14 REFINEMENT OF A PRECONDITIONED ACCELERATED OVERRELAXATION (AOR) ITERATIVE METHOD

[1]Ndanusa, Abdulrahman, [2]Mustapha, Aliyu Umar, [3]Ibrahim, Ismail Gidado and [4]Isah, Ibrahim Onimisi
[1]Department of Mathematics, Federal university of Technology, Minna, Nigeria
[2,3]Department of Statistics, Federal Polytechnic, Offa, Nigeria
[4]Department of Mathematical Sciences, Kogi State University, Anyigba, Kogi State, Nigeria
Email: as.ndanusa@futminna.edu.ng

Abstract
Iterative methods for solution of linear algebraic systems abound literature. Of primary concern to any acceptable iteration method is its efficiency in terms of rate of convergence; the faster the convergence the better. This present paper is an attempt to investigate the effect of refinement on a fast-converging preconditioned AOR iteration, in order to further improve on its convergence. The resulting iteration, the refinement of preconditioned AOR (RPAOR) method is subjected to validation through numerical experiments, and it was revealed that the RPAOR attains convergence rates that surpass those of the classical AOR, the preconditioned AOR and the refinement of AOR methods, in all cases.

Keywords: *Accelerated Overrelaxation, Convergence, Refinement, Preconditioning, Spectral Radius.*

Introduction

A fundamental problem in numerical linear algebra is that of solving the large sparse linear algebraic system

$$Bx = c \quad (1)$$

where $B \in \mathbb{R}^{n,n}$ is a nonsingular square matrix with nonvanishing diagonal entries, $x \in \mathbb{R}^n$ and $c \in \mathbb{R}^n$ are unknown and known vectors respectively. A usual splitting of the coefficient matrix B of system (1) is obtained as

$$(D_B - E_B - F_B)x = c \quad (2)$$

where D_B is the diagonal part of B, $-E_B$ and $-F_B$ its strictly lower and strictly upper parts, respectively. The system (1) is transformed into the form

$$Ax = b \quad (3)$$

with the splitting

$$(I - E - F)x = b \quad (4)$$

where $I = {D_B}^{-1}D_B$, $E = {D_B}^{-1}E$, $F = {D_B}^{-1}F$ and $b = {D_B}^{-1}c$. It is noteworthy that systems (1) and (2) have the same solution. The basic linear iteration formula for the solution of (1) or (3) is written as

$$x^{(n+1)} = Lx^{(n)} + k \; n = 0,1,2,\cdots \quad (5)$$

Both L and k are obtained when a regular splitting of A is written as $A = M - N$. Arising therefrom the matrix $L(= M^{-1}N)$, referred to as the iteration matrix of the method, and the column vector $k(= M^{-1}b)$.
The AOR iterative method for solving (3) is given by

$$x^{(n+1)} = \mathcal{L}_{r,\omega}x^{(n)} + (I - rE)^{-1}\omega b \; n = 0,1,2,\cdots \quad (6)$$

where $\mathcal{L}_{r,\omega} = (I - rE)^{-1}[(1-\omega)I + (\omega - r)E + \omega F]$ is the AOR iteration matrix. The linear stationary iteration $x^{(n+1)} = Lx^{(n)} + k$ will ordinarily be convergent for any initial guess $x^{(0)}$ if and only if $\rho(L) < 1$, where $\rho(L)$ is the spectral radius of iteration matrix of the method (Ames, 1977). However, the success of most iterations depends on how fast the convergence is. Preconditioning is a special technique introduced to hasten the convergence of iterative methods through the application of a nonsingular matrix P known as the preconditioner to system (3) thus:

$$PAx = Pb \quad (7)$$

Several preconditioners have been introduced by researchers. These include, Wu and Huang (2007), Yun and Kim (2008), Dehghan and Hajarian (2009), Darvishi *et al.* (2011), Ndanusa and Adeboye (2012), Faruk and Ndanusa (2019), Mayaki and Ndanusa (2019), Wang (2019) and Abdullahi and Ndanusa (2020). Iterative refinement is another technique for accelerating the rate of convergence of iterative methods. It consists of performing iterations on the system whose right-hand side is the residual vector for successive approximations until satisfactory accuracy is attained. The refinement of AOR method (6) is defined by Vatti *et al.* (2018) as

$$x^{(n+1)} = [(I - rE)^{-1}\{(1-\omega)I + (\omega - r)E + \omega F\}]^2 x^{(n)} + \omega[I + (I - rE)^{-1}\{(1-\omega)I + (\omega - r)E + \omega F\}](I - rE)^{-1}b, n = 0,1,2,\cdots$$

Or more compactly,

$$x^{(n+1)} = {L_{r,\omega}}^2 x^{(n)} + \omega\left[I + L_{r,\omega}\right](I - rE)^{-1}b \, , n = 0,1,2,\cdots$$

That is,

$$x^{(n+1)} = R_{L_{r,\omega}}x^{(n)} + d \, , n = 0,1,2,\cdots \quad (8)$$

where $R_{L_{r,\omega}} = [(I - rE)^{-1}\{(1-\omega)I + (\omega - r)E + \omega F\}]^2$ is the Refinement of AOR (RAOR) iterative matrix, and $d = \omega[I + (I - rE)^{-1}\{(1-\omega)I + (\omega - r)E + \omega F\}](I - rE)^{-1}b$. Some recent studies in the refinement of various iteration methods include the works of Dafchahi (2008), Vatti and Gonfa (2011), Vatti and Eneyew (2011), Kyurkchiev and Iliev (2013), Laskar and Behera (2014), Gonfa (2016), Kebede (2017), Muleta and Gofe (2018), Vatti *et al.* (2018), Eneyew *et al.* (2019) and Eneyew *et al.* (2020). This present work aims to undertake iterative refinement of a preconditioned AOR iteration method so as to increase its rate of convergence.

Materials And Methods

Formulation of RPAOR Iteration Method

Consider a generic 3×3 irreducible $L-$matrix with weak diagonal dominance corresponding to the coefficient matrix of a linear system thus.

$$A_1=\begin{pmatrix}1 & a_{12} & a_{13}\\ a_{21} & 1 & a_{23}\\ a_{31} & a_{32} & 1\end{pmatrix} \quad (9)$$

Then the AOR iterative matrix $\mathcal{L}_{r,\omega}$ corresponding to the matrix A_1 is computed to be

$$\begin{aligned}\mathcal{L}_{r,\omega}=\big[&[1-\omega,-\omega a_{12},-\omega a_{13}],[-ra_{12}(1-\omega)-(\omega-r)a_{21},ra_{21}\omega a_{12}-\omega+1,ra_{21}\omega a_{13}\\&-\omega a_{23}],[(r^2a_{21}a_{32}-ra_{31})(1-\omega)+ra_{32}(\omega-r)a_{21}-(\omega-r)a_{31},-(r^2a_{21}a_{32}-ra_{31})\omega a_{12}\\&-a_{32}(1-\omega)-(\omega-r)a_{32},-(r^2a_{21}a_{32}-ra_{31})\omega a_{13}+ra_{32}\omega a_{23}+1-\omega]\big]\end{aligned}$$

And the Refinement of AOR iterative matrix (RAOR), $R_{L_{r,\omega}}$, is also computed as

$$\begin{aligned}R_{L_{r,\omega}}=&\\ \big[&[1-\omega,-\omega a_{12},-\omega a_{13}],[-ra_{12}(1-\omega)-(\omega-r)a_{21},ra_{21}\omega a_{12}-\omega+1,ra_{21}\omega a_{13}-\omega a_{23}],[(r^2a_{21}a_{32}-\\&ra_{31})(1-\omega)+ra_{32}(\omega-r)a_{21}-(\omega-r)a_{31},-(r^2a_{21}a_{32}-ra_{31})\omega a_{12}-a_{32}(1-\omega)-\\&(\omega-r)a_{32},-(r^2a_{21}a_{32}-ra_{31})\omega a_{13}+ra_{32}\omega a_{23}+1-\omega]\big]\times\big[[1-\omega,-\omega a_{12},-\omega a_{13}],[-ra_{12}(1-\omega)-\\&(\omega-r)a_{21},ra_{21}\omega a_{12}-\omega+1,ra_{21}\omega a_{13}-\omega a_{23}],[(r^2a_{21}a_{32}-ra_{31})(1-\omega)+ra_{32}(\omega-r)a_{21}-(\omega-\\&r)a_{31},-(r^2a_{21}a_{32}-ra_{31})\omega a_{12}-a_{32}(1-\omega)-(\omega-r)a_{32},-(r^2a_{21}a_{32}-ra_{31})\omega a_{13}+ra_{32}\omega a_{23}+1-\omega]\big].\end{aligned}$$

The preconditioner of Abdullahi and Ndanusa (2020) is expressed as $P=I+\hat{S}$, where,

$$\hat{S}=\bar{S}+S'=\begin{cases}-a_{1n} & \\ -a_{i1}\,, & i=2,\cdots,n\\ -a_{i,i+1}\,, & i=1,\cdots,n-1\\ 0\,, & \text{otherwise}\end{cases} \quad (10)$$

This preconditioner is then applied to system (7) thereby producing the iterative matrix of the Preconditioned AOR (PAOR) method, $L_{P(r,\omega)}$, as

$$L_{P(r,\omega)}=\left[\begin{array}{l}\left[-\frac{(1-\omega)(-a_{12}a_{21}-a_{13}a_{31}+1)}{a_{12}a_{21}+a_{13}a_{31}-1},-\frac{\omega a_{13}a_{32}}{a_{12}a_{21}+a_{13}a_{31}-1},-\frac{\omega a_{12}a_{23}}{a_{12}a_{21}+a_{13}a_{31}-1}\right],\\ \left[\begin{array}{c}\frac{ra_{23}a_{31}(1-\omega)(-a_{12}a_{21}-a_{13}a_{31}+1)}{(a_{12}a_{21}+a_{13}a_{31}-1)(a_{12}a_{21}+a_{23}a_{32}-1)}-\frac{(\omega-r)a_{23}a_{31}}{a_{12}a_{21}+a_{23}a_{32}-1},\\ \frac{ra_{23}a_{31}\omega a_{13}a_{32}}{(a_{12}a_{21}+a_{13}a_{31}-1)(a_{12}a_{21}+a_{23}a_{32}-1)}-\frac{(1-\omega)(-a_{12}a_{21}-a_{23}a_{32}+1)}{a_{12}a_{21}a_{23}a_{32}-1},\\ \frac{ra_{23}^2a_{31}\omega a_{12}}{(a_{12}a_{21}+a_{13}a_{31}-1)(a_{12}a_{21}+a_{23}a_{32}-1)}-\frac{\omega a_{21}a_{13}}{a_{12}a_{21}+a_{23}a_{32}-1}\end{array}\right],\\ \left[\begin{array}{c}-\frac{r^2a_{23}a_{31}(a_{12}a_{31}-a_{32})(1-\omega)(-a_{12}a_{21}-a_{13}a_{31}+1)}{(a_{12}a_{21}+a_{13}a_{31}-1)(a_{12}a_{21}+a_{23}a_{32}-1)(a_{13}a_{31}-1)}\\ +\frac{r(a_{12}a_{31}-a_{32})(\omega-r)a_{23}a_{31}}{(a_{13}a_{31}-1)(a_{12}a_{21}+a_{23}a_{32}-1)},\\ -\frac{r^2a_{23}a_{31}(a_{12}a_{31}-a_{32})\omega a_{13}a_{32}}{(a_{12}a_{21}+a_{13}a_{31}-1)(a_{12}a_{21}+a_{23}a_{32}-1)(a_{13}a_{31}-1)}\\ +\frac{r(a_{12}a_{31}-a_{32})(1-\omega)(-a_{12}a_{21}-a_{23}a_{32}+1)}{(a_{13}a_{31}-1)(a_{12}a_{21}+a_{23}a_{32}-1)}-\frac{(\omega-r)(a_{12}a_{31}-a_{32})}{a_{13}a_{31}-1},\\ \frac{r^2a_{23}^2a_{31}(a_{12}a_{31}-a_{32})\omega a_{12}}{(a_{12}a_{21}+a_{13}a_{31}-1)(a_{12}a_{21}+a_{23}a_{32}-1)(a_{13}a_{31}-1)}\\ +\frac{r(a_{12}a_{31}-a_{32})\omega a_{21}a_{13}}{(a_{13}a_{31}-1)(a_{12}a_{21}+a_{23}a_{32}-1)}-\frac{(1-\omega)(-a_{13}a_{31}+1)}{a_{13}a_{31}-1}\end{array}\right]\end{array}\right]$$

Correspondingly, the iterative matrix of Refinement of Preconditioned AOR (RPAOR), i.e., $R_{L_{P(r,\omega)}}$ is computed as

$$R_{L_{P(r,\omega)}}=\left[\left[-\frac{(1-\omega)(-a_{12}a_{21}-a_{13}a_{31}+1)}{a_{12}a_{21}+a_{13}a_{31}-1},-\frac{\omega a_{13}a_{32}}{a_{12}a_{21}+a_{13}a_{31}-1},-\frac{\omega a_{12}a_{23}}{a_{12}a_{21}+a_{13}a_{31}-1}\right],\right.$$

$$[\frac{ra_{23}a_{31}(1-\omega)(-a_{12}a_{21}-a_{13}a_{31}+1)}{(a_{12}a_{21}+a_{13}a_{31}-1)(a_{12}a_{21}+a_{23}a_{32}-1)}-\frac{(\omega-r)a_{23}a_{31}}{a_{12}a_{21}+a_{23}a_{32}-1},$$
$$\frac{ra_{23}a_{31}\omega a_{13}a_{32}}{(a_{12}a_{21}+a_{13}a_{31}-1)(a_{12}a_{21}+a_{23}a_{32}-1)}-\frac{(1-\omega)(-a_{12}a_{21}-a_{23}a_{32}+1)}{a_{12}a_{21}a_{23}a_{32}-1},$$
$$\frac{ra_{23}^2a_{31}\omega a_{12}}{(a_{12}a_{21}+a_{13}a_{31}-1)(a_{12}a_{21}+a_{23}a_{32}-1)}-\frac{\omega a_{21}a_{13}}{a_{12}a_{21}+a_{23}a_{32}-1}],$$
$$[-\frac{r^2a_{23}a_{31}(a_{12}a_{31}-a_{32})(1-\omega)(-a_{12}a_{21}-a_{13}a_{31}+1)}{(a_{12}a_{21}+a_{13}a_{31}-1)(a_{12}a_{21}+a_{23}a_{32}-1)(a_{13}a_{31}-1)}$$
$$+\frac{r(a_{12}a_{31}-a_{32})(\omega-r)a_{23}a_{31}}{(a_{13}a_{31}-1)(a_{12}a_{21}+a_{23}a_{32}-1)},$$
$$-\frac{r^2a_{23}a_{31}(a_{12}a_{31}-a_{32})\omega a_{13}a_{32}}{(a_{12}a_{21}+a_{13}a_{31}-1)(a_{12}a_{21}+a_{23}a_{32}-1)(a_{13}a_{31}-1)}$$
$$+\frac{r(a_{12}a_{31}-a_{32})(1-\omega)(-a_{12}a_{21}-a_{23}a_{32}+1)}{(a_{13}a_{31}-1)(a_{12}a_{21}+a_{23}a_{32}-1)}-\frac{(\omega-r)(a_{12}a_{31}-a_{32})}{a_{13}a_{31}-1},$$
$$\frac{r^2a_{23}^2a_{31}(a_{12}a_{31}-a_{32})\omega a_{12}}{(a_{12}a_{21}+a_{13}a_{31}-1)(a_{12}a_{21}+a_{23}a_{32}-1)(a_{13}a_{31}-1)}$$
$$+\frac{r(a_{12}a_{31}-a_{32})\omega a_{21}a_{13}}{(a_{13}a_{31}-1)(a_{12}a_{21}+a_{23}a_{32}-1)}-\frac{(1-\omega)(-a_{13}a_{31}+1)}{a_{13}a_{31}-1}]$$
$$\times[-\frac{(1-\omega)(-a_{12}a_{21}-a_{13}a_{31}+1)}{a_{12}a_{21}+a_{13}a_{31}-1},$$
$$-\frac{\omega a_{13}a_{32}}{a_{12}a_{21}+a_{13}a_{31}-1},-\frac{\omega a_{12}a_{23}}{a_{12}a_{21}+a_{13}a_{31}-1}],$$
$$[\frac{ra_{23}a_{31}(1-\omega)(-a_{12}a_{21}-a_{13}a_{31}+1)}{(a_{12}a_{21}+a_{13}a_{31}-1)(a_{12}a_{21}+a_{23}a_{32}-1)}-\frac{(\omega-r)a_{23}a_{31}}{a_{12}a_{21}+a_{23}a_{32}-1},$$
$$\frac{ra_{23}a_{31}\omega a_{13}a_{32}}{(a_{12}a_{21}+a_{13}a_{31}-1)(a_{12}a_{21}+a_{23}a_{32}-1)}-\frac{(1-\omega)(-a_{12}a_{21}-a_{23}a_{32}+1)}{a_{12}a_{21}a_{23}a_{32}-1},$$
$$\frac{ra_{23}^2a_{31}\omega a_{12}}{(a_{12}a_{21}+a_{13}a_{31}-1)(a_{12}a_{21}+a_{23}a_{32}-1)}-\frac{\omega a_{21}a_{13}}{a_{12}a_{21}+a_{23}a_{32}-1}],$$
$$[-\frac{r^2a_{23}a_{31}(a_{12}a_{31}-a_{32})(1-\omega)(-a_{12}a_{21}-a_{13}a_{31}+1)}{(a_{12}a_{21}+a_{13}a_{31}-1)(a_{12}a_{21}+a_{23}a_{32}-1)(a_{13}a_{31}-1)}$$
$$+\frac{r(a_{12}a_{31}-a_{32})(\omega-r)a_{23}a_{31}}{(a_{13}a_{31}-1)(a_{12}a_{21}+a_{23}a_{32}-1)},$$
$$-\frac{r^2a_{23}a_{31}(a_{12}a_{31}-a_{32})\omega a_{13}a_{32}}{(a_{12}a_{21}+a_{13}a_{31}-1)(a_{12}a_{21}+a_{23}a_{32}-1)(a_{13}a_{31}-1)}$$
$$+\frac{r(a_{12}a_{31}-a_{32})(1-\omega)(-a_{12}a_{21}-a_{23}a_{32}+1)}{(a_{13}a_{31}-1)(a_{12}a_{21}+a_{23}a_{32}-1)}-\frac{(\omega-r)(a_{12}a_{31}-a_{32})}{a_{13}a_{31}-1},$$
$$\frac{r^2a_{23}^2a_{31}(a_{12}a_{31}-a_{32})\omega a_{12}}{(a_{12}a_{21}+a_{13}a_{31}-1)(a_{12}a_{21}+a_{23}a_{32}-1)(a_{13}a_{31}-1)}$$
$$+\frac{r(a_{12}a_{31}-a_{32})\omega a_{21}a_{13}}{(a_{13}a_{31}-1)(a_{12}a_{21}+a_{23}a_{32}-1)}-\frac{(1-\omega)(-a_{13}a_{31}+1)}{a_{13}a_{31}-1}]$$

Numerical Experiment

The formulation of the RPAOR iteration is validated with the following sample matrix which can be found in Abdullahi and Ndanusa (2020).

Let the coefficient matrix A of the linear system (3) be given by

$$A=\begin{pmatrix} 1 & -1/6 & -1/7 & -1/8 & -1/6 & -1/7 \\ -1/8 & 1 & -1/6 & -1/7 & -1/8 & -1/6 \\ -1/6 & -1/8 & 1 & -1/6 & -1/7 & -1/8 \\ -1/7 & -1/6 & -1/8 & 1 & -1/6 & -1/7 \\ -1/8 & -1/7 & -1/6 & -1/8 & 1 & -1/6 \\ -1/6 & -1/8 & -1/7 & -1/6 & -1/8 & 1 \end{pmatrix}$$

The results of computations are presented in Tables 1, 2 and 3 of the next section.

Results and Discussion

A computation of the spectral radii of iteration matrices of AOR, Preconditioned AOR, Refinement of AOR (RAOR) and Refinement of Preconditioned AOR (RPAOR) methods is undertaken under similar conditions using Maple 2019 mathematical software package. The following representations are adopted for ease of use: $L_{r,\omega}$ = Iteration matrix of the AOR method; $L_{P(r,\omega)}$ = Iteration matrix of preconditioned AOR method of Abdullahi and Ndanusa (2020); $R_{L_{r,\omega}}$ = Iteration matrix of Refinement of AOR method; $R_{L_{P(r,\omega)}}$ = Iteration matrix of Refinement of Preconditioned AOR method of Abdullahi and Ndanusa (2020); $\rho(L_{r,\omega})$ = Spectral radius of $L_{r,\omega}$; $\rho(L_{P(r,\omega)})$ = Spectral radius of $L_{P(r,\omega)}$; $\rho(R_{L_{r,\omega}})$ = Spectral radius of $R_{L_{r,\omega}}$; $\rho\left(R_{L_{P(r,\omega)}}\right)$ = Spectral radius of $R_{L_{P(r,\omega)}}$; $R(G)$ =Rate of convergence of the iteration method whose iteration matrix is defined by G.

Table 1: Spectral radii of various iteration matrices

ω	r	$\rho(L_{r,\omega})$	$\rho(L_{P(r,\omega)})$	$\rho(R_{L_{r,\omega}})$	$\rho(R_{L_{P(r,\omega)}})$
0.95	0.85	0.6205255277	0.4820339009	0.3850519305	0.2323566801
0.90	0.80	0.6518574112	0.5268521180	0.4249180806	0.2775731577
0.80	0.70	0.7083014149	0.6059612644	0.5016908907	0.3671890584
0.70	0.65	0.7516743194	0.6652871216	0.5650142820	0.4426069515
0.60	0.50	0.8026767336	0.7352605039	0.6442899392	0.5406080078
0.50	0.40	0.8429614522	0.7897395742	0.7105840063	0.6236885955
0.40	0.30	0.8796743773	0.8391415217	0.7738270125	0.7041584949
0.30	0.20	0.9133536720	0.8843014020	0.8342149333	0.7819889721
0.20	0.10	0.9444202400	0.9258516890	0.8919295900	0.8572013474
0.10	0.05	0.9727210830	0.9636211849	0.9461863097	0.9285657895

Table 2: Rates of convergence of various iterative methods

ω	r	$R(L_{r,\omega})$	$R(L_{P(r,\omega)})$	$R(R_{L_{r,\omega}})$	$R(R_{L_{P(r,\omega)}})$
0.95	0.85	0.2072403474	0.3169224172	0.4144806949	0.6338448373
0.90	0.80	0.1858473925	0.2783112697	0.3716947889	0.5566225340
0.80	0.70	0.1497818907	0.2175551369	0.2995637845	0.4351102685
0.70	0.65	0.1239702870	0.1769908834	0.2479405743	0.3539817694
0.60	0.50	0.0954593253	0.1335587625	0.1909186503	0.2671175256
0.50	0.40	0.0741922848	0.1025160988	0.1483845718	0.2050321972
0.40	0.30	0.0556780577	0.0761647890	0.1113561141	0.1523295771
0.30	0.20	0.0393610209	0.0533996864	0.0787220401	0.1067993715
0.20	0.10	0.0248347141	0.0334585768	0.0496694280	0.0669171550
0.10	0.05	0.0120116710	0.0160936607	0.0240233400	0.0321873208

Table 3: Ratios of rates of convergence

ω	r	$\frac{R(R_{L_{P(r,\omega)}})}{R(L_{r,\omega})}$	$\frac{R(R_{L_{P(r,\omega)}})}{R(L_{P(r,\omega)})}$	$\frac{R(R_{L_{P(r,\omega)}})}{R(R_{L_{r,\omega}})}$
0.95	0.85	3.0585011329	2.0000000092	1.5292505661
0.90	0.80	2.9950516201	1.9999999806	1.4975257943
0.80	0.70	2.9049591140	1.9999999756	1.4524795420
0.70	0.65	2.8553758966	2.0000000147	1.4276879466
0.60	0.50	2.7982339573	2.0000000045	1.3991169809
0.50	0.40	2.7635245060	1.9999999961	1.3817622325
0.40	0.30	2.7358996235	1.9999999882	1.3679498277
0.30	0.20	2.7133282892	1.9999999757	1.3566641739
0.20	0.10	2.6945007191	2.0000000418	1.3472503650
0.10	0.05	2.6796705304	1.9999999627	1.3398353768

Four different iteration processes are performed with the matrix A; these are, AOR, Preconditioned AOR, Refinement of AOR and Refinement of Preconditioned AOR iterations. The spectral radii of iterative matrices of the various iterations, along with varied values of relaxation and acceleration parameters ω and r respectively, are computed and presented in Table 1. It reveals that, $\rho\left(R_{L_{P(r,\omega)}}\right) < \rho(R_{L_{r,\omega}}) < \rho(L_{P(r,\omega)}) < \rho(L_{r,\omega})$, which indicates the efficiency of the RPAOR iteration over the other methods. Also, convergence is observed to be faster when the relaxation and acceleration parameters, ω and r are closer to 1. In Tables 2 and 3, the rates of convergence of the 4 iterations are compared, where it is shown that in the best case scenario, the RPAOR converges thrice as fast as the AOR method, the RPAOR converges twice as fast as the PAOR method, and the RPAOR converges one and a half times as fast as the RAOR method.

Conclusion

The iterative refinement of a fast-converging preconditioned AOR method is undertaken. A comparative analysis of the spectral radii of four different iteration processes is done. The results proved the RPAOR method to be more effective and efficient than the other methods as it converges thrice as fast as the AOR method.

Acknowledgements

The authors express their utmost gratitude and appreciation to the anonymous reviewers for their positive criticisms of the work.

References

Ames, W. F. (1977). Numerical methods for partial differential equations. (2nd ed.). Great Britain: Thomas Nelson & Sons Ltd..

Abdullahi, I. and Ndanusa A. (2020). A new modified preconditioned accelerated overrelaxation (AOR) iterative methods for L-matrix linear algebraic systems. *Science World Journal*, 15(2), pp. 45-50.

Dafchahi, F. N. (2008). A new refinement of Jacobi method for solution of linear system equations $Ax = b$. *International Journal of Contemporary Mathematical Sciences*, 3(17): 819-827

Darvishi, M. T., Hessari, P. and Shin, B. C. (2011). Preconditioned modified AOR method for systems of linear equations. *International Journal for numerical methods in biomedical engineering,* 27: 758-769.

Dehghan, M. and Hajarian, M. (2009). Improving preconditioned SOR-type iterative methods for L-matrices. *International Journal for Numerical Methods in Biomedical Engineering*, 27, pp. 774-784.

Eneyew, T. K., Awgichew, G., Haile, E. and Abie, G. D. (2019). Second refinement of Jacobi iterative method for solving linear system of equations. *International Journal of Computing Science and Applied Mathematics*, 5(2): 41-47.

Eneyew, T. K., Awgichew, G., Haile, E. and Abie, G. D. (2020). Second refinement of Gauss-Seidel iterative method for solving linear system of equations. *Ethiopian Journal of Science and Technology,* 13(1): 1-15.

Faruk, A. I. and Ndanusa, A. (2019). Improvements of successive overrelaxation (SOR) methods for L-matrices. *Savanna Journal of Basic and Applied Sciences*, 1(2), pp. 218-223.

Gonfa, G. G. (2016). Refined iterative method for solving system of linear equations. *American Journal of Computational and Applied Mathematics*, 6(3): 144-147.

Kebede, T. (2017). Second degree refinement Jacobi iteration method for solving system of linear equation. *International Journal of Computing Science and Applied Mathematics*, 3(1): 5-10.

Kyurkchiev, N. and Iliev, A. (2013). A refinement of some over-relaxation algorithms for solving a system of linear equations. *Serdica Journal of Computing,* 7(3): 245–256.

Laskar, A. H. and Behera, S. (2014). Refinement of iterative methods for the solution of system of linear equations $Ax = b$. *IOSR Journal of Mathematics (IOSR-JM)*, 10(3): 70-73.

Mayaki, Z. and Ndanusa, A. (2019). Modified successive overrelaxation (SOR) type methods for M-matrices. *Science world Journal*, 14(4), pp. 1-5.

Muleta, H. and Gofe, G. (2018). Refinement of generalized accelerated over relaxation method for solving system of linear equations based on the Nekrassov-Mehmke1-method. *Ethiopian Journal of Education and Science,* 13(2): 1-18.

Ndanusa, A. and Adeboye, K. R. (2012). Preconditioned SOR iterative methods for L-matrices. *American Journal of Computational and Applied Mathematics*, 2(6), pp. 300-305.

Vatti, V. B. K. and Eneyew, T. K. (2011). A refinement of Gauss-Seidel method for solving of linear system of equations. *International Journal of ContemporaryMathematical Sciences*, 6(3): 117-121.

Vatti, V. B. K. and Gonfa, G. G. (2011). Refinement of Generalized Jacobi (RGJ) method for solving system of linear equations. *International Journal of Contemporary Mathematical Sciences*, 6(3): 109-116.

Vatti, V. B. K., Sri, R. and Mylapalli, M. S. K. (2018). A refinement of accelerated over relaxation method for the solution of linear systems. *International Journal of Pure and Applied Mathematics,* 118(18): 1571-1577.

Wang, H. (2019). A preconditioned AOR iterative scheme for systems of linear equations with L-matrices. *Open Mathematics*, 17: 1764–1773.

Wu, S. L. and Huang, T. (2007). A modified AOR-type iterative method for L-matrix linear systems. *Anziam Journal,* 49: 281 – 292.

Yun, J. H. and Kim, S. W. (2008). Convergence of the preconditioned AOR method for irreducible L-matrices. *Applied Mathematics and Computation*, 201: 56-64.