}$$
$$-\frac{19125}{9883}y_{n+1}+\frac{18075}{9883}y_{n+2}-y_{n+3}+\frac{14208}{9883}y_{n+1/2}=\frac{3275}{9883}y_n-\frac{5970}{9883}hf_{n+3}+\frac{450}{9883}hf_{n+4}$$

$$-\frac{204750}{125888}y_{n+1}+\frac{31675}{125888}y_{n+2}-\frac{5838}{125888}y_{n+3}+y_{n+1/2}=-\frac{53025}{125888}y_n-\frac{95520}{125888}hf_{n+1/2}-\frac{450}{125888}hf_{n+4} \quad (21)$$

Equation (21) constitute the members of a zero -stable block integrators of order $(5,5,5,5,5)^T$ with $C_6=\left[-\frac{49}{995},-\frac{227}{47760},\frac{241}{23880},-\frac{283}{9552},\frac{1561}{305664}\right]^T$ as the error constants respectively. To start the integration process with n=0, we use (21) and this produces $y_1, y_{1/2}, y_2, y_3$ and y_4 simultaneously without the need of any starting method (predictor).

For $K=6$, the general form of the method upon addition of one off grid point is expressed as;

$$\bar{y}(x)=\alpha_1(x)y_n+\alpha_2(x)y_{n+1}+\alpha_3(x)y_{n+2}+\alpha_4(x)y_{n+3}+\alpha_5(x)y_{n+4}+\alpha_6(x)y_{n+5}+\alpha_7(x)y_{n+\frac{1}{2}}+h\beta_0(x)f_{n+6}$$

(22)

Recall from (7), $Da=F$

The matrix D of the proposed method i s expressed as:

$$D=\begin{bmatrix}1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7\\ 1 & x_n+h & (x_n+h)^2 & (x_n+h)^3 & (x_n+h)^4 & (x_n+h)^5 & (x_n+h)^6 & (x_n+h)^7\\ 1 & x_n+2h & (x_n+2h)^2 & (x_n+2h)^3 & (x_n+2h)^4 & (x_n+2h)^5 & (x_n+2h)^6 & (x_n+2h)^7\\ 1 & x_n+3h & (x_n+3h)^2 & (x_n+3h)^3 & (x_n+3h)^4 & (x_n+3h)^5 & (x_n+3h)^6 & (x_n+3h)^7\\ 1 & x_n+4h & (x_n+4h)^2 & (x_n+4h)^3 & (x_n+4h)^4 & (x_n+4h)^5 & (x_n+4h)^6 & (x_n+4h)^7\\ 1 & x_n+5h & (x_n+5h)^2 & (x_n+5h)^3 & (x_n+5h)^4 & (x_n+5h)^5 & (x_n+5h)^6 & (x_n+5h)^7\\ 1 & x_n+\frac{1}{2}h & (x_n+\frac{1}{2}h)^2 & (x_n+\frac{1}{2}h)^3 & (x_n+\frac{1}{2}h)^4 & (x_n+\frac{1}{2}h)^5 & \left(x_n+\frac{1}{2}h\right)^6 & \left(x_n+\frac{1}{2}h\right)^7\\ 0 & 1 & 2x_n+12h & 3(x_n+6h)^2 & 4(x_n+6h)^3 & 5(x_n+6h)^4 & 6(x_n+6h)^5 & 7(x_n+6h)_6\end{bmatrix} \quad (23)$$

The matrix D in equation (23) which when solved by matrix inversion technique or Gaussian Elimination method will yield the continuous coefficients substituted in (22) to obtain continous form of the six step block hybrid BDF with one off step interpolation point.

$$\bar{y}(x) = \frac{1}{h^7}\left(\frac{1}{1250640}Py_n + \frac{1}{13896}Qy_{n+1} + \frac{1}{20844}Ry_{n+2}\right) + \frac{1}{h^7}\left(\frac{1}{312660}Sy_{n+3} + \frac{1}{194544}Ty_{n+4} + \frac{1}{625320}Uy_{n+5}\right) + \frac{1}{h^7}\left(\frac{128}{547155}Vy_{n+1/2} + \frac{1}{20844}Wf_{n+6}\right) \quad (24)$$

where

$P = (1250640h^7 + 3254x_n^7 + 71281x_n^6h + 624077x_n^5h^2 + 2798515x_n^4h^3 + 68333441x_n^3h^4 + 8892484x_n^2h^5 + 5552148x_nh^6) +$
$(427686x_n^5h + 3120385x_n^4h^2 + 1194060x_n^3h^3 + 20500323x_n^2h^4 + 17784968x_nh^5 + 5552148h^6 + 22778x_n^6)x + (8892484h^5 + 1069215x_n^4h + 6240770x_n^3h^2 + 16791090x_n^2h^3 + 20500323x_nh^4 + 68334x_n^5)x^2 - (6833441h^4 + 11194060x_nh^3 + 6240770x_n^2h^2 + 1425620x_n^3h + 113890x_n^4)x^3$
$+5(559703h^3 + 624077x_nh^2 + 213843x_n^2h + 22778x_n^3)x^4 - (624077h^2 + 427686x_nh + 68334x_n^2)x^5 + (10183h + 3254x_n)x^6 - (3254)x^7$

$Q = (4475x_n^6h + 36586x_n^5h^2 + 147569x_n^4h^3 + 302152x_n^3h^4 + 283124x_n^2h^5 + 82320x_nh^6 + 214x_n^7) - 2(13425x_n^5h + 91465x_n^4h^2 + 295138x_n^3h^3 + 453228x_n^2h^4 + 283124x_nh^5 + 41160h^6 + 749x_n^6)x$
$+\left((283124h^5 + 67125x_n^4h + 365860x_n^3h^2 + 885414x_n^2h^3 + 906456x_nh^4 + 4494x_n^5)\right)x^2$
$-2(151076h^4 + 295138x_nh^3 + 182930x_n^2h^2 + 44750x_n^3h + 3745x_n^4)x^3$
$+(147569h^3 + 182930x_nh^2 + 67125x_n^2h + 7490x_n^3)x^4$
$-2(18293h^2 + 13425x_nh + 2247x_n^2)x^5 + (4475h + 1498x_n)x^6 - (214)x^7$

$R = -(5219x_n^6h + 39868x_n^5h^2 + 145934x_n^4h^3 + 259330x_n^3h^4 + 198767x_n^2h^5 + 50460x_nh^6 + 262x_n^7) + 2(15657x_n^5h + 99670x_n^4h^2 + 291868x_n^3h^3 + 388995x_n^2h^4 + 198767x_nh^5 + 25230h^6 + 917x_n^6)x$
$-\left((198767h^5 + 78285x_n^4h + 398680x_n^3h^2 + 875604x_n^2h^3 + 777990x_nh^4 + 5502x_n^5)\right)x^2$
$+2(129665h^4 + 291868x_nh^3 + 199340x_n^2h^2 + 52190x_n^3h + 4585x_n^4)x^3$
$+(145934h^3 + 199340x_nh^2 + 78285x_n^2h + 9170x_n^3)x^4$
$+2(19934h^2 + 15657x_nh + 2751x_n^2)x^5 - (5219h + 1834x_n)x^6 + (262)x^7$

$S = (57449x_n^6h + 410920x_n^5h^2 + 1384805x_n^4h^3 + 2240896x_n^3h^4 + 1603076x_n^2h^5 + 390480x_nh^6 + 3034x_n^7) - 2(172347x_n^5h + 1027300x_n^4h^2 + 2769610x_n^3h^3 + 3361344x_n^2h^4 + 1603076x_nh^5 + 195240h^6 + 10619x_n^6)x$
$+\left((1603076h^5 + 861735x_n^4h + 4109200x_n^3h^2 + 8308830x_n^2h^3 + 6722688x_nh^4 + 63714x_n^5)\right)x^2$
$-2(1120448h^4 + 2769610x_nh^3 + 2954600x_n^2h^2 + 574490x_n^3h + 53095x_n^4)x^3$
$+5(276961h^3 + 410920x_nh^2 + 172347x_n^2h + 21238x_n^3)x^4$
$-2(205460h^2 + 172347x_nh + 31857x_n^2)x^5 + 7(8207h + 3034x_n)x^6 - (3034)x^7$

$T = -(16855x_n^6h + 113399x_n^5h^2 + 358609x_n^4h^3 + 551291x_n^3h^4 + 381184x_n^2h^5 + 91020x_nh^6 + 938x_n^7) + (101130x_n^5h + 566995x_n^4h^2 + 1434436x_n^3h^3 + 1653873x_n^2h^4 + 762368x_nh^5 + 91020h^6 + 6566x_n^6)x$

$$-\left((381184h^5 + 252825x_n^4h + 1133990x_n^3h^2 + 2151654x_n^2h^3 + 1653873x_nh^4 + 19698x_n^5)\right)x^2$$
$$+(551291h^4 + 1434436x_nh^3 + 1133990x_n^2h^2 + 337100x_n^3h + 32830x_n^4)x^3$$
$$-(358609h^3 + 566995x_nh^2 + 252825x_n^2h + 32830x_n^3)x^4$$
$$+(113399h^2 + 101130x_nh + 19698x_n^2)x^5 \quad -(16855h + 6566x_n)x^6 + (938)x^7$$

$$U = (12287x_n^6h + 78574x_n^5h^2 + 238385x_n^4h^3 + 355672x_n^3h^4 + 241268x_n^2h^5 + 56976x_nh^6 + 718x_n^7) - 2(36861x_n^5h + 196435x_n^4h^2 + 476770x_n^3h^3 + 533508x_n^2h^4 + 241268x_nh^5 + 28488h^6 + 2513x_n^6)x$$
$$+\left((241268h^5 + 184305x_n^4h + 785740x_n^3h^2 + 1430310x_n^2h^3 + 1067016x_nh^4 + 15078x_n^5)\right)x^2$$
$$-2(177836h^4 + 476770x_nh^3 + 392870x_n^2h^2 + 122870x_n^3h + 12565x_n^4)x^3$$
$$+5(47677h^3 + 78574x_nh^2 + 36861x_n^2h + 5026x_n^3)x^4 \quad -2(39287h^2 + 36861x_nh + 7539x_n^2)x^5$$
$$+(12287h + 5026x_n)x^6 - (718)x^7$$

$$V = -(1049x_n^6h + 8875x_n^5h^2 + 37715x_n^4h^3 + 84076x_n^3h^4 + 91916x_n^2h^5 + 37680x_nh^6 + 49x_n^7) + (6294x_n^5h + 44375x_n^4h^2 + 150860x_n^3h^3 + 252228x_n^2h^4 + 183832x_nh^5 + 37680h^6 + 343x_n^6)x$$
$$-\left((91916h^5 + 15735x_n^4h + 88750x_n^3h^2 + 226290x_n^2h^3 + 252228x_nh^4 + 1029x_n^5)\right)x^2$$
$$+(84076h^4 + 150860x_nh^3 + 88750x_n^2h^2 + 20980x_n^3h + 1715x_n^4)x^3$$
$$-(7543h^3 + 8875x_nh^2 + 3147x_n^2h + 343x_n^3)x^4 \quad +(8875h^2 + 6294x_nh + 1029x_n^2)x^5$$
$$-(1049h + 343x_n)x^6 + (49)x^7$$

$$W = -(31x_n^6h + 185x_n^5h^2 + 535x_n^4h^3 + 773x_n^3h^4 + 514x_n^2h^5 + 120x_nh^6 + 2x_n^7) + (186x_n^5h + 925x_n^4h^2 + 2140x_n^3h^3 + 2319x_n^2h^4 + 1028x_nh^5 + 120h^6 + 14x_n^6)x$$
$$-\left((514h^5 + 465x_n^4h + 1850x_n^3h^2 + 3210x_n^2h^3 + 2319x_nh^4 + 42x_n^5)\right)x^2$$
$$+(773h^4 + 2140x_nh^3 + 1850x_n^2h^2 + 620x_n^3h + 70x_n^4)x^3$$
$$-5(107h^3 + 185x_nh^2 + 93x_n^2h + 14x_n^3)x^4 \quad +(185h^2 + 186x_nh + 42x_n^2)x^5$$
$$-(31h + 14x_n)x^6 + (2)x^7$$

Evaluating (24) at points $x = x_{n+6}$ and its derivative at $x = x_{n+5}, x = x_{n+4}, x = x_{n+3}, x = x_{n+2}, x = x_{n+1}, x = x_{n+1/2}$ yields the following seven discrete hybrid schemes which are used as block integrator;

$$-\frac{968}{193}y_{n+1} + \frac{3025}{579}y_{n+2} - \frac{9680}{1737}y_{n+3} + \frac{6050}{1351}y_{n+4} - \frac{4840}{1737}y_{n+5} + y_{n+6} + \frac{40960}{12159}y_{n+1/2} = \frac{1210}{1737}y_n - \frac{220}{579}hf_{n+6}$$

$$-y_{n+1} - \frac{119280}{106785}y_{n+2} + \frac{46172}{106785}y_{n+3} - \frac{15390}{106785}y_{n+4} + \frac{2821}{106785}y_{n+5} + \frac{217088}{106785}y_{n+1/2} = \frac{24626}{106785}y_n - \frac{145908}{106785}hf_{n+1} + \frac{168}{106785}hf_{n+6}$$

$$\frac{74340}{6510}y_{n+1} - y_{n+2} - \frac{47096}{6510}y_{n+3} + \frac{11790}{6510}y_{n+4} - \frac{1932}{6510}y_{n+5} - \frac{36864}{6510}y_{n+1/2} = -\frac{6272}{6510}y_n - \frac{60795}{6510}hf_{n+2} - \frac{105}{6510}hf_{n+6}$$

$$-\frac{574875}{322616}y_{n+1} + \frac{940800}{322616}y_{n+2} - y_{n+3} - \frac{366075}{322616}y_{n+4} + \frac{45395}{322616}y_{n+5} + \frac{342016}{322616}y_{n+\frac{1}{2}} = \frac{64645}{322616}y_n - \frac{729540}{322616}hf_{n+3} + \frac{2100}{322616}hf_{n+6}$$

$$\frac{758520}{703305}y_{n+1} - \frac{931980}{703305}y_{n+2} + \frac{1448048}{703305}y_{n+3} - y_{n+4} - \frac{183848}{703305}y_{n+5} - \frac{483328}{703305}y_{n+\frac{1}{2}} = -\frac{95893}{703305}y_n - \frac{729540}{703305}hf_{n+4} - \frac{5880}{703305}hf_{n+6}$$

$$-\frac{713475}{458521}y_{n+1} + \frac{781200}{458521}y_{n+2} - \frac{914340}{458521}y_{n+3} + \frac{930150}{458521}y_{n+4} - y_{n+5} + \frac{471040}{458521}y_{n+\frac{1}{2}} = \frac{96054}{458521}y_n - \frac{243180}{458521}hf_{n+5} + \frac{12600}{458521}hf_{n+6}$$

$$-\frac{83679750}{54281216}y_{n+1}+\frac{22799700}{54281216}y_{n+2}-\frac{10588116}{54281216}y_{n+3}+\frac{3779325}{54281216}y_{n+4}-\frac{716870}{54281216}y_{n+5}+y_{n+\frac{1}{2}}=$$
$$-\frac{14124495}{54281216}y_n-\frac{31127040}{54281216}hf_{n+\frac{1}{2}}-\frac{44100}{54281216}hf_{n+6} \qquad (25)$$

Equation (25) constitute the members of a zero -stable block integrators of order $(7,7,7,7,7,7,7)^T$ with $C_8=\left[-\frac{605}{16212},-\frac{1869}{8},\frac{951}{8},-\frac{29355}{16},\frac{14469}{4},-\frac{35955}{8},\frac{2143291}{32}\right]^T$ as the error constants respectively. To start the integr ation process with n=0, we use (25) and this produces $y_1, y_{1/2}, y_2, y_3, y_4$, , y_5, and y_6 simultaneously without the need of any starting method (predictor).

Stability Analysis

Following Fatunla (1992;1994), that defined the block method to be zero-stable provided the roots $R_{ij}=1(1)k$ of the first characteristic polynomial $\rho(R)$ specified as

$$\rho(R)=det\left|\sum_{i=0}^{k}A^{(i)}R^{k-i}\right|=0 \qquad (26)$$

satisfies $|R_j|\leq 1$,the multiplicity must not exceed 2.

The block methods proposed in equations (21) for $k=4$ are put in the matrix equation form and for easy analysis the result was normalized to obtain

$$A^0=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}, \qquad A^1=\begin{bmatrix}0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\end{bmatrix}$$

And

$$B^0=\begin{bmatrix}\frac{242}{225}&\frac{-4807}{5760}&\frac{2197}{5760}&\frac{-1427}{9600}&\frac{151}{5760}\\\frac{2008}{1575}&\frac{-179}{360}&\frac{119}{360}&\frac{-79}{600}&\frac{59}{2520}\\\frac{1856}{1575}&\frac{4}{45}&\frac{41}{45}&\frac{-16}{75}&\frac{11}{315}\\\frac{216}{175}&\frac{-3}{40}&\frac{63}{40}&\frac{51}{200}&\frac{3}{280}\\\frac{256}{225}&\frac{8}{45}&\frac{52}{45}&\frac{88}{75}&\frac{16}{45}\end{bmatrix}, B^1=\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$$

The first characteristic polynomial of the block method is given by $\rho(R)=\det(RA^0-A^1)$ .Substituting the A^0 and A^1 into the function above gives

$$\rho(R)=\det\left[R\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}-\begin{bmatrix}0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\end{bmatrix}\right]$$

$$=det\left[\begin{bmatrix}R&0&0&0&0\\0&R&0&0&0\\0&0&R&0&0\\0&0&0&R&0\\0&0&0&0&R\end{bmatrix}-\begin{bmatrix}0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\end{bmatrix}\right]$$

$$=det\begin{bmatrix}R&0&0&0&-1\\0&R&0&0&-1\\0&0&R&0&-1\\0&0&0&R&-1\\0&0&0&0&R-1\end{bmatrix}$$

$$=R^3(R(R-1))-0=0$$

$$R=0\,(four\ times)\ or\ R=1$$

The first characteristic polynomial of the block method is given by $\rho(R) = \det(RA^0 - A^1)$ .Substituting the A^0 and A^1 into the function above gives

$$\rho(R) = \det\left[R\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix} - \begin{bmatrix}0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\end{bmatrix}\right]$$

(27)

$$= det\left[\begin{bmatrix}R&0&0&0&0\\0&R&0&0&0\\0&0&R&0&0\\0&0&0&R&0\\0&0&0&0&R\end{bmatrix} - \begin{bmatrix}0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\end{bmatrix}\right] = det\begin{bmatrix}R&0&0&0&-1\\0&R&0&0&-1\\0&0&R&0&-1\\0&0&0&R&-1\\0&0&0&0&R-1\end{bmatrix}$$

$$\Rightarrow R^3(R(R-1)) - 0 = 0$$

$$R_1 = R_2 = R_3 = R_4 = 0 \; or \; R_5 = 1$$

From (3.0) and equation (27) the hybrid method is zero stable and consistent since the order of the method $p = 5 > 1$. And by Henrici (1962); the hybrid method is convergent. Also, the block methods proposed in equations (25) for $k = 6$ are put in the matrix equation form and for easy analysis the result was normalized to obtain

$$\begin{pmatrix}\frac{40960}{12159} & -\frac{968}{193} & \frac{3025}{579} & \frac{-9680}{1737} & \frac{6050}{1351} & \frac{-4840}{1737} & 1\\ \frac{217088}{106785} & -1 & \frac{-119280}{106785} & \frac{46172}{106785} & \frac{-15390}{106785} & \frac{2831}{106785} & 0\\ \frac{-56864}{6510} & \frac{74340}{6510} & -1 & \frac{-47096}{6510} & \frac{11790}{6510} & \frac{-1932}{6510} & 0\\ \frac{342016}{322616} & \frac{-574875}{322616} & \frac{940800}{322616} & -1 & \frac{-366075}{322616} & \frac{45395}{322616} & 0\\ \frac{-483328}{703305} & \frac{758520}{705305} & \frac{-931980}{703305} & \frac{1448048}{703305} & -1 & \frac{-133848}{703305} & 0\\ \frac{471040}{458521} & \frac{-713475}{458521} & \frac{781200}{458521} & \frac{-914840}{458521} & \frac{930150}{458521} & -1 & 0\\ 1 & \frac{-88679750}{54281216} & \frac{22799700}{54281216} & \frac{-10588116}{54281216} & \frac{3779325}{54281216} & \frac{-716870}{54281216} & 0\end{pmatrix}\begin{pmatrix}y_{n+\frac{1}{2}}\\y_{n+1}\\y_{n+2}\\y_{n+3}\\y_{n+4}\\y_{n+5}\\y_{n+6}\end{pmatrix} =$$

$$\begin{pmatrix}0&0&0&0&0&0&\frac{1210}{1737}\\0&0&0&0&0&0&\frac{24626}{106785}\\0&0&0&0&0&0&\frac{-6272}{6510}\\0&0&0&0&0&0&\frac{64645}{322616}\\0&0&0&0&0&0&\frac{-95893}{703305}\\0&0&0&0&0&0&\frac{96054}{458521}\\0&0&0&0&0&0&\frac{-14124495}{54281216}\end{pmatrix}\begin{pmatrix}y_{n-6}\\y_{n-5}\\y_{n-4}\\y_{n-3}\\y_{n-2}\\y_{n-1}\\y_n\end{pmatrix} +$$

$$h\begin{pmatrix}0&0&0&0&0&0&-\frac{220}{579}\\0&-\frac{145908}{106785}&0&0&0&0&\frac{168}{106785}\\0&0&\frac{-60795}{6510}&0&0&0&-\frac{105}{6510}\\0&0&0&-\frac{729540}{322616}&0&0&\frac{2100}{322616}\\0&0&0&0&-\frac{729540}{703305}&0&-\frac{5880}{703305}\\0&0&0&0&0&\frac{-243180}{458521}&\frac{12600}{458521}\\\frac{-31127040}{54281216}&0&0&0&0&0&\frac{-44100}{54281216}\end{pmatrix}\begin{pmatrix}f_{n+\frac{1}{2}}\\f_{n+1}\\f_{n+2}\\f_{n+3}\\f_{n+4}\\f_{n+5}\\f_{n+6}\end{pmatrix} +$$

$$h\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix}\begin{pmatrix}f_{n-6}\\f_{n-5}\\f_{n-4}\\f_{n-3}\\f_{n-2}\\f_{n-1}\\f_n\end{pmatrix}$$

}.................................. (28)

The first characteristic polynomial of the block method is given by $\rho(R) = \det(RA^0 - A^1)$ .Substituting the A^0 and A^1 into the function above gives

$$\rho(R) = \det\left[R\begin{bmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\end{bmatrix} - \begin{bmatrix}0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\end{bmatrix}\right] \quad \text{.................(29)}$$

$$= det\left[\begin{bmatrix}R&0&0&0&0&0&0\\0&R&0&0&0&0&0\\0&0&R&0&0&0&0\\0&0&0&R&0&0&0\\0&0&0&0&R&0&0\\0&0&0&0&0&R&0\\0&0&0&0&0&0&R\end{bmatrix} - \begin{bmatrix}0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\end{bmatrix}\right]$$

$$= det\begin{bmatrix}R&0&0&0&0&0&-1\\0&R&0&0&0&0&-1\\0&0&R&0&0&0&-1\\0&0&0&R&0&0&-1\\0&0&0&0&R&0&-1\\0&0&0&0&0&R&-1\\0&0&0&0&0&0&R-1\end{bmatrix}$$

$= R^5(R(R-1)) - 0 = 0$

$\Rightarrow R_1 = R_2 = R_3 = R_4 = R_5 = R_6 = 0 \; or \; R_7 = 1$

From (3.0) and equation (29) the hybrid method is zero stable and consistent since the order of the method $p = 7 > 1$. And by Henrici (1962); the hybrid method is convergent.

Numerical Example

To illustrate the performance of our propose d methods we will compare their performance with exact results. Consider the initial value problem

$y' = \lambda(y - x) + 1 \; , y(0) = 1$

The problem is stiff in nature for negative λ values and it has exact solution $y(x) = e^{\lambda x} + x$.

This problem is considered for $\lambda = -5, and \; \lambda = -20$ with step length $h = 0.01$. The problem is solved using the Block Hybrid Backward Differentiation Formulae (BHBDF) for $k = 4$ and $k = 6$.

Table : Proposed (BHBDF) for $k = 4, \lambda = -5$

N	X	Exact value	Approximate value	Error
0	0.00	1.000000000	1.000000000	0
1	0.01	0.961229424	0.961229424	5.03E − 10
2	0.02	0.924837418	0.9248374 18	3.9E − 11
3	0.03	0.890707976	0.890707976	4.27E − 10
4	0.04	0.858730753	0.858730753	8E − 11
5	0.05	0.828800783	0.828800783	7.4E − 11
6	0.06	0.80081822	0.800818221	3.17E − 10
7	0.07	0.774688089	0.77468809	2.8E − 10
8	0.08	0.750320046	0.750320046	3.7E − 11
9	0.09	0.727628151	0.727628152	3.77E − 10
10	0.1	0.70653 0659	0.706530667	2.86E − 10

Table **2: Proposed (BHBDF) for** $k = 4, \lambda = -20$

N	X	Exact value	Approximate value	Error
0	0.00	1.000000000	1.000000000	0
1	0.01	0.828730753	0.828769259	$3.8506896E - 05$
2	0.02	0.690320046	0.690432454	$1.12408945E - 04$
3	0.03	0.578811636	0.578993079	$1.81443769E - 04$
4	0.04	0.489328964	0.489416887	$8.7923593E - 04$
5	0.05	0.417879441	0.417989621	$1.10180194E - 04$
6	0.06	0.361194211	0.361365196	$1.70985282E - 04$
7	0.07	0.316596963	0.316826495	$2.29531969E - 04$
8	0.08	0.281896518	0..282023466	$1.26948875E - 04$
9	0.09	0.255298888	0. 255440879	$1.4199104E - 04$
10	0.1	0.235335283	0.235532264	$1.96981522E - 04$

Table **3: Propose d (BHBDF) for** $k = 6, \lambda = -5$

N	X	Exact value	Approximate value	Error
0	0.00	1.000000000	1.000000000	0
1	0.01	0.961229424	0.961229424	0
2	0.02	0.924837418	0.924837418	0
3	0.03	0.890707976	0.890707976	0
4	0.04	0.858730753	0.858730753	0
5	0.05	0.82880078 3	0.828800783	0
6	0.06	0.80081822	0.80081822	0
7	0.07	0.774688089	0.774688089	0
8	0.08	0.750320046	0.750320046	0
9	0.09	0.727628151	0.727628151	0
10	0.1	0.706530659	0.706530659	0

Table **4: Proposed (BHBDF) for** $k = 6, \lambda = -20$

N	X	Exact value	Approximate value	Error
0	0.00	1.000000000	1.000000000	0
1	0.01	0.828730753	0.828730765	$1. 2E - 08$
2	0.02	0.690320046	0.690320059	$1.3 E - 08$
3	0.03	0.578811636	0.578811645	$9.0E - 09$
4	0.04	0.489328964	0.489328973	$9.0E - 09$
5	0.05	0.417879441	0.417879445	$4. -09$
6	0.06	0.3611942 11	0.361194228	$1 .7E - 08$
7	0.07	0.316596963	0.316596981	$1E - 08$
8	0.08	0.281896518	0..281896533	$1.5 E - 08$
9	0.09	0.255298888	0. 2552989	$1. 2E - 08$
10	0.1	0.235335283	0.235335293	$1. 0E - 08$

It would be observed from the results in tables 1 and 2 that the error alternates, that's for BHBDF 4 for $\lambda = -5$ and $\lambda = -20$ respectively. The BHBDF6 with $\lambda = -5$ is the best in terms of performance and accuracy as indicated in table 3.In table 4, the BHBDF6 with $\lambda = -20$ shows an alternate in error, but is of hig her accuracy in comparison with results in table 2. However all the block methods produce accurate results when compared with exact results.

Conclusions

We have derived the hybrid form of the Backward Differentiation Formulae (BDF) for $k = 4$ and $k = 6$. The idea of Multistep Collocation (MC) was used to reformulate the derived hybrid formulae into continous form which were employed as block methods for direct solution of $y' = f(x, y)$. A convergence analysis of the discrete hybrid methods to establish their effectiveness and reliability is presented. The methods were tested on some stiff IVP and shown to perform satisfactorily without the requirement of any starting method.

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