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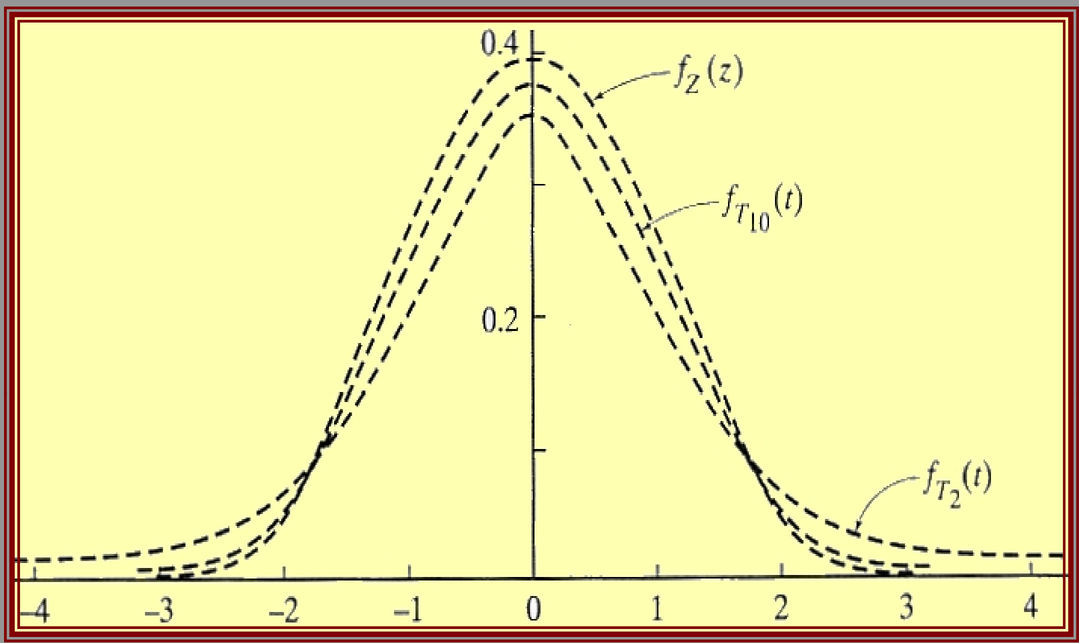

JOURNAL OF PROBABILITY AND STATISTICAL SCIENCE

Volume 19 Number 1 February 2021

ISSN 1726-3328

JPSS

Journal of Probability and Statistical Science

Volume 19 Number 1 February 2021

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Published by: Susan Rivers' Cultural Institute, Hsinchu City, Taiwan, ROC.

Journal of Probability and Statistical Science
19(1), 67-81, Feb. 2021

Topp Leone Exponentiated Lomax Distribution and Its Application to Breast and Bladder Cancer Data

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ABSTRACT In this study, a new four-parameter lifetime distribution called the Topp Leone exponentiated Lomax distribution was introduced. An expansion for the probability distribution function and cumulative density function was carried out which was used to derive the moments and moment generating function of the new model. Some mathematical properties of the distribution such as the moments, moment generating function, quantile function, survival, hazard, reversed hazard and odds functions were presented. The estimation of the parameters by maximum likelihood method was discussed. Two real life data sets were used to show the fit and exibility of the new distribution over some lifetime distributions in literature. The results showed that the new distribution fits better in the two datasets considered.

Keywords Linear representation; Reliability function; Real-life, Reversed-J shaped, Symmetrical distributions.

1. Introduction

The Lomax distribution (Pareto distribution of the second kind) introduced in [13] has, in recent years, assumed a position of importance in the field of life testing because of its uses to fit business failure data but it is now been used in the field of reliability and life testing [6]. It has also found application in the area of biological sciences and even for modelling the distribution of sizes of computer files on servers [7]. The two parameter Lomax distribution has the cumulative distribution function (cdf) and probability density function (pdf) given respectively as

Received September 2020, revised December 2020, in final form January 2021.
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 JPSS: ISSN 1726-3328

$$H(x;\beta,\lambda)=1-(1+\beta x)^{-\lambda}, \tag{1}$$

$$h(x;\beta,\lambda)=\lambda\beta(1+\beta x)^{-(\lambda+1)}, \tag{2}$$

$x\geq 0,\ \ \beta,\lambda>0,$ where β is the scale parameter and λ is the shape parameter.

The Topp Leone distribution proposed in [20] is a J-shaped distribution. Hu *et al*. [8] note that the J-shaped distribution creates some fundamental statistical problems that is, the mean, a common statistic used for calculating future sales, product quality, consumer satisfaction and other metrics only works for unimodal distributions (distributions with one hump). The J-shaped distribution is bimodal (two humps); when a mean is calculated from a bimodal distribution, it essentially becomes meaningless. For details on J-shape distribution see [2, 3, 10, 19]. The Topp Leone distribution was used to develop the Topp Leone family of distributions in [18]. Developing families of distribution is a new trend in probability distribution theory; these families of distributions are proposed purposely for developing new compound distributions which of course are expected to be better than their existing parent distribution.

Topp-Leone family of distribution has been used by a number of authors: Topp Leone generated Weibull distribution in [4], Topp Leone inverse Weibull distribution in [1] and Topp Leone Burr XII distribution in [16], the Topp Leone odd log-logistic family of distributions in [5], the transmuted Topp Leone G family of distributions in [21], the exponentiated Topp Leone distribution in [14] and the Topp Leone exponentiated-G family of distributions in [9].

In this context, we proposed an extension of the Lomax distribution based on [9] which stems from the following general construction: if H denotes the baseline cumulative function of a random variable, then a generalized class of distributions can be defined by

$$F(x;\alpha,\theta,\xi)=\left[1-\{1-[H(x;\xi)]^{\alpha}\}^{2}\right]^{\theta}. \tag{3}$$

The pdf corresponding to (3) is given as

$$f(x;\alpha,\theta,\xi)=(2\alpha\theta)h(x;\xi)[H(x;\xi)]^{\alpha-1}\{1-[H(x;\xi)]^{\alpha}\}\left[1-\{1-[H(x;\xi)]^{\alpha}\}^{2}\right]^{\theta-1}, \tag{4}$$

$x\geq 0,\ \ \alpha,\theta,\xi>0,$ where $\alpha,\theta>0$ are two additional shape parameters which govern skewness and tail weights and $\xi>0$ is another parameter.

The aim of this paper is to extend the Lomax distribution by using the Topp Leone exponentiated-G family of distributions proposed in [9].

The rest of the paper is outlined as follows. In Section 2, we define the Topp Leone exponentiated Lomax distribution and provide expansions for its cdf and pdf. Some mathematical properties of this distribution are considered in Sections 3. Order statistics of the distribution is presented in Section 4. Maximum likelihood estimation is performed in Section 5. In Section 6, real-life application of the distribution to data sets is provided. Finally, concluding remarks are presented in Section 7.

2. The Topp Leone Exponentiated Lomax (TLExLx) Distribution

In this section, we derive a new life distribution called TLExLx distribution which extends the Lomax distribution. The cdf of the new distribution is obtained by inserting (1) into (3) to get

$$F(x;\alpha,\theta,\beta,\lambda)=\left[1-\{1-[1-(1+\beta x)^{-\lambda}]^{\alpha}\}^{2}\right]^{\theta}. \tag{5}$$

The pdf corresponding to (5) is given as

$$f(x;\alpha,\theta,\beta,\lambda)=2\alpha\theta\lambda\beta(1+\beta x)^{-(\lambda+1)}[1-(1+\beta x)^{-\lambda}]^{\alpha-1}\{1-[1-(1+\beta x)^{-\lambda}]^{\alpha}\}$$
$$\times\left[1-\{1-[1-(1+\beta x)^{-\lambda}]^{\alpha}\}^{2}\right]^{\theta-1}, \tag{6}$$

$x\geq 0,\ \alpha,\theta,\beta,\lambda>0,$ where α,θ,λ are the shape parameters and β is the scale parameter.

3. Expansion for Densities

In this section, linear representation of the cdf and pdf of the TLExLx distribution are presented. Using the generalized binomial expansion given as

$$(1-y)^{b-1}=\sum_{i=0}^{\infty}\frac{(-1)^{i}\Gamma(b)}{(i!)\Gamma(b-i)}y^{i}. \tag{7}$$

Using (5) in relation to (7), we have

$$\left[1-\{1-[1-(1+\beta x)^{-\lambda}]^{\alpha}\}^{2}\right]^{\theta}=\sum_{i=0}^{\infty}\frac{(-1)^{i}\Gamma(\theta+1)}{(i!)\Gamma(\theta+1-i)}\{1-[1-(1+\beta x)^{-\lambda}]^{\alpha}\}^{2i}, \tag{8}$$

$$\{1-[1-(1+\beta x)^{-\lambda}]^{\alpha}\}^{2i}=\sum_{j=0}^{\infty}\frac{(-1)^{j}\Gamma(2i+1)}{(j!)\Gamma(2i+1-j)}[1-(1+\beta x)^{-\lambda}]^{\alpha j}, \tag{9}$$

$$[1-(1+\beta x)^{-\lambda}]^{\alpha j}=\sum_{k=0}^{\infty}\frac{(-1)^{k}\Gamma(\alpha j+1)}{(k!)\Gamma(\alpha j+1-k)}(1+\beta x)^{-\lambda k}, \tag{10}$$

and

$$F(x)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+j+k}\Gamma(\theta+1)\Gamma(2i+1)\Gamma(\alpha j+1)}{(i!)(j!)(k!)\Gamma(\theta+1-i)\Gamma(2i+1-j)\Gamma(\alpha j+1-k)}(1+\beta x)^{-\lambda k}. \tag{11}$$

Also, using the last term in (6) in relation to (7), we have

$$\left[1-\{1-[1-(1+\beta x)^{-\lambda}]^{\alpha}\}^{2}\right]^{\theta-1}=\sum_{i=0}^{\infty}\frac{(-1)^{i}\Gamma(\theta)}{(i!)\Gamma(\theta-i)}\{1-[1-(1+\beta x)^{-\lambda}]^{\alpha}\}^{2i}, \tag{12}$$

$$\{1-[1-(1+\beta x)^{-\lambda}]^{\alpha}\}^{2i+1}=\sum_{j=0}^{\infty}\frac{(-1)^{j}\Gamma[2(i+1)]}{(j!)\Gamma[2(i+1)-j]}[1-(1+\beta x)^{-\lambda}]^{\alpha j}, \tag{13}$$

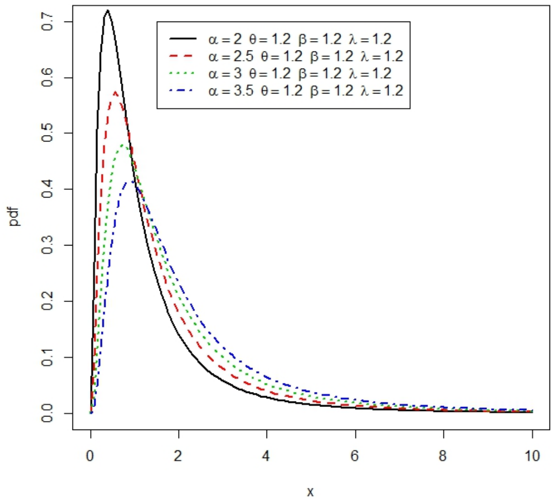
$$[1-(1+\beta x)^{-\lambda}]^{\alpha(j+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma[\alpha(j+1)]}{(k!)\Gamma[\alpha(j+1)-k]}(1+\beta x)^{-\lambda k}, \tag{14}$$

and

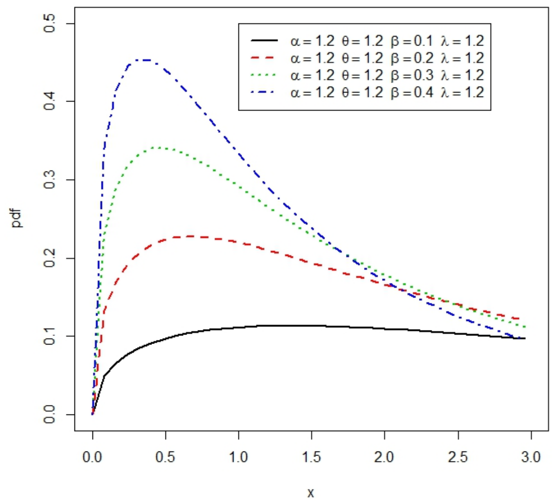
$$f(x) = 2\alpha\theta\lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{i+j+k}\Gamma(\theta)\Gamma[2(i+1)]\Gamma[\alpha(j+1)]}{(i!)(j!)(k!)\Gamma(\theta-i)\Gamma[2(i+1)-k]\Gamma[\alpha(j+1)-k]}(1+\beta x)^{-\lambda(1+k)-1}. \tag{15}$$

Equation (15) is the linear representation of cdf in (6) from which some properties of the distribution can be derived.

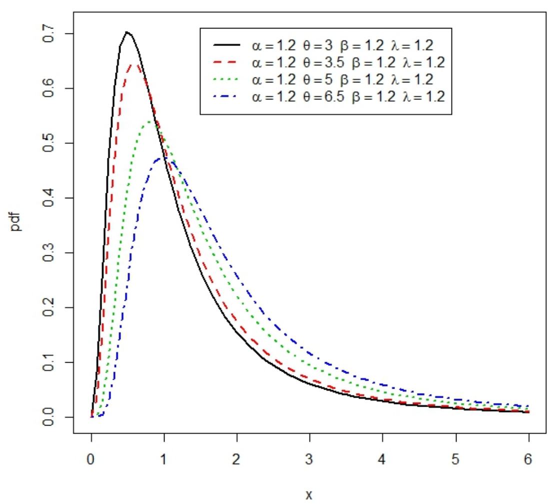
Plots of the pdf of the TLExLx distribution for different parameter values are given in Figure 1.



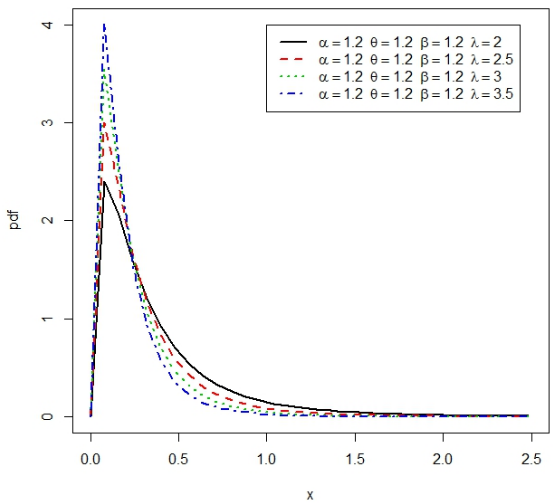
(a) Parameter α varies, other parameters fixed at 1.2



(b) Parameter β varies, other parameters fixed at 1.2



(c) Parameter θ varies, other parameters fixed at 1.2



(d) Parameter λ varies, other parameters fixed at 1.2

Figure 1 Plots of the pdf of the TLExLx distribution for different parameter values

4. Properties

4.1 Moments

Assume Y is a Lomax distributed random variable with parameters β and λ, then the rth moment of Y is given as

$$E(Y^r)=\left(\frac{\lambda}{\beta^r}\right)B(r+1,\lambda-r). \tag{16}$$

Let X be a random variable having the TLExLx distribution. Using the expansion in (15), it is easy to obtain the rth moment of X as

$$E(Y^r)=2\alpha\theta\lambda\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left[\frac{(-1)^{i+j+k}\Gamma(\theta)\Gamma[2(i+1)]\Gamma[\alpha(j+1)]}{(i!)(j!)(k!)\Gamma(\theta-i)\Gamma[2(i+1)-k]\Gamma[\alpha(j+1)-k]}\right.$$

$$\left.\left(\frac{\lambda(i+1)}{\beta^r}\right)B[r+1,\lambda(i+1)-r]\right]. \tag{17}$$

4.2 Moment Generating Function

$$M_X(t)=\int_0^{\infty}e^{tx}f(x)dx. \tag{18}$$

Since the series expansion for e^{tx} is given as

$$e^{tx}=\sum_{m=0}^{\infty}\frac{(tx)^m}{m!}, \tag{19}$$

$$M_X(t)=2\alpha\theta\lambda\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\left[\frac{(-1)^{i+j+k}\Gamma(\theta)\Gamma[2(i+1)]\Gamma[\alpha(j+1)](t^m)}{(i!)(j!)(k!)(m!)\Gamma(\theta-i)\Gamma[2(i+1)-k]\Gamma[\alpha(j+1)-k]}\right.$$

$$\left.\left(\frac{\lambda(i+1)}{\beta^m}\right)B[m+1,\lambda(i+1)-m]\right]. \tag{20}$$

4.3 Reliability Function

The reliability function is also known as survival function, which is the probability of an item not failing prior to some time. It can be defined as

$$R(x;\alpha,\theta,\beta,\lambda)=1-F(x;\alpha,\theta,\beta,\lambda), \tag{21}$$

$$R(x;\alpha,\theta,\beta,\lambda)=1-\left[1-\{1-[1-(1+\beta x)^{-\lambda}]^{\alpha}\}^2\right]^{\theta}. \tag{22}$$

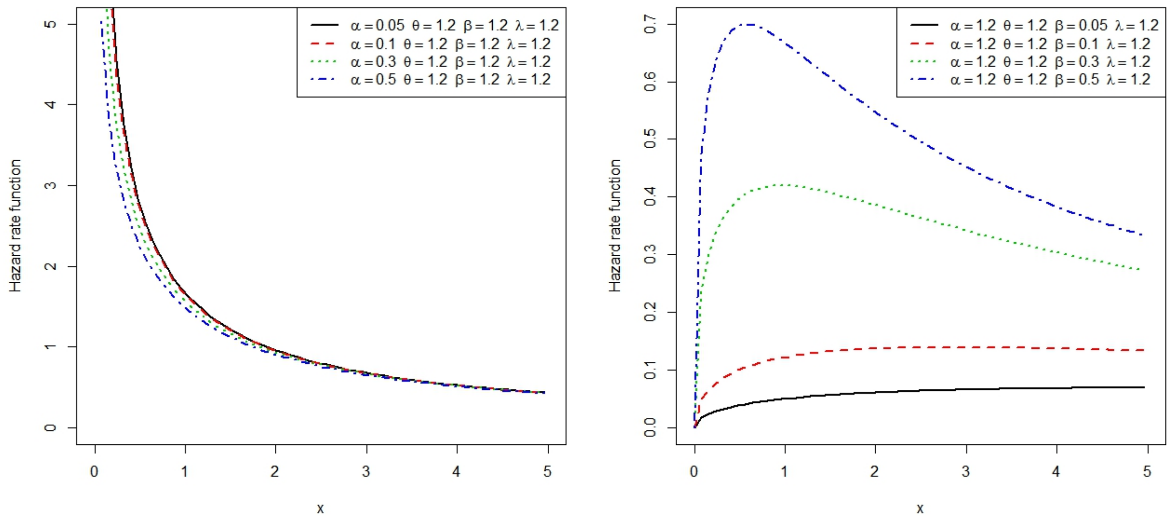
4.4 Hazard Rate Function

The hazard rate function (hrf) is given as

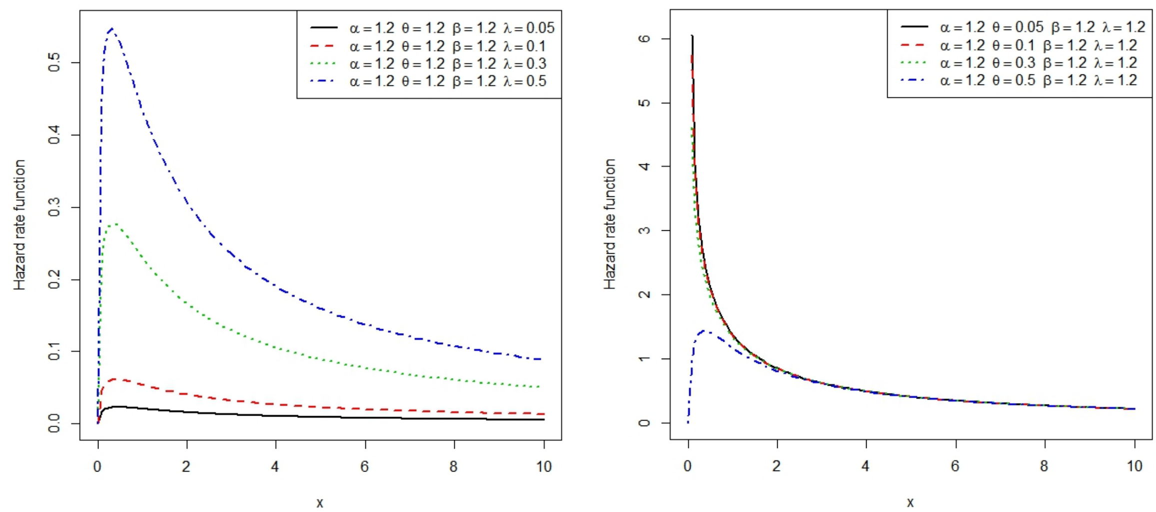
$$\tau(x;\alpha,\theta,\beta,\lambda)=\frac{f(x;\alpha,\theta,\beta,\lambda)}{R(x;\alpha,\theta,\beta,\lambda)}, \tag{23}$$

$$\tau(x;\alpha,\theta,\beta,\lambda)=\frac{1}{1-\left[1-\{1-[1-(1+\beta x)^{-\lambda}]^{\alpha}\}^{2}\right]^{\theta}}\Big[2\alpha\theta\lambda\beta(1+\beta x)^{-(\lambda+1)}[1-(1+\beta x)^{-\lambda}]^{\alpha-1}$$

$$\times\{1-[1-(1+\beta x)^{-\lambda}]^{\alpha}\}\left[1-\{1-[1-(1+\beta x)^{-\lambda}]^{\alpha}\}^{2}\right]^{\theta-1}\Big]. \tag{24}$$



(a) Parameter α varies, other parameters fixed at 1.2 (b) Parameter β varies, other parameters fixed at 1.2



(c) Parameter λ varies, other parameters fixed at 1.2 (d) Parameter θ varies, other parameters fixed at 1.2

Figure 2 Plots of the hrf for different parameter values

4.5 Reversed Hazard Rate Function

The reversed hazard rate function for the TLExLx distribution is obtained by dividing (6) by (5) to give

$$r(x) = \frac{f(x)}{R(x)}, \tag{25}$$

$$r(x) = \frac{1}{\left[1-\{1-[1-(1+\beta x)^{-\lambda}]^{\alpha}\}^{2}\right]^{\theta}}\Big[2\alpha\theta\lambda\beta(1+\beta x)^{-(\lambda+1)}[1-(1+\beta x)^{-\lambda}]^{\alpha-1}$$

$$\times\{1-[1-(1+\beta x)^{-\lambda}]^{\alpha}\}\left[1-\{1-[1-(1+\beta x)^{-\lambda}]^{\alpha}\}^{2}\right]^{\theta-1}\Big]. \tag{26}$$

4.6 Odds Function

The odds function for the TLExLx distribution is obtained by dividing (5) by (22) to give

$$Q(x) = \frac{F(x)}{R(x)}, \tag{27}$$

$$Q(x) = \frac{\left[1-\{1-[1-(1+\beta x)^{-\lambda}]^{\alpha}\}^{2}\right]^{\theta}}{1-\left[1-\{1-[1-(1+\beta x)^{-\lambda}]^{\alpha}\}^{2}\right]^{\theta}}. \tag{28}$$

4.7 Quantile Function

The quantile function is defined as the inverse of the cdf and it is given as $Q(x) = F^{-1}(u)$. Using the cdf of TLExLx distribution in (5), we have

$$F(x;\alpha,\theta,\beta,\lambda) = \left[1-\{1-[1-(1+\beta x)^{-\lambda}]^{\alpha}\}^{2}\right]^{\theta} = u$$

and

$$x = Q(u) = \frac{1}{\beta}\left[\{1-[1-(1-u^{-1/\theta})^{1/2}]^{1/\alpha}\}^{-1/\lambda} - 1\right]. \tag{29}$$

The median of the TLExLx distribution can be derived by substituting $u = 0.5$ in (29) as follows:

$$x_m = Q(0.5) = \frac{1}{\beta}\left[\{1-[1-(1-(0.5)^{-1/\theta})^{1/2}]^{1/\alpha}\}^{-1/\lambda} - 1\right]. \tag{30}$$

5. Distribution of Order Statistic

Let $X_1, X_2, \ldots, X_n$ be n independent random variable from the TLExLx distributions and let $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ be their corresponding order statistic. Let $F_{r:n}(x)$ and $f_{r:n}(x)$,

$r=1,2,\ldots,n$ denote the cdf and pdf of the rth order statistics $X_{r:n}$ respectively. The pdf of the rth order statistic of $X_{r:n}$ is given as

$$f_{r:n}(x)=\frac{1}{B(r,n-r+1)}\sum_{i=0}^{n-r}(-1)^i[F(x)]^{r+i-1}f(x). \tag{31}$$

Using the cdf and pdf of TLExIEx distribution in (5) and (6), we have

$$\begin{aligned}
f_{r:n}(x)&=\frac{1}{B(r,n-r+1)}\sum_{i=0}^{n-r}\Big[(-1)^i\left[1-\{1-[1-(1+\beta x)^{-\lambda}]^{\alpha}\}^2\right]^{\theta(r+i-1)}(2\alpha\theta\lambda\beta)(1+\beta x)^{-(\lambda+1)}\\
&\quad\times[1-(1+\beta x)^{-\lambda}]^{\alpha-1}\{1-[1-(1+\beta x)^{-\lambda}]^{\alpha}\}\left[1-\{1-[1-(1+\beta x)^{-\lambda}]^{\alpha}\}^2\right]^{\theta-1}\Big]\\
&=\frac{(2\alpha\theta\lambda\beta)}{B(r,n-r+1)}\sum_{i=0}^{n-r}\Big[(-1)^i\left[1-\{1-[1-(1+\beta x)^{-\lambda}]^{\alpha}\}^2\right]^{\theta(r+i-1)}(1+\beta x)^{-(\lambda+1)}\\
&\quad\times[1-(1+\beta x)^{-\lambda}]^{\alpha-1}\{1-[1-(1+\beta x)^{-\lambda}]^{\alpha}\}\left[1-\{1-[1-(1+\beta x)^{-\lambda}]^{\alpha}\}^2\right]^{\theta-1}\Big]\\
&=\frac{2\alpha\theta\lambda\beta}{B(r,n-r+1)}\sum_{i=0}^{n-r}\sum_{j,k,l,m=0}^{\infty}\left[(-1)^{i+j+k+l+m}\binom{\theta(r+i)-1}{j}\binom{2j+1}{k}\binom{\alpha(k+1)-1}{l}\right.\\
&\quad\left.\times\binom{-\lambda(k+1)-1}{m}(\beta x)^m\right].
\end{aligned} \tag{32}$$

Eq. (32) is the rth order statistic of the TLExLx distribution. Therefore, the pdf of the minimum and maximum order statistics of the TLExLx distribution are obtained by setting $r=1$ and $r=n$ in (32) respectively.

6. Estimation

In this section, we estimate the parameters of the TLExLx distribution using maximum likelihood estimation (MLE). For a random sample $X_1,X_2,\ldots,X_n$ of size n from the TLExLx$(\alpha,\beta,\theta,\lambda)$, the log-likelihood function $L(\alpha,\beta,\theta,\lambda)$ of (6) is given as

$$\begin{aligned}
\log L&=n\log 2+n\log\alpha+n\log\lambda+n\log\beta+n\log\theta-(\lambda+1)\sum_{i=1}^{n}\log(1+\beta x_i)\\
&\quad+(\alpha-1)\sum_{i=1}^{n}[1-(1+\beta x_i)^{-\lambda}]+\sum_{i=1}^{n}\log\{1-[1-(1+\beta x_i)^{-\lambda}]^{\alpha}\}\\
&\quad+(\theta-1)\sum_{i=1}^{n}\log\left[1-\{1-[1-(1+\beta x_i)^{-\lambda}]^{\alpha}\}^2\right].
\end{aligned} \tag{33}$$

Differentiating the log-likelihood with respect to $\alpha,\beta,\theta,\lambda$ and setting the result equals to zero, we have

$$\frac{\partial \log L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}[1-(1+\beta x_i)^{-\lambda}] + \sum_{i=1}^{n}\frac{[1-(1+\beta x_i)^{-\lambda}]^{\alpha}\log[1-(1+\beta x_i)^{-\lambda}]}{1-[1-(1+\beta x_i)^{-\lambda}]^{\alpha}}$$
$$+(\theta-1)\sum_{i=1}^{n}\frac{2\alpha\{1-[1-(1+\beta x_i)^{-\lambda}]^{\alpha}\}[1-(1+\beta x_i)^{-\lambda}]^{\alpha}\log[1-(1+\beta x_i)^{-\lambda}]^{\alpha}}{1-\{1-[1-(1+\beta x_i)^{-\lambda}]^{\alpha}\}^{2}} = 0, \quad (34)$$

$$\frac{\partial \log L}{\partial \beta} = \frac{n}{\beta} - (\lambda+1)\sum_{i=1}^{n}\frac{x_i}{1+\beta x_i} + (\alpha-1)\sum_{i=1}^{n}\frac{\lambda(1+\beta x_i)^{-\lambda-1}x_i}{1-(1+\beta x_i)^{-\lambda}}$$
$$+\sum_{i=1}^{n}\frac{\alpha[1-(1+\beta x_i)^{-\lambda}]^{\alpha-1}\lambda(1+\beta x_i)^{-\lambda-1}x_i}{1-[1-(1+\beta x_i)^{-\lambda}]^{\alpha}}$$
$$+(\theta-1)\sum_{i=1}^{n}\frac{2\{1-[1-(1+\beta x_i)^{-\lambda}]^{\alpha}\}\alpha[1-(1+\beta x_i)^{-\lambda}]^{\alpha-1}\lambda(1+\beta x_i)^{-\lambda-1}x_i}{1-\{1-[1-(1+\beta x_i)^{-\lambda}]^{\alpha}\}^{2}} = 0, \quad (35)$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n}\log(1+\beta x_i) + (\alpha-1)\sum_{i=1}^{n}\frac{(1+\beta x_i)^{-\lambda}\log[(1+\beta x_i)^{-\lambda}]}{1-(1+\beta x_i)^{-\lambda}}$$
$$+\sum_{i=1}^{n}\frac{\alpha[1-(1+\beta x_i)^{-\lambda}]^{\alpha-1}(1+\beta x_i)^{-\lambda}\log[(1+\beta x_i)^{-\lambda}]}{1-[1-(1+\beta x_i)^{-\lambda}]^{\alpha}}$$
$$+(\theta-1)\sum_{i=1}^{n}\frac{2\{1-[1-(1+\beta x_i)^{-\lambda}]^{\alpha}\}\alpha[1-(1+\beta x_i)^{-\lambda}]^{\alpha-1}(1+\beta x_i)^{-\lambda}\log[(1+\beta x_i)^{-\lambda}]}{1-\{1-[1-(1+\beta x_i)^{-\lambda}]^{\alpha}\}^{2}} = 0, \quad (36)$$

and

$$\frac{\partial \log L}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\log\left[1-\{1-[1-(1+\beta x_i)^{-\lambda}]^{\alpha}\}^{2}\right] = 0. \quad (37)$$

Now, equations (34) to (37) do not have a simple form and are therefore intractable. As a result, we have to resort to non-linear estimation of the parameters using iterative procedures with a package in R called Adequacy Model.

7. Application to Real-Life Data Set

In this section, two real-life data sets are applied to the TLExLx distribution to assess the exibility of the new distribution. The two data sets are fitted to the TLExLx distribution and other extensions of the Lomax distributions such as the Topp Leone Lomax (TLLx), Exponentiated Lomax (ExLx) and Lomax (Lx) distributions.

- TLLx distribution: $f(x;\theta,\lambda,\beta) = 2\theta\lambda\beta(1+\beta x)^{-(2\lambda+1)}[1-(1+\beta x)^{-2\lambda}]^{\theta-1}$. (38)
- ExLx distribution: $f(x;\theta,\lambda,\beta) = \theta\lambda\beta(1+\beta x)^{-(\lambda+1)}[1-(1+\beta x)^{-\lambda}]^{\theta-1}$. (39)
- Lx distribution: $f(x;\lambda,\beta) = \lambda\beta(1+\beta x)^{-(\lambda+1)}$. (40)

Data Set 1: The first data set was given in [11] and it represents the survival times of one hundred and twenty-one (121) patients with breast cancer obtained from a large hospital in a period from 1929 to 1938. It has also been applied in [15]. The data set is as follows:

0.3, 0.3, 4.0, 5.0, 5.6, 6.2, 6.3, 6.6, 6.8, 7.4, 7.5, 8.4, 8.4, 10.3, 11.0, 11.8, 12.2, 12.3, 13.5, 14.4, 14.4, 14.8, 15.5, 15.7, 16.2, 16.3, 16.5, 16.8, 17.2, 17.3, 17.5, 17.9, 19.8, 20.4, 20.9, 21.0, 21.0, 21.1, 23.0, 23.4, 23.6, 24.0, 24.0, 27.9, 28.2, 29.1, 30.0, 31.0, 31.0, 32.0, 35.0, 35.0, 37.0, 37.0, 37.0, 38.0, 38.0, 38.0, 39.0, 39.0, 40.0, 40.0, 40.0, 41.0, 41.0, 41.0, 42.0, 43.0, 43.0, 43.0, 44.0, 45.0, 45.0, 46.0, 46.0, 47.0, 48.0, 49.0, 51.0, 51.0, 51.0, 52.0, 54.0, 55.0, 56.0, 57.0, 58.0, 59.0, 60.0, 60.0, 60.0, 61.0, 62.0, 65.0, 65.0, 67.0, 67.0, 68.0, 69.0, 78.0, 80.0, 83.0, 88.0, 89.0, 90.0, 93.0, 96.0, 103.0, 105.0, 109.0, 109.0, 111.0, 115.0, 117.0, 125.0, 126.0, 127.0, 129.0, 129.0, 139.0, 154.0.

Data Set 2: The second data set was given in [12] and it represents the remission times (in months) of a random sample of one hundred and twenty-eight (128) bladder cancer patients. The data set is as follows:

0.08, 2.09, 3.48, 4.87, 6.94 , 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46 , 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

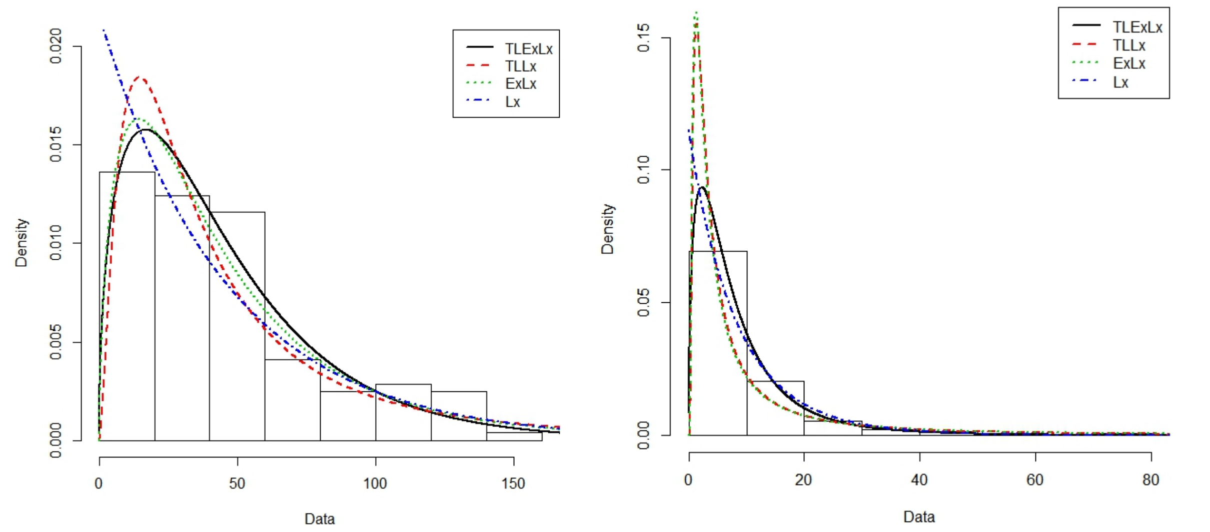
The maximum likelihood estimates and the Information Criteria values for the fitted distributions are reported in Table 1 and Table 2. The results show that the TLExLx distribution provides a significantly better fit than the other three models considered base on the values of Akaike Information Criterion (AIC) in [17], Consistent Akaikes Information Criterion) (CAIC), Bayesian Information Criterion (BIC) and Hannan-Quinn information criterion (HQIC), and the plotted graphs of the fitted models in Figure 3.

Table 1 The MLEs and Information Criteria of the models based on Data Set 1

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	$-l$	AIC	CAIC	BIC	HQIC
TLExLx	2.1540	0.0023	0.7082	7.3091	580.2862	1168.5720	1168.9170	1179.7560	1173.114
TLLx	-	0.0331	2.7213	0.9616	592.7610	1191.5220	1191.7270	1199.9090	1194.9280
ExLx	-	0.0048	1.6403	6.6481	582.6266	1171.2530	1171.4580	1179.8410	1174.9660
Lx	-	0.0002	-	114.6219	585.3535	1174.7070	1174.8090	1180.2980	1176.9780

Table 2 The MLEs and Information Criteria of the models based on Data Set 2

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	$-l$	AIC	CAIC	BIC	HQIC
TLExLx	1.9181	0.0576	0.8017	2.0768	409.9765	827.9531	828.2979	839.1363	832.4950
TLLx	-	4.7328	11.4809	0.4412	437.8107	881.6213	881.8265	890.0087	885.0278
ExLx	-	9.3867	16.8280	0.8267	440.8362	887.6724	887.8776	896.0598	885.0278
Lx	-	0.0086	-	13.4397	413.8335	831.6670	831.7687	896.0598	891.9379



(a) Histogram of data and fitted pdfs for the data set 1 (b) Histogram of data and fitted pdfs for the data set 2

Figure 3 Histogram of data and fitted pdfs for the two data sets.

8. Conclusion

A new distribution called the Topp Leone exponentiated Lomax distribution which extends the Lomax distribution has successfully been established. An expansion for the cdf and pdf was derived which was now used to generate some properties of the distribution such as the moments and moment generating function. Some other properties were derived such as the survival function, hazard rate and reverse hazard rate functions, odds function, quantile function, the median and order statistics. The estimation of parameters by the method of the maximum likelihood was carried out using a package in R known as Adequacy Model. Application of the Topp Leone exponentiated Lomax distribution to two real datasets show from Table 1 and Table 2 that the Topp Leone exponentiated Lomax distribution is quite effective and superior in fitting the two datasets considered. Future research can be done on the characterization and also check for the asymptotic behaviour of the model.

Acknowledgements

The authors are grateful to the Editor-in-Chief, the Associate Editor and anonymous reviewers for their constructive comments and suggestions which led to remarkable improvement of the paper.

Authors' contributions: S.I proposed the Topp Leone exponentiated lomax distribution and wrote the initial draft of the manuscript. The authors S.I.D, A.I and H.M.J. finalized this work. All authors read and approved the final manuscript.

Funding: Not applicable.

Availability of data and materials: The Topp Leone exponentiated lomax distribution is derived from the family of distributions proposed in [1]. All the data sets such survival times of one hundred and twenty-one (121) patients with breast cancer reported in [16] and the remission times (in months) of a random sample of one hundred and twenty-eight (128) bladder cancer patients are already available online.

Competing interests: The authors declare that they have no competing interests.

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Appendix A

Validity check for the TLExLx distribution: The TLExLx distribution is a valid pdf. It suffices to show that

$$\int_0^\infty 2\alpha\theta\lambda\beta(1+\beta x)^{-(\lambda+1)}[1-(1+\beta x)^{-\lambda}]^{\alpha-1}\{1-[1-(1+\beta x)^{-\lambda}]^{\alpha}\}\left[1-\{1-[1-(1+\beta x)^{-\lambda}]^{\alpha}\}^2\right]^{\theta-1}dx = 1.$$

Let $t = 1-(1+\beta x)^{-\lambda}$, then $dx = \dfrac{dt}{\lambda\beta(1+\beta x)^{-(\lambda+1)}}$. Thus, it suffices to show that

$$2\alpha\theta\lambda\beta\int_0^1 (1+\beta x)^{-(\lambda+1)} t^{\alpha-1}(1-t^{\alpha})[1-(1-t^{\alpha})^2]^{\theta-1}\frac{dt}{\lambda\beta(1+\beta x)^{-(\lambda+1)}} = 1,$$

i.e.,

$$2\alpha\theta\int_0^1 t^{\alpha-1}(1-t^{\alpha})[1-(1-t^{\alpha})^2]^{\theta-1}dt=1.$$

Let $m=1-t^{\alpha}$, then $dt=\frac{-dm}{\alpha t^{\alpha-1}}$. When $t=0$, $m=1$; when $t=1$, $m=0$. Thus, it suffices to show that

$$2\alpha\theta\int_0^1 t^{\alpha-1}m(1-m^2)^{\theta-1}\frac{(-dm)}{\alpha t^{\alpha-1}}=1,$$

i.e., $2\theta\int_0^1 m(1-m^2)^{\theta-1}dm=-1$. Let $z=1-m^2$, then $dm=\frac{-dz}{2m}$. When $m=0$, $z=1$; when $m=1$, $z=0$. Thus, it suffices to show that

$$2\theta\int_0^1 mz^{\theta-1}\frac{-dz}{2m}=-\theta\int_0^1 z^{\theta-1}dz=-1,$$

i.e.,

$$\theta\int_0^1 z^{\theta-1}dz=z^{\theta}\Big|_0^1=1^{\theta}-0^{\theta}=1-0=1.$$

Therefore,

$$f(x)=2\alpha\theta\lambda\beta(1+\beta x)^{-(\lambda+1)}[1-(1+\beta x)^{-\lambda}]^{\alpha-1}\{1-[1-(1+\beta x)^{-\lambda}]^{\alpha}\}\left[1-\{1-[1-(1+\beta x)^{-\lambda}]^{\alpha}\}^2\right]^{\theta-1}$$

is a valid pdf.

Appendix B

R Codes for estimation of the parameters:

```
# MLE for TLExLx distribution
Dat=c( 0.3, 0.3, 4.0, 5.0, 5.6, 6.2, 6.3, 6.6, 6.8, 7.4, 7.5, 8.4, 8.4, 10.3, 11.0, 11.8, 12.2, 12.3,
13.5, 14.4, 14.4, 14.8, 15.5, 15.7, 16.2, 16.3, 16.5, 16.8, 17.2, 17.3, 17.5, 17.9, 19.8, 20.4, 20.9,
21.0, 21.0, 21.1, 23.0, 23.4, 23.6, 24.0, 24.0, 27.9, 28.2, 29.1, 30.0, 31.0, 31.0, 32.0, 35.0, 35.0,
37.0, 37.0, 37.0, 38.0, 38.0, 38.0, 39.0, 39.0, 40.0, 40.0, 40.0, 41.0, 41.0, 41.0, 42.0, 43.0, 43.0,
43.0, 44.0, 45.0, 45.0, 46.0, 46.0, 47.0, 48.0, 49.0, 51.0, 51.0, 51.0, 52.0, 54.0, 55.0, 56.0, 57.0,
58.0, 59.0, 60.0, 60.0, 60.0, 61.0, 62.0, 65.0, 65.0, 67.0, 67.0, 68.0, 69.0, 78.0, 80.0, 83.0, 88.0,
89.0, 90.0, 93.0, 96.0, 103.0, 105.0, 109.0, 109.0, 111.0, 115.0, 117.0, 125.0, 126.0, 127.0,
129.0, 129.0, 139.0, 154.0)
```

```
pdf_tlexlx <−function(x; alpha; beta; theta; lambda)
{
2∗alpha∗theta∗lambda∗beta∗(1+beta∗x)∗∗(−(lambda+1))∗(1−(1+beta∗x)∗∗(−lambda
))∗∗(alpha−1)(1−(1−(1+beta∗x)∗∗(−lambda))∗∗alpha)∗(1−(1−(1−(1+beta∗x)
∗∗(−lambda))∗∗alpha)∗∗2)∗∗(theta−1)
}
L_tlexlx <−function(par)
{
alpha = par[1]
beta = par[2]
theta = par[3]
lambda = par[4]
R=pdf tlexlx(x=dat, alpha, beta, theta, lambda)
-sum(log(R))
}
result_1 <−optim(c(1, 1, 1, 1), L_tlexlx, method =
"CG")
result_1
AIC = 2∗result_1value+2*4
AIC
CAIC=AIC+((2*4*(4+1))/(121-4-1))
CAIC
BIC=2*result_1value + 4∗log(121)
BIC
HQIC = 2∗result_1value+2*4*log(log(121))
HQIC
```

P.S. R Codes of MLE for TLLx, ExLx, and Lx distributions will be available upon request from the corresponding author of this paper.

JPSS: ISSN 1726-3328