Direct and Indirect Transmission Dynamics of Typhoid Fever Model by Differential Transform Method

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ABSTRACT

The aim of this paper is to apply the Differential Transformation Method (DTM) to solve typhoid fever model for a given constant population. This mathematical model is described by nonlinear first order ordinary differential equations. First, we find the solution of this model by using the differential transformation method (DTM). In order to show the efficiency of the method, we compare the solutions obtained by DTM and RK4. We illustrated the profiles of the solutions, from which we speculate that the DTM and RK4 solutions agreed well.

Keywords: Typhoid Fever, Differential Transform Method, Runge-Kutta Method.

INTRODUCTION

Typhoid fever is one of the infectious diseases which is endemic in most part of the world. It is systemic infection caused by Salmonella typhi (S typhi). The bacteria is transmitted through food and water contaminated with faces and urine of an infected patient or a carrier. Once the bacteria enters the body they travel in the human intestines, and then enter to the bloodstream. Merrell andFalkow, 2004).

Modelling the transmission dynamics of typhoid is an important and interesting topic for a lot of Computational mathematical researchers (LaSalle, 1976). Mathematical models and computer simulations have become useful in analysing the spread and control of infectious disease. Many advances have been made towards the fight against typhoid fever such as treatment with drugs, vaccination and environmental sanitation. Various studies including mathematical model for the control of typhoid disease fever disease, dynamic model for analyzing and predicting process of typhoid fever among others been conducted by many researchers.

Typhoid fever infects 21 million people and kills 200,000 worldwide every year. Asymptomatic carriers are believed to play a major role in the evolution and global transmission dynamics of Typhoid fever, and their presence greatly hinders the eradication of Typhoid fever using treatment and vaccination. (Naresh et al., 2008).

When dealing with large populations, as in the case of Typhoid fever, compartmental mathematical models are used. In the deterministic model, individuals in the population are assigned to different subgroups, each representing a specific stage of the epidemic. "Typhoid fever has continued to be a health problem in developing countries where there is poor sanitation, poor standard of personal hygiene and prevalence of contaminated food. It is endemic in many parts of the developing world, illness do occur around the world in span of a day". (Lifshitz, 1996)

Treatment of typhoid is based on antibiotic susceptibility of the patient blood culture. The oral chloramphenicol, amoxicillin may be used if the strain is sensitive. The chronic carrier state may be eradicated using oral therapy , ciprooxacin or noroxacin. Multi-drug resistant strains of S.Typhi are increasingly common worldwide which makes treatment by antibiotics more difficult and costly. Typhoid symptoms vary widely and are very much similar to the symptoms of other microbial infections. Here are some of the common typhoid fever symptoms: Variable degrees of high grade fever in about 75% of cases, Muscle pains and body aches, Chills, Decreased appetite, Headaches, Nose bleeds, Pain in the abdomen in 20 to 40% of cases, Dizziness, Rose spots (rashes) over the skin, Weakness and fatigue, Constipation or diarrhoea, sore throat and a cough. (Lifshitz, 1996). Several mathematical models on the transmission dynamics of typhoid fever have been developed these includes (Adetunde, 2008), (Date et al, 2015),(Cvjetanovic et al, 2014), (Kalajdzievska, 2011), (Lauria et al 2009), (Moatlhodl and Gosaamang, 2017), (Moffact, 2014), (Muhammad, et al 2015), (Mushayabasa, 2011), (Mushayabasa, 2017), (Nthiiri, 2016), (Virginia et al, 2014), (Watson and Edmunds, 2015), (Peter et al, 2017), (Ibrahim et al, 2017). But none has incorporated both direct and indirect transmission dynamics in typhoid fever. We will like to complement and extend the existing works in the literature.

The aim of this paper is to present the application of Differential Transform Method to the proposed model and to verify the validity of the Differential Transform Method in solving the model using computer in-built Maple 18 classical fourth-order Runge-Kutta method as a base.

"The concept of the differential transform was first proposed by (Zhou, 1986) and its main applications therein are to solve both linear and nonlinear initial value problems in electric circuit analysis. Differential Transform Method (DTM) is proved to be an excellent tool to investigate analytical and numerical solutions of nonlinear ordinary differential equations". The concept of differential transformation is briefly presented and some well-known properties of this DTM are rewritten in a more generalized forms. The Differential Transformation Method is one of the semi- analytical method commonly used for solving ordinary and partial differential equations in the forms of polynomials as approximations of exact solution.

In this study, we employ the Differential Transformation Method (DTM) to the system of non-linear differential equations which describe our model and approximating the solutions in a sequence of time intervals. In other to illustrate the accuracy of the DTM, the obtained results are compared with classical fourth-order Runge-Kutta Method.

FORMULATION OF MODEL

In this section, a deterministic, compartmental mathematical model to describe the transmission dynamics of typhoid fever is formulated to extend and complement the ones existing in the literature. The model subdivides the human population into four compartments: susceptible S(t), infected I(t), infected carrier Ic(t), and recovered R(t). The models assume direct transmission of typhoid from infected individuals to susceptible individuals. However, typhoid is largely contacted from environmental bacteria through contaminated water or food and drinks and transmission of typhoid through direct person-to-person contact. To incorporate this real biological phenomenon, we consider an additional compartment, W(t), which represents bacteria in the environment. We assume that susceptible individuals get infected with typhoid at a rate proportional to the susceptible population, S. Individuals in the infected class, can recover from typhoid at the rate δ. The Infected carrier and infected individuals both excrete bacteria into the environment. However, the rate of excretion by the infectious group ε_2 is higher than that of the carrier group ε_1 this is because infectious carrier do not show any signs of infection. The constant recruitment rate into the susceptible human is represented by θ , while the natural death rate of human is represented by μ. Several mathematical models have been developed to explain the dynamics of typhoid fever but none has incorporated both direct and indirect transmission dynamics in typhoid fever model. We considered an indirect transmission of typhoid by addition of environmental bacteria concentration. We assume the existence of both direct transmission of typhoid from infected individuals to susceptible and indirect transmission of bacteria from the environment to the susceptible individuals

MODEL EQUATIONS

From the assumptions, descriptions and the compartmental diagram in figure1, we formulate the following system of differential equations

$$\left.\begin{aligned}
\frac{dS}{dt} &= \Theta - \mu_1 S - \lambda S \\
\frac{dIc}{dt} &= \rho\lambda S - (\mu_2 + \varepsilon_1) I_c \\
\frac{dI}{dt} &= (1-\rho)\lambda S - (\mu_3 + \delta + \varepsilon_2) I \\
\frac{dR}{dt} &= \delta I - \mu_4 R \\
\frac{dW}{dt} &= \varepsilon_1 I_c + \varepsilon_2 I - \mu_b W
\end{aligned}\right\} \quad (1)$$

Where

$\lambda = \beta_1 I_c + \beta_2 I + \beta_3 W$

Substituting the value of force of infection

$$\left.\begin{aligned}
\frac{dS}{dt} &= \Theta - \mu_1 S - S(\beta_1 I_c + \beta_2 I + \beta_3 W) \\
\frac{dIc}{dt} &= \rho S(\beta_1 I_c + \beta_2 I + \beta_3 W) - (\mu_2 + \varepsilon_1) I_c \\
\frac{dI}{dt} &= (1-\rho) S(\beta_1 I_c + \beta_2 I + \beta_3 W) - (\mu_3 + \delta + \varepsilon_2) I \\
\frac{dR}{dt} &= \delta I - \mu_4 R \\
\frac{dW}{dt} &= \varepsilon_1 I_c + \varepsilon_2 I - \mu_b W
\end{aligned}\right\} \quad (2)$$

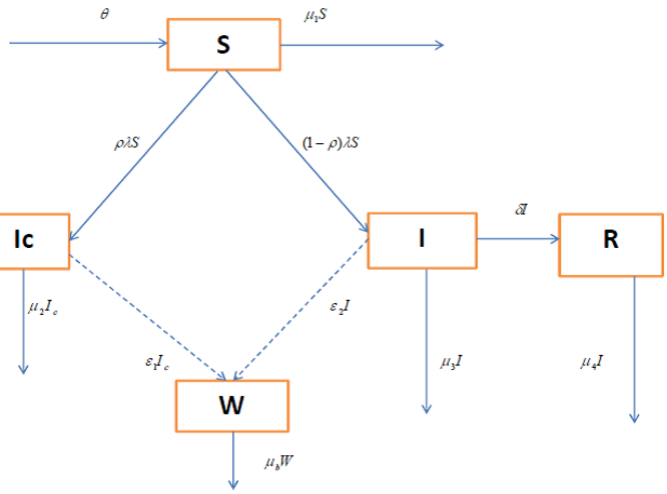

Figure 1: Pictorial Representation of Model

Table 1: Description of Variables and Parameters for Model

Variables	**Description**
$S(t)$	susceptible individuals at time t
$Ic(t)$	carrier infectious individuals at time t
$I(t)$	infectious individuals at time t
$R(t)$	recovered individuals at time t
$W(t)$	environmental bacteria concentration,
Parameters	**Interpretation**
θ	recruitment rate of susceptible individuals
μ_1	natural death rate
μ_2	natural rate for I_c class and disease induced death rate
μ_3	natural death rate for I class and disease induced death rate
μ_4	natural death rate
ε_1	bacteria sheeding rate for I_c
ε_2	bacteria sheeding rate for I
ρ	probability that newly infected individuals are asymtomatic/carrier
β_1	transmission rate between S and I_c
β_2	transmission rate between S and I
β_3	transmission rate between S and W
δ	recovery rate for infectious class
λ	force of infection

EXISTENCE AND UNIQUENESS OF SOLUTION

The validity of any mathematical model depends on whether the given system of equations has a solution, and if the solution is unique. We shall use the Lipchitz condition to verify the existence and uniqueness of solution for the system of equations 2

Theorem 1 **(Derrick and Groosman, 1976)**

Let D denote the region

$|t-t_0|\le a, \|x-x_0\|\le 1, x=(x_1,x_2,\dots\dots,x_n), x_o=(x_{10},x_{20},\dots\dots x_{no})$

And suppose that $f(t,x)$ satisfies the Lipchitz condition

$$\|f(t,x_1)-f(t,x_2)\|\le k\|x_1-x_2\|$$

whenever the pairs (t,x_1) and (t,x_2) belong to D where k is a positive constant. Then, there is a constant $\delta\ge 0$ such that there exists a unique continuous vector solution of $x(t)$ of the system in the interval $t-t_o\le\delta$. It is important to note that the condition is satisfied by the requirement that $\frac{\partial f_i}{\partial x_j}, i, j=1,2,\cdots,$ be continuous and bounded in D Considering the model equations 2, we are interested in the region $0\le\alpha\le R.$ We

Let the system of equations of model 2 be as follows

$$\begin{aligned}
F_1 &= \Theta-\mu_1 S-S(\beta_1 I_c+\beta_2 I+\beta_3 W)\\
F_2 &= \rho S(\beta_1 I_c+\beta_2 I+\beta_3 W)-(\mu_2+\varepsilon_1)I_c\\
F_3 &= (1-\rho)S(\beta_1 I_c+\beta_2 I+\beta_3 W)-(\mu_3+\delta+\varepsilon_2)I\\
F_4 &= \delta I-\mu_4 R\\
F_5 &= \varepsilon_1 I_c+\varepsilon_2 I-\mu_b W
\end{aligned}$$

look for the bounded solution in the region and whose partial derivatives satisfy $f\le\alpha\le 0.$ where α and δ are positive constants.

Theorem 2

Let D denote the region $0\le\alpha\le R$, then equation 2 have a unique solution. We show that

$\frac{\partial f_i}{\partial x_j}, i, j=1,2,3,4,5$

Are continuous and bounded in D For F_1

$$\begin{aligned}
F_1 &= \Theta-\mu_1 S-S(\beta_1 I_c+\beta_2 I+\beta_3 W)\\
F_2 &= \rho S(\beta_1 I_c+\beta_2 I+\beta_3 W)-(\mu_2+\varepsilon_1)I_c\\
F_3 &= (1-\rho)S(\beta_1 I_c+\beta_2 I+\beta_3 W)-(\mu_3+\delta+\varepsilon_2)I\\
F_4 &= \delta I-\mu_4 R\\
F_5 &= \varepsilon_1 I_c+\varepsilon_2 I-\mu_b W
\end{aligned}$$

For F_1

$$\left|\frac{\partial f_1}{\partial S}\right|=\left|\ -\mu_1-(\beta_1 I+\beta_2 I_c+\beta_3 W)\ \right|<\infty,\quad \left|\frac{\partial f_1}{\partial I_c}\right|=\left|\ -S\beta_1\ \right|<\infty,\quad \left|\frac{\partial f_1}{\partial I}\right|=\left|\ -S\beta_2\ \right|<\infty$$

$$\left|\frac{\partial f_1}{\partial W}\right|=\left|\ -S\beta_3\ \right|<\infty\quad \left|\frac{\partial f_1}{\partial R}\right|=0<\infty$$

For F_2

$$\left|\frac{\partial f_2}{\partial S}\right|=\left|\ \rho(\beta_1 I_c+\beta_2 I+\beta_3 W)\ \right|<\infty,\quad \left|\frac{\partial f_2}{\partial I_c}\right|=\left|\ \rho S\beta_1-(\mu_2+\varepsilon_1)\ \right|<\infty,\quad \left|\frac{\partial f_2}{\partial I}\right|=\left|\ \rho S\beta_2\ \right|<\infty$$

$$\left|\frac{\partial f_2}{\partial R}\right|=0<\infty,\quad \left|\frac{\partial f_2}{\partial W}\right|=\left|\ \rho S\beta_3\ \right|<\infty$$

These partial derivative exist, continuous and are bounded, similarly for F_3 through to F_5. Hence, by theorem 2, the model (2) has a unique solution.

DIFFERENTIAL TRANSFORM METHOD

With reference to (Benhammouda et al, 2014), (Hassan, 2008), (Akinboro et al, 2014). (Peter and Akinduko, 2018), and (Peter and Ibrahim, 2017). The process involved in DTM is as follows: Given an arbitrary function of m, suppose $y(m)$ is a non-linear funtion of m, then $y(m)$ can be expanded in a Taylor series about a point $x=0$ as

$$y(m) = \sum_{k=0}^{\infty} m^k \frac{1}{k!}\left[\frac{d^k}{dm^k} y(m)\right]_{m=0}$$

Thus,the differential Transform of $y(x)$ is given as:

$$Y(k) = \frac{1}{k!}\left[\frac{d^k}{dm^k} y(x)\right]_{m=0}$$

and the inverse differential Transform is given as

$$y(m) = \sum_{k=0}^{\infty} Y(k) m^k$$

The table 1 below gives the operational properties of differential Transform Method. Given two un-correlated arbitary functions of m as $c(m)$ and $d(m)$ where $C(m)$ and $D(m)$ are the transformed functions respectively.

Table 2: Basic operation properties of the DTM

S/No	Original Function	Transformed Function
1	$y(m) = c(m) \pm d(m)$	$Y(k) = C(k) \pm D(k)$
2	$y(m) = \alpha r(m)$	$Y(k) = \alpha C(k),\ \alpha$ is a constant
3	$y(m) = \dfrac{dr(m)}{dm}$	$Y(k) = (k+1)C(k+1)$
4	$y(m) = \dfrac{d^2 r(m)}{dm^2}$	$Y(k) = (k+1)(k+2)C(k+2)$
5	$y(m) = \dfrac{d^n r(m)}{dm^n}$	$Y(k) = (k+1)(k+2)\cdots(k+n)C(k+n)$
6	$y(m) = 1$	$Y(k) = \delta(k)$
7	$y(m) = m$	$Y(k) = \delta(k-1)$, δ is the Kronecker delta
8	$y(m) = e^{(\lambda m)}$	$Y(k) = \dfrac{\lambda^k}{k!}$
9	$y(x) = c(m)d(m)$	$Y(k) = \sum_{n=0}^{k} D(n)C(k-n)$
10	$y(m) = (1+m)^n$	$Y(k) = \dfrac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$

SOLUTION OF THE MODEL EQUATIONS

In this section, we apply the steps involved in differential transform method to the model as follows: We recall the system of equations governing the typhoid model as applying the operational properties in Table (2) to the system of differential equations of the model system, we transform the model equations into its differential transform equivalent as follows where

$K_2 = \mu_3 + \delta,\ K_1 = \alpha + \mu_2,\ K_3 = \mu_3 + \delta$

$$S(k+1) = \frac{1}{k+1}[\theta H(k,0) - \sum_{l=0}^{k} S(l)(\beta_1 I_c(k-l) + \beta_2 I(k-l) + \beta_3 W(k-l)) - \mu_1 S(k)]$$

$$I_c(k+1) = \frac{1}{k+1}[\rho \sum_{l=0}^{k} S(l)(\beta_1 I_c(k-l) + \beta_2 I(k-l) + \beta_3 W(k-l)) - K_1 I_c(k)]$$

$$I(k+1) = \frac{1}{k+1}[(1-\rho)\sum_{l=0}^{k} S(l)(\beta_1 I_c(k-l) + \beta_2 I(k-l) + \beta_3 W(k-l)) + \alpha I_c(k) - K_2 I(k)]$$

$$R(k+1) = \frac{1}{k+1}[\delta I(k) - \mu_4 R(k)]$$

$$W(k+1) = \frac{1}{k+1}[\varepsilon_1 I_c(k) + \varepsilon_2 I(k) - \mu_b W(k)]$$

Subject to the initial conditions S(0) = 60, Ic(0) = 40, I(0) = 20, R(0) = 10, W(0) = 200. Using the initial conditions and the parameter values in table (1) S

Hence, when $k = 5$ the solution to the system (2) in closed form is obtained as

$$S(t) = \sum_{n=0}^{k} S(k)t^k = 60 + 9.99811480 \times 10^{10} t - 1.570777935 \times 10^6 t^2 + 3.720184350 \times 10^5 t^3$$
$$- 5.622311122 \times 10^9 t^4 + 1.085598330 \times 10^{10} t^5 + \cdots$$

$$Ic(t) = \sum_{n=0}^{k} Ic(k)t^k = 40 + 70t + 7.498781600t^2 - 2.738138330t^3$$
$$+ 2.8111202358 \times 10^9 t^4 - 5.713612288 \times 10^9 t^5 + \cdots$$

$$I(t) = \sum_{n=0}^{k} I(k)t^k = 20 + 83t + 7.498667350 \times 10^5 t^2 - 3.113041231 \times 10^5 t^3$$
$$- 2.811202358 \times 10^9 t^4 - 5.71361228 \times 10^9 t^5 + \cdots$$

$$R(t) = \sum_{n=0}^{k} R(k)t^k = 10 + 13.580t + 30.16082000t^2 + 1.874652561 \times 10^5 t^3$$
$$- 65024.53968t^4 4.216822002 \times 10^8 t^5 + \cdots$$

$$W(t) = \sum_{n=0}^{k} W(k)t^k = 200 - 24t + 34.6300000t^2 + 2.249614284 \times 10^5 t^3$$
$$- 66856.80228t^4 - 5.060150244 \times 10^8 t^5 + \cdots$$

NUMERICAL SIMULATION AND GRAPHICAL ILLUSTRATION OF THE MODEL

We demonstrated the numerical simulation which illustrate the analytical results for the proposed Model. This is achieved by using some set of parameter values given in the table (2) below and whose source are mainly from literature and well as assumptions . We considered different initial conditions for the human populations. $S(0) = 60$, $I_c(0) = 40$, $I(0) = 20$, $R(0) = 10$ and That of baterial populations $W(t) = 200$ The DTM is demostrated against mapple buit-in fourth order Runge-Kutta Procedure for the solution of the model. Figure (2) to (6) shows the combined plots of the solutions of $S(t)$, $I_c(t)$, $I(t)$, $R(t)$ and $W(t)$ by DTM and RK4

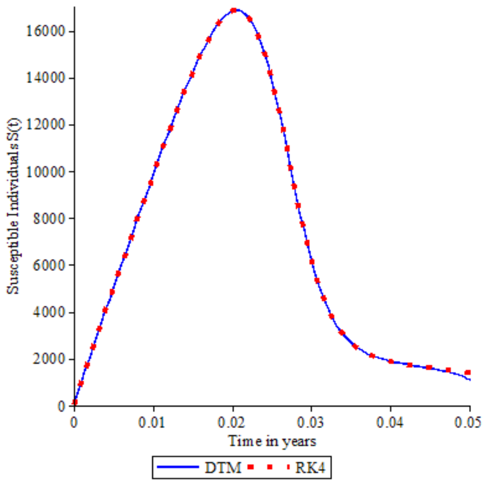

Figure 2: Solution of Susceptible Population by DTM and RK4

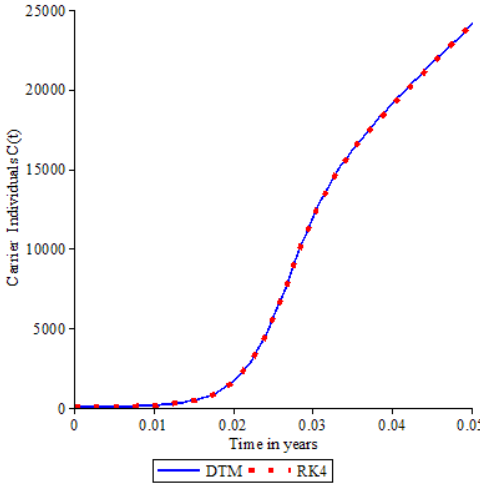

Figure 3: Solution of Carrier Population by DTM and RK4

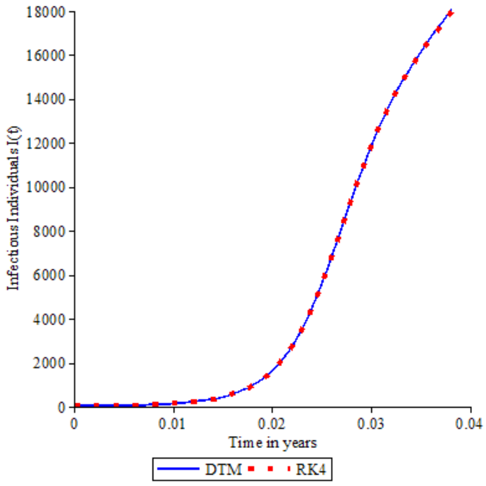

Figure 4: Solution of Infected Population by DTM and RK4

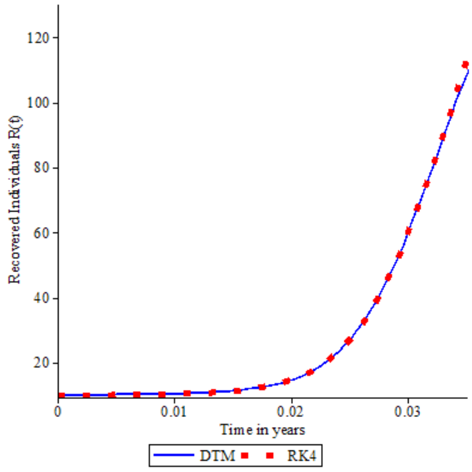

Figure 5: Solution of Recovered Population by DTM and RK4

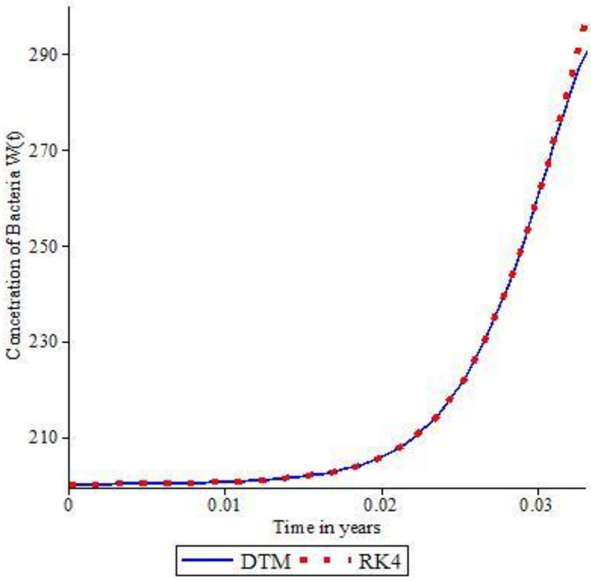

Figure 6: Solution of Bacteria Population by DTM and RK4

Table 3: Parameters values for model

Parameter	Initial Value	Source
μ_2	0.2	Assumed
μ_1	0.142	Mushayabasa, (2011)
μ_3	0.2	Assumed
μ_4	0.142	Mushayabasa, (2011)
ρ	0.5	Assumed
β_1	0.02	Assumed
β_2	0.01	Assumed
β_3	0.01	Assumed
δ	0.75	Assumed
θ	10^6	Lauria *et al.*, (2009)
ε_1	0.4	Estimated
ε_2	0.5	Estimated
μ_b	0.01	Mushayabasa, (2017)

DISCUSSION OF RESULTS FOR DIFFERENTIAL TRANSFORM METHOD

The solutions obtained by using Differential Transform Method with given initial conditions compared favourably with the solution obtained by using classical fouth-other Runge-Kuta method. The solutions of the two methods follows the same pattern and behaviour. This shows that Differential Transform Method is suitable and efficient to conduct the analysis of typhoid models.

CONCLUSION

We present a deterministic model on the analysis of direct and indirect transition dynamics of typhoid fever model. We tested for the existence and uniqueness of solution for the model using the Lipchitz condition to ascertain the efficacy of the models Differential Transform Method (DTM) is employed to attempt the series solution of the model. Numerical simulations were carried out to compare the results obtained by Differential Transform Method with the result of classical fourth-order Runge-Kutta method. The results of the simulations were displayed graphycally. Based on the results obtained from this study,The results obtained confirm the accuracy and potential of the DTM to cope with the analysis of modern epidemics.

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