STABILITY ANALYSIS OF THE RECRUITMENT DYNAMICS AND CONTROL OF TERRORISM

[1]YUSUF, I., [2]IBRAHIM M. O., [3]ABDULRAHMAN, S, [4]ENAGI, A. I., [5]USMAN, I. G., [6]ISA, N., [7]GARBA, Z. U, [8]ABDULLAHI, U. K. & [8]MAYAKI, Z. I.

[1]*Department of Computer Science, Niger State College of Education Minna, Nigeria*
[2]*Department of Mathematics, University of Ilorin, Ilorin, Nigeria*
[3]*Department of Mathematics, Federal University Birnin Kebbi, Nigeria*
[4]*Department of Mathematics, Federal University of Technology Minna, Nigeria*
[5]*Department of Mathematics, Zamfara State College of Education Maru, Nigeria*
[6]*Academic Planning Unit, Federal University Gusau, Zamfara State, Nigeria*
[7]*Department of Mathematics, Shehu Shagari College of Education Sokoto, Nigeria*
[8]*Department of Mathematics, Usmanu Danfodiyo University Sokoto, Nigeria*

ABSTRACT

In this paper, terrorist membership is treated as an infection that spreads through the host community by social contact. We developed and analysed a new mathematical model for the dynamics of terrorism in a population, incorporating the effect of extreme ideology, public enlightenment campaign, Reality Therapy (RT), and Aggression and Replacement Training (ART) as control measures. We obtained the effective recruitment number (R_{eff}) *which can be used to control the recruitment of terrorism and hence, established the conditions for local and global stability of the terrorist-free equilibrium. Bifurcation analysis was carried out using centre manifold theory which revealed a condition for forward bifurcation for the model. Numerical simulations validated the analytical results and further reveals that sensitisation only at any coverage rate does not have much impact on the control of terrorism. Instead, Reality Therapy (RT) should target more of the dissidents that are not yet exposed to extreme ideology.*

1. INTRODUCTION

A terrorist is somebody who uses violence means not limited to bombing, kidnapping, and assassination, to create fears in the minds of others (usually innocent and helpless), often for political purposes. Dempsey (2006) posit that 'terrorism' is a failure of political process that begins with inequalities, corruption and injustice in a given political system, and moves from a frustrated attempts at reform that breed fear and anger, to political confrontation and conspicuously erupted into violence.

The word 'terrorism' was first coined in the 1970s during the French Revolution, which gave the label "The Reign of Terror" to the period between 1793 to 1794 and since then, terrorist activities have been increasing globally. It was the terrorist act of the 28th June 1914, which lead to the assassination of the Archduke of Australia and his wife that precipitated the World War I (Martel, 2003). The world has recorded at least 120,000 terrorist attacks in the span of 43 years from 1970 to 2013 (Global Terrorism Database in 2014).

Within the year 2013 alone, there was a record of about 10,000 terrorist attacks in the world which resulted in the death of more than 17,000 innocent people. Over 70% of

these terrorist incidents occurred in Iraq, Afghanistan, Pakistan, Nigeria, and Syria. The five countries mentioned above have suffered high morbidity and mortality in 2013. When compared with the terrorist records of 2012, the world's terrorist incidents increased by more than 40% in the year 2013. This serves as a bad indicator for global peace (Global Terrorism Index, 2014).
A first step toward hindering terrorist recruitment is to understand how terrorist organization work — where they recruit, what tools they use, whom they target, and why they terrorize. A clearer picture of this recruitment process could help to develop strategies and interventions to counter terrorist groups' ability to replenish and increase their numbers. The development of strategies and public policies to stem the rising crime rate is vital and therein lays the strength of mathematical modeling. Mathematical modeling and numerical simulation augment the traditional approaches to research since they help organize existing data, identify areas with missing data and relatively inexpensive and more practical than carry out an actual experiment (Kaplan and Brandeau, 1994)

Policy makers are susceptible to what Ball (2003) calls linear thinking and this has led to the development of linear models of human behaviour. Yet human behavior is inherently non-linear (Brown, 1995) and thus we assume that terrorist behaviour and crime may be best described by non-linear system. The choice of an infections disease model is motivated by research suggesting that the best model for violence may be that of a socially infectious disease (Bingenheimer, 2005). Terrorist membership is treated as an infection that multiplies due to social interaction or contagion whereby terrorist members co-opt the vulnerable and dissidents through verbal and non-verbal communications.

During the last decade, Udwadia *et al.* (2006), Erika and Christian (2007), Saperstein (2008), Gutfraind (2010), Charlinda and Todd (2013), Cherif *et al.* (2010), Choucri *et al.* (2013) have designed mathematical models on terrorism and provided long-term predictions regarding terrorism prevalence and control in various regions. Considering the work of all the authors mentioned above, we developed a new mathematical model improving on their works by incorporating the following factors which are vital in the transmission and control of terrorism especially in countries where the disease is endemic.

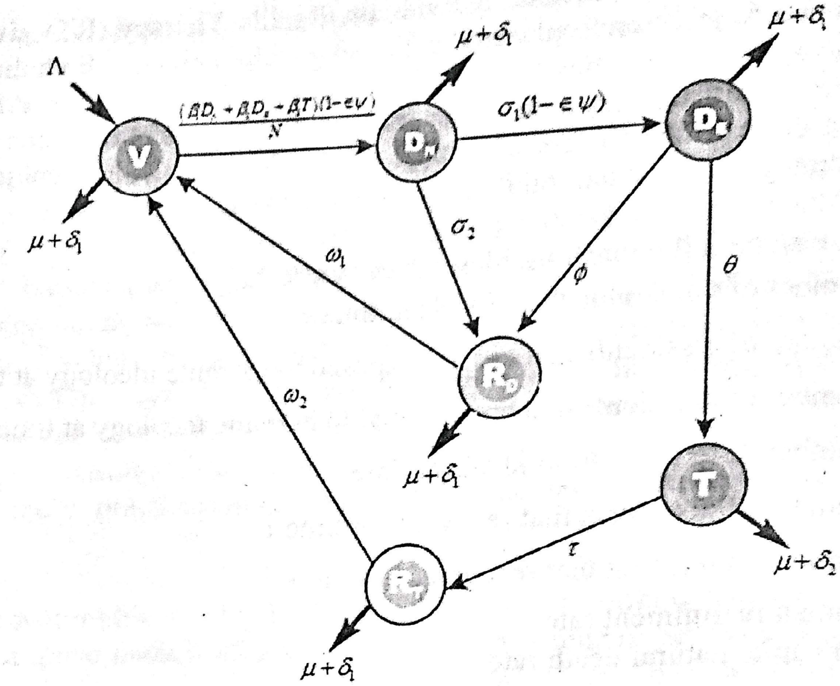
i) Sensitization coverage enhanced by public enlightenment campaign;
ii) Counter-extremism strategies (different from counter-terrorism strategies) enhanced by Reality Therapy (RT) and Aggression Replacement Training (ART)
iii) Standard incidence function;
iv) Terrorist induced death due to suicide bombing and/or counter-terrorism activities;
v) Exposure to extreme ideologies

2. MODEL FORMULATION

We formulate a model for the spread and control of terrorism in the human population with the total population size at time, t given by $N(t)$. The total population is compartmentalized into 6 epidemiological classes. The model incorporated sensitization coverage given to only vulnerable individuals, Reality Therapy (RT) given to dissident individuals and Aggression Replacement Training (ART) given to both dissidents that are exposed to extreme ideologies and the terrorists in different ways as ART1 and ART2 respectively. Terrorist induced death occurs in all the classes at the same rates except for the terrorist class which suffers additional deaths due to suicide bombing and counter-terrorism strategies, while natural death occurs equally in all classes.

The model has the following variables and parameters:

- $V(t)$ Number of vulnerable individuals at time t
- $D_N(t)$ Number of dissidents that are not exposed to extreme ideology at time t
- $D_E(t)$ Number of dissidents that are exposed to extreme ideology at time t
- $T(t)$ Number of terrorist individuals at time t
- $R_D(t)$ Number of dissidents that recovered at time t
- $R_T(t)$ Number of terrorist that recovered at time t
- Λ Human recruitment rate
- μ Per capita natural death rate
- δ_1 Terrorist-induced death rates
- δ_2 Additional death due to suicide bombing and counter-terrorism activities
- β_1 Effective social contact rate between V and D_N
- β_2 Effective social contact rate between V and D_E
- β_3 Effective social contact rate between V and T
- σ_1 Rate of progression from D_N to D_E
- σ_2 Rate of applying Reality Therapy (RT) to D_N
- ϕ Rate of applying Aggression Replacement Training $(ART1)$ to D_E
- θ Rate of progression from D_E to T
- τ Rate of applying Aggression Replacement Training combined with amnesty $(ART2)$ to T
- ω_1 Loss of RT and $ART1$ immunities by R_D
- ω_2 Loss of $ART2$ immunity by R_T
- ψ Sensitization rate which is enhanced by public enlightenment campaign (efficacy and compliance) on terrorist recruitment and therefore $\in\psi$ is the effective sensitization rate
- $\in$ Sensitization efficacy



Figu

The corresponding mathematical equations of the schematic diagram can be described by a system of Ordinary Differential Equations (ODEs) given below:

$$\left.\begin{aligned}
\frac{dV}{dt} &= \Lambda - \frac{(\beta_1 D_N + \beta_2 D_E + \beta_3 T)(1-\in\psi)V}{N} + \omega_1 R_D + \omega_2 R_T - (\mu+\delta_1)V \\
\frac{dD_N}{dt} &= \frac{(\beta_1 D_N + \beta_2 D_E + \beta_3 T)(1-\in\psi)V}{N} - (\sigma_1(1-\in\psi) + \sigma_2 + \mu + \delta_1)D_N \\
\frac{dD_E}{dt} &= \sigma_1(1-\in\psi)D_N - (\phi+\theta+\mu+\delta_1)D_E \\
\frac{dT}{dt} &= \theta D_E - (\tau+\mu+\delta_2)T \\
\frac{dR_D}{dt} &= \sigma_2 D_N + \phi D_E - (\omega_1+\mu+\delta_1)R_D \\
\frac{dR_T}{dt} &= \tau T - (\omega_2+\mu+\delta_1)R_T
\end{aligned}\right\} \quad (1)$$

where

$$N(t) = V(t) + D_N(t) + D_E(t) + T(t) + R_N(t) + R_E(t) \quad (2)$$

So that

$$\frac{dN}{dt} = \Lambda - (\mu+\delta_1)N + (\delta_1 - \delta_2)T \quad (3)$$

in the biological-feasible region:

$$\Omega = \left\{ \begin{pmatrix} V \\ D_N \\ D_E \\ T \\ R_D \\ R_T \end{pmatrix} \in R_+^6 \left| \begin{array}{c} V \geq 0, \\ D_N \geq 0, \\ D_E \geq 0, \\ T \geq 0, \\ R_D \geq 0, \\ R_T \geq 0, \\ V + D_N + D_E + T + R_D + R_T \leq N \end{array} \right. \right\} \quad (4)$$

Setting

$$\left. \begin{array}{l} k_1 = \mu + \delta_1 \\ k_2 = \sigma_1 \vartheta + \sigma_2 + \mu + \delta_1 \\ k_3 = \phi + \theta + \mu + \delta_1 \\ k_4 = \tau + \mu + \delta_2 \\ k_5 = \omega_1 + \mu + \delta_1 \\ k_6 = \omega_2 + \mu + \delta_1 \\ \vartheta = 1 - \in \psi \end{array} \right\} \quad (5)$$

System (1) becomes

$$\left. \begin{array}{l} \dfrac{dV}{dt} = \Lambda - \dfrac{(\beta_1 D_N + \beta_2 D_E + \beta_3 T)\vartheta V}{N} + \omega_1 R_D + \omega_2 R_T - k_1 V \\ \dfrac{dD_N}{dt} = \dfrac{(\beta_1 D_N + \beta_2 D_E + \beta_3 T)\vartheta V}{N} - k_2 D_N \\ \dfrac{dD_E}{dt} = \sigma_1 \vartheta D_N - k_3 D_E \\ \dfrac{dT}{dt} = \theta D_E - k_4 T \\ \dfrac{dR_D}{dt} = \sigma_2 D_N + \phi D_E - k_5 R_D \\ \dfrac{dR_T}{dt} = \tau T - k_6 R_T \end{array} \right\} \quad (6)$$

which can be shown to be positively invariant with respect to the system.

3. MODEL ANALYSIS

We now determine the existence of equilibrium points; computing the effective recruitment number; and establishing the conditions for stability of the equilibrium points.

3.1 Existence of Terrorist Free Equilibrium State, (E_f)

At the terrorist free equilibrium state we have absence of terrorist. Thus, all the infected classes will be zero and the entire population will comprise of vulnerable individuals. At equilibrium state the rate of change of each of the state variable is equal to zero, i.e.,

$$\frac{dV}{dt}=\frac{dD_N}{dt}=\frac{dD_E}{dt}=\frac{dT}{dt}=\frac{dR_H}{dt}=\frac{dR_T}{dt}=0 \quad (7)$$

Let

$$E_f=\left(V^0, D_N^0, D_E^0, T^0, R_H^0, R_T^0\right) \quad (8)$$

At terrorist-free (TF) equilibrium state. Thus, substituting (7) into (6) with $D_N^0=D_E^0=T^0=0$, we obtained the terrorist-free equilibrium state given by:

$$\begin{pmatrix} V^0 \\ D_N^0 \\ D_E^0 \\ T^0 \\ R_H^0 \\ R_T^0 \end{pmatrix}=\begin{pmatrix} \frac{\Lambda}{\mu+\delta_1} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (9)$$

3.2 Effective Recruitment Number, (R_{eff})

Using the approach of Diekmann and Heesterbeek (2000), we obtained the effective recruitment number (R_{eff}) of the system (6) which is the spectral radius of the next generation matrix, G, i.e $R_{eff}=\rho G$, where $G=FV^{-1}$.

Now

$$F=\begin{pmatrix} \beta_1\vartheta & \beta_2\vartheta & \beta_3\vartheta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (10)$$

and

$$V=\begin{pmatrix} k_2 & 0 & 0 \\ -\sigma_1\vartheta & k_3 & 0 \\ 0 & -\theta & k_4 \end{pmatrix} \quad (11)$$

Then

$$V^{-1} = \begin{pmatrix} \frac{1}{k_2} & 0 & 0 \\ \frac{\sigma_1 \vartheta}{k_2 k_3} & \frac{1}{k_3} & 0 \\ \frac{\sigma_1 \theta \vartheta}{k_2 k_3 k_4} & \frac{\theta}{k_3 k_4} & \frac{1}{k_4} \end{pmatrix} \qquad (12)$$

Thus

$$R_{eff} = \frac{(k_3 k_4 \beta_1 + k_4 \sigma_1 \beta_2 \vartheta + \sigma_1 \beta_3 \theta \vartheta)\vartheta}{k_2 k_3 k_4} \qquad (13)$$

3.3 Local Stability of Terrorist-Free Equilibrium Point (E_f)

We used the Jacobian stability technique to determine the local stability of the system. Linearization of the system (6) at terrorist-free equilibrium point (E_f), gives the Jacobian matrix

$$J(E_f) = \begin{pmatrix} -k_1 & -\beta_1 \vartheta & -\beta_2 \vartheta & -\beta_3 \vartheta & \omega_1 & \omega_2 \\ 0 & -(k_2 - \beta_1 \vartheta) & \beta_2 \vartheta & \beta_3 \vartheta & 0 & 0 \\ 0 & \sigma_1 \vartheta & -k_3 & 0 & 0 & 0 \\ 0 & 0 & \theta & -k_4 & 0 & 0 \\ 0 & \sigma_2 & \phi & 0 & -k_5 & 0 \\ 0 & 0 & 0 & \tau & 0 & -k_6 \end{pmatrix} \qquad (14)$$

But

$$k_2 = \frac{\beta_1 \vartheta V^0}{N^0} + \frac{\beta_2 \vartheta D_E{}^0 V^0}{D_N{}^0 N^0} + \frac{\beta_3 \vartheta T^0 V^0}{D_N{}^0 N^0} \qquad (15)$$

Then

$$k_2 > \beta_1 \vartheta \qquad (16)$$

Using elementary row-transformation on (15), we have

$$J(E_f) = \begin{pmatrix} -k_1 & -\beta_1 \vartheta & -\beta_2 \vartheta & -\beta_3 \vartheta & \omega_1 & \omega_2 \\ 0 & -(k_2 - \beta_1 \vartheta) & \beta_2 \vartheta & \beta_3 \vartheta & 0 & 0 \\ 0 & 0 & -M_1 & -M_2 & 0 & 0 \\ 0 & 0 & 0 & -M_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -k_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & -k_6 \end{pmatrix} \qquad (17)$$

where

$$\left.\begin{aligned} M_1 &= \frac{k_3\beta_1\vartheta - k_2k_3 + \sigma_1\beta_3\vartheta^2}{\beta_1\vartheta - k_2} \\ M_2 &= \frac{\sigma_1\beta_3\vartheta^2}{\beta_1\vartheta - k_2} \\ M_3 &= \frac{\begin{pmatrix} k_4\sigma_1\beta_2\vartheta^2 - k_2k_3k_4 \\ +\sigma_1\beta_3\theta\vartheta^2 + k_3k_4\beta_1\vartheta \end{pmatrix}}{\sigma_1\beta_2\vartheta^2 - k_2k_3 + k_3\beta_1\vartheta} \end{aligned}\right\} \tag{18}$$

and clearly, the eigenvalues are

$$\left.\begin{aligned} \lambda_1 &= -k_1 < 0 \\ \lambda_2 &= -(k_2 - \beta_1\vartheta) \\ \lambda_3 &= -\left(\frac{k_3(k_2 - \beta_1\vartheta) + \sigma_1\beta_2\vartheta^2}{(k_2 - \beta_1\vartheta)}\right) \\ \lambda_4 &= -M_3 \\ \lambda_5 &= -k_5 < 0 \\ \lambda_6 &= -k_6 < 0 \end{aligned}\right\} \tag{19}$$

$\lambda_2 < 0, \lambda_3 < 0$ since from (16), $k_2 > \beta_1\vartheta$

For λ_4 to be negative, then

$$-\left(\frac{k_4\sigma_1\beta_2\vartheta^2 - k_2k_3k_4 + \sigma_1\beta_3\theta\vartheta^2 + k_3k_4\beta_1\vartheta}{\sigma_1\beta_2\vartheta^2 - k_2k_3 + k_3\beta_1\vartheta}\right) < 0 \tag{20}$$

and

$$\frac{R_{eff} - 1}{k_3(k_2 - \beta_1\vartheta) - \sigma_1\beta_2\vartheta^2} < 0 \tag{21}$$

$$R_{eff} < 1 \tag{22}$$

Obviously, λ_4 is negative if $R_{eff} < 1$, is negative, implying all the eigenvalues have negative real parts, we thus, established the following result.

Theorem 1: The terrorist-free equilibrium of system (1) is locally asymptotically stable (LAS) if $R_{eff} < 1$.

The epidemiological implication of the theorem is that terrorist can be eliminated (control) from the population when $R_{eff} < 1$, if the initial size of the sub-populations of the model are in the basin of attraction of the terrorist-free equilibrium point.

3.4 Global Stability of Terrorist-Free Equilibrium Point (E_f)

In order to ensure that terrorism is independent of the initial size of the sub-populations of the model (1), it is necessary to show that the terrorist-free equilibrium (TFE) is globally-asymptotically stable (GAS). One common approach in studying the global asymptotic stability of the TFE is to construct an appropriate Lyapunov function (Garba and Gumel, 2010).

Theorem 2: The terrorist-free equilibrium (E_f) of the model is GAS in Ω if $R_{eff} \leq 1$.

Proof: To prove the global asymptotic stability of the model, we construct an appropriate Lyapunov function as shown:

$$L = \theta\sigma_1\vartheta D_N + k_2\theta D_E + k_2k_3T \tag{23}$$

The derivative of (23) along the solutions of the model equations is:

$$L' = \left\{ \begin{array}{l} \theta\sigma_1\vartheta\left[\dfrac{(\beta_1 D_N + \beta_2 D_E + \beta_3 T)\vartheta V}{N} - k_1 D_N\right] \\ +k_2\theta(\sigma_1\vartheta D_N - k_3 D_E) \\ +k_2k_3(\theta D_E - k_4 T) \end{array} \right\} \tag{24}$$

From (6), we have

$$\left.\begin{array}{l} D_N = \dfrac{k_3k_4T}{\sigma_1\vartheta\theta} \\ D_E = \dfrac{k_4T}{\theta} \end{array}\right\} \tag{25}$$

Substituting (25) into (24) and simplifying gives

$$L' = k_2k_3k_4\left[\frac{(k_3k_4\beta_1 + k_4\sigma_1\beta_2\vartheta + \beta_3\sigma_1\theta\vartheta)\vartheta V}{k_2k_3k_4N} - 1\right]T \tag{26}$$

Now, since $\dfrac{V}{N} \leq \dfrac{V^0}{N^0}$, we have that

$$L' \leq k_2k_3k_4\left[\frac{(k_3k_4\beta_1 + k_4\sigma_1\beta_2\vartheta + \beta_3\sigma_1\theta\vartheta)\vartheta V}{k_2k_3k_4N} - 1\right]T \tag{27}$$

i.e

$$L' \leq k_2k_3k_4(R_{eff} - 1)T \tag{28}$$

Since all parameters of the model are nonnegative, it follows that when $R_{eff} < 1$, $L' \leq 0$; the equality $L' = 0$ holds when $R_{eff} = 1$ and $J = 0$. Therefore, the largest compact invariant set $\{(V, D_N, D_E, R_D, R_T, T) \in \Re_6^+ : L' = 0\}$ is the singleton $\{E_f\}$. Hence, by the LaSalle invariance principle (LaSalle, 1976), E_f is overall globally asymptotically stable in $\Re_6^+$ and hence, the proof is complete.

The above theorem shows that corruption will be under control regardless of the initial profile of the subpopulation in the community if R_{eff} can be brought down to a level less than unity.

3.5 Existence of Terrorist Endemic Equilibrium State, (E^{**})

The endemic equilibrium state is the state in which terrorists persist. That is the coordinates should satisfy the conditions:

$$E^{**} = \left\{ \begin{pmatrix} V \\ D_N \\ D_E \\ T \\ R_D \\ R_T \end{pmatrix} \in R_+^6 \left| \begin{array}{c} V \geq 0, \\ D_N \geq 0, \\ D_E \geq 0, \\ T \geq 0, \\ R_D \geq 0, \\ R_T \geq 0, \\ V + D_N + D_E + T + R_D + R_T \leq N \end{array} \right. \right\} \quad (29)$$

At the endemic equilibrium state, let

$$E^{**} = \begin{pmatrix} V^{**} \\ D_N^{**} \\ D_E^{**} \\ T^{**} \\ R_D^{**} \\ R_T^{**} \\ N^{**} \end{pmatrix} \quad (30)$$

Substituting (30) into system (7) and solving gives

$$V^{**} = \frac{\Lambda}{k_1} + \left(\frac{\left\{ \begin{array}{l} k_3k_4\sigma_2\omega_1 + k_4\sigma_1\vartheta\phi\omega_1 \\ +k_5\tau\omega_2\sigma_1\theta\vartheta - k_2k_3k_4k_5k_6 \end{array} \right\}}{\sigma_1\theta\vartheta k_1k_5k_6} \right) T^{**}. \quad (31)$$

$$D_N^{**} = \frac{k_3k_4}{\sigma_1\vartheta\theta} T^{**} \quad (32)$$

$$D_E^{**} = \frac{k_4}{\theta} T^{**} \quad (33)$$

$$R_D^{**} = \left(\frac{k_3k_4\sigma_2 + k_4\sigma_1\vartheta\phi}{\sigma_1\vartheta\theta k_5} \right) T^{**} \quad (34)$$

$$R_T^{**} = \frac{\tau}{k_6} T^{**} \tag{35}$$

$$T^{**} = \left\{ \frac{(R_{eff} - 1)\begin{pmatrix} k_2k_3k_4k_5k_6 - k_3k_4\sigma_2\omega_1 \\ +k_4\sigma_1\vartheta\phi\omega_1 + k_5\tau\omega_2\sigma_1\theta\vartheta \end{pmatrix}}{k_1k_2k_3k_4k_5k_6\sigma_1\theta\vartheta N^{**} \\ +\Lambda k_5k_6\sigma_1\theta\vartheta\left(k_3k_4\beta_1\vartheta + k_4\sigma_1\vartheta\beta_2\vartheta^2 + \sigma_1\theta\beta_3\vartheta^2\right)} \right\} \tag{36}$$

Since all parameters are assumed to be nonnegative, it follows that $T^{**} > 0$ whenever $R_{eff} > 1$ which resulted into an equilibrium state where each of the sub-population is greater than zero. Thus, we established the following result.

Lemma 1: The model system (1) has a unique endemic (positive) equilibrium whenever $R_{eff} > 1$.

3.6 Bifurcation Analysis

We used the centre manifold theory as described in Castillo-Chavez and Song (2004) for bifurcation analysis. In order to apply the theorem, we make the following change of variables. Let

$$\left.\begin{aligned} V &= x_1 \\ D_N &= x_2 \\ D_E &= x_3 \\ T &= x_4 \\ R_D &= x_5 \\ R_T &= x_6 \end{aligned}\right\} \tag{37}$$

further by using the vector notation, we have

$$X = (x_1, x_2, x_3, x_4, x_5, x_6)^T \tag{38}$$

the model system (1) can be written in the form

$$\frac{dX}{dt} = F = (f_1, f_2, f_3, f_4, f_5, f_6)^T \tag{39}$$

such that

$$\left.\begin{aligned}\frac{dx_1}{dt}&=f_1=\Lambda-\frac{(\beta_1x_2+\beta_2x_3+\beta_3x_4)\vartheta x_1}{N}+\omega_1x_5+\omega_2x_6-k_1x_1\\\frac{dx_2}{dt}&=f_2=\frac{(\beta_1x_2+\beta_2x_3+\beta_3x_4)\vartheta x_1}{N}-k_2x_2\\\frac{dx_3}{dt}&=f_3=\sigma_1\vartheta x_2-k_3x_3\\\frac{dx_4}{dt}&=f_4=0x_3-k_4x_4\\\frac{dx_5}{dt}&=f_5=\sigma_2x_2+\phi x_3-k_5x_5\\\frac{dx_6}{dt}&=f_6=\tau x_4-k_6x_6\end{aligned}\right\}\qquad(40)$$

Where

$$N=x_1+x_2+x_3+x_4+x_5+x_6$$

(41)

Now, the Jacobian of the system (40) above at the terrorist-free is given by:

$$J(E_f)=\begin{pmatrix}-k_1 & -\beta_1\vartheta & -\beta_2\vartheta & -\beta_3\vartheta & \omega_1 & \omega_2\\ 0 & -(k_2-\beta_1\vartheta) & \beta_2\vartheta & \beta_3\vartheta & 0 & 0\\ 0 & \sigma_1\vartheta & -k_3 & 0 & 0 & 0\\ 0 & 0 & 0 & -k_4 & 0 & 0\\ 0 & \sigma_2 & \phi & 0 & -k_5 & 0\\ 0 & 0 & 0 & \tau & 0 & -k_6\end{pmatrix}\qquad(42)$$

Let $\beta_1=\beta_1^*$ be a bifurcation parameter and if we consider the case $R_{eff}=1$ and solve for $\beta_1=\beta_1^*$ from (13) since using R_{eff} directly as a bifurcation parameter is often inconvenient.

Hence

$$\beta_1^*=\frac{k_2k_3k_4-(k_4\beta_2+\beta_3 0)\sigma_1\vartheta}{\vartheta k_3k_4}\qquad(43)$$

The system (6) with $\beta_1=\beta_1^*$ has a simple zero eigenvalues, hence, the centre manifold theory will be used to analyze the dynamics of the system near $\beta_1=\beta_1^*$.

Let V and W be the corresponding left and right eigenvectors associated with the zero eigenvalues of the Jacobian matrix (42) at $\beta_1=\beta_1^*$ (denoted by $J_{\beta_1^*}$) chosen such that $VJ(E_f)=0$ and $J(E_f)W=0$ with $VW=1$ where $V=[v_1,v_2,v_3,v_4,v_5,v_6]$, and $W=[w_1,w_2,w_3,w_4,w_5,w_6]^T$.

Computing the left eigenvectors, we have

$$\left.\begin{aligned}
&-v_1k_1=0\\
&-v_1\beta_1\vartheta-v_2(k_2-\beta_1\vartheta)+v_3\sigma_1\vartheta+v_5\sigma_2=0\\
&-v_1\beta_2\vartheta+v_2\beta_2\vartheta-v_3k_3+v_4\theta+v_5\phi=0\\
&-v_1\beta_3\vartheta+v_2\beta_3\vartheta-v_4k_4+v_6\tau=0\\
&v_1\omega_1-v_5k_5=0\\
&v_1\omega_2-v_6k_6=0
\end{aligned}\right\}\quad(44)$$

Solving (44), gives

$$\left.\begin{aligned}
&v_1=v_5=v_6=0\\
&v_2=\frac{k_4v_4}{\beta_3\vartheta}>0\\
&v_3=\frac{k_4(k_2-\beta_1\vartheta)v_4}{\beta_3\sigma_1\vartheta^2}>0\\
&v_4>0\ \text{(a free left eigenvector)}
\end{aligned}\right\}\quad(45)$$

Similarly, we have the right eigenvectors as

$$\left.\begin{aligned}
&w_1=\frac{(Q_1-Q_2)w_4}{k_1k_5k_6\sigma_1\theta\vartheta}\\
&w_2=\frac{k_3k_4w_4}{\sigma_1\theta\vartheta_2\vartheta}>0\\
&w_3=\frac{k_4w_4}{\theta}>0\\
&w_4>0\text{(a free right eigenvector)}\\
&w_5=\frac{(k_3k_4\sigma_2+k_4\sigma_1\phi\vartheta)w_4}{k_5\sigma_1\theta\vartheta}>0\\
&w_6=\frac{\tau w_4}{k_6}>0
\end{aligned}\right\}\quad(46)$$

Where

$$\left.\begin{aligned}
&Q_1=k_4k_6\omega_1(k_3\sigma_2+\sigma_1\vartheta\phi)+k_5\sigma_1\theta\vartheta\tau\omega_2\\
&Q_2=k_4k_5k_6\vartheta(k_3\beta_1+\sigma_1\beta_2\vartheta)+k_5k_6\sigma_1\theta\vartheta^2\beta_3
\end{aligned}\right\}\quad(47)$$

Computation of a and b:

For a, we have

$$a=\sum_{k,i,j=1}^{n}v_kw_iw_j\frac{\partial^2f_k}{\partial x_i\partial x_j}(0,0)\quad(48)$$

Computing the associated non-zero partial derivatives of F at the E_f for the sign of a, we have:

$$a = \frac{2v_2\vartheta}{N^*}(w_2\beta_1 + w_3\beta_2 + w_4\beta_3)w_1 \tag{49}$$

Thus, $a < 0$ when $Q_1 < Q_2$

Similarly, for b, we have

$$b = \sum_{k,i=1}^{n} v_k w_i \frac{\partial^2 f_k}{\partial x_i \partial \phi}(0,0) \tag{50}$$

And thus, computing the associated non-zero partial derivatives of F at the E_f for the sign of b, we have:

$$b = \frac{\vartheta x_1}{N^*} > 0 \tag{51}$$

Thus, $a < 0$ and $b > 0$. So by Theorem 2 of Castillo-Chavez and Song (2004) the following result is established.

Theorem 3: The positive endemic equilibrium state of system (1) is locally asymptotically stable (LAS) when $R_{eff} > 1$ but close to 1.

4.0 RESULTS AND DISCUSSION

In this section, we presented some numerical simulation to monitor the dynamics of the full model (1) for various values of the associated effective recruitment number in order to confirm our analytical results on the global stability of the equilibrium points as well as the occurrence of a forward.

4.1 Variables and Population-dependent Parameters Values

A major difficulty in modeling is estimating values for the model parameters in order to determine model prediction. Owing to the difficulties in obtaining police and sociological records due to the sensitive nature of this research (Sooknanan *et al.*, 2012), hypothetical values are used where necessary.

Table 4.1 Variables and population-dependent parameter values

S/N	Variable/ Parameter	Values
1	V	6,100
2	D_N	3,000
3	D_E	800
4	T	100
5	R_D	0
6	R_T	0
7	Λ	150
8	μ	0.019
9	β_1	0.35
10	β_2	0.15

11	β_3	0.08
12	N	10,000

Table 4.2 Values of population-independent parameter

S/N	Parameter	Value
1	σ_1	0.15
2	ω_1	0.33
3	ω_2	0.20
4	θ	0.05
5	τ, ϕ, σ_2, and $\in$	(0,1)
6	δ_1	0.00012
7	δ_2	0.15

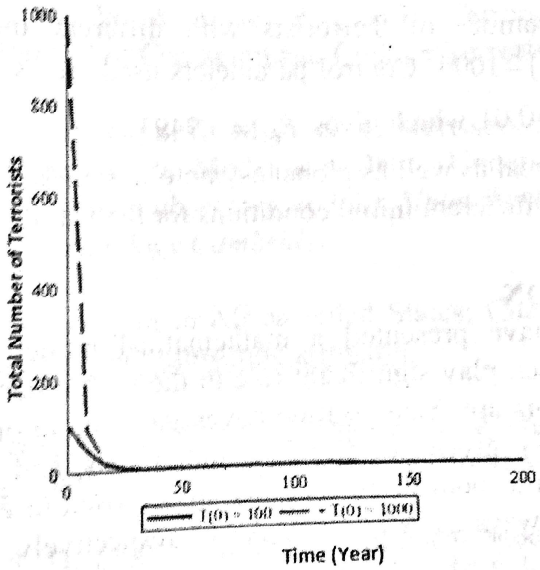


Figure 2: Total number of Terrorists with different initial variables conditions: $T(0)=100$ and $T(0)=1000$. Control parameters used are as in Table 4.1 and Table 4.2 with $\in = \sigma_2 = \phi = \tau = 0.3$ which gives $R_{eff} = 0.7076$.

We clearly see from Figure 2 that the solution profile converges to the terrorist-free equilibrium in all cases. This confirms our analytical result of local as well as global asymptotical stability of terrorist-free equilibrium for $R_{eff} = 0.7076$ and different initial conditions for the variables as in Table 4.1.

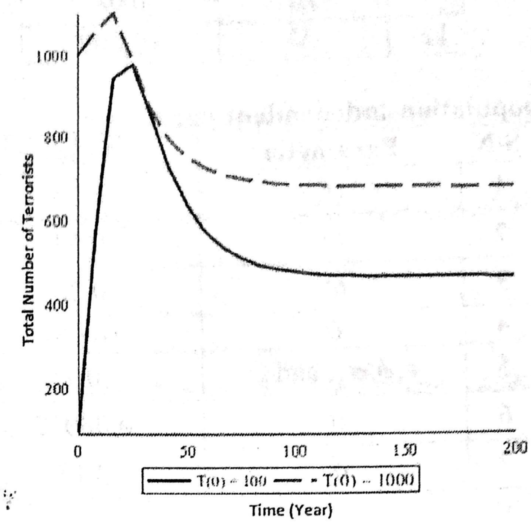


Figure 3: Total number of Terrorists with different initial variables conditions: $T(0)=100$ and $T(0)=1000$. Control parameters used are as in Table 4.1 and Table 4.3 with $\in=\sigma_2=\phi=\tau=0.01$ which gives $R_{eff}=3.9493$.

Figure 3 shows the local as well as global asymptotic stability of the endemic equilibrium for $R_{eff}=3.9493$ and different initial conditions for the variables as in Table 4.1.

5.0 CONCLUSION

In this paper, we have presented a mathematical model which incorporated very important factors which play significant role in the recruitment dynamics and control of terrorism. These factors are: Sensitization coverage given to only vulnerable individuals, Reality Therapy (RT) given to dissident individuals and Aggression Replacement Training (ART) given to both dissidents that are exposed to extreme ideologies and the terrorists in different ways as ART1 and ART2 respectively. The effective recruitment number (R_{eff}) was obtained and the analysis revealed that for $R_{eff}\leq 1$, the terrorist-free equilibrium is globally asymptotically stable even though the cancerous nature of terrorism cannot allow it to be totally eradicated, but it can be curbed to a bearable minimum. And for whatever reason that make $R_{eff}>1$, the terrorist-free equilibrium point is unstable and the endemic equilibrium emerges.

Finally, there is the need to apply the model to a country of interest incorporating High Counter-Terrorism Feasible Reforms (HCFR), which include but not limited to justice which is the soul of peace, public awareness on the dangers and consequences of terrorism, transparency and accountability. Though, HCFR is possible only when the government has the political will to curb terrorism and there is moderate standard of living. The authors have started a work titled "Effect of Different Control Strategies on the Spread of Terrorism in Nigeria"

REFERENCES

Ball, P. (2003). The Physical Modelling of Human Social Systems, *Complexus, 1*, 190 - 206

Bingenheimer, J.B. Brennan, R.T. and Earls, F.J. (2005). Firearm Violence Exposure and Serious Violent Behavior. Science, 308, 1323-1326.

Brown, C. (1995). Serpents in the Sand: Essays on the Nonlinear Nature of Politics and Human Destiny. University of Michigan.

Castillo-Chavez, C., & Song, B. (2004). Dynamical model of tuberculosis and their applications. *Mathematical Biosciences and Engineering, 1*, 361-404.

Charlinda, S. & Todd, S. (2013). An Empirical Study of Suicide Terrorism: A Global Analysis. *Southern Economic Journal 2014.*

Cherif, A., Yoshioka, H., Ni, W. & Bose, P. (2010). Terrorism: Mechanism of Radicalization Process, Control of Contagion and Counter-Terrorist Measures.

Choucri, N., Goldsmith, D., Madnick, S. E., Mistree, D., Morrison, J. B. & Siegel, M. D. (2007). Using System Dynamics to Model and Better Understand State Stability. *Composite Information Systems Laboratory (CISL) Sloan School of Management, Massachusetts Institute of Technology Cambridge.*

Dempsey, T. (2006). Counterterrorism in Africa failed States: Challenges and Potential Solutions. http://www.strategicstudiesinstitute.army.mil/.

Dieckmann, O., & Heesterbeek, J. A. P. (2000). *Mathematical epidemiology of infectious diseases: Model building, analysis and integration*. New York: Wiley.

Erika, A. S. & Christian, J. D. (2007). Utilizing Biological Models to Determine the Recruitment of the Irish Republican Army. *International Journal of Human and Social Sciences 1;3*. http://www.waset.org

Garba, S. M. and Gumel, A. B. (2010). Mathematical Recipe for HIV Elimination in Nigeria. Journal of the Nigeria Mathematical Society. 29:51-112.

Global Terrorism Database (2014). *National Consortium for the Study of Terrorism and Responses to Terrorim,* University of Maryland. http://www.start.umd.edu

Global Terrorism Database (2014). *National Consortium for the Study of Terrorism and Responses to Terrorim,* University of Maryland. http://www.start.umd.edu

Gutfraind, A. (2010). Mathematical Terrorism. (Unpublished). Cornell University. Postgraduate Research.

Kaplan, E., & Brandeau, M. (1994). *Modelling AIDS and the AIDS epidemic*. New York: Raven.

LaSalle, J. P. (1976). The Stability of Dynamical Systems. *Regional Conference Series in Applied Mathematics,* SIAM: Philadeiphia, PA.

Martel, G. (2003). *The Origins of the First World War*, (3rd ed.). London: Pearson Longman, 18 – 21.

Saperstein, A. M. (2008). Mathematical Modeling of the Interaction between Terrorism and Counter-Terrorism and its Policy Implications. *Wiley Periodicals, Inc.*, 14(1). http://www.wiley.com

Sooknanan, J. Bhalt. (2012). B and Commissioning D. M. G. life and Death in a Gang- A Mathematical Model of gang Membership, Int. Journ. of pure and Appl. Mathematics (4)10-15.

Udwadia, F., Leitmann, G. & Lambertini, L. (2006). A Dynamical Model of Terrorism. *Discrete Dynamics in Nature and Society.*