Agreement between the Homotopy Perturbation Method and Variation Iterational Method on the Analysis of One-Dimensional Flow Incorporating First Order Decay

BY

***[1]Jimoh, O. R., [2]Aiyesimi, Y. M. and [3]Jiya, M.**
[1,2,3,4]Department of Mathematics,
Federal University of Technology, Minna, Nigeria.
*[1]razaq.jimoh@futminna.edu.ng
[2]yomiaiyesimi2007@yahoo.co.uk
[3]jiyason2010@gmail.com
*Corresponding author

Abstract

In this paper, a comparative study of reactive contaminant flow for constant initial concentration in one dimension is presented. The adsorption term is modeled by Freudlich Isotherm. An approximation of the one-dimensional contaminant flow model was obtained using homotopy-perturbation transformation and the resulting linear equations were solved semi-analytically by homotopy-perturbation method (HPM) and Variational Iteration Method (VIM). Graphs were plotted using the solution obtained from the methods and the results presented and discussed. The analysis of the results obtained show that the concentration of the contaminant decreases with time and distance as it moves away from the origin.

Keywords: Homotopy-perturbation, contaminant, advection, diffusion, adsorption, Variational Iteration Method

1.0 Introduction

Virtually every day, the Society generates a large number of pollutants which often find their ways into the environment. These happen intentionally during agricultural practices or unintentionally resulting from leakages of municipal wastes disposal sites.

Consequently, as many of these chemicals or pollutants enter the food chain, contamination of both surface and subsurface water supplies has become a critical issue. A high percentage of drinking water in Nigeria comes from groundwater. The effect of contamination of groundwater systems as a subject of discourse cannot be over-emphasized in assessing the hazards of public health. In order to predict the fate of such pollutants during their transport, an arduous task for hydro-geologists and scientists emerges. The problem involves defining the flow path of groundwater in the aquifers, the travel time of the water along the flow lines and to predict the chemical reaction which alters the concentrations during transport.

2.0 Literature Review

Several researchers have been on the field in the study of hydrodynamic dispersion with various initial and boundary conditions. Aiyesimi (2004), Gideon and Aiyesimi (2005) and Gideon (2011) studied one dimensional contaminant flow problems by method of perturbation. Clint (1993) and Makinde and chinyoka (2010) employed numerical methods and Laplace Transform method (Singh *et al.*,2010) and Yadav and Kumar, 2011). Another well-known method, the Homotopy perturbation method was used to solve wide range of physical problems, eliminating the limitations of perturbation method (Rezania *et al.*,2009, Muhammad, 2010; Rajabi *et al.*, 2007; and Jiya, 2010). A semi-analytical solution of one-dimensional contaminant flow problem was handled by method of weighted residual method and the results approximate the behavior of the contaminant in the finite flow domain (Jimoh *et al.*, 2017). Most of the researches done in the past either neglects the non-linear term or considers it as a constant, and they are mostly nonreactive. In this paper, we provide a comparative analysis of the non-linear reactive contaminant transport problem with initial continuous point source by Homotopy-Perturbation method and Variational Iteration method.

3 Methodology

3.1 Formulation of the Model

We consider an incompressible fluid flow through a homogeneous, saturated porous medium where the fluid is not solute-free, i.e. contaminated with solute of concentration C(x,t). The following assumptions are made (i) The flow is steady (ii) the solute transport is described by advection, molecular diffusion and mechanical dispersion (iii) the flow is one dimensional and in x-direction.

Under these assumptions, mass conservation of the contaminant may be combined with a mathematical expression of the relevant process to obtain a differential equation describing flow Jimoh *et al.*, 2017. Following (Bear, 1997), (Yadav, *et al.* 2011), (Jimoh and Aiyesimi, 2012 and 2013), Ramakanta and Mehta (2010), Singh (2013) and Singh *et al*. (2015), the one dimensional partial differential equation describing hydrodynamic dispersion in adsorbing, homogeneous and isotropic porous medium can be written as

$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+\frac{pb}{\theta}\frac{\partial S}{\partial t}-D\frac{\partial^2 C}{\partial x^2}+\alpha C=0,\ 0<x<\infty,\ t>0 \quad (1)$$

where $S(x,t)$ is the mass of the contaminant absorbed on the solid matrix per unit mass of the solid, $p_b > 0$ is the bulk density of the porous medium, $n > 0$ is the porosity, $D > 0$ accounts for both molecular diffusion and mechanical dispersion, u is the fluid velocity and α is the reaction parameter. We consider the non-linear flow equation with the associated boundary and initial conditions:

$$\begin{cases}\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+\frac{pb}{\theta}\frac{\partial S}{\partial t}-D\frac{\partial^2 C}{\partial x^2}+\alpha C=0,\\ C(0,t)=B,\ C(\infty,t)=0,\\ C(x,0)=Be^{-\lambda x}\\ \alpha>0,\ 0<x<\infty\end{cases} \quad (2)$$

Equation (2) can be rewritten as

$$\frac{\partial C}{\partial t}+\frac{\partial f(C)}{\partial t}+u\frac{\partial C}{\partial x}-D\frac{\partial^2 C}{\partial x^2}+\alpha C=0 \quad (3)$$

which can also be expressed in the form:

$$\frac{\partial C}{\partial t}+\frac{\partial f}{\partial C}.\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}-D\frac{\partial^2 C}{\partial x^2}+\alpha C=0 \quad (4)$$

This can again be expressed as

$$\frac{\partial C}{\partial t}+\varepsilon\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}-D\frac{\partial^2 C}{\partial x^2}+\alpha C=0 \quad (5)$$

where $\varepsilon=\frac{\partial f}{\partial C}$, the perturbation parameter.

That is,

$$(1-\varepsilon)\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}-D\frac{\partial^2 C}{\partial x^2}+\alpha C=0 \quad (6)$$

3.2 Principle of Homotopy-Perturbation Method (HPM)

In order to explain the method of homotopy-perturbation, we consider the function:

$$A(u)-f(r)=0,\ r\in\Omega \quad (7)$$

with the boundary conditions:

$$B\left(u,\frac{\partial u}{\partial n}\right)=0,\ r\in\Gamma, \quad (8)$$

where $A, B,$ f(r) and Γ are a general differential operator, a boundary operator, a known analytical function and a boundary of the domain respectively. The operator A can be divided into two parts L and N where L is linear and N is nonlinear. Equation (7) can therefore be rewritten as follows:

$$L(u)+N(u)-f(r)=0,\ r\in\Omega \qquad (9)$$

By homotopy-perturbation method, we form a homotopy:

$$v(r,p):\Omega\times[0,1]\rightarrow R \qquad (10)$$

which satisfies

$$H(v,p)=(1-p)(L(u)-L(u_0))+p(A(v)-f(r))=0,\ p\in[0,1],\ r\in\Omega \qquad (11)$$

where $p\in[0,1]$, is an embedding parameter, while u_0 is an initial approximation of (7), which satisfies the boundary conditions (He, 2000 and 2005). From equation (9), we have:

$$H(v,0)=L(v)-L(u_0)=0 \qquad (12)$$

$$H(v,1)=A(v)-f(r)=0 \qquad (13)$$

According to HPM, we can first use the embedding parameter p as a “small parameter”, and assume that the solutions of equation (9) can be written as a power series in p :

$$v=v^{(0)}+pv^{(1)}+p^2v^{(2)}+p^3v^{(3)}+\ldots \qquad (14)$$

and the best approximate solution is

$$u=\lim_{p\rightarrow 1} v=v^{(0)}+pv^{(1)}+p^2v^{(2)}+p^3v^{(3)}+\ldots \qquad (15)$$

The convergence of the above solution was discussed in Abdul-Sattar (2011).

3.3 Solution of the Contaminant flow problem by Homotopy perturbation method

By the Homotopy-perturbation transformation equation:

$$H(v,p)=(1-p)\left((1+\varepsilon)\frac{\partial C}{\partial t}\right)+p\left((1+\varepsilon)\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}-D\frac{\partial^2 C}{\partial x^2}+\alpha C\right)=0 \quad (16)$$

Defining,

$$C(x,t)=v^{(0)}+pv^{(1)}+p^2v^{(2)}+p^3v^{(3)}+\ldots \quad (17)$$

Equation (18) is substituted in (17), we have

$$\begin{aligned}H(v,p)&=(1-p)\left((1+\varepsilon)\frac{\partial}{\partial t}\left(v^{(0)}+pv^{(1)}+p^2v^{(2)}+p^3v^{(3)}+\ldots\right)\right)\\&+p\left(\begin{array}{l}(1+\varepsilon)\frac{\partial}{\partial t}\left(v^{(0)}+pv^{(1)}+p^2v^{(2)}+p^3v^{(3)}+\ldots\right)+u\frac{\partial}{\partial x}\left(v^{(0)}+pv^{(1)}+p^2v^{(2)}+p^3v^{(3)}+\ldots\right)\\-D\frac{\partial^2}{\partial x^2}\left(v^{(0)}+pv^{(1)}+p^2v^{(2)}+p^3v^{(3)}+\ldots\right)+\alpha\left(v^{(0)}+pv^{(1)}+p^2v^{(2)}+p^3v^{(3)}+\ldots\right)\end{array}\right)=0\end{aligned} \quad (18)$$

The following equations result from equating coefficients of corresponding terms on both sides of equation (18).

$$p^0:(1+\varepsilon)\frac{\partial}{\partial t}v^{(0)}(x,t)=0 \quad (19)$$

$$p^1:(1+\varepsilon)\frac{\partial}{\partial t}v^{(1)}(x,t)+u\frac{\partial}{\partial x}v^{(0)}(x,t)-D\frac{\partial^2}{\partial x^2}v^{(0)}(x,t)+\alpha v^{(0)}(x,t)=0 \quad (20)$$

$$p^2:(1+\varepsilon)\frac{\partial}{\partial t}v^{(2)}(x,t)+u\frac{\partial}{\partial x}v^{(1)}(x,t)-D\frac{\partial^2}{\partial x^2}v^{(1)}(x,t)+\alpha v^{(1)}(x,t)=0 \quad (21)$$

$$p^3:(1+\varepsilon)\frac{\partial}{\partial t}v^{(3)}(x,t)+u\frac{\partial}{\partial x}v^{(2)}(x,t)-D\frac{\partial^2}{\partial x^2}v^{(2)}(x,t)+\alpha v^{(2)}(x,t)=0 \quad (22)$$

$$p^4:(1+\varepsilon)\frac{\partial}{\partial t}v^{(4)}(x,t)+u\frac{\partial}{\partial x}v^{(3)}(x,t)-D\frac{\partial^2}{\partial x^2}v^{(3)}(x,t)+\alpha v^{(3)}(x,t)=0 \quad (23)$$

and so on.

The above equations (19)-(23) were solved successively and obtained the following results.

$$v^{(0)}(x,t) = Be^{-\lambda x} \tag{24}$$

$$v^{(1)}(x,t) = \frac{Be^{-\alpha x}\left(-\alpha + u\lambda + D\lambda^2\right)t}{1+\varepsilon} \tag{25}$$

$$v^{(2)}(x,t) = \frac{1}{2}\frac{Be^{-\lambda x}t^2\left(-\alpha + u\lambda + D\lambda^2\right)^2}{(1+\varepsilon)^2} \tag{26}$$

$$v^{(3)}(x,t) = \frac{1}{6}\frac{Be^{-\lambda x}t^3\left(-\alpha + u\lambda + D\lambda^2\right)^3}{(1+\varepsilon)^3} \tag{27}$$

$$C(x,t) = \lim_{p\to 1}\left(v^{(0)} + pv^{(1)} + p^2v^{(2)} + p^3v^{(3)} + ..\right) \tag{28}$$

Therefore,

$$C(x,t) = Be^{-\lambda x} + \frac{Be^{-\alpha x}\left(-\alpha + u\lambda + D\lambda^2\right)t}{1+\varepsilon} + \frac{1}{2}\frac{Be^{-\lambda x}t^2\left(-\alpha + u\lambda + D\lambda^2\right)^2}{(1+\varepsilon)^2} + \frac{1}{6}\frac{Be^{-\lambda x}t^3\left(-\alpha + u\lambda + D\lambda^2\right)^3}{(1+\varepsilon)^3} \tag{29}$$

3.4 Basic Idea of Variational iteration Method (VIM)

In order to throw light on the concept of Variational iteration Method (VIM), the following differential equation is considered:

$$Lu + Nu = g(t)$$ (30

where L is a linear operator, N is a non-linear operator and $g(t)$ an inhomogeneous term. By VIM, the correctional functional can be constructed as follows:

$$U_{n+1}(t) = U_n(t) + \int_s^t \lambda\left(LU_n(s) + N\tilde{U}_n(s) - g(s)\right)ds \tag{31}$$

Where λ is the Lagrange multiplier (He, 2000), which can be obtained using the variational theory. The subscript n is the nth approximation and $\tilde{U}_n(s)$ is considered as the restricted variation (He, 2005).

3.5 Solution of the Contaminant Flow Problem by Variation of Parameter Method (VIM).

The transformed version of the equation (6) is solved by VIM as follows:

$$C_{n+1}(x,t)=C_n(x,t)+\int_0^t \lambda\begin{pmatrix}(1+\varepsilon)\dfrac{\partial}{\partial s}C_n(x,s)-D\dfrac{\partial^2}{\partial x^2}C_n(x,s)\\ +u\dfrac{\partial}{\partial x}C_n(x,s)+\alpha C_n(x,s)\end{pmatrix}ds \tag{32}$$

$$C_0(x,t)=Be^{-\lambda x} \tag{33}$$

The Lagrange multiplier is $\lambda=-1$ as obtained from the variational theory.

The iterations in equation (32) yield the following results:

$$C_1(x,t)=C_0(x,t)+\int_0^t (-1)\begin{pmatrix}(1+\varepsilon)\dfrac{\partial}{\partial s}C_0(x,s)-D\dfrac{\partial^2}{\partial x^2}C_0(x,s)\\ +u\dfrac{\partial}{\partial x}C_0(x,s)+\alpha C_0(x,s)\end{pmatrix}ds \tag{34}$$

i.e.,

$$C_1(x,t)=Be^{-\lambda x}+DB\lambda^2e^{-\lambda x}t+uB\lambda e^{-\lambda x}t-\alpha B\lambda e^{-\lambda x}t \tag{35}$$

Further iterations give

$$C_2(x,t)=\frac{1}{2}Be^{-\lambda x}\begin{pmatrix}2+2D\lambda^2t+2u\lambda t-2\alpha t+t^2D^2\lambda^4+2t^2Du\lambda^3-2t^2D\alpha\lambda^2\\ +t^2u^2\lambda^2-2t^2u\alpha\lambda+t^2\alpha^2-2t\varepsilon D\lambda^2-2t\varepsilon u\lambda+2t\varepsilon\alpha\end{pmatrix} \tag{36}$$

$$C_3(x,t)=\frac{1}{6}Be^{-\lambda x}\begin{pmatrix}6+3t^3u^2\lambda^4D+3t^3u\lambda^5D^2+3t^3u\lambda\alpha^2-3t^3u^2\lambda^2\alpha\\ +3t^3D\lambda^2\alpha^2-3t^3D^2\lambda^4\alpha-6t^2D^2\lambda^4\varepsilon-6t^2\varepsilon u^2\lambda^2\\ +6t\varepsilon^2D\lambda^2+6t\varepsilon^2u\lambda-t^3\alpha^3-12t^2D\lambda^3\varepsilon u+12t^2\varepsilon u\alpha\lambda\\ +12t^2D\lambda^2\varepsilon\alpha-6t^3u\lambda^3D\alpha+6D\lambda^2t+6u\lambda t+3t^2D^2\lambda^4\\ +3t^2u^2\lambda^2+6t\varepsilon\alpha-6\alpha t+3t^2\alpha^2+6t^2Du\lambda^3-6t^2D\alpha\lambda^2\\ -6t^2u\alpha\lambda-6t\varepsilon D\lambda^2-6t\varepsilon u\lambda+t^3u^3\lambda^3+t^3D^3\lambda^6\\ +6t^2\varepsilon\alpha^2-6t\varepsilon^2\alpha\end{pmatrix} \tag{37}$$

$$C_n(x,t)=\frac{1}{6}Be^{-\lambda x}\begin{pmatrix}6+3t^3u^2\lambda^4D+3t^3u\lambda^5D^2+3t^3u\lambda\alpha^2-3t^3u^2\lambda^2\alpha\\+3t^3D\lambda^2\alpha^2-3t^3D^2\lambda^4\alpha-6t^2D^2\lambda^4\varepsilon-6t^2\varepsilon u^2\lambda^2\\+6t\varepsilon^2D\lambda^2+6t\varepsilon^2u\lambda-t^3\alpha^3-12t^2D\lambda^3\varepsilon u+12t^2\varepsilon u\alpha\lambda\\+12t^2D\lambda^2\varepsilon\alpha-6t^3u\lambda^3D\alpha+6D\lambda^2t+6u\lambda t+3t^2D^2\lambda^4\\+3t^2u^2\lambda^2+6t\varepsilon\alpha-6\alpha t+3t^2\alpha^2+6t^2Du\lambda^3-6t^2D\alpha\lambda^2\\-6t^2u\alpha\lambda-6t\varepsilon D\lambda^2-6t\varepsilon u\lambda+t^3u^3\lambda^3+t^3D^3\lambda^6\\+6t^2\varepsilon\alpha^2-6t\varepsilon^2\alpha\end{pmatrix} \qquad (38)$$

Therefore,

$$C(x,t)=\lim_{n\to\infty}C_n(x,t)$$

$$=\frac{1}{6}Be^{-\lambda x}\begin{pmatrix}6+3t^3u^2\lambda^4D+3t^3u\lambda^5D^2+3t^3u\lambda\alpha^2-3t^3u^2\lambda^2\alpha\\+3t^3D\lambda^2\alpha^2-3t^3D^2\lambda^4\alpha-6t^2D^2\lambda^4\varepsilon-6t^2\varepsilon u^2\lambda^2\\+6t\varepsilon^2D\lambda^2+6t\varepsilon^2u\lambda-t^3\alpha^3-12t^2D\lambda^3\varepsilon u+12t^2\varepsilon u\alpha\lambda\\+12t^2D\lambda^2\varepsilon\alpha-6t^3u\lambda^3D\alpha+6D\lambda^2t+6u\lambda t+3t^2D^2\lambda^4\\+3t^2u^2\lambda^2+6t\varepsilon\alpha-6\alpha t+3t^2\alpha^2+6t^2Du\lambda^3-6t^2D\alpha\lambda^2\\-6t^2u\alpha\lambda-6t\varepsilon D\lambda^2-6t\varepsilon u\lambda+t^3u^3\lambda^3+t^3D^3\lambda^6\\+6t^2\varepsilon\alpha^2-6t\varepsilon^2\alpha\end{pmatrix} \qquad (39)$$

4. Results and discussion

The graphs in figures 1-6 were obtained using equations (29) and (39).

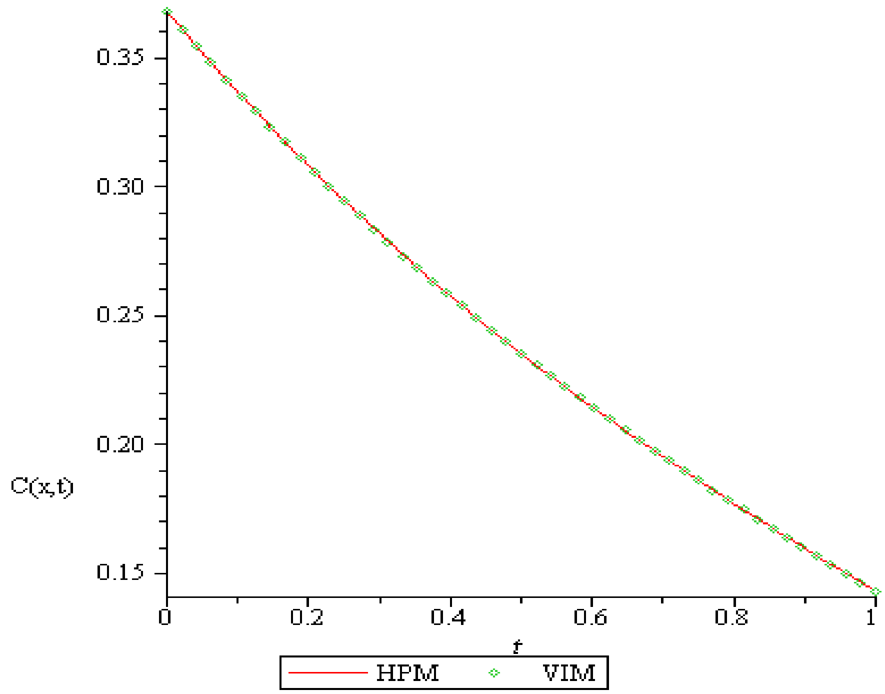

Figure 1: Graph of Concentration against time for the solution

obtained by VIM and HPM

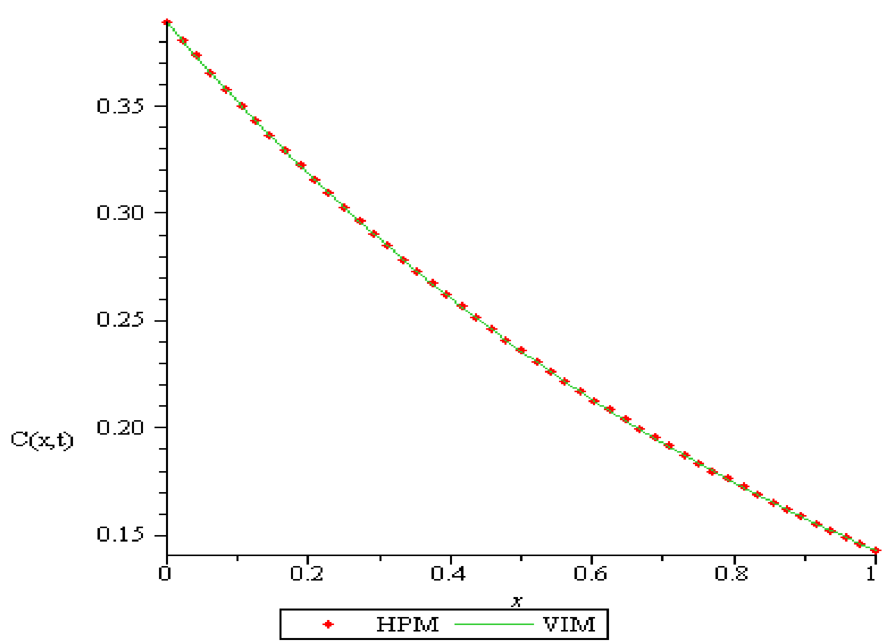

Figure 2: Graph of Concentration against distance for the solution obtained by VIM and HPM

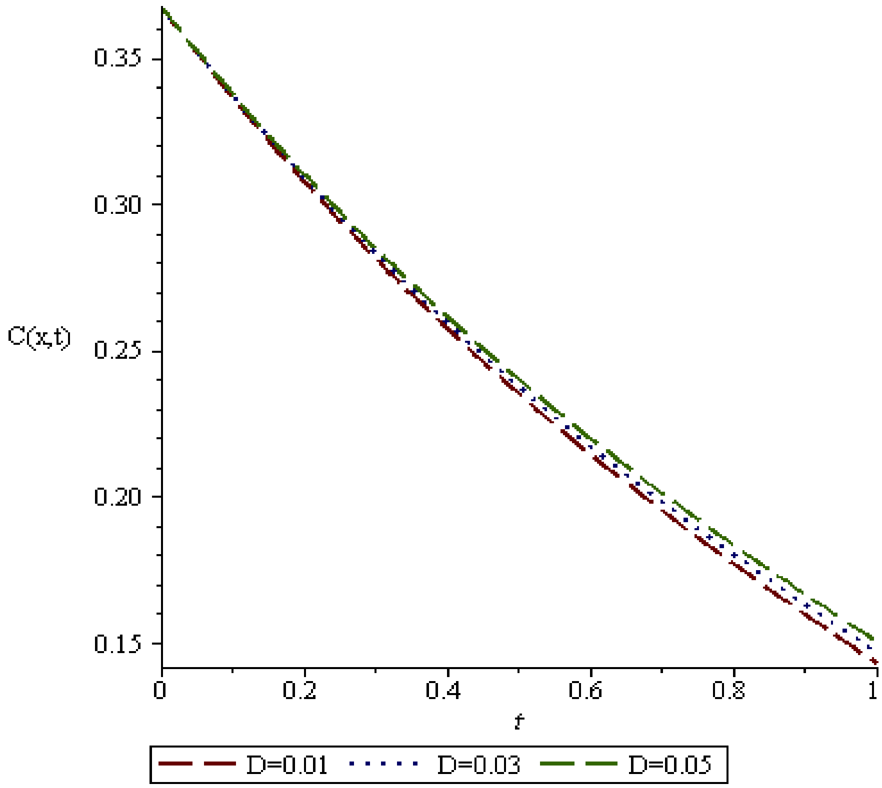

Figure 3: Concentration profile with time for varying dispersion coefficients

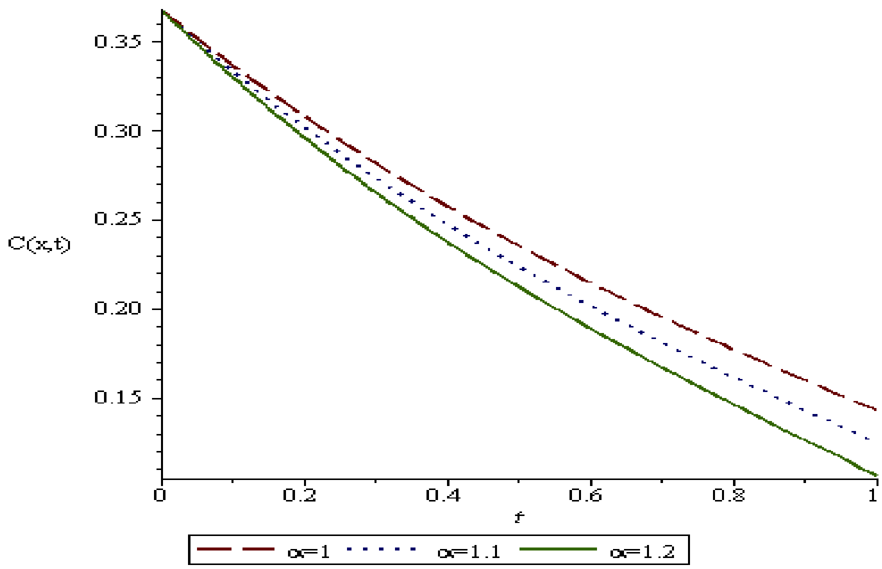

Figure 4: Concentration profile with time for varying decay parameter

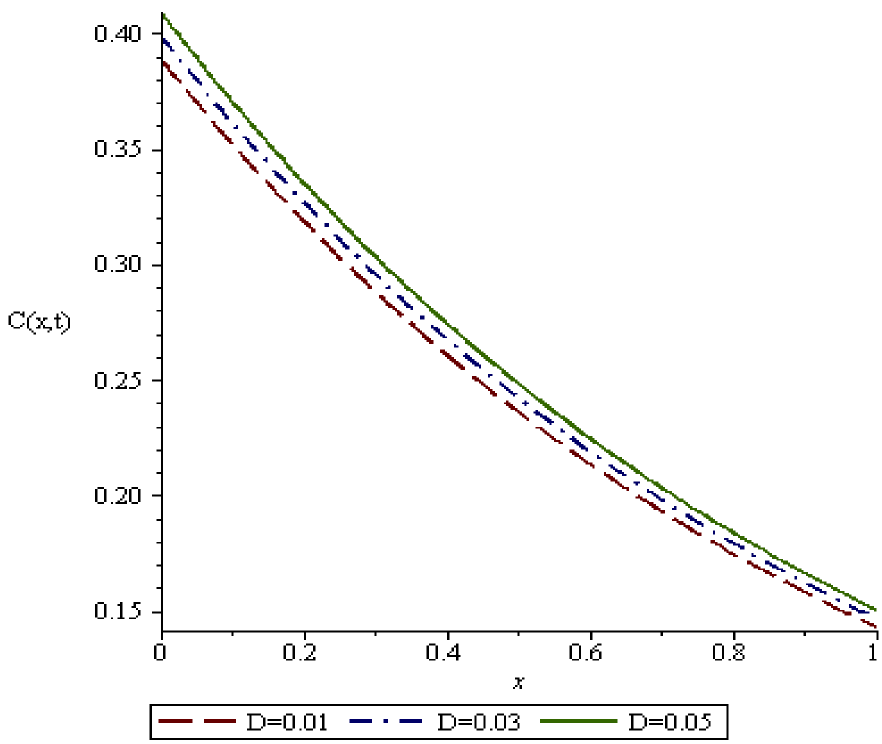

Figure 5: concentration profile with distance for varying dispersion coefficients

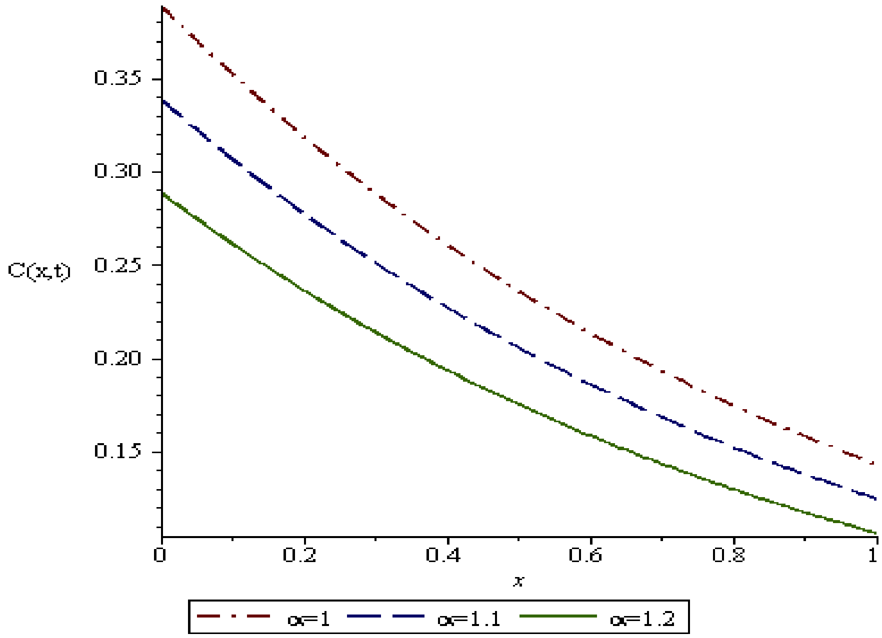

Figure 6: concentration profile with distance for varying dispersion coefficient

The graphs in figures 1 and 2 were plotted using the solutions obtained using HPM and VIM. The concentration profiles show how the two results overlapped. These show that the application of VIM performs equally as good as that of HPM. In both cases, the concentration of the contaminant decrease with increase in time and distance.

Figure 3 is the concentration profile with time for varying dispersion coefficient. The graph shows that as the dispersion coefficient decreases, the contaminant concentration decreases faster with time. Figure 4 is the graph of contaminant concentration against time for varying decay parameter. The graph shows that as the decay parameter increases, the contaminant concentration decreases faster with time.

Similarly, the concentration profile of the contaminant with distance for varying dispersion parameter is shown in figure 5. The graph shows that as the dispersion coefficient decreases, the contaminant concentration decreases with time. Lastly, figure 6 is the concentration profile of contaminant with distance for varying decay parameter. This shows the effect of varying the decay parameter. It shows that as the decay parameter decreases, the concentration of the contaminant decreases.

5. **Conclusion**

A one-dimensional contaminant flow model with non-zero initial concentration has been solved semi-analytically using the homotopy perturbation method and variational Iteration method. The study shows that the result obtained from variational iteration method is as accurate as that of homotopy perturbation method. This can easily be seen from figures 1 and 2 as the graphs overlapped completely. The effects of change in the first order decay coefficients and the dispersion coefficient on the contaminant concentration along transient groundwater were shown in the figures 3 to 6. The study reveals that the contaminant concentration decreases with increase in time and distance from the origin.

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