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EFFECT OF TRANSVERSE RELAXATION RATE ON TIME - DEPENDENT MAGNETIC RESONANCE IMAGING

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ABSTRACT

Magnetic Resonance Imaging (MRI) developed from Nuclear Magnetic Resonance involves a non invasive medical approach towards studying the anatomy, physiology and pathology of human living tissues. In this study, attempt is made at expressing mathematically the processes involved in MRI for diagnosis and possible treatment of diseases within the human body. A time - dependent second order non-homogenous linear differential equation from the Bloch (NMR) equations is evolved. The parameters in the equations are equilibrium magnetization M_o, *radio frequency* $rfB_1(x,t)$ *field, gyro-magnetic ratio of blood spin* γ *as well as* T_1 *and* T_2 *relaxation times. The solution obtained will be examined when the system is under an influence of a driving force,* $F_o \cos wt$ *and* $\gamma B_1(t) = \cos wt$ *is the radio frequency field. However, for the purpose of this study, only* T_2 *relaxation times are varied and analyzed for the measurement of the signals in relation to its effect on human anatomy.*

INTRODUCTION

Nuclear Magnetic Resonance, NMR, measures how much electromagnetic radiation of a specific frequency is absorbed by an atomic nucleus that is placed in a strong magnetic field. Its objective is to visualize the atomic and molecular structure of chemical compounds - Edward (2006). NMR is produced when a radio frequency field is imposed at right angles to a much larger static magnetic field to perturb the orientation of nuclear magnetic moments generated by spinning electrically charged atomic nuclei.

The procedure requires that a substance be placed in a strong magnetic field. This strong magnetic field affects the spin of the atomic nuclei of elements, for example hydrogen molecules. These have an angular momentum arising from their inherent property of spin. NMR is inherently a three-dimensional phenomenon. The spatial resolution of a three-dimensional set of data is usually equal in all three directions. The basic requirements for Nuclear Magnetic Resonance spectroscopy are that the magnetic field be homogenous over the volume of the sample; that there be a radio frequency field rotating in a plane perpendicular to the static field and that there be a means of detecting the interaction of the frequency field with the sample.

Magnetic Resonance Imaging was developed from the knowledge gained in the study of Nuclear Magnetic Resonance. It will be correct to refer to it as Nuclear Magnetic Resonance Imaging (NMRI), however the word 'nuclear' connotes radiation, and to prevent it from being mistaken for radiation exposure, which is not one of the safety concerns for MRI, the word 'nuclear' was deleted from it. Scientists still use NMR when discussing non-medical devices operating on the same principle - Dada (2006).
The improvement made on Nuclear Magnetic Resonance was on the precision of the radio waves in order to gauge the resonant signals with more accuracy. In addition, means of applying the magnetic fields was enhanced. Mathematical methods by which radio signals could be analyzed and transformed into a useful image that would show precise distinctions between different areas of living tissue was also developed - Mansfield (2006).

METHODOLOGY

The main goal of this study is to establish a methodology of using mathematical techniques so that the accurate measurement of blood flow in human physiological and pathological conditions can be carried out non-invasively and become simple to implement in medical clinics. The objective of this research work is to formulate mathematically a time – dependent second-order non-homogeneous differential equation from the Felix Bloch's Nuclear Magnetic Resonance equations:

$$\frac{dM_x}{dt} = -\frac{M_x}{T_2} \quad 1$$

$$\frac{dM_y}{dt} = \gamma M_z B_1(x) - \frac{M_y}{T_2} \quad 2$$

$$\frac{dM_y}{dt} = -\gamma M_y B_1(x) - \frac{(M_0 - M_z)}{T_1} \quad 3$$

- Felix (2006)

3.0 Mathematical Analysis of Time-Dependent Bloch (NMR) Flow Equations

NMR spins are always in motion therefore, it is pertinent to treat them with reference to their motion since they change position with time. This motion is very much pronounced in fluids. From the kinetic theory of moving fluids, given a property M of the fluid, then the rate at which this property changes with respect to a point moving along with the fluid be the total derivative:

$$\frac{dM}{dt} = \frac{\partial M}{\partial t} + \frac{\partial M}{\partial x}V_x + \frac{\partial M}{\partial y}V_y + \frac{\partial M}{\partial z}V_z$$

$$\Rightarrow \frac{dM}{dt} = \frac{\partial M}{\partial t} + V.\nabla M$$

Therefore, the three Bloch equations above become:

$$\frac{dM_x}{dt} = \frac{\partial M_x}{\partial t} + V.\nabla M_x = -\frac{M_x}{T_2} \quad 4$$

$$\frac{dM_y}{dt} = \frac{\partial M_y}{\partial t} + V.\nabla M_y = \gamma M_z B_1(x) - \frac{M_y}{T_2} \quad 5$$

$$\frac{dM_z}{dt} = \frac{\partial M_z}{\partial t} + V.\nabla M_z = -\gamma M_y B_1(x) - \frac{(M_z - M_o)}{T_1} \quad 6$$

If the flow is along horizontal x direction, partial derivatives along y and z directions are zero. Note that for blood flow analysis, it is assumed the blood spins to be flowing along the x-direction hence the flow is independent of y and z components. Flow against the gravity is made possible by one - way valves, located several centimeters apart in the veins - Setaro (2006).

For a flow that is independent of the space coordinate, x, that is the magnetization does not change appreciably over a large x for a very long time, then all partial derivatives with respect to x could be set to zero (time - dependent). Hence equations (4 - 6) become:

$$\frac{dM_x}{dt} = -\frac{M_x}{T_2} \quad 7$$

$$\frac{dM_y}{dt} = \gamma M_z B_1(t) - \frac{M_y}{T_2} \quad 8$$

$$\frac{dM_z}{dt} = -\gamma M_y B_1(t) - \frac{(M_z - M_o)}{T_1} \quad 9$$

From equations 8 and 9, we have

$$\frac{d^2 M_y}{dt^2} + \left(\frac{1}{T_1} + \frac{1}{T_2}\right)\frac{dM_y}{dt} + .\left(\gamma^2 B_1^2(t) + \frac{1}{T_1 T_2}\right)M_y = \frac{M_o \gamma B_1(t)}{T_1} \quad 10$$

Equation (10) is the **Time - Dependent** Bloch NMR flow equation - Awojoyogbe (2004).

4.0 Solution of Time-Dependent Bloch Nuclear Magnetic Resonance Flow Equations

Let $k = \frac{1}{T_1 T_2}$; $T_o = \frac{1}{T_1} + \frac{1}{T_2}$; $F_o = \frac{M_o}{T_1}$ and $\gamma B_1(t) = \cos wt$ then equation (10) becomes:

$$\frac{dM_y^2}{dt^2} + T_o \frac{dM_y}{dt} + kM_y = F_o \cos wt \qquad 11$$

Equation 11 can be expressed as

$$y'' + \frac{c}{m} y' + \frac{p}{m} y = \frac{1}{m} r(t) \qquad 12$$

We can assume: $T_o = \frac{c}{m}$; $k = \frac{p}{m}$ and $F_o = \frac{M_o}{T_1} = \frac{1}{m}$

also let $w = \sqrt{\frac{p}{m}} \Rightarrow w^2 = \frac{p}{m}$

Note that $F_o \cos wt$ is an input or a driving force then if zero, it implies a freely vibrating system.

By Kreyszig (1988) the linear differential equation:

$$my'' + cy' + py = F_o \cos wt \qquad 13$$

admits the result

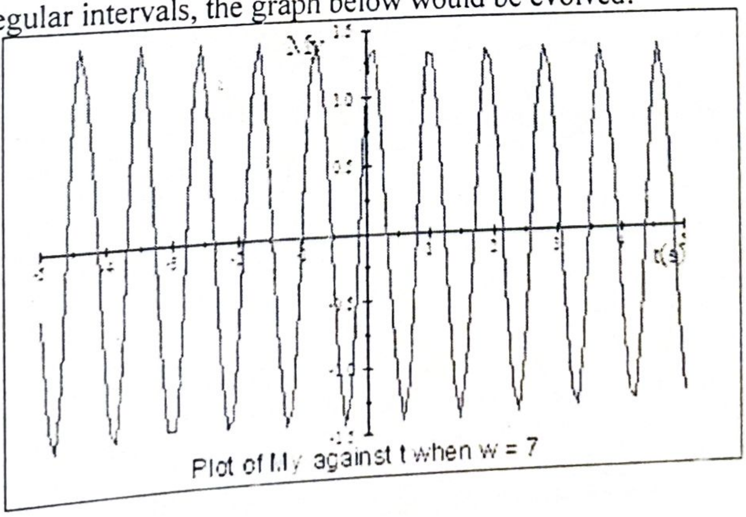
$$y(t) = e^{-\alpha t}(A\cos wt + B\sin wt) + \{F_o \frac{k - w^2}{(k-w^2)^2 + (wT_o)^2} \cos wt + F_o \frac{wT_o}{(k-w^2)^2 + (wT_o)^2} \sin wt\} \qquad -14$$

4.1 Plot of 2-Dimensional Time-Dependent Flow Equation

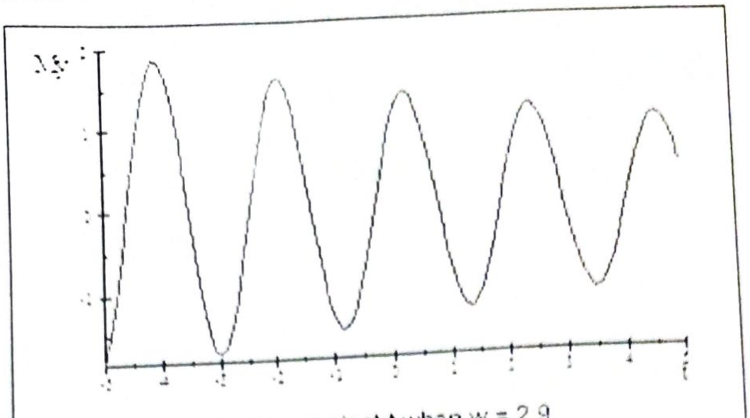
For the purpose of this research work, T_2 values will be varied as 0.02, 0.07, 0.12, 0.17,……, 0.47 while keeping T_1=1 constant. Thereafter, the corresponding values for w will be deduced. Recall the solution of time – dependent from equation 14, the complementary part is

$$y_h(t) = e^{-\alpha t}(a\cos wt + b\sin wt)$$

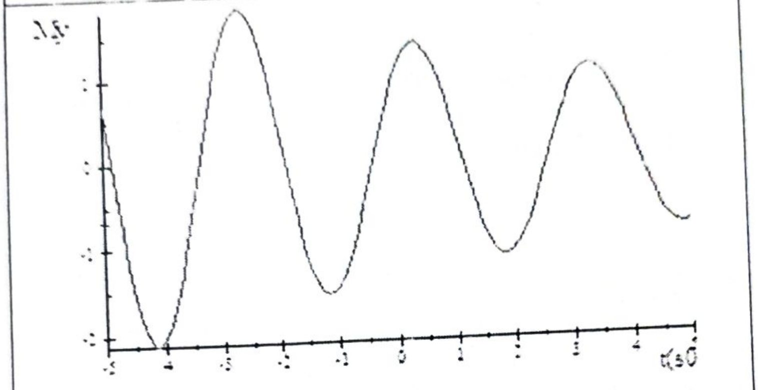
Assume $a = b = 1$; $\alpha = \frac{T_o}{2}$ and $w = \sqrt{\frac{1}{T_1 T_2}}$ and by keeping $T_1 = 1$ (constant) and varying T_2 between 0.02 and 0.52 at regular intervals, the graph below would be evolved:



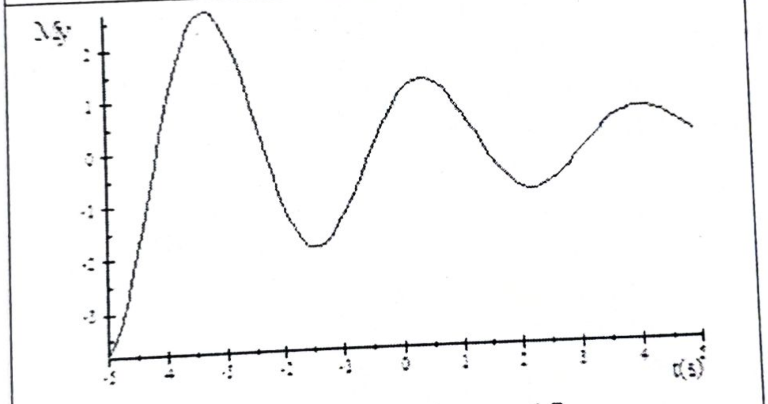
Plot of My against t when w = 7



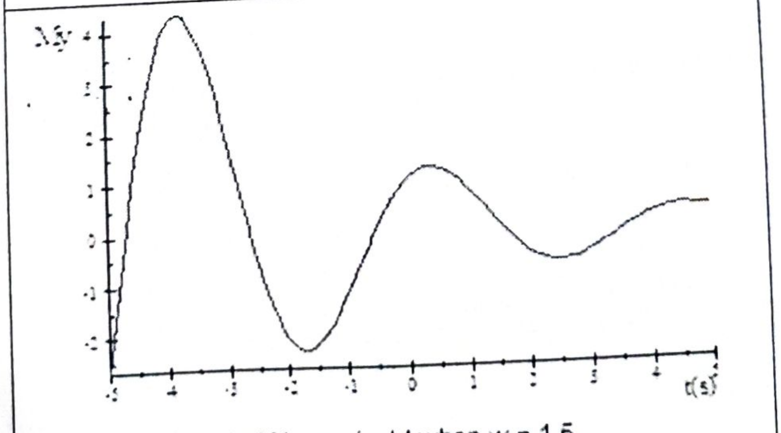
Plot of My against t when w = 2.9



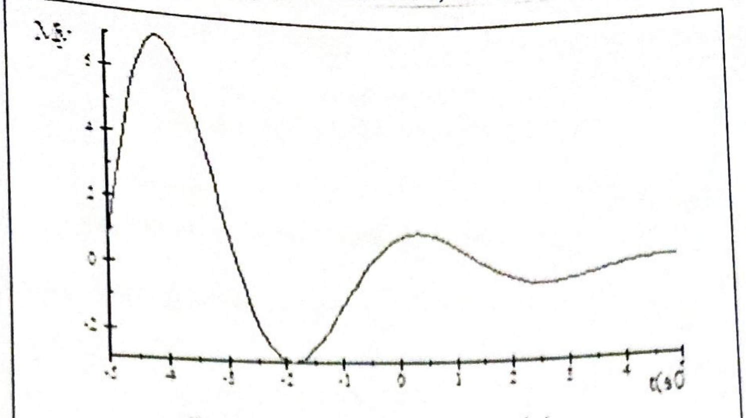
Plot of My against t when w = 2.1



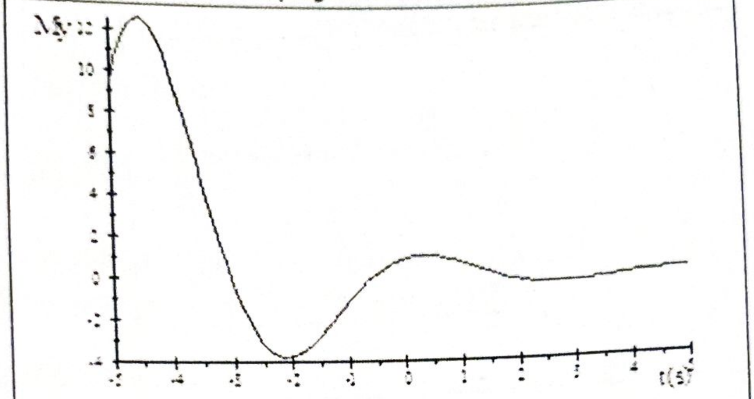
Plot of My against t when w = 1.7



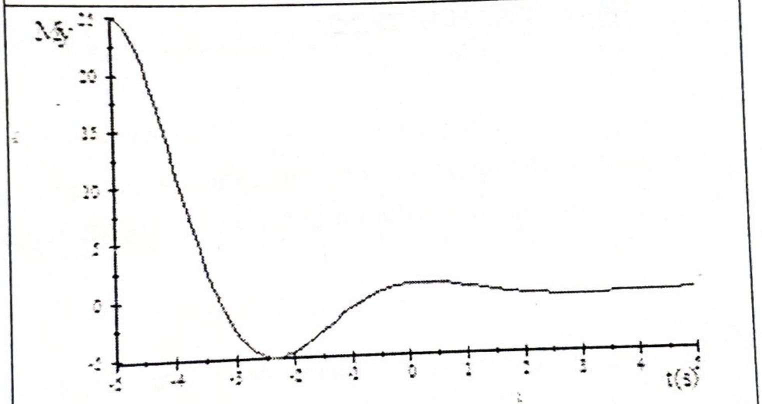
Plot of My against t when w = 1.5



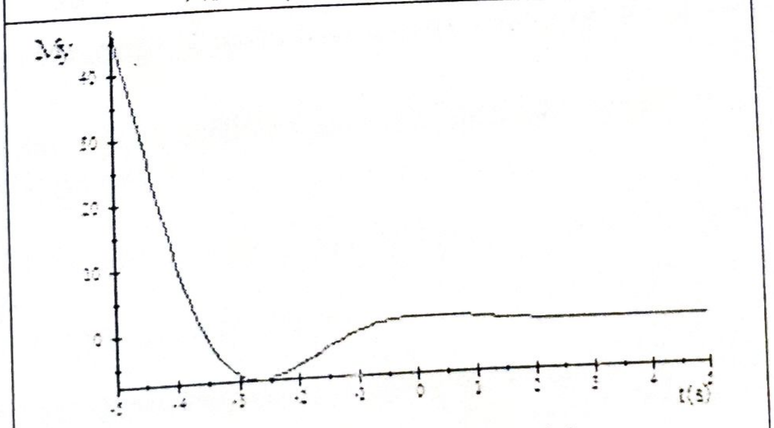
Plot of My against t when w = 1.4



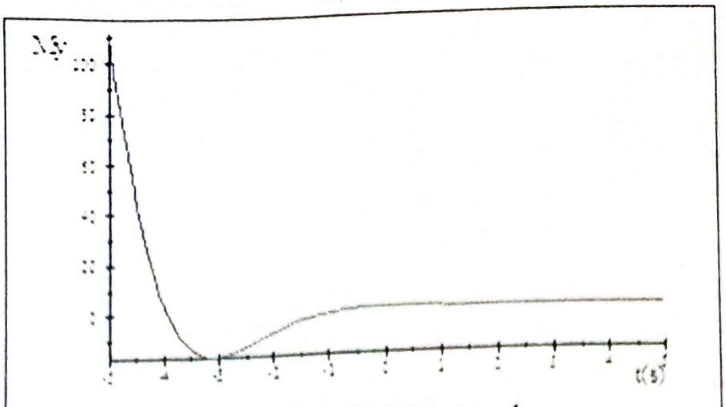
Plot of My against t when w = 1.3



Plot of My against t when w = 1.2



Plot of My against t when w = 1.1


Plot of My against t when w = 1

Plot of My against t when w = 0.9

Fig. 1

Analysis of Results

$e^{-\alpha t}(A\cos wt + B\sin wt)$ approaches zero as t approaches infinity practically after a sufficiently long time. This complementary function represents the transient solution as seen from figure 1. This can be considered under the following headings:

Over-Damping

If $T_o^{\,2} > 4k$ then there is over damping. This also implies presence of large amount of friction. The free motion described by the complementary function is given by $M_y(t) = Ae^{m_1 t} + Be^{m_2 t}$. After a sufficiently long time and in the absence of external force, transverse magnetization M_y terminates to zero eventually as time approaches infinity.

In NMR flow, when $rfB_1(t)$ is withdrawn, the transverse magnetization, $M_y(t)$ reduces to zero as time increases i.e.

$$Ae^{m_1 t} + Be^{m_2 t} = 0 \qquad 15$$

$$\Rightarrow e^{(m_1 - m_2)t} = -\frac{B}{A}, A \neq 0$$

If $A and B$ are of opposite signs, and since a real exponential function must always be positive then there is only one value of t that can satisfy the above equation.

Critical Damping

If $T_o^{\,2} = 4k$ then the roots m_1, m_2 of the characteristic equation are real and equal $m_1 = m_2 = -\frac{T_o}{2}$. This is called critical damping. In this case, the free motion described by the complementary function is given by $M_y(t) = Ae^{m_1 t} + Bte^{m_1 t}$. If $B = 0$, there is no value of t for which $M_y(t) = 0$, but in all other cases there is one and only one value of t for which $M_y(t)=0\{-\infty<t<0\}$. This is physically irrelevant. Since exponential function is never zero and the coefficients can have at most one positive zero. It follows that the motion can have at most one passage through the equilibrium position.

Under-Damping

If $T_o^2 < 4k$ then the motion is under-damped. It is not periodic but there are regularly spaced passages through the equilibrium position at intervals of $\frac{\pi}{\varepsilon}$, where

$$\varepsilon = \frac{1}{2\sqrt{4k - T_o^2}} \qquad 16$$

This implies imaginary period $= \frac{2\pi}{\varepsilon}$ and imaginary NMR frequency $\frac{w_d}{2\pi} = \frac{\varepsilon}{2\pi} Hz$.

The *second part* of the solution is

$$M_{yp}(t) = F_o \frac{k - w^2}{(k-w^2)^2 + (wT_o)^2}\cos wt + F_o \frac{wT_o}{(k-w^2)^2+(wT_o)^2}\sin wt \qquad 17$$

which is the only part contributing to the transverse magnetization resulting in sinusoidal plot. This will be discussed under two cases i.e. $T_o = 0$ (un-damped) and $T_o > 0$ (damped).

Undamped Forced Oscillation

This occurs when $T_o = 0$.

Assuming $w^2 \neq w_o^2$ where $w_o^2 = \frac{k}{m}$

$$M_{yp}(t) = \frac{F_o}{m(w_o^2 - w^2)}\cos wt = \frac{F_o}{k[(1-\frac{w}{w_o})^2]}\cos wt$$

From the above and using the expression, the general solution will be:

$$M_y(t) = C\cos(w_o t - \delta) + \frac{F_o}{m(w_o^2 - w^2)}\cos wt$$

This implies a superposition of two harmonic oscillations whose frequencies are the natural frequency $\frac{w_o}{2\pi}$ i.e. the frequency of the free un-damped motion of the system and the frequency $\frac{w}{2\pi}$ of the output. This phenomenon which occurs as w approaches w_o is called *resonance* and it forms the basis of operation in Nuclear Magnetic Resonance.

However, if w is close to w_o, the particular solution is

$$M_{yp}(t) = \frac{F_o}{m(w_o^2 - w^2)}(\cos wt - \cos w_o t) \qquad 18$$

corresponding to the initial condition $y(0) = 0$ and $y'(0) = 0$.

By using the relation $\cos v - \cos u = 2\sin\frac{u+v}{2}\sin\frac{u-v}{2}$,

Equation 18 may be written as $M_{yp}(t) = \frac{2F_o}{m(w_o^2 - w^2)}\sin\frac{w_o + w}{2}t\sin\frac{w_o - w}{2}t$

This will result in beats as the difference between the input and natural frequencies is small.

Damped Forced Oscillation

The second case happens when $T_o > 0$. Implying that there is damping.

Recalling the general solution, $M_y(t)$

$$= e^{-\alpha t}(A\cos wt + B\sin wt) + \{F_o \frac{k - w^2}{(k - w^2)^2 + (wT_o)^2} \cos wt + F_o \frac{wT_o}{(k - w^2)^2 + (wT_o)^2} \sin wt\}$$

The first part which is the complementary function $e^{-\alpha t}(A\cos wt + B\sin wt)$ approaches zero as t approaches infinity hence, after a sufficiently long time, the output corresponding to a purely sinusoidal input will practically be a harmonic solution whose frequency is that of the input. In this case, the amplitude will always be *finite* as against the situation when it is un-damped in which case the amplitude is *infinite* as w approaches w_o.

DISCUSSION

In this study, the meaning and evolution of Bloch Nuclear Magnetic Resonance and also its modification to Magnetic Resonance Imaging have been explained. The analytic solution has been presented under the heading- time dependent. The solution which has the complementary part and the particular integral were further examined under several instances namely - over-damping; critical damping; under-damping as well as un-damped and damped forced oscillation. In the complementary solution, T_1 spin – lattice relaxation time was kept constant while T_2 spin – spin relaxation time varied. Graphs of these different stages were plotted and the results obtained analyzed.

CONCLUSION

From the various graphs plotted above for M_y with respect to the variations in T_2 values the **complementary function** $e^{-\alpha t}(A\cos wt + B\sin wt)$ approaches zero as t approaches infinity practically after a sufficiently long time. This complementary function represents the transient solution. On the whole, it can be concluded that the particular integral is the only part contributing to the transverse magnetization. This is the reason why the plot is sinusoidal.

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