Comparative Analysis of a Non-Reactive Contaminant Flow Problem for Constant Initial Concentration in Two Dimensions by Homotopy-Perturbation and Variational Iteration Methods.

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ABSTRACT

In this paper, we present a comparative analysis of non-reactive contaminant flow problem for constant initial concentration in two dimensions by homotopy-perturbation and Variational Iteration method. We provide an approximation of this equation using homotopy-perturbation transformation and solve the resulting linear equations analytically by homotopy-perturbation method (HPM) and Variational Iteration Method (VIM). Graphs are plotted using the solution obtained from the method and the results are presented and discussed.

(Keywords: Homotopy-perturbation, contaminant, advection, diffusion, adsorption)

INTRODUCTION

In recent times, modeling of contaminant transport in porous media remain a critical issue in the field of hydrology and environmental sciences because contaminant frequently penetrate the subsurface, subsoil, aquifer and groundwater either intentionally or intentionally, and the contaminant residues constitute a threat to the environment and by extension the groundwater. The advection-dispersion equation is the most common method of modeling contaminant transport in porous media. The partial differential equation describing the contaminant transport is characterized by advective transport with flowing groundwater, molecular diffusion, hydrodynamic dispersion and adsorption.

A number of analytical solutions have been achieved to describe one dimensional advective-dispersion contaminant transport with varying initial and boundary conditions. Clint (1993) and Makinde and Chinyoka (2010) studied the non-linear contaminant transport equations using numerical methods; Aiyesimi (2004), Gideon and Aiyesimi (2005) and Gideon (2001) employed perturbation method in their studies. Other researchers like Singh, Singh, and Singh (2010) and Yadav and Kumar (2011) adopted the Laplace Transformation Method. Essa, et al. (2007) investigated the dispersion of pollutant from a point source, analytically taking into consideration the vertical variation of both wind speed and eddy diffusivity. Massabo, et al. (2006) gave analytical solutions for a two-dimensional advection equation with anisotropic dispersion.

Other well-known methods are the Homotopy Perturbation Method (HPM) and the Variational Iteration Method (VIM). HPM was used to solve wide range of physical problems, eliminating the limitations of perturbation method Rajabi, Ganji, and Taherian (2007) and Jiya (2010). Most of the researches done in the past either neglects the non-linear term or considers it as a constant.

In this paper, we present a comparative analysis of non-reactive contaminant flow problem for constant initial concentration in two dimensions by Homotopy-Perturbation and Variational Iteration method. This two-Dimensional case is chosen for this study because contaminant dispersion occurs in the direction where there is concentration gradient. It could be in x and y directions.

MATERIALS AND METHODS

The contaminant flow equation was modeled using the advection-dispersion terms mass conservation principle (Bear, 1997). In two dimensions, we consider the advection and dispersion in both x and y-directions and by mass conservation law, we have the governing equation of the two-dimensional flow equation as:

$$C_t + \frac{pb}{n} S_t + UC_x + VC_y - D_L C_{xx} - D_T C_{yy} = 0; \quad (1)$$
$$0 < x < \infty, 0 < y < \infty, t > 0.$$

Equation (1) can be rewritten as:

$$\frac{\partial C}{\partial t} + \frac{\partial \emptyset(C)}{\partial t} + U\frac{\partial C}{\partial x} + V\frac{\partial C}{\partial y} - D_L\frac{\partial^2 C}{\partial x^2} - D_T\frac{\partial^2 C}{\partial y^2} = 0, (2)$$

This can further be written as:

$$\frac{\partial C}{\partial t} + \frac{\partial \emptyset}{\partial C}\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} + V\frac{\partial C}{\partial y} - D_L\frac{\partial^2 C}{\partial x^2} - D_T\frac{\partial^2 C}{\partial y^2} = 0\,. (3)$$

Defining $\frac{\partial \emptyset}{\partial C} = \varepsilon$, then:

$$\frac{\partial C}{\partial t} + \varepsilon\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} + V\frac{\partial C}{\partial y} - D_L\frac{\partial^2 C}{\partial x^2} - D_T\frac{\partial^2 C}{\partial y^2} = 0, \quad (4)$$

Simplifying Equation (4), we obtain:

$$(1+\varepsilon)\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} + V\frac{\partial C}{\partial y} - D_L\frac{\partial^2 C}{\partial x^2} - D_T\frac{\partial^2 C}{\partial y^2} = 0 \quad (5)$$

$$C(0,y,t) = A, C(\infty,y,t) = 0,$$
$$C(x,0,t) = A, C(x,\infty,t) = 0, t > 0,.$$
$$C(x,y,0) = Ae^{-\lambda xy} \quad (6)$$
$$0 < x < \infty, 0 < y < \infty,$$

where U is the flow horizontal velocity, V is the vertical velocity of flow, $\mathrm{D_L}$ is the horizontal dispersion, $\mathrm{D_T}$ is the vertical dispersion, $C(x,y,t)$ is the concentration of the contaminant, x and y are the horizontal and the vertical distance from the source respectively, t the time and ε is the perturbation parameter.

Basic Idea of Homotopy-Perturbation Method (HPM)

In order to explain the method of homotopy-perturbation, we consider the function:

$$\mathrm{A(u)} - \mathrm{f(r)} = 0, \mathrm{r} \in \Omega \quad (7)$$

having the boundary conditions:

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, r \in \Gamma,$$

where $A, B, \mathrm{f(r)}$ and Γ are a general differential operator, a boundary operator, a known analytical function and a boundary of the domain respectively. The operator A can be divided into two parts L and N where L is linear and N is nonlinear. Equation (7) can therefore be rewritten as follows:

$$\mathrm{L}(u) + \mathrm{N}(u) - f(r) = 0, r \in \Omega \quad (8)$$

By homotopy-perturbation method, we form a homotopy:

$$v(\mathrm{r,p}): \Omega \times [0,1] \longrightarrow \mathrm{R}$$

which satisfies

$$\mathrm{H}(v,p) = (1-p)[\mathrm{L}(u) - \mathrm{L}(u_0)] + p[\mathrm{A}(v) - f(r)] = 0 \quad (9)$$
$$p \in [0,1], \mathrm{r} \in \Omega,$$

where $p \in [0,1]$ is an embedding parameter, while $\mathrm{u_0}$ is an initial approximation of (7), which satisfies the boundary conditions. From Equation (9), we have:

$$\mathrm{H}(v,0) = \mathrm{L}(v) - \mathrm{L}(u_0) = 0, \quad (10)$$

$$\mathrm{H}(v,1) = \mathrm{A}(v) - \mathrm{f(r)} = 0. \quad (11)$$

According to HPM, we can first use the embedding parameter p as a "small parameter", and assume that the solutions of Equation (9) can be written as a power series in p:

$v = v_0 + pv_1 + p_2v^2 + \cdots$, and the best

approximate solution is:

$$u = \lim_{\mathrm{p}\to 1} v = v^{(0)} + pv^{(1)} + p^2v^{(2)} + \cdots. \quad (12)$$

The convergence of the above solution is discussed in Abdul-Sattar, et al. (2011).

Solution of Two-Dimensional Contaminant Flow Problem by HPM

By applying homotopy-perturbation transformation to the linearized form of Equation (1) (i.e., Equation (5)), we have:

$$H(v,p) = (1-p)\left[(1+\varepsilon)\frac{\partial C}{\partial t}\right] + p\left[(1+\varepsilon)\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} + V\frac{\partial C}{\partial y} - D_L\frac{\partial^2 C}{\partial x^2} - D_T\frac{\partial^2 C}{\partial y^2}\right] = 0 \qquad (13)$$

$$Let\ (x,y,t) = v^{(0)} + pv^{(1)} + p^2v^{(2)} + p^3v^{(3)} + \cdots \qquad (14)$$

By substituting Equation (14) in (13), we have the following set of equations:

$$(1+\varepsilon)\frac{\partial}{\partial t}v^{(0)}(x,y,t) = 0 \qquad (15)$$

$$(1+\varepsilon)\frac{\partial}{\partial t}v^{(1)}(x,y,t) + U\frac{\partial}{\partial x}v^{(0)}(x,y,t) + V\frac{\partial}{\partial y}v^{(0)}(x,y,t) - D_L\frac{\partial^2}{\partial x^2}v^{(0)}(x,y,t) - D_T\frac{\partial^2}{\partial y^2}v^{(0)}(x,y,t) = 0 \qquad (16)$$

$$(1+\varepsilon)\frac{\partial}{\partial t}v^{(2)}(x,y,t) + U\frac{\partial}{\partial x}v^{(1)}(x,y,t) + V\frac{\partial}{\partial y}v^{(1)}(x,y,t) - D_L\frac{\partial^2}{\partial x^2}v^{(1)}(x,y,t) - D_T\frac{\partial^2}{\partial y^2}v^{(1)}(x,y,t) = 0 \qquad (17)$$

$$(1+\varepsilon)\frac{\partial}{\partial t}v^{(3)}(x,y,t) + U\frac{\partial}{\partial x}v^{(2)}(x,y,t) + V\frac{\partial}{\partial y}v^{(2)}(x,y,t) - D_L\frac{\partial^2}{\partial x^2}v^{(2)}(x,y,t) - D_T\frac{\partial^2}{\partial y^2}v^{(2)}(x,y,t) = 0 \qquad (18)$$

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$$(1+\varepsilon)\frac{\partial}{\partial t}v^{(n)}(x,y,t) + U\frac{\partial}{\partial x}v^{(n-1)}(x,y,t) + V\frac{\partial}{\partial y}v^{(n-1)}(x,y,t) - D_L\frac{\partial^2}{\partial x^2}v^{(n-1)}(x,y,t) - D_T\frac{\partial^2}{\partial y^2}v^{(n-1)}(x,y,t) = 0 \qquad (19)$$

We now solve the above set of equations by method of homotopy-perturbation. Equation (14) accepts a solution of the form: $v^{(0)}(x,y,t) = Ae^{-\lambda xy}$, and

$v^{(0)}(x,y,0) = Ae^{-\lambda xy}$; $v^{(0)}(0,y,t) = A$, $v^{(0)}(\infty,y,t) = 0, v^{(0)}(x,0,t) = A,\ v^{(0)}(x,\infty,t) = 0, A > 0$.

By solving Equations (16) and (17), we have the following:

$$v^{(1)}(x,y,t) = \frac{A\lambda e^{-\lambda xy}(D_L\lambda y^2 + D_T\lambda x^2 + Uy + Vx)t}{1+\varepsilon} \qquad (20);$$

$$v^{(2)}(x,y,t) = \frac{1}{2}\frac{1}{(1+\varepsilon)^2}\big(A\lambda e^{-\lambda xy}t^2\big(\lambda x^2V^2 + \lambda^3y^4{D_L}^2 + 4D_L\lambda D_T + \lambda^3x^4{D_T}^2 + \lambda y^2U^2 - 2UV + 2U\lambda^2y^3D_L + 2U\lambda^2yD_Tx^2 + 2U\lambda yVx - 4U\lambda D_Tx + 2V\lambda^2xD_Ly^2 + 2V\lambda^2x^3D_T - 4V\lambda D_Ly + 2D_L\lambda^3y^2D_Tx^2 - 8D_L\lambda^2yD_Tx\big). \qquad (21)$$

Therefore, the solution of the nonreactive two-dimensional contaminant flow equation is:

$$C(x,y,t) = \lim_{p\to 1}\left(v^{(0)} + pv^{(1)} + p^2v^{(2)} + p^3v^{(3)} + \cdots\right) \qquad (22)$$

That is

$$C(x,y,t) = Ae^{-\lambda xy} + \frac{A\lambda e^{-\lambda xy}(D_L\lambda y^2 + D_T\lambda x^2 + Uy + Vx)t}{1+\varepsilon} + \frac{1}{2}\frac{1}{(1+\varepsilon)^2}\big(A\lambda e^{-\lambda xy}t^2\big(\lambda x^2V^2 + \lambda^3y^4{D_L}^2 + 4D_L\lambda D_T + \lambda^3x^4{D_T}^2 + \lambda y^2U^2 - 2UV + 2U\lambda^2y^3D_L + 2U\lambda^2yD_Tx^2 + 2U\lambda yVx - 4U\lambda D_Tx + 2V\lambda^2xD_Ly^2 + 2V\lambda^2x^3D_T - 4V\lambda D_Ly + 2D_L\lambda^3y^2D_Tx^2 - 8D_L\lambda^2yD_Tx\big) \quad . \qquad (23)$$

Basic Idea of Variational Iteration Method (VIM)

To explain the concept of VIM, consider the following differential equation:

$$Lu + Nu = g(t) \qquad (24)$$

where L is a linear operator, N a nonlinear operator and g(t) an inhomogeneous term. By VIM, the correction functional can be constructed as follows:

$$U_{n+1(t)} = U_n(t) + \int_s^t \lambda(LU_n(s) + N\tilde{U}_n(s) - g(s))ds,$$

where λ is the Lagrange multiplier He (2000) which can be obtained optimally using the variational theory. The subscript n depicts the nth approximation and $\tilde{U}_n$ is considered as a restricted variation He, 1998.

Solution of Two-Dimensional Contaminant Flow Problem by VIM

The transformed version of the contaminant flow equation (i.e., Equation (5)) is solved by VIM as follows:

$$(1+\varepsilon)\frac{\partial C}{\partial t}+U\frac{\partial C}{\partial x}+V\frac{\partial C}{\partial y}-D_L\frac{\partial^2 C}{\partial x^2}-D_T\frac{\partial^2 C}{\partial y^2}=0;\, C(0,y,t)=A,\, C(\infty,y,t)=0,$$
$$C_0(x,y,t)=Ae^{-\lambda xy},$$

and

$$\frac{\partial C_0(x,y,0)}{\partial t}=0.$$

The initial approximation is:

$$C_0(x,y,0)+\frac{\partial C_0(x,y,0)}{\partial t} \quad (25)$$

$$C_{n+1}(x,y,t)=C_n(x,y,t)+\int_0^t\lambda\left((1+\varepsilon)\frac{\partial C_n(x,y,s)}{\partial t}+U\frac{\partial C_n(x,y,s)}{\partial x}+V\frac{\partial C_n(x,y,s)}{\partial y}-D_L\frac{\partial^2 C_n(x,y,s)}{\partial x^2}-D_T\frac{\partial^2 C_n(x,y,s)}{\partial y^2}\right)ds \quad (26)$$

The Lagrange Multiplier is obtained from variational theory as λ=-1.

From Equation (26),

$$C_1(x,y,t)=Ae^{-\lambda xy}+\int_0^t(-1)\left((1+\varepsilon)\frac{\partial C_0(x,y,s)}{\partial t}+U\frac{\partial C_0(x,y,s)}{\partial x}+V\frac{\partial C_0(x,y,s)}{\partial y}-D_L\frac{\partial^2 C_0(x,y,s)}{\partial x^2}-D_T\frac{\partial^2 C_0(x,y,s)}{\partial y^2}\right)ds \quad (27)$$

Therefore from Equation (28), we have:

$$C_1(x,y,t)=Ae^{-\lambda xy}+D_LA\lambda^2y^2e^{-\lambda xy}t+UA\lambda ye^{-\lambda xy}t+D_TA\lambda^2x^2e^{-\lambda xy}t+VA\lambda xe^{-\lambda xy}t \quad (28)$$

The next iteration gives,

$$C_2(x,y,t)=\tfrac{1}{2}Ae^{-\lambda xy}(-2t\varepsilon\lambda x-2t\varepsilon D_T\lambda^2x^2-2t\varepsilon U\lambda y-2t\varepsilon D_L\lambda^2y^2+2t^2D_LU\lambda^3y^3-4t^2D_LV\lambda^2y+2t^2D_TV\lambda^3x^3-4t^2UD_T\lambda^2x+2+2t^2UD_T\lambda^3x^2y+2t^2D_LV\lambda^3xy^2+2t^2UV\lambda^2xy+2t^2D_LD_T\lambda^4x^2y^2-8t^2D_LD_T\lambda^3xy+t^2{D_L}^2\lambda^4y^4+t^2{D_T}^2\lambda^4x^4+t^2\lambda^2U^2y^2-2t^2UV\lambda+4t^2D_LD_T\lambda^2+t^2\lambda^2V^2x^2+2U\lambda yt+2D_T\lambda^2x^2t+2V\lambda xt+2D_L\lambda^2y^2t \quad (29)$$

Equation (23), on simplification gives equation (29). This show that the results obtained from HPM agrees with that of VIM.

RESULTS AND DISCUSSION

The graph in Figure 1 is obtained using the solutions (23) and (29) with the aid of input data: λ=1, U=0.1, V=0.01, $\varepsilon=0.001$, $D_L=0.01$, $D_T=0.0001$, x=1 ,y=1 and 0< $t<1$, the concentration is plotted against time. Similarly, with the aid of input data: λ=1, U=0.1, V=0.01, $\varepsilon=0.001$, $D_L=0.01$, $D_T=0.0001$, t=1 ,y=1 and 0< $x<1$, the concentration is plotted against distance (x) and the graph is presented in Figure 2. Finally, the 3-dimensional graphs of concentration against distance (x) and time(t) are plotted to show the relationship between concentration, distance and time using the solution (23) from HPM and (29) from VIM separately and the graphs are shown in Figures 3 and 4.

Figures 3 and 4 are plotted with the aid of input data: λ=1, U=0.1, V=0.01, $\varepsilon=0.001$, $D_L=0.01$, $D_T=0.0001$,y=1 0< $t<1$ and 0< $x<1$. In all the graphs plotted, the dispersion coefficients is assumed proportional to seepage velocity.

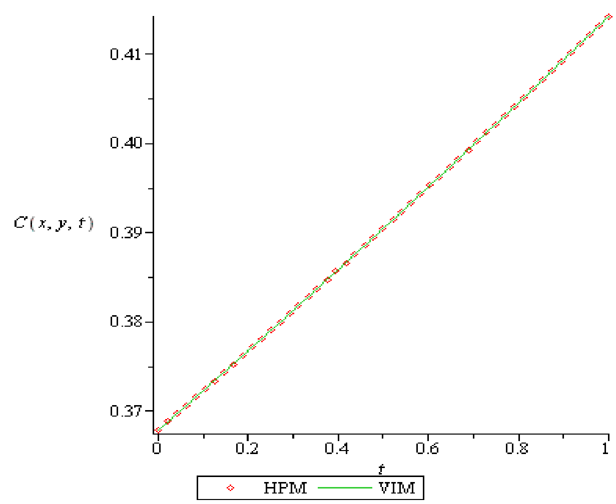

Figure 1: Graph of Concentration against Time.

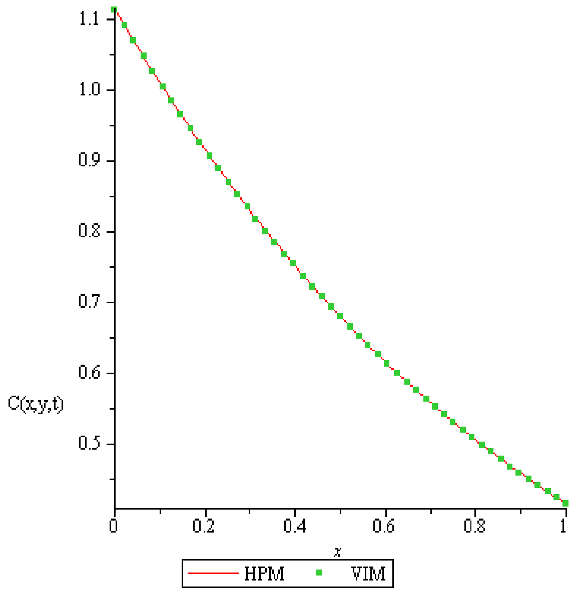

Figure 2: Graph of Concentration against Distance.

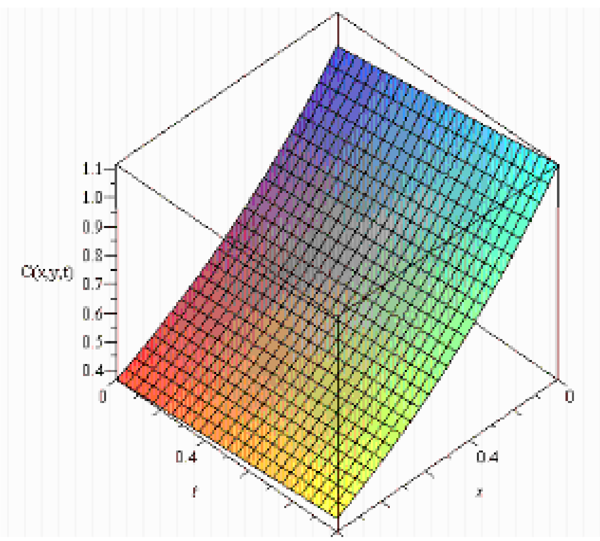

Figure 3: Three-D Graph of Concentration against Distance and Time from HPM Solution.

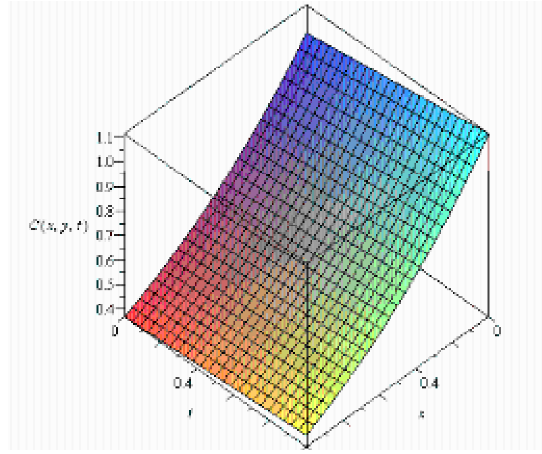

Figure 4: Three-D Graph of Concentration against Distance and Time from VIM solution.

CONCLUSION

In this work, analytical solutions are obtained for nonreactive two-dimensional contaminant flow problem. These research findings reveal that the solution we obtained from VIM agree completely with that of HPM in all ramifications as clearly shown in the graphs. From the graphs plotted for the nonreactive contaminant flow solutions, it is obvious that the concentration of the contaminant increases with time and also the concentration decreases as the distance increases.

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