AN ANALYTIC INVESTIGATION OF THE BLOCH NUCLEAR MAGNETIC RESONANCE FLOW EQUATIONS FOR THE ANALYSIS OF GENERAL FLUID FLOW

***S. I. Yusuf**
Works Department, Federal University of Technology, Minna, Nigeria.
Y. M. Aiyesimi
Department of Mathematics, Federal University of Technology, Minna, Nigeria.
O. B. Awojoyogbe
Department of Physics, Federal University of Technology, Minna, Nigeria.

ABSTRACT

Nuclear Magnetic Resonance (NMR) techniques now referred to as Magnetic Resonance Imaging (MRI) are used to study the anatomy, physiology and pathology of human living tissues. In this study, the mathematical processes involved are examined. A second order time independent non-homogenous linear differential equation from the Bloch NMR equations is evolved. The parameters in the equations are equilibrium magnetization M_o, radio frequency $rfB_1(x,t)$ field, gyro-magnetic ratio of blood spin γ, velocity V as well as T_1 and T_2 relaxation times. A general method of solution of each equation under the influence of radio frequency magnetic field ($rfB_1(x,t) \neq 0$) and in the absence of radio frequency magnetic field ($rfB_1(x,t) = 0$) is evolved. For the purpose of this study, the influence of the relaxation times T_1 and T_2 are examined.

1. INTRODUCTION

Nuclear Magnetic Resonance, NMR, measures how much electromagnetic radiation of a specific frequency is absorbed by an atomic nucleus that is placed in a strong magnetic field. Its objective is to visualize the atomic and molecular structure of chemical compounds - Edward (1945). NMR is produced when a radio frequency field is imposed at right angles to a much larger static magnetic field to perturb the orientation of nuclear magnetic moments generated by spinning electrically charged atomic nuclei.

The procedure requires that a substance be placed in a strong magnetic field. This strong magnetic field affects the spin of the atomic nuclei of elements, for example hydrogen molecules. These have an angular momentum arising from their inherent property of spin. NMR is inherently a three-dimensional phenomenon. The spatial resolution of a three-dimensional set of data is usually equal in all three directions. The basic requirements for Nuclear Magnetic Resonance spectroscopy are that the magnetic field be homogenous over the volume of the sample; that there be a radio frequency field rotating in a plane perpendicular to the static field and that there be a means of detecting the interaction of the frequency field with the sample.

The main goal of this study is to establish a methodology of using mathematical techniques so that an accurate measurement of blood flow in human physiological and pathological conditions can be carried out non-invasively and become simple to implement in medical clinics. The objective of this research work is to formulate mathematically a non-transient second order non-homogenous linear differential equation from the Felix Bloch's Nuclear Magnetic Resonance equations:

$$\frac{dM_x}{dt} = -\frac{M_x}{T_2} \qquad (1.1)$$

$$\frac{dM_y}{dt} = \gamma M_z B_1(x) - \frac{M_y}{T_2} \qquad (1.2)$$

$$\frac{dM_z}{dt} = -\gamma M_y B_1(x) - \frac{(M_0 - M_z)}{T_1} \qquad (1.3)$$

-Bloch (2006).

The components of magnetization of fluid flow are given as stated above and are fundamental to understanding Magnetic Resonance Images with:

M_o = equilibrium magnetization

M_x = component of magnetization along the x-axis

M_y = motion of transverse magnetization (y-axis)

M_z = component of magnetization along the field (z-axis)

γ = gyro-magnetic ratio of blood spin

$rfB_1(x,t)$ = radio-frequency field

T_1 = Longitudinal or spin lattice relaxation time

T_2 = Transverse or spin-spin relaxation time

V = the flow velocity

It will be observed that at the initial state, $M_z = M_o$ and $M_y = 0$ whereas the net magnetization M_{xy} perpendicular to the field is zero. Following any perturbation of this magnetization, e.g. following the application of $rfB_1(x,t)$ field, precesses take place whereby M_z and M_{xy} return to their equilibrium values of M_o and zero, respectively.

Blood is also assumed to be magnetized by the static B_o field to an equilibrium magnetization, M_o and that resonance condition exists at Larmor frequency: $f_o = \gamma B - \omega = 0$. The Larmor theorem states that the motion of a magnetic moment in a magnetic field $\sim B_o$ is a precession around that field. The precession frequency is also called Larmor frequency.

The solutions so obtained will be examined under these two different conditions namely:

(i) $\gamma^2 B_1^2 << K$ i.e. radio frequency field is negligible i.e. zero and

(ii) $\gamma^2 B_1^2 >> K$ i. e. radio frequency is introduced.

2. MATHEMATICAL FORMULATION AND SOLUTIONS OF TIME - INDEPENDENT BLOCH NMR FLOW EQUATIONS

It is assumed that blood is a Newtonian fluid - Ayeni (1993). It is magnetized by the static B_o field to an equilibrium magnetization and that resonance condition exists at Larmor frequency: $f_o = \gamma B - \omega = 0$. For blood flow analysis, it is also assumed the blood is flowing along the x-direction hence the flow is independent of y and z components. This implies that the flow is constant along y and z directions. The NMR signal is the electro-motive force, e.m.f. induced by the precessing transverse magnetization M_y of the flowing spins and is dependent on the flow velocity V, T_1 and T_2 relaxation parameters. M_y results from the combined effect of B_o and $rfB_1(x,t)$ on blood spins. For steady flow, blood or fluid flows through a blood vessel of uniform cross section with velocity V. It is also assumed that resonance condition exists within the excitor as well as the detector coils.

For blood flow analysis, it is assumed the blood spins to be flowing along the x-direction hence the flow is independent of y and z components. This implies that the flow is constant along y and z directions. Flow against the gravity is made possible by one - way valves, located several centimeters apart in the veins – Setaro (2006).

NMR spin are always in motion. Therefore, it is pertinent to treat them with reference to their motion since they change position from time to time - Dada (2006). This motion is very much pronounced in fluids. From the kinetic theory of moving fluids, given a property M of the fluid, then the rate at which this property changes with respect to a point moving along with the fluid be the total derivative:

$$\frac{dM}{dt}=\frac{\partial M}{\partial t}+\frac{\partial M}{\partial x}V_x+\frac{\partial M}{\partial y}V_y+\frac{\partial M}{\partial z}V_z$$

$$\Rightarrow \frac{dM}{dt}=\frac{\partial M}{\partial t}+V.\nabla M$$

Therefore, the three Bloch equations (1 – 3) above become:

$$\frac{dM_x}{dt}=\frac{\partial M_x}{\partial t}+V.\nabla M_x=-\frac{M_x}{T_2} \qquad (2.1)$$

$$\frac{dM_y}{dt}=\frac{\partial M_y}{\partial t}+V.\nabla M_y=\gamma M_z B_1(x)-\frac{M_y}{T_2} \qquad (2.2)$$

$$\frac{dM_z}{dt}=\frac{\partial M_z}{\partial t}+V.\nabla M_z=-\gamma M_y B_1(x)-\frac{M_o-M_z}{T_1} \qquad (2.3)$$

With (2.2), equation (2.3) becomes:

$$V^2\frac{\partial^2 M_y}{\partial x^2}+2V\frac{\partial^2 M_y}{\partial x\partial t}+V\left(\frac{1}{T_1}+\frac{1}{T_2}\right)\frac{\partial M_y}{\partial x}+\left(\frac{1}{T_1}+\frac{1}{T_2}\right)\frac{\partial M_y}{\partial t}+\frac{\partial^2 M_y}{\partial t^2}+\left\{\frac{1}{T_1T_2}+\gamma^2 B_1^2(x,t)\right\}M_y=\frac{\gamma B_1(x,t)M_o}{T_1} \qquad (2.4)$$

Equation (2.4) is a general second order differential equation which can be applied to any fluid flow – Awojoyogbe (2002).

Now to evolve *Time-Independent flow equation* from equations 4 – 6, all partial derivatives with respect to time can be set to zero (i.e. $\frac{\partial}{\partial t}=0$, a condition necessary for steady flow). This then leads us to the equation

$$\Rightarrow V\frac{dM_y}{dx}\left(V\frac{d}{dx}+\frac{1}{T_1}\right)+\frac{M_y}{T_2}\left[V\left(\frac{d}{dx}+\frac{1}{T_1}\right)\right]=-\gamma^2 M_y B_1^2(x)+\frac{\gamma B_1(x)M_o}{T_1}$$

$$\Rightarrow \frac{d^2 M_y}{dx^2} + \frac{1}{V}\left(\frac{1}{T_1} + \frac{1}{T_2}\right)\frac{\partial M_y}{\partial x} + \frac{1}{V^2}\{\gamma^2 B_1^2(x) + \frac{1}{T_1 T_2}\}M_y = \frac{M_0 \gamma B_1(x)}{V^2 T_1} \quad (2.5)$$

This equation (2.5) can be re written as

$$\frac{d^2 M_y}{dx^2} + \frac{T_o}{V}\frac{dM_y}{dx} + \frac{1}{V^2}\{\gamma^2 B_1^2 + K\}M_y = \frac{F_0 \gamma B_1}{V^2 T_1} \quad (2.6)$$

Where $$T_o = \frac{1}{T_1} + \frac{1}{T_2}; K = \frac{1}{T_1 T_2} and F_o = \frac{M_o}{T_1}$$

Equation (2.6) is the **Time - Independent** Bloch NMR flow equation - Awojoyogbe (2004).

3. SOLUTIONS OF TIME – INDEPENDENT BLOCH NMR FLOW EQUATIONS

Case I:

Considering $\gamma^2 B_1^2 << K$ i.e. radio frequency field $\gamma^2 B_1^2$ is negligible or totally removed implying the ground state of the protons, then equation 9 becomes

$$\frac{d^2 M_y}{dx^2} + \frac{T_o}{V}\frac{dM_y}{dx} + \frac{K}{V^2}M_y = \frac{F_o \gamma B_1}{V^2 T_1} \quad (3.1)$$

Solving (3.1) using the method of variation of parameters by Erwin Kreyszig (1988) gives the solution -

$$M_y(x) = A_1(x)e^{-\frac{x}{\lambda}} + A_2(x)e^{-\frac{x}{\lambda}} + \frac{F_o \lambda^2 \gamma B_1}{(V^2 - T_o V\lambda + k\lambda^2)} \quad (3.2)$$

Case II: When $Y^2 B^2{}_1 >> K$ i.e radio-frequency field is introduced.

Equation 9 becomes

$$\frac{d^2 M_y}{dx^2} + \frac{T_o}{V}\frac{dM_y}{dx} + \frac{\gamma^2 B_1^2}{V^2}M_y = \frac{F_o \gamma B_1}{V^2 T_1} \quad (3.3)$$

which gives the solution

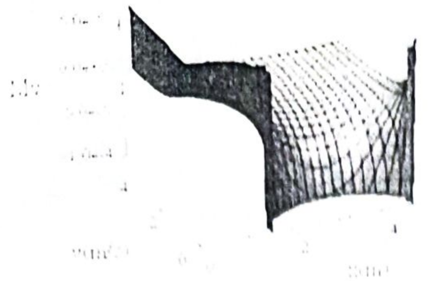
$$\Rightarrow M_y(x) = C_1 e^{-\frac{x}{\lambda}} + C_2 e^{-\frac{x}{\lambda}} + \frac{F_o \lambda^2 \gamma B_1}{(V^2 - T_o V\lambda + \gamma^2 B_1^2 \lambda^2)} \quad (3.4)$$

3.1 Plot of a 3-Dimensional Time-Independent Flow Equation

The solution to the equation as solved above is

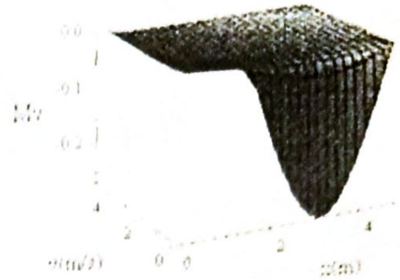
$$M_y(x) = A_1(x)e^{-\frac{x}{\lambda}} + A_2(x)e^{-\frac{x}{\lambda}} + \frac{F_o \lambda^2 \gamma B_1}{(V^2 - T_o V\lambda + k\lambda^2)}$$ for the condition ($\gamma^2 B_1^2 << K$)

Now assuming $A_1 = A_2 = 1$; $F_o = 1$ and radio frequency field $\gamma B_1(x) = \cos\frac{x}{\lambda}$ and also keeping $T_1 = 1$ (constant). Varying T_2 as 0.02, 0.07, 0.12, 0.17,......, 0.47, the graphs plotted below would be obtained:



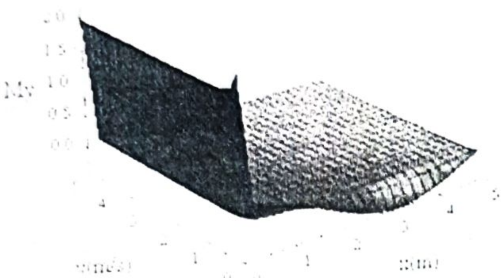
Plot of My against v and x at T2=0.02

a.



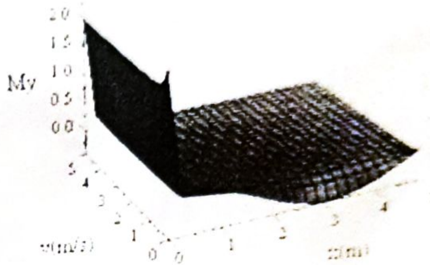
Plot of My against v and x at T2=0.07

b.



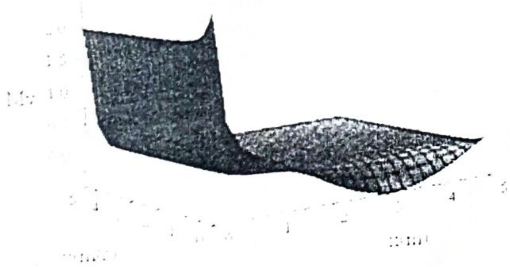
Plot of My against v and x at T2=0.12

c.



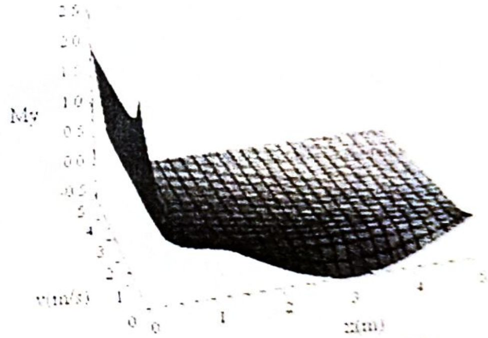
Plot of My against v and x at T2=0.17

d.



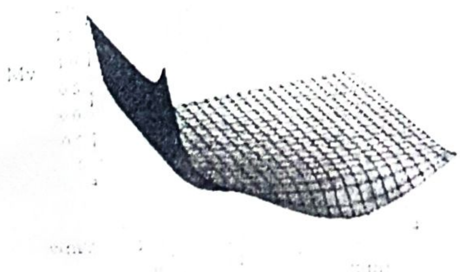
Plot of My against v and x at T2=0.22

e.



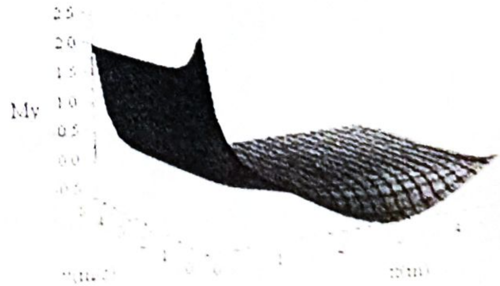
Plot of My against v and x at T2=0.27

f.



Plot of My against v and x at T2=[illegible]

g.



Plot of My against v and x at T2=0.37

h.

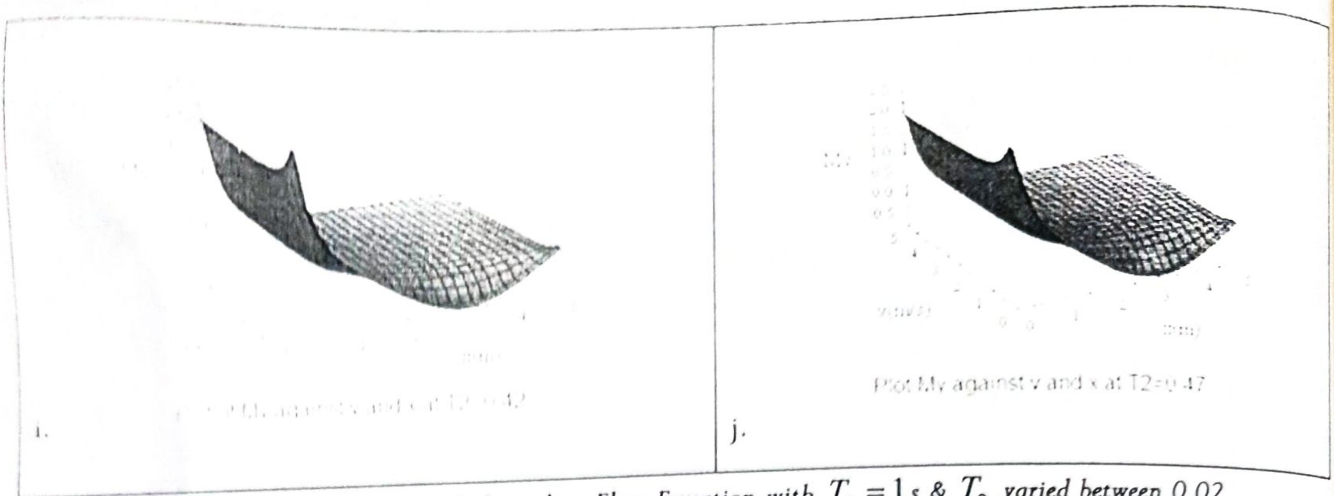
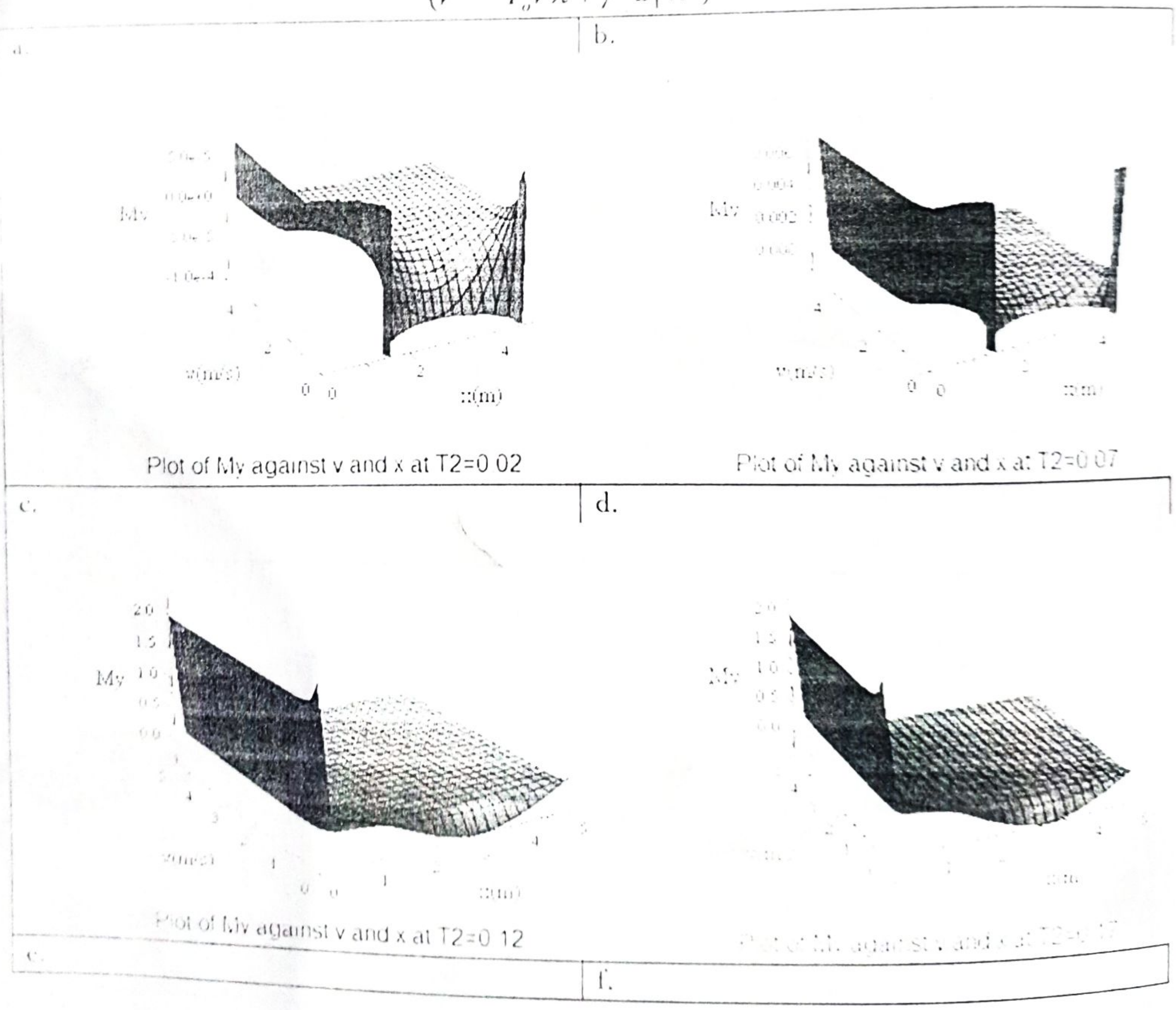
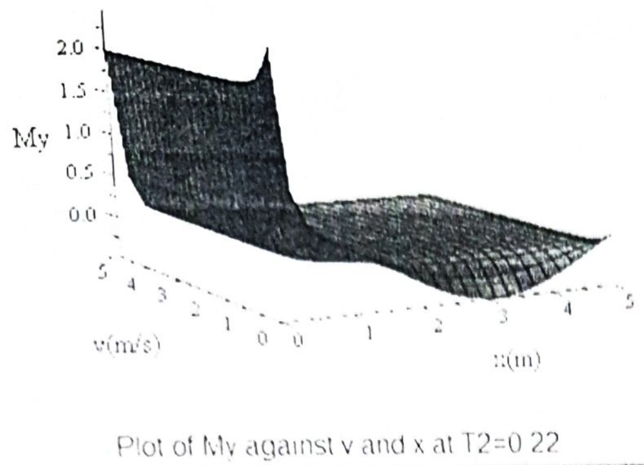


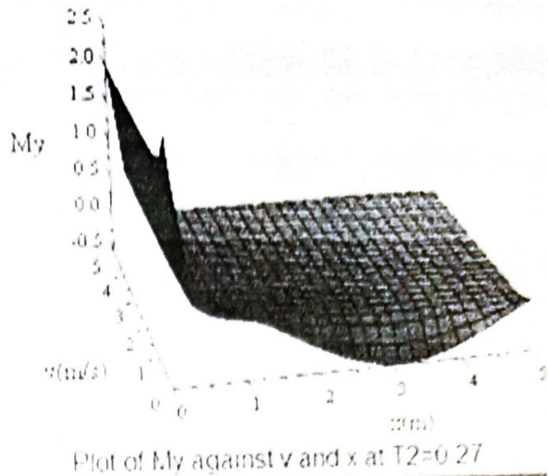
Fig. 1- Plot of a 3-Dimensional Time-Independent Flow Equation with $T_1 = 1s$ *&* T_2 *varied between 0.02 and 0.47 when* $\gamma^2 B_1^2 << K$.

Similarly, by using the same assumptions and plotting

$$M_y(x) = A_1(x)e^{\cdots} + A_2(x)e^{\cdots} + \frac{F_o \lambda^2 \gamma B_1}{(V^2 - T_o V \lambda + \gamma^2 B_1^2 \lambda^2)}$$ for condition ($\gamma^2 B_1^2 >> K$)

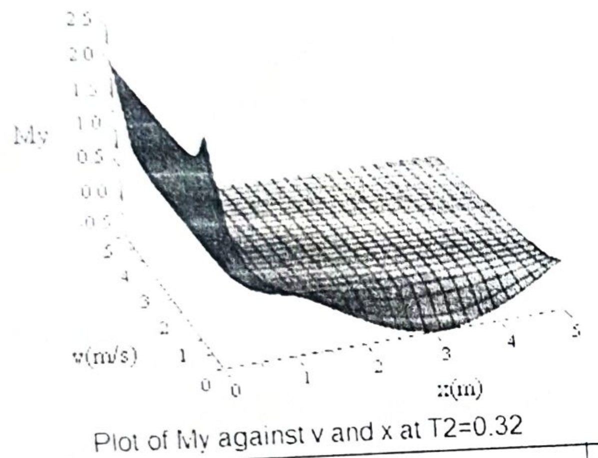


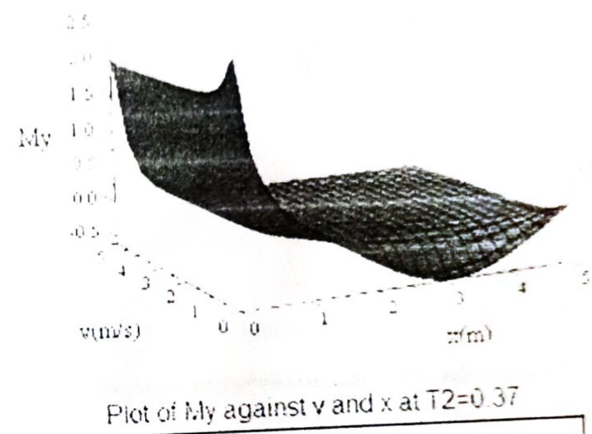




g.

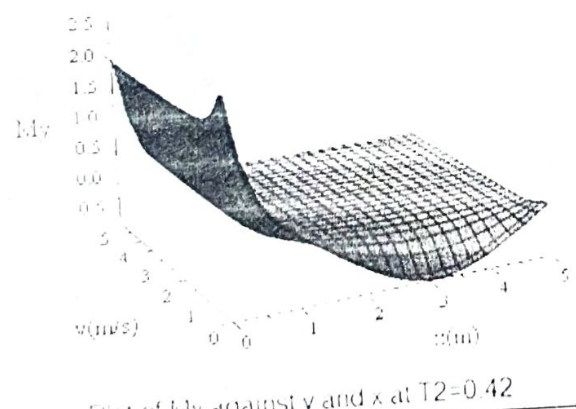
h.





i.

j.



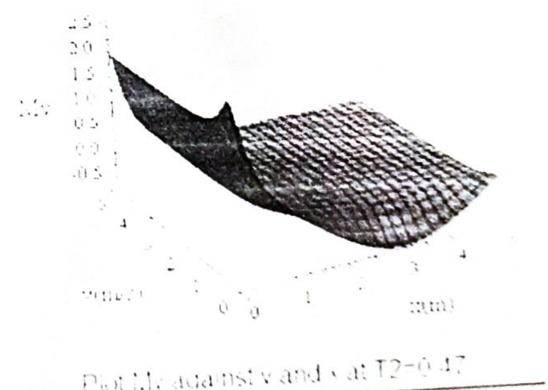


Fig.2 - Plot of a 3-Dimensional Time-Independent Flow Equation with $T_1 = 1s$ & T_2 varied between 0.02 and 0.47 when $\gamma^2 B_1^2 >> K$

Now if T_2 is kept constant at 0.3 and T_1 is varied as 0.8, 0.85, 0.9, 0.95, 1.0, 1.05, 1.10 and 1.15, then the graphs plotted below would be obtained for

$$M_y(x) = A_1(x)e^{-\frac{x}{\lambda}} + A_2(x)e^{-\frac{x}{\lambda}} + \frac{F_o \lambda^2 \gamma B_1}{(V^2 - T_o V\lambda + k\lambda^2)}, \qquad (\gamma^2 B_1^2 << K)$$

b.

a.

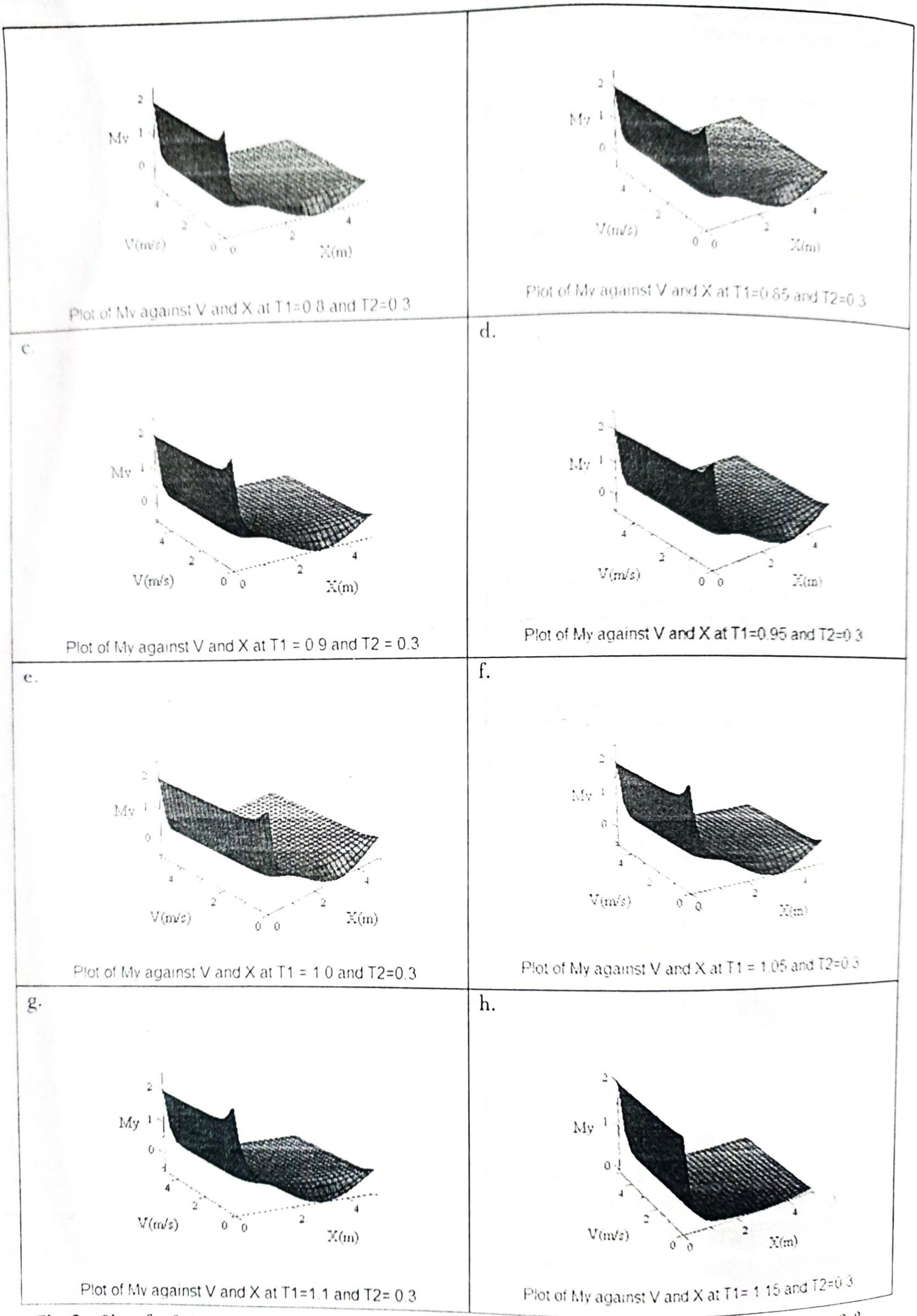
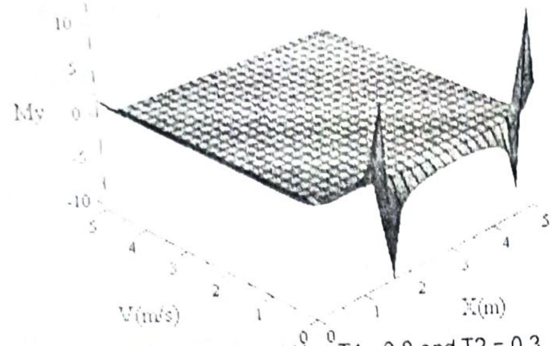


Fig. 3 - Plot of a 3-Dimensional Time-Independent Flow Equation with $T_2 = 0.3\,s$ & T_1 *varied between 0.8 and 1.15 when* $\gamma^2 B_1^{\,2} << K$.

Similarly by using the same assumptions and plotting

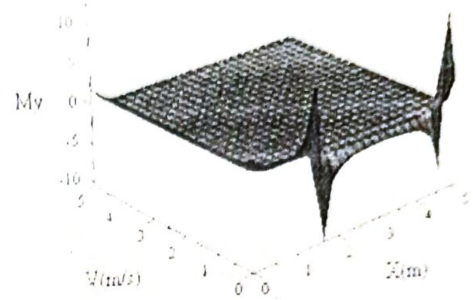
$$M_y(x) = A_1(x)e^{-\frac{x}{\lambda}} + A_2(x)e^{-\frac{x}{\lambda}} + \frac{F_o\lambda^2\gamma B_1}{(V^2 - T_oV\lambda + k\lambda^2)}$$ for the condition $(\gamma^2 B_1^2 >> K)$

a.



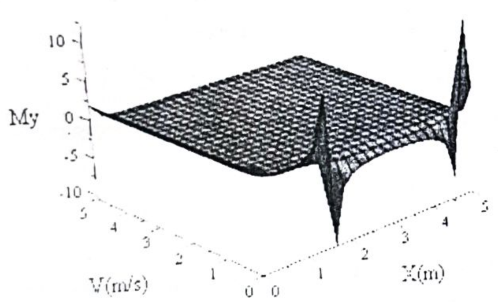
Plot of My against V and X at T1 =0.8 and T2 = 0.3

b.



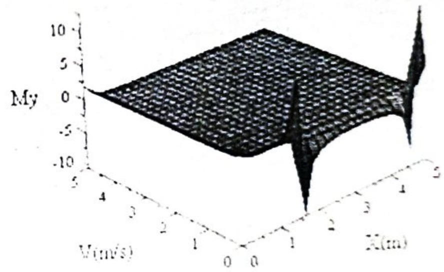
Plot of My against V and X at T1=0.85 and T2 =0.3

c.



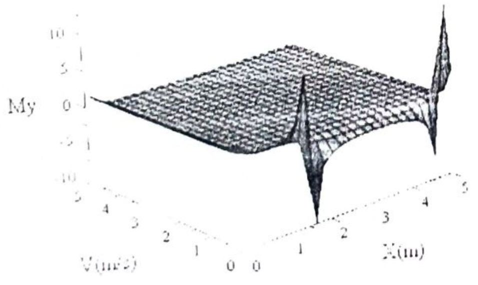
Plot of My against V and xX at T1 = 0.9 and T2=0.3

d.



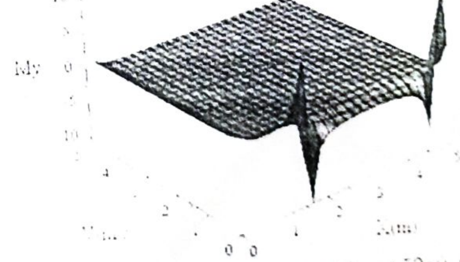
Plot of My against V and X at T1=0.95 and T2=0.3

e.



Plot of My against V and X at T1=1.0 and T2 = 0.3

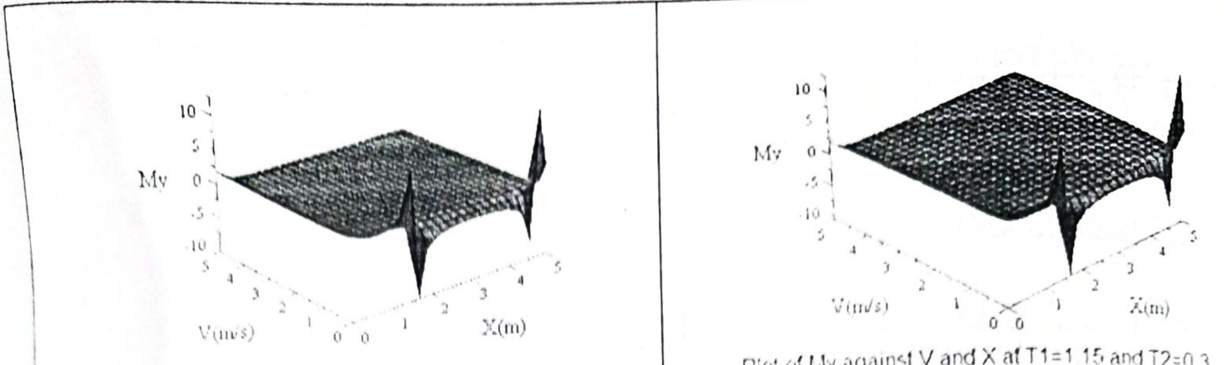
f.



Plot of My against V and X at T1=1.05 and T2=0.3

g.

h.



Plot of My against V and X at T1=1.1 and T2=0.3

Plot of My against V and X at T1=1.15 and T2=0.3

Fig.4 *Plot of a 3-Dimensional Time-Independent Flow Equation with* $T_2 = 0.3\,s$ & T_1 *varied between 0.8 and 1.15 when* $\gamma^2 B_1^{\;2} >> K$.

4. ANALYSIS OF RESULTS

From figure 1, where radio frequency ($\gamma^2 B_1^{\;2} << k = \frac{1}{T_1 T_2}$) is negligible and T_2 varied, the effect is that between $0.02 - 0.07$ both the transverse magnetization M_y and the velocity v show appreciable difference. The implication is that the strength of the signal is high with high velocity as there is rapid interaction between the molecules and its environment. Recall that transverse magnetization M_y carry information about the atoms and their environment. However, the response of the magnetization and velocity do not show significant difference between 0.12 and 0.47 implying the strength of the signal is low.

5. CONCLUDING REMARKS

In this study, the analytic solution of Bloch NMR equation has been presented under the heading - time independent. Two instances were considered - the free precision of the nuclei or negligible field and the presence of an exciting field or driving force. In each of these instances, T_1 spin – lattice relaxation time was kept constant while T_2 spin – spin relaxation time varied and the process reversed. Graphs of these different stages were plotted and the results obtained analyzed. It is now known that Nuclear Magnetic Resonance measures how much electromagnetic radiation of a specific frequency is absorbed by an atomic nucleus that is placed in a strong magnetic field. The improvement made on Nuclear Magnetic Resonance was on the precision of the radio waves in order to gauge the resonant signals with more accuracy alongside the development made in applying the magnetic field now made it to be referred to Magnetic Resonance Imaging. Various mathematical methods by which radio signals could be analyzed and transformed into a useful image that would show precise distinctions between different areas of living tissue have been developed - Mansfield (2003). It has further been discovered that while spin – lattice relaxation time (T_1) is constant for blood, the spin – spin relaxation time (T_2) shows remarkable difference

From the various graphs plotted above for M_y with respect to the separate variations of T_1 and T_2, It is generally obvious from the plots in time independent equations (Figures 1 to 4) that the NMR system as developed mathematically is more sensitive to change in the spin – spin relaxation time (T_2) than the spin – lattice relaxation time (T_1). This is evident in all the cases with T_2-constant, where the curves for each set of the varying T_1 do not show significant difference from one another.

* Correspondence address - Email: amaams5@yahoo.com Tel.: 08029030039.

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