APPLICATION OF DIFFERENTIAL TRANSFORMATION METHOD FOR SOLVING MATERNALLY-DERIVE –IMMUNITY SUSCEPTIBLE INFECTED RECOVERED (MSIR) MODEL OF A MEASLES DISEASE

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INTRODUCTION

Measles is an airborne disease which spreads easily through the coughs and sneezes of those infected. It may also be spread through contact with saliva or nasal secretions. Nine out of ten people who are not immune and share living space with an infected person will likely catch it. People are infectious to others from four days before to four days after the start of the rash. People usually do not get the disease more than once in a life time; indicating that once recovered from the disease, the person become permanently immune (Atkinson, 2011).

INTRODUCTION CONT'D

According to WHO, measles is one of the leading causes of death among young children even though a safe and cost-effective vaccine is available. In 2015, there were 134 200 measles deaths globally – about 367 deaths every day or 15 deaths every hour. Measles vaccination resulted in a 79% drop in measles deaths between 2000 and 2015 worldwide. In 2015, about 85% of the world's children receive 'xone dose of measles vaccine by their first birthday through routine health services – up from 73% in 2000. During 2000-2015, measles vaccination prevented an estimated 20.3 million deaths making measles vaccine one of the best buys in public health.

INTRODUCTION CONT'D

Differential Transformation Method (DTM) is one of the method use to solve linear and nonlinear differential equations. It was first proposed by Zhou, (1986), for solving linear and nonlinear initial value problems in electrical circuit analysis. The DTM construct a semi-analytical numerical technique that uses Taylor series for the solution of differential equations in the form of a polynomial. DTM is a very effective and powerful tool for solving different kinds of differential equations. The main advantage of this method is that it can be applied directly to linear and nonlinear Ordinary Differential Equations (ODEs) without linearization, discretization or perturbation.

WHAT WE HAVE DONE

In this paper:

- We used DTM to solve the MSIR model of the measles disease
- The DTM solution was validated with Runge-Kutta 4-5th order in Maple software
- The DTM solutions were presented graphically

MODEL EQUATIONS

$$\frac{dM}{dt} = \beta N - (\theta + \mu)M \qquad (2.1)$$

$$\frac{dS}{dt} = \theta M - \frac{\alpha SI}{N} - (\mu + v)S \qquad (2.2)$$

$$\frac{dI}{dt} = \frac{\alpha SI}{N} - (\gamma + \mu + \delta)I \qquad (2.3)$$

$$\frac{dR}{dt} = \gamma I - \mu R + vS \qquad (2.4)$$

Where,

$$N = M + S + I + R.$$

MODEL EQUATIONS CONT'D

- Definition of Variables and Parameter of the Model

M = passively immune infants

S = Susceptible class

I = Infected class

R = Recovered class

N = Total Population

α = Contact rate

Λ = recruitment rate

μ = natural death rate

δ = death rate due to disease

θ = loss of temporal immunity period

γ = recovery rate

v = vaccination rate

β = Birth rate

SOLUTION OF THEMODEL

- **Solution of the Model Equations using Differential Transformation Method (DTM)**

In this section we are going to apply Differential Transformation Method to the Model equation and solve.

Let the model equation be a function $h(t)$, $h(t)$ can be expanded in Taylor series about a point $t = 0$ as

$$h(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left[\frac{d^k h}{dt^k} \right]_{t=0} \tag{2.4}$$

Where,

$$h(t) = \{m(t), s(t), i(t), \ r(t)\} \tag{2.5}$$

SOLUTION OF THE MODEL CONT'D

The differential transformation of $h(t)$ is defined as

$$H(t)=\frac{1}{k!}\left[\frac{d^k h}{dt^k}\right]_{t=0} \qquad (2.6)$$

Where,

$$H(t)=\{M(t),\ S(t),\ I(t),\ R(t)\} \qquad (2.7)$$

Then the inverse differential transform is

$$h(t)=\sum_{k=0}^{\infty} t^k H(t) \qquad (2.8)$$

SOLUTION OF THE MODEL CONT'D

Using the fundamental operations of differential transformation method, we obtain the following recurrence relation of equation (3.1) to (3.4) as

$$M(k+1)=\frac{1}{k+1}\left[\beta N-(\mu+\theta)M\right]$$

$$S(k+1)=\frac{1}{k+1}\left[\theta M-\frac{\alpha}{N}\sum_{m=0}^{k}S(m)I(k-m)-(v+\mu)S(k)\right]$$

$$I(k+1)=\frac{1}{k+1}\left[\frac{\alpha}{N}\sum_{m=0}^{k}S(m)I(k-m)-(\delta+\gamma+\mu)I(k)\right] \qquad (2.9)$$

$$I(k+1)=\frac{1}{k+1}\left[\gamma I-\mu R+vS\right]$$

SOLUTION OF THE MODEL CONT'D

With initial conditions

$M(0)=82{,}010{,}000,\ \ S(0)=7{,}099{,}464{,}364,\ \ I(0)=254{,}918,\ \ R(0)=118270718$

(2.10)

The parameter values are

$N=7{,}300{,}000{,}000, \alpha_1=0.005,\ \alpha=0.9,\ \beta=0.019,\ \ \delta=0.53,\ \mu=0.0$

$v=0.85,\ \ \gamma=0.47,\ \theta=0.39$

(2.11)

We consider $k=0,\ 1,\ 2,\ 3$

We are going to considered three cases, varying different values of vaccination rate, v

SOLUTION OF THE MODEL CONT'D

Case 1: $v = 0.85$

Substituting (2.10) and (2.11) into (2.9) for $k = 0, 1, 2, 3$ gives

$$M(1) = 106060020,\ M(2) = 48244056,\ M(3) = 3983255,$$
$$M(4) = 30711621$$
$$S(1) = -6059579648,\ S(2) = 262035140,\ S(3) = -743157523,$$
$$S(4) = 16327345$$
$$I(1) = -33834,\ I(2) = -92976,\ I(3) = 39990,\ I(4) = 7466$$
$$R(1) = 6033718355,\ R(2) = -2599464175,\ R(3) = 749350235,$$
$$R(4) = -159414975$$

(2.12)

SOLUTION OF THE MODEL CONT'D

Substituting (2.12) into (2.8) for $k = 0,\ 1,\ 2,\ 3$ gives

$$m(t) = \sum_{k=0}^{\infty} m(k)t^k = 82010000 + 106060020t + 48244056t^2 + 3983255t^3$$
$$+ 30711621t^4$$

$$s(t) = \sum_{k=0}^{\infty} s(k)t^k = 7099464364 - 6059579648t + 262035140t^2 - 74315752t^3$$
$$+ 16327345t^4$$

$$i(t) = \sum_{k=0}^{\infty} i(k)t^k = 254918 - 33834t - 92976t^2 + 39990t^3 + 7466t^4$$

$$r(t) = \sum_{k=0}^{\infty} r(k)t^k = 118270718 + 6033718355t - 2599464175t^2 + 749350235t^3$$
$$- 159414975t^4$$

(2.13)

SOLUTION OF THE MODEL CONT'D

Case 2: $v = 0.50$

Substituting (2.10) and (2.11) into (2.9) for $k = 0,\ 1,\ 2,\ 3$ gives

$$\left.\begin{array}{l} M(1)=106060020,\ M(2)=48244056,\ M(3)=3983255,\\ \qquad M(4)=30711621\\ S(1)=-357476121,\ S(2)=928743534,\ S(3)=-150994477,\\ \qquad S(4)=16327345\\ I(1)=-33834,\ I(2)=-53929,\ I(3)=17085,\ I(4)=3220\\ R(1)=3548905828,\ R(2)=-907895354,\ R(3)=157203194,\\ \qquad R(4)=-19186708 \end{array}\right\}$$

(2.14)

SOLUTION OF THE MODEL CONT'D

Substituting (2.14) into (2.8) for $k = 0,\ 1,\ 2,\ 3$ gives

$$\left.\begin{aligned}
m(t) &= \sum_{k=0}^{\infty} m(k)t^k = 82010000 + 106060020t + 48244056t^2 + 3983255t^3 \\
&\qquad + 30711621t^4 \\
s(t) &= \sum_{k=0}^{\infty} s(k)t^k = 7099464364 - 357476121t + 928743534t^2 - 150994477t^3 \\
&\qquad + 16327345t^4 \\
i(t) &= \sum_{k=0}^{\infty} i(k)t^k = 2549918 - 33834t - 53929t^2 + 17085t^3 + 3220t^4 \\
r(t) &= \sum_{k=0}^{\infty} r(k)t^k = 118270718 + 3548905828t - 907895354t^2 + 157203194t^3 \\
&\qquad - 19186708t^4
\end{aligned}\right\} \qquad (2.15)$$

SOLUTION OF THE MODEL CONT'D

Case 3: $v = 0.25$

Substituting (2.10) and (2.11) into (2.9) for $k = 0, 1, 2, 3$ gives

$$\left.\begin{aligned}
&M(1)=106060020,\ M(2)=48244056,\ M(3)=3983255,\\
&\qquad M(4)=30711621\\
&S(1)=-1799901030,\ S(2)=252912028,\ S(3)=-15476262,\\
&\qquad S(4)=4879493\\
&I(1)=-33834,\ I(2)=-263039,\ I(3)=6304,\ I(4)=850\\
&R(1)=17740339737,\ R(2)=-232091739,\ R(3)=21690834,\\
&\qquad R(4)=-1009907
\end{aligned}\right\} \quad (2.16)$$

SOLUTION OF THE MODEL CONT'D

Substituting (2.16) into (2.8) for $k = 0, 1, 2, 3$ gives

$$m(t)=\sum_{k=0}^{\infty} m(k)t^{k} = 82010000+106060020t+48244056t^{2}+3983255t^{3}$$

$$+30711621t^{4}$$

$$s(t)=\sum_{k=0}^{\infty} s(k)t^{k} = 7099464364-1799901030t+252912028t^{2}-15476262t^{3}$$

$$+4879493$$

$$i(t)=\sum_{k=0}^{\infty} i(k)t^{k} = 2549918-33834t-263039t^{2}+6304t^{3}+850t^{4}$$

$$r(t)=\sum_{k=0}^{\infty} r(k)t^{k} = 118270718+17740339737t-232091739t^{2}+21690834t^{3}$$

$$-1009907t^{4}$$

(2.17)

RESULTS AND DISCUSSION

- **Numerical Solution of DTM and Runge-Kutta**
  We compared the DTM solutions with the Runge-Kutta impeded in Maple software

Table 1: Numerical solution of Maternally-Derived-Immunity

t	DTM	RUNGE-KUTTA
0	82010000.0000	82010000.0000
0.1	93141346.6771	92407714.9923
0.2	105519568.4736	102399727.9392
0.3	119494224.9551	112001868.6768
0.4	135488583.5776	21229349.5002
0.5	153999619.6875	130096788.8131
0.6	175598016.5216	138618235.0603
0.7	200928165.2071	146807188.2767
0.8	230708164.7616	154676621.6537
0.9	265729822.0931	162239002.6453
1.0	306858652.0000	169506311.5421

RESULTS AND DISCUSSION CONT'D

Table 2: Numerical solution of Susceptible

t	DTM	RUNGE-KUTTA
0	7099464364.0000	7099464364.00000
0.1	6518983083.0328	6518973604.2711
0.2	5986678467.7888	5986595001.9527
0.3	5498679357.5788	5498357705.5832
0.4	5051506448.0128	5050617796.2627
0.5	4642072291.0000	4640031884.8469
0.6	4267681294.7488	4263531508.1332
0.7	3926029723.7668	3918301369.1918
0.8	3615205698.8608	3601758125.8419
0.9	3333689197.1368	3311531021.7912
1.0	3080352052.0000	3045445194.6477

RESULTS AND DISCUSSION CONT'D

Table 3: Numerical solution of Infected

t	DTM	RUNGE-KUTTA
0	254918.0000	254918.0000
0.1	250645.5766	250645.5030
0.2	244764.0256	244760.7979
0.3	237540.1646	237516.1576
0.4	229258.7296	229160.7508
0.5	220222.3750	219933.2495
0.6	210751.6736	210056.2946
0.7	201185.1166	199732.8615
0.8	191879.1136	189144.0930
0.9	183207.9926	178448.3956
1.0	175564.0000	167781.5011

RESULTS AND DISCUSSION CONT'D

Table 4: Numerical solution of Recovered/Immune

t	DTM	RUNGE-KUTTA
0	118270718.0000	118270718.0000
0.1	696381320.4875	696381424.3608
0.2	1226775559.9200	1226781149.5879
0.3	1713375643.7975	1713424746.6791
0.4	2159721183.6800	2159940743.3293
0.5	2568969195.1875	2569657728.8486
0.6	2943894098.0000	2945629940.3248
0.7	3286887715.8575	3290659005.3171
0.8	3599959276.5600	3607315135.0971
0.9	3884735411.9675	3897956474.5273
1.0	4142460158.0000	4164745776.5776

RESULTS AND DISCUSSION CONT'D

- **Graphical Solution of the Differential Transformation Solution**

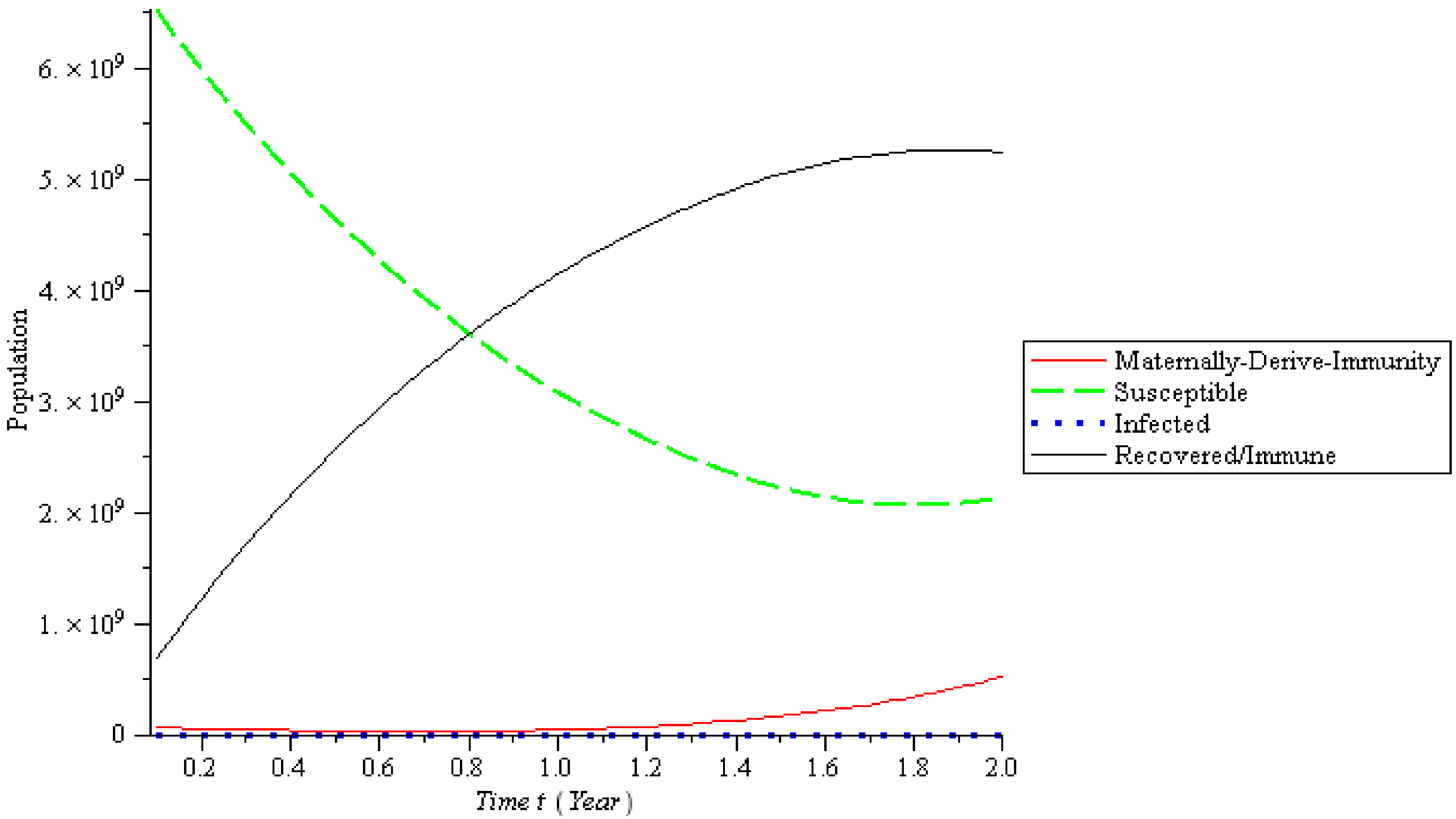

Figure 1: The Population against time of High Vaccination rate

RESULTS AND DISCUSSION CONT'D

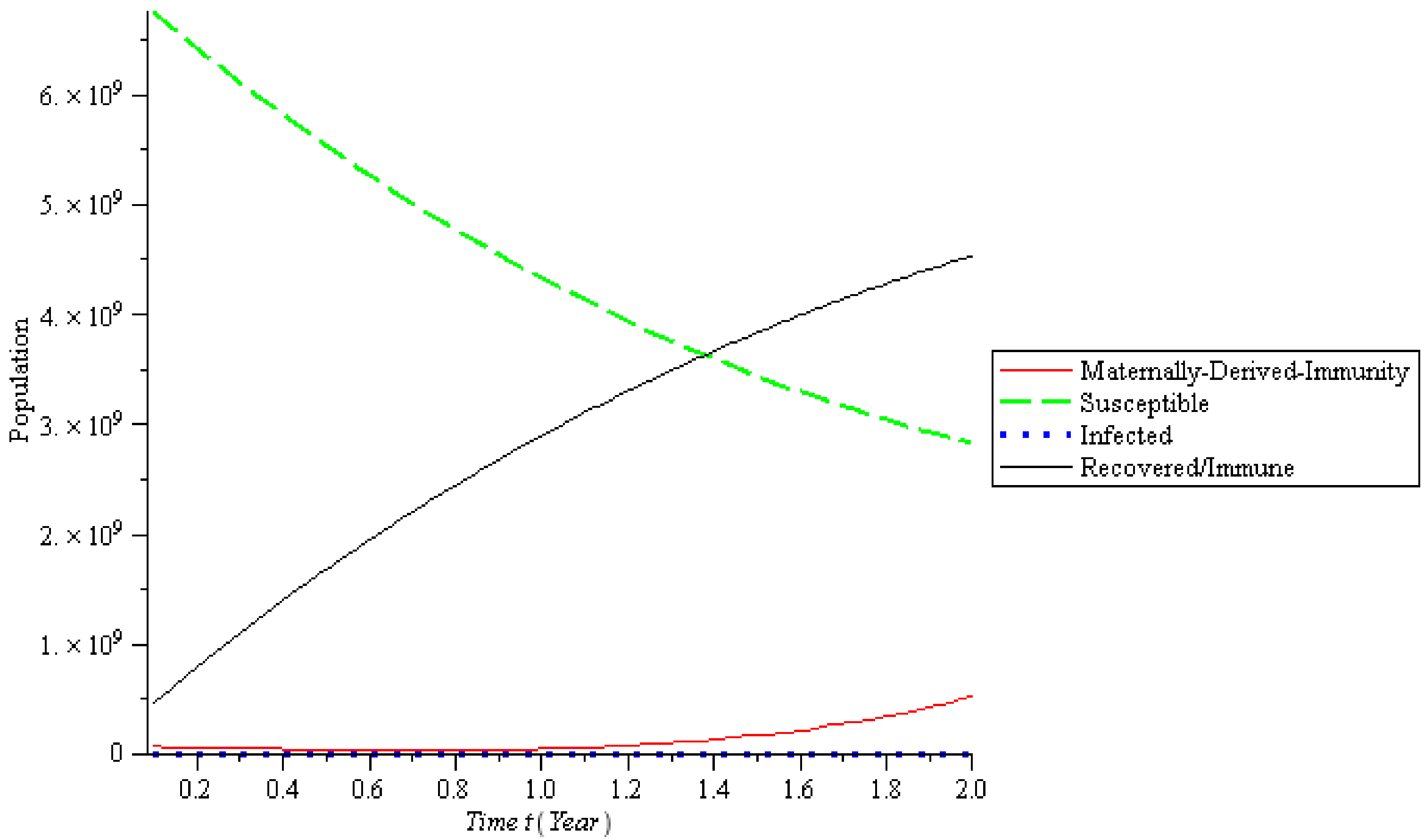

Figure 2: The Population against time of High Moderate Vaccination rate

RESULTS AND DISCUSSION CONT'D

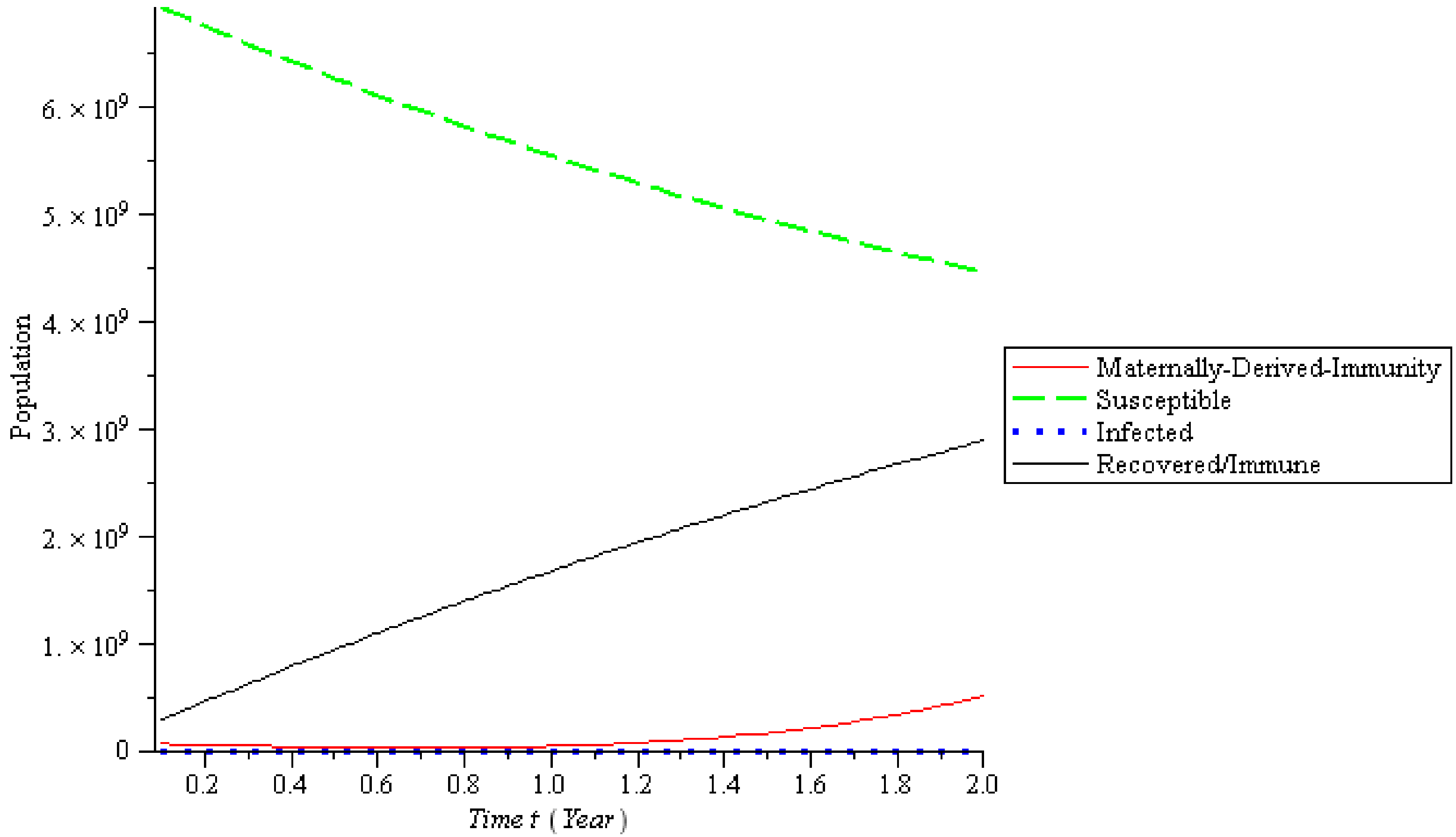

Figure 3: The Population against time of Low Vaccination rate

CONCLUSION

The method of solution gives us better understanding of the model. Immunization plays a vital role in preventing the outbreak of disease in the population. Attention should be given to immunization in order eradicate measles from the population.

The numerical solutions shows that the Differential Transformation Method is in agreement with the Runge-Kutta.

THANK YOU FOR LISTENING