EFFICIENCY OF SPLIT-PLOT RESPONSE SURFACE DESIGNS IN THE PRESENCE OF MISSING OBSERVATIONS

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Abstract

In most experimental designs, situations often arise where some observations are missing due to some unforeseen factors. In such situations, some properties like optimality, orthogonality, and rotatability, which are performance criteria of a design, are destroyed. In the presence of missing observations, efficiency of completely randomized response surface designs has been extensively studied. However, completely randomized response surface designs become inadequate, especially in most industrial experiments, where some factors consist of levels that are difficult and/or expensive to change, which are termed hard-to-change (HTC) factors, and some with levels that are easy to change, termed easy-to-change (ETC) factors. An appropriate approach to such experiments restricts the randomization of the HTC factor levels and this leads to a split-plot structure, for which the designs depend on relative magnitude (d) of model's variance components. Relatively little or no work has been done on investigating the impact of missing observations on efficiency of response surface designs conducted with a split-plot structure. Therefore, this study examines the impact of pairs of missing observations of factorial point (f), whole-plot axial point(α), subplot axial point(β), and center point(c), on efficiency of split-plot response surface designs in terms of trace(A), maximum (G), integrated average(V) prediction variances optimality criteria under different values of d. At $d = 0.5$, maximum A-efficiency losses of $19.1, 10.6, 15.7\%$ were observed, due to missing pairs: ff, $\beta\beta$, $f\beta$, respectively; maximum G- and V-efficiency losses of $10.1, 0.1, 16.1, 0.1\%$ and $0.1, 0.1, 1.1, 0.2\%$ were also observed, due, respectively, to missing pairs $ff, \alpha\alpha, \beta\beta, cc$. An efficiency was robust to missing $cc, \alpha\alpha, \alpha c, fc, f\alpha$ while G and Vefficiencies were robust to missing $\alpha\alpha$. As d increases, the efficiency losses became insignificant.

Keywords: Response Surface Designs, Splitplots, Missing observations, Efficiency, Optimality criteria

1.0 Introduction

Response surface methodology (RSM) is an efficient statistical tool, introduced by Box and Wilson (1951), which is used for modeling and optimizing the performance of designed experiments. Historically, RSM assumed that all the factors are equally easy to change and as a result, most of the early works in the area have tended to ignore the split-plot structure that results when some of the factors are significantly harder to change than others. Most industrial experimental situations consist of two sets of factors – those with levels that are difficult, time-consuming, and sometimes even impossible to change termed hard-to-change (HTC) factors, and those with levels that are easy to change (ETC factors). Typical examples of HTC factors include temperature, pressure, power, gas flow, etc. (Goos and Vandebroek, 2004). In such situations **split-plot** designs are used, in which the experimental runs are performed in groups, where, in a group, the levels of the HTC factors are not reset from run to run. This creates dependence among the runs in one group, thereby leading to clusters of correlated errors and responses.

Research works on impact of a split-plot structure on response surface designs began in the 1990s and most of these work focused on two-level fractional factorial designs run as split-plots. Letsinger, Myers, and Lentner (1996) is the first major paper to address second-order response surface designs within a split-plot structure. The authors investigate the efficiency of various second-order completely randomized response surface designs when conducted with a split-plot structure. Vining, Kowalski and Montgomery (2005), hereafter referred to as VKM, modified the completely randomized Central Composite designs (Box and Wilson, 1951) and Box-Behnken designs (Box and Behnken, 1960) to accommodate the split-plot structure.

There are two separate randomizations for every split-plot design- whole plot factor levels are randomly assigned to the whole plots using a different randomization for each block; subplot factor levels are randomly assigned within each whole plot using a separate randomization for each whole plot. This leads to two error terms for effects comparison, one for the whole-plot treatments (σ_γ^2), and one for the subplot treatments (σ_ε^2) as well as the interaction between whole-plot treatments and subplot treatments.

Split-plot central composite designs (CCDs) consist of four different categories of points. These include the factorial portion (f), which consists of n_f equally-spaced points that contribute to the estimation of linear and interaction terms in the model, axial point (whole-plot(α) and subplot(β)), which consists of points lying on the coordinate axis of each input variable, which allow for efficient estimation of pure quadratic terms in the model and center (c) points, which provide an internal estimate of error (i.e., the pure error), and efficiently provide information about the existence of curvature in the system. If curvature is found in the system, the addition of axial points allows for efficient estimation of the pure quadratic terms. A design matrix for one such design with 3 factors (one whole-plot and two subplot factors) is given by Table 1.1 in the APPENDIX, with the number of points (n) and their categories.

1.1 Model and Notations

The generalized least squares (GLS) model for a split-plot response surface design is

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\boldsymbol{\gamma} + \boldsymbol{\varepsilon} \qquad \textbf{(1.1)}$$

where **y** is the N x 1 vector of responses, **X** is the N x p overall model matrix, **β** is the p x 1 vector of regression coefficients, Z is an N x b incidence matrix assigning observations to each of the b whole plots; γ is the N x 1 vector of whole-plot error terms, ε is the N x 1 vector of subplot error terms. It is assumed that

$\gamma_i \sim N(0, \sigma_\gamma^2), \quad \varepsilon_{ij} \sim N(0, \sigma_\varepsilon^2), \quad cov(\gamma_i, \varepsilon_{ij}) = 0.$

The variance - covariance matrix for the observation vector **y** is

$Var(y) = V = \sigma_\varepsilon^2 I_n + \sigma_\gamma^2 ZZ'$

$$= \sigma_\varepsilon^2(I_n + dZZ')$$

where $d = \frac{\sigma_\gamma^2}{\sigma_\varepsilon^2}$ gives the relative magnitude of the two variance components. The matrix ZZ' is a block diagonal matrix with diagonal matrices of $J_{n1}, J_{n2}, \ldots, J_{nz}$, where J_{ni} is an $n_i \times n_i$ matrix of 1's and n_i is the number of observations in the ith whole-plot.

The generalized least squares (GLS) estimates are

$$\widehat{\beta}_{GLS} = (X'V^{-1}X)^{-1}X'V^{-1}y$$

$$Var(\widehat{\beta}_{GLS}) = (X'V^{-1}X)^{-1}$$

$$\hat{y} = X(X'V^{-1}X)^{-1}X'V^{-1}y = Hy \qquad (1.2)$$

where **X** is the model matrix, **y** is the vector of responses and H is the 'hat' matrix.

1.2 Missing Observations

Missing observations can hardly be avoided during experimentation due to some uncontrollable reasons. Missing observations can create a big problem by making the results of a response surface experiment quite misleading, thereby adversely affecting the inference. Thus the estimates of the parameters will be misleading. Besides, unavailability of some observations destroys some useful properties like orthogonality, rotatability, optimality, and efficiency, which are performance criteria of an experimental design. There is therefore a serious need for experimental designs which guard against (or are insensitive to) the effect of missing observations.

Extensive studies have been undertaken concerning impact of missing observations on efficiency of response surface designs with complete randomization in terms of some given criteria. Ahmad and Gilmour (2010) study the robustness of subset response surface designs to a missing value in terms of ratio of prediction variance criterion. The authors compute the ratio of prediction variances for the design with a missing observation to the prediction variance for the full design. They observed that the minimum ratio of prediction variances were quite robust to missing design points for almost all designs and for all types of missing design points except few. However, relatively little or no research has been conducted on investigating efficiency of split-plot response surface designs in terms of given optimality criteria when some observations are missing. In this work, *A*-, *G*-, and *V*-efficiency of split-plot central composite designs (CCDs) constructed using VKM (2005) format, were investigated for missing pairs of observations of the design points.

1.3 Optimality Criteria

Design optimality criteria are single-valued summaries for quality properties of the design such as the precision with which the model parameters are estimated or the uncertainty associated with prediction in the design region. The four commonly-used optimality criteria include the *D, A, G,* and *V*- optimality criteria.

The notion of optimality criteria in connection with split-plot response surface experiments was first discussed by Letsinger et al. (1996). The authors noted that a *D*-optimal design is one in which $|\mathbf{M}| = |\mathbf{X}'\mathbf{V}^{-1}\mathbf{X}|$ is maximized, or equivalently, $|(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}|$ is minimized. An *A*-optimal design is one in which $Trace(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}$ is minimized. A *G*-Optimal design minimizes the maximum $v(z,x)$, where

$$v(z,x) = \frac{\mathrm{N}}{\mathrm{K}}\int_{\Omega} f(z,x)'(X'R^{-1}X)^{-1} f(z,x)\mathrm{dzdx}$$

is the scaled prediction variance (SPV), and R is the correlation matrix of the responses. Letsinger et al. (1996) also addressed the integrated variance of prediction criterion for split-plot response surface designs and gave the criterion as

$$V \rightarrow \underset{x}{Min}\frac{1}{\mathrm{K}}\int_{\Omega} v(z,\mathrm{x})\mathrm{dzdx}$$

$$= \underset{x}{Min}\frac{N}{K}tr\left[(X'R^{-1}X)^{-1}\int_{\Omega} f(z,x)f(z,x)'dzdx\right]$$

$$= Min\,tr\left\{(X'R^{-1}X)^{-1}\left[\frac{N}{K}\int_{\Omega} f(z,x)f(z,x)'dzdx\right]\right\}$$

$$= Min\,tr\left\{(X'R^{-1}X)^{-1}B\right\}$$

Where $K = \int_{\Omega} dzdx$ and $B = \left[\frac{1}{K}\int_{x.} f(z,x)f(z,x)'dzdx\right]$ are, respectively, the volume of the design region (Ω), and the moment matrix; R is the correlation matrix.

Each of these criteria strongly depends on the unknown variance components only through their ratio, *d*. Webb, Lucas and Borkowski (2004) described an experiment with variance ratio 6.92 in a computer component manufacturing company. Kowalski, Cornell and Vining (2002) studied a mixture experiment with process variables where the estimated variance ratio is 0.82.

The impact of correlated observations on efficiency of experimental designs has received considerable attention in literature and it turns out that the presence of correlation between observations can be beneficial to the design efficiency.

Bradley and Christopher (2009) noted that two alternative split-plot response surface designs d_1 and d_2 can be compared by computing their relative efficiencies in terms of a given criterion. The authors gave the relative *D*_efficiency of a design d_1, relative to another design d_2 as

$$RE_D(d_1, d_2) = \frac{|X_1'V_1^{-1}X_1|}{|X_2'V_2^{-1}X_2|} \qquad (1.3)$$

while the relative *V*_efficiency of d_1, relative to d_2 was also given by the authors as

$$RE_V(d_1, d_2) = \frac{Trace\left([X_2'V_2^{-1}X_2]^{-1}B\right)}{Trace\left([X_1'V_1^{-1}X_1]^{-1}B\right)} = \frac{Trace\left([X_2'R_2^{-1}X_2]^{-1}B\right)}{Trace\left([X_1'R_1^{-1}X_1]^{-1}B\right)} \qquad (1.4)$$

In this work, relative *A*-, *G*-, and *V*-efficiencies of reduced split-plot response surface designs (due to missing observations) relative to the full designs were investigated using specific values of d = 0.5, 1.0, 5.0 and 10.

2.0 Methodology

Three split-plot CCDs of different sizes were constructed using the VKM (2005) format and were used throughout the study to validate the formulated efficiency functions. For each design, the whole-plot and subplot axial points were fixed at equal distance of 1 (i.e., $\alpha = \beta = 1$) and it is assumed that missing observations occur at the subplot level only. The designs include the 4-factor D(2,2) CCD, which consists of two whole-plot and two subplot factors, the 3-factor D(1,2) CCD with one whole-plot and two subplot factors, and the 4-factor D(1,3) CCD with one whole-plot and three subplot factors.
In order to examine the efficiency of the reduced split-plot response surface designs (due to missing observations) relative to the corresponding full design, the following relative efficiency functions were formulated:

$$RE_A = \frac{\mathrm{Trace}\left[(X'V^{-1}X)^{-1}\right]}{\mathrm{Trace}[(X'V^{-1}X)^{-1}]_{reduced}} \quad (2.1)$$

where $Trace[(X'V^{-1}X)^{-1}]$ and $Trace[(X'V^{-1}X)^{-1}]_{reduced}$ are respectively the A-criterion for the full and reduced designs due to a pair of missing observations.

The relative G- and V- efficiencies are respectively:

$$RE_G = \frac{MAX_{Z,X\epsilon R}[v(z,x)]}{MAX_{Z,X\epsilon R}v(z,x)_{reduced}} \quad (2.2)$$

$$RE_V = \frac{Trace\{(M(x))^{-1}B\}}{Trace\{(M(x))^{-1}B\}_{reduced}} \quad (2.3)$$

where $v(z,x)$ is the scaled prediction variance, $(M(\zeta))^{-1}$ is the covariance matrix and $B = \left[\frac{1}{K}\int_{\Omega} f(z,x)f'(z,x)dzdx\right]$ is the moment matrix for a given split-plot response surface design; $\mathbf{f(z,x)}$ is the general form of the 1 x p model vector.
Here we note that

(i) Relative efficiency greater than 1 indicates that the missing observation or combination of observations has little or no adverse effect on the design in terms of the criterion, which implies that the criterion was, to some extent, robust to the missing points.

(ii) Relative efficiency smaller than 1 indicates that the missing point or combination of points has large adverse effect on the design in terms of the criterion.

There are ten possible groups of pairs that are formed from factorial (f), whole-plot axial (α), subplot axial (β), and center points (c). These groups are ff, $\alpha\alpha$, $\beta\beta$, cc, fα, fβ, fc, $\alpha\beta$, αc, and βc. A-, G-, and V-criterion values for the full and corresponding reduced split-plot CCDs due to missing observations of these pairs were first computed and tabulated range of d. However, this criterion was highly affected by the missing pairs of each of factorial (ff) and center (cc) point observations at low values of d, though the effect continue to reduce as d increases.

(2) Four-Factor D(1,3) Split-plot CCD

The scaled prediction variances, G-criterion locations and the V-criterion values were given in Table 3.5. The relative G and V-efficiency plots were given in Figures 3.5a and 3.5b respectively.

From Figure 3.5a, we observed that the G-efficiency was adversely affected by missing pairs of observations of ff and $\beta\beta$ for $d \leq 0.92$ and $d \leq 3.51$ respectively. The effect continues to reduce as d increases. We also observed that this criterion was robust to missing pairs of whole plot axial and center point observations for the whole range of d. Figure 3.5b shows that the highest adverse effect on the relative V-efficiency was due to missing pairs of the subplot axial observations ($\beta\beta$) and the center observations (cc) for small values of d. However, as d increases, the criterion improves. This criterion was observed to be quite robust to the missing pairs of observations of the factorial points (ff) and the whole plot axial points ($\alpha\alpha$) for the whole range of d.

Table 3.1: $A-criterion\ values$ $((X'V^{-1}X)^{-1})$ for complete design and for reduced designs due to a pair of missing observations in D(2,2) under different d

D	tr(M[illegible]) due to missing										
	None	ff	αα	Bβ	Cc	Fα	fβ	Fc	Aβ	αc	βc
0.5	2.724	3.016	2.785	3.412	2.743	2.847	3.163	2.832	3.072	2.753	3.058
1	3.925	4.234	3.989	4.62	3.945	4.05	4.377	4.035	4.284	3.955	4.269
5	13.53	13.86	13.59	14.23	13.55	13.65	13.99	13.64	13.9	13.56	13.88
10	25.53	25.87	25.61	26.24	25.56	25.66	26.01	25.65	25.91	25.56	25.89

Table 3.2. A − criterion values $((X'V^{-1}X)^{-1})$ for complete design and for reduced designs due to a pair of missing observations in **D(1,2)** CCD

D	tr(M[illegible]) due to missing										
	None	ff	αα	Bβ	Cc	Fα	fβ	Fc	Aβ	αc	βc
0.5	1.799	2.223	1.816	2.016	2.132	2.016	2.267	2.120	1.909	1.916	2.014
1	2.616	3.094	2.634	2.839	2.949	2.849	3.111	2.953	2.731	2.733	2.836
5	9.153	9.738	9.171	9.385	9.486	9.408	9.683	9.513	9.275	9.270	9.379
10	17.32	17.93	17.34	17.55	17.65	17.58	17.86	17.68	17.44	17.44	17.552

Table 3.3. A − criterion value $((X'V^{-1}X)^{-1})$ for complete design and for reduced designs due to a pair of missing observations in **D(1,3)** CCD

D	tr(M[illegible]) due to missing

	None	ff	αα	Bβ	Cc	Fα	fβ	Fc	Aβ	αc	βc
0.5	1.483	1.665	1.486	1.589	1.536	1.559	1.625	1.581	1.537	1.507	1.558
1	2.219	2.408	2.222	2.328	2.272	2.297	2.364	2.318	2.274	2.243	2.296
5	8.105	8.303	8.108	8.216	8.158	8.184	8.252	8.205	8.162	8.128	8.183
10	15.46	15.66	15.46	15.57	15.51	15.54	15.61	15.56	15.52	15.49	15.54

Table 3.4. SPV properties and G-criterion location for the full design and for the design with a pair of missing observations for **D(1,2)** split-plot CCD

D	$v_{(z,x)}$ due to missing	Design point ±1	α(1.732)	β(1.732)	0	G – location z_1	x_1	x_2	V
0.5									
	Full	12.786	10.166	14.094	11.999	0.000	1.732	0.000	8.217
	ff	15.338	9.473	15.141	10.999	1.000	1.000	1.000	8.559
	αα	11.782	9.352	12.963	10.999	0.000	0.000	1.732	7.598
	ββ	11.853	9.382	15.097	10.999	0.000	1.732	0.000	8.223
	cc	11.72	9.319	12.92	14.666	0.000	0.000	0.000	9.27
1	-	11.839	12.708	13.904	14.999	0.000	0.000	0.000	9.517
	ff	13.75	11.792	14.589	13.749	0.000	1.732	0.000	9.606
	αα	10.9	11.674	12.778	13.749	0.000	0.000	0.000	8.775
	ββ	10.961	11.7	14.384	13.749	0.000	0.000	1.732	9.263
	cc	10.852	11.649	12.745	16.499	0.000	0.000	0.000	10.028
5	-	9.946	13.523	17.791	20.999	0.000	0.000	0.000	12.119
	ff	10.205	16.375	13.127	19.249	0.000	0.000	0.000	11.476
	αα	9.134	16.317	12.407	19.249	0.000	0.000	0.000	11.126
	ββ	9.158	16.327	12.945	19.249	0.000	0.000	0.000	11.298
	cc	9.117	16.308	12.396	20.166	0.000	0.000	0.000	11.543
10	-	9.516	18.946	13.436	22.363	0.000	0.000	0.000	12.71
	ff	9.331	17.406	12.73	20.499	0.000	0.000	0.000	11.86
	αα	8.732	17.372	12.323	20.499	0.000	0.000	0.000	11.66
	ββ	8.746	17.378	12.616	20.499	0.000	0.000	0.000	11.755
	cc	8.723	17.367	12.316	20.999	0.000	0.000	0.000	11.888

Table 3.5. Scaled prediction variance ($v_{(z,x)}$) properties and G-criterion location for the full design and for the design with a pair of missing observations for **D(1,3)** split-plot CCD

d'	$v_{(z,x)}$	Design point	G – location	V

d	due to missing	±1	α(2.00)	β(2.00)	0	Z_1	X_1	X_2	X_3	
0.5										
	Full	22.597	16.985	24.934	19.167	0.000	2.000	0.000	0.000	11.837
	ff	27.842	16.327	25.081	18.333	1.000	1.000	1.000	1.000	11.723
	αα	21.640	16.255	23.859	18.333	0.000	2.000	0.000	0.000	11.341
	ββ	21.737	16.276	29.691	18.333	0.000	0.000	2.000	0.000	11.933
	cc	21.615	16.246	23.850	19.555	0.000	0.000	0.000	2.000	11.893
1	-	21.221	22.924	24.289	25.875	0.000	0.000	0.000	0.000	15.099
	ff	25.127	21.991	24.168	24.750	1.000	1.000	1.000	1.000	14.757
	αα	20.317	21.934	23.240	24.750	0.000	0.000	0.000	0.000	14.456
	ββ	20.389	21.952	27.675	24.750	0.000	2.000	0.000	0.000	14.913
	cc	20.298	21.928	23.233	25.666	0.000	0.000	0.000	0.000	14.870
5	-	18.464	34.803	22.992	39.291	0.000	0.000	0.000	0.000	21.622
	ff	19.330	33.312	22.309	37.583	0.000	0.000	0.000	0.000	20.792
	αα	17.668	33.292	21.995	37.583	0.000	0.000	0.000	0.000	20.686
	ββ	17.692	33.299	23.496	37.583	0.000	0.000	0.000	0.000	20.844
	cc	17.661	33.290	21.993	37.888	0.000	0.000	0.000	0.000	20.824
10	-	17.837	37.503	22.697	42.340	0.000	0.000	0.000	0.000	23.104
	ff	17.977	35.884	21.883	40.500	0.000	0.000	0.000	0.000	22.160
	αα	17.065	35.873	21.711	40.500	0.000	0.000	0.000	0.000	22.102
	ββ	17.078	35.877	22.532	40.500	0.000	0.000	0.000	0.000	22.189
	cc	17.062	35.872	21.710	40.666	0.000	0.000	0.000	0.000	22.178

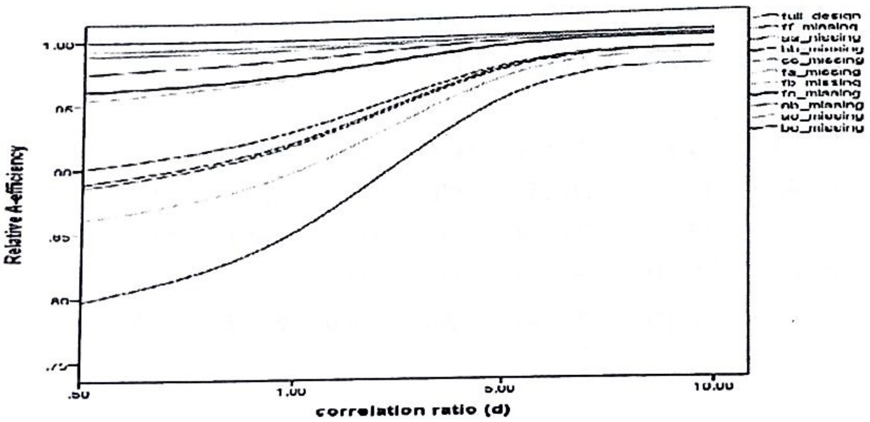

Fig. 3.1. Relative A-efficiency curves for the reduced and full split-plot D(2,2) CCDs under different correlation ratios

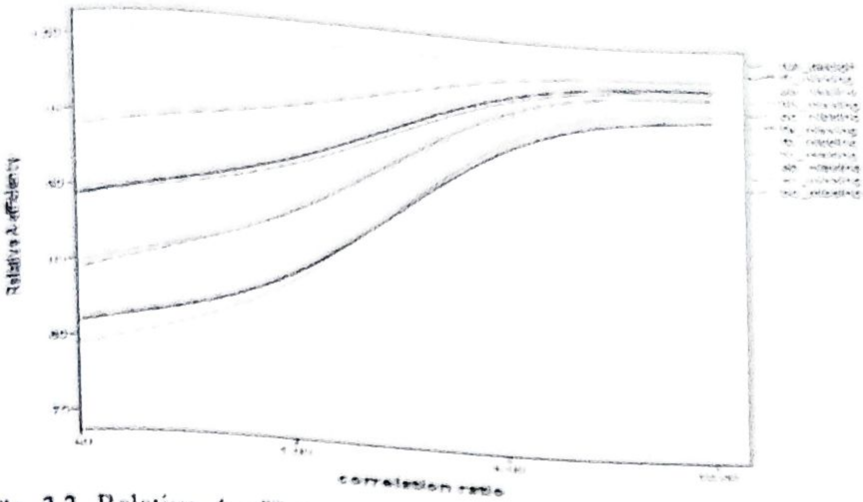

Fig. 3.2. Relative A-efficiency curves for the reduced and full split-plot D(1,2) CCDs under different variance ratios

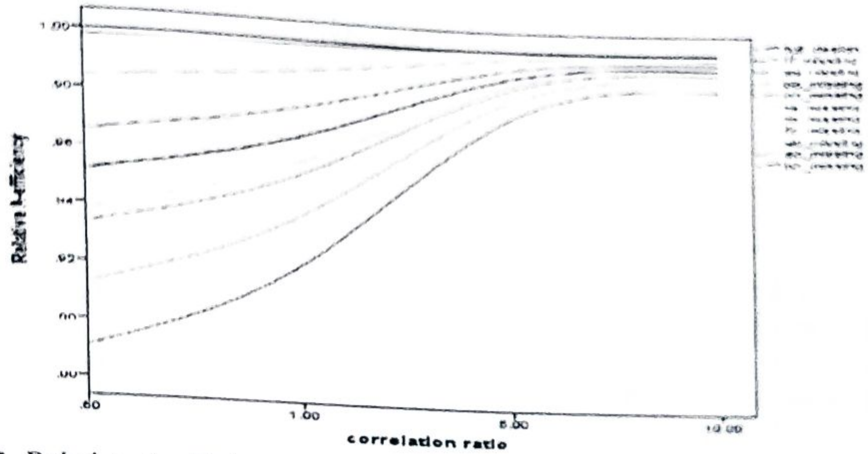

Fig. 3.3. Relative A-efficiency curves for the reduced and full split-plot D(1,3) CCDs under different variance ratios

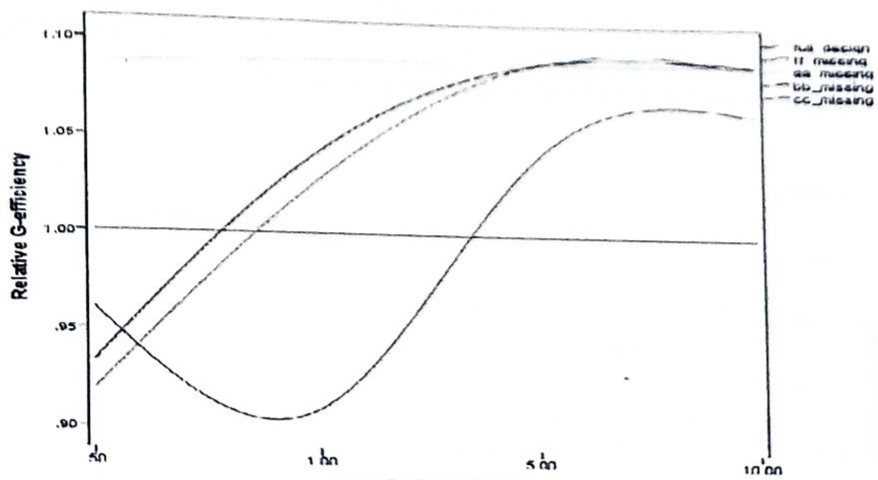

Fig. 3.4a. Relative G-efficiency curves of the reduced split-plot CCDs for the full quadratic model in **one** whole plot and **two** subplot variables under different correlation ratios

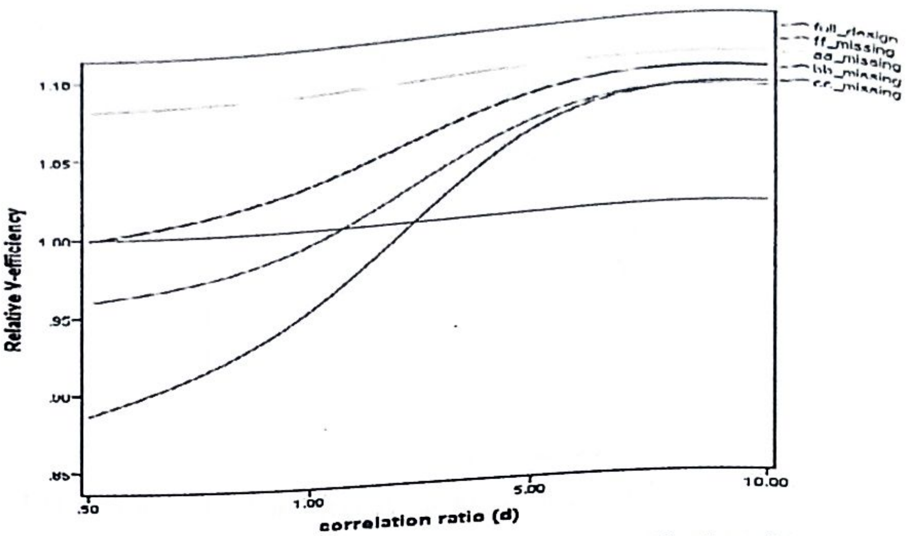

Fig. 3.4b. Relative V-efficiency curves of the reduced split-plot CCDs for the full quadratic model in **one** whole plot and **two** subplot variables for various degrees of correlation

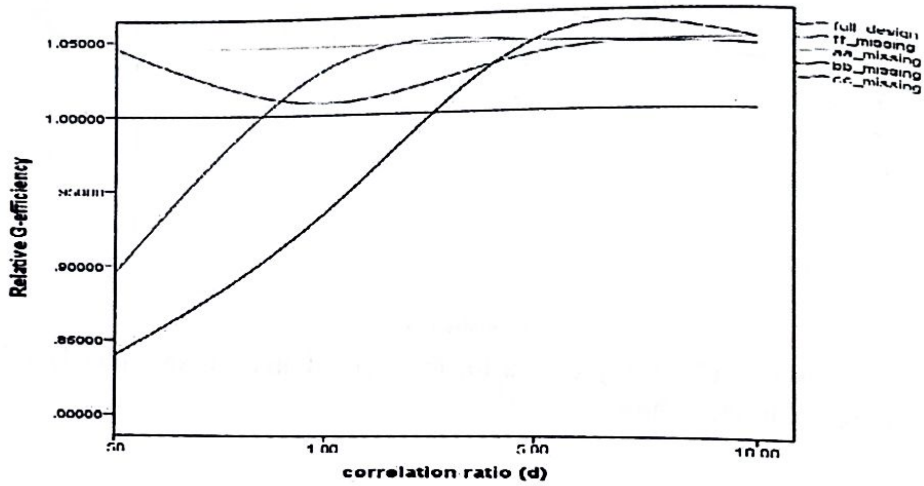

Fig. 3.5a. Relative G-Efficiency curves of the reduced split-plot CCDs for the full quadratic model in D(1,3) CCD under different variance ratios

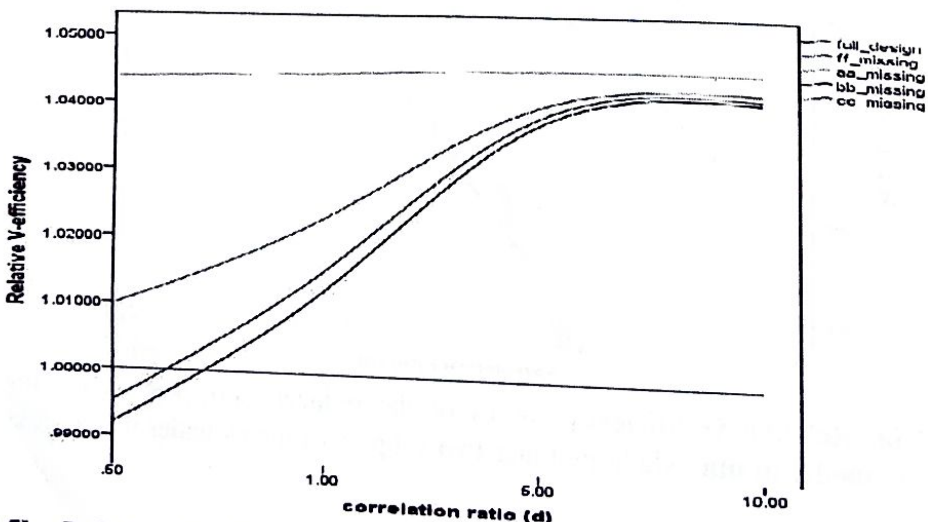

Fig. 3.5b. Relative V-efficiency curves of the reduced split-plot CCDs for the full quadratic model in D(1,3) CCD under different variance ratios

4.0 Conclusion

The study revealed that relative efficiency of these designs in terms of the given criteria was adversely affected by missing pairs of observations and that the effect depends strongly on the category of points that constitute the pairs and also on the correlation ratio (d). This implies that some points are more influential than others.

It was shown that the relative A-efficiency appears to be slightly robust to the missing pairs of observations of cc, αα, αc, fc, and fα, and adversely affected at low values of d, by the missing pairs of ββ, ff and fβ. This implies that care should be taken in handling observations of such influential points during experimentation.

For the relative G and V-efficiencies, the study shows that missing pairs of the whole-plot axial point (αα) was not as influential as those of ff, ββ, and cc, as each of the relative efficiencies was robust to the missing αα, and adversely affected by each of the missing ff, ββ, and cc.

The study revealed that even for the missing points with considerable effects on the efficiency, their effects continue to disappear as the value of d increases, which indicates that the presence of correlation between observations was beneficial to the design efficiency and that the higher this correlation, the more efficient the design in terms of the given criteria.

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APPENDIX

Table 1.1. Design Matrix for VKM (2005) D(1,2) split-plot CCD

Wp	z	x1	x2	n	category
1	-1	-1	-1	4	factorial
	-1	1	-1		
	-1	-1	1		
	-1	1	1		
2	1	-1	-1	4	factorial
	1	1	-1		
	1	-1	1		
	1	1	1		
3	-α	0	0	4	Whole-plot axial
	- α	0	0		
	- α	0	0		
	- α	0	0		
4	+α	0	0	4	Whole-plot axial
	+α	0	0		
	+α	0	0		
	+α	0	0		
5	0	- β	0	4	Subplot axial
	0	+β	0		
	0	0	- β		
	0	0	+β		
6	0	0	0	4	center
	0	0	0		
	0	0	0		
	0	0	0		