Journal of science, education and technology,
Volume 2 Number 1

April 2009
ISSN 1596-9770

Control of Diffusion Equation Using Finite Element Formulation

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Abstract
The finite method is a numerical method which can be used for the accurate solution of a complex boundary-value problem and other problem in Engineering. This paper dealt with the solution to an optimization problem for the diffusion equation using the finite element method. It was discovered that the nodal temperature $(u(x,t))$ shows cooling effect from the left to the right side of the roda, and subsequently the output $(z(x,t))$ decreases, it is symmetric in nature.
Key words: Finite element, boundary-value problem, optimization problem, and diffusion equation.

1. Introduction

The related applications of optimization methods to equations of mathematical physics have become obvious in most literatures. Also, the theory that has been accumulated on optimization techniques for solving continuous optimum control problems characterized by linear and integral quadratic objective functional has become quite extensive. For example, Curtain and Pritchad (1977) provided the analytical solution to the optimization problem of minimization of an integral quadratic cost functional subject to the one-dimensional heat diffusion problem with a source. More on the control of diffusion problem have also been solved by Ibiejugba (1980) via a Ritz penalty technique and Reju (1991).

Diffusion Model

The governing equation is given by

$$\frac{\partial z_n(x,t)}{\partial t} = \frac{\partial^2 z_n(x,t)}{\partial x^2} + u_n(x,t) \quad 1.1$$

where u is the control function, which is diffusion process with source.

Considering the problem as follows:

$$Min \;\; J(z,u) = \int_0^1 \int_0^1 [z_n^2(x,t) + u_n^2(x,t)]dxdt \quad 1.2a$$

Subject to

$$\frac{\partial z_n(x,t)}{\partial t} = \frac{\partial^2 z_n(x,t)}{\partial x^2} + u_n(x,t)$$

$$\left.\begin{array}{ll} z(x,0) = z_o(0), & 0 \le x \le 1 \\ z(0,t) = z(1,t) = 0, & 0 \le t \le 1 \end{array}\right\} \quad 1.2b$$

Laying emphasis on quadratic functional, we easily see that the functional can be written as:

$$\int_0^1 \int_0^1 [z_n^2(x,t) + u_n^2(x,t)]dxdt$$

$$= \int_0^1 \int_0^1 [u_n(x,t) + iz_n(x,t)][u_n(x,t) - iz_n(x,t)]dxdt \quad 1.3$$

where $i = \sqrt{(-1)}$ is the complex unit.

Setting

$$\left.\begin{array}{l} w(x,t) = u(x,t + iz(x,t)) \\ or\; w(x,t) = z(x,t) + iu(x,t) \end{array}\right\} \quad 1.4$$

we have the following equation

$$\int_0^1 \int_0^1 [z^2(x,t) + u^2(x,t)]dxdt$$

$$= \int_0^1 \int_0^1 [w(x,t)\overline{w}(x,t)]dxdt$$

$$= \int_0^1 \int_0^1 w^2(x,t)dxdt \quad 1.5$$

$$u^2(x,t) + z^2(x,t) = w^2(x,t) \quad 1.6$$

Following the Pythagorean principle, equation (1.6) suggests that $u(x,t)$ and $z(x,t)$ are orthogonal planes in (x,t) while the set $\{u(x,t), z(x,t), w(x,t)\}$ forms a Pythagorean triple. The function $u(x,t)$ is a source (negative sink) term that adds heat to the unit rod at a rate $u(x,t)$ per unit time, per unit length. Thus, according to Boyce and Diprima (1977), we have

$u(x,t) = u(z,x,t) > 0$ 1.7

This shows that $u(x,t)$ also depends on $z(x,t)$. Since $u(x,t)$ and $z(x,t)$ are planes in (x,t), it is then easy to see that $u(x,t)$ and $z(x,t)$ are parabolic surfaces instead of curves.

We see that with $z^2(x,t) + u^2(x,t) = [z(x,t) + u(x,t)][z(x,t) - iu(x,t)]$ the roots of cost integrand are the imaginary planes $+iu(x,t)$.

Invoking the result due to Krasnov et al (1984) which states that the functional

$J[y(x)] = \int_0^1 [x^2 + y^2]dx$ 1.8

attains a strict minimum on the curve $y(x) = 0$, we will have analogous but a more generalized formulation result.

2. Methodology

We recall (1.2a) subject to (1.2b).The Hamilton for this problem is related to that of Singh and Titli (1978) written as:

$$H = z^2(x,t) + u^2(x,t) + \lambda^T\left[\frac{\partial^2 z(x,t)}{\partial x^2} + u(x,t)\right] \quad 2.1$$

where

$\lambda^T = \lambda^T(t)$ 2.2

Setting

$$f(z,u) = \frac{\partial^2 u}{\partial x^2} + u \quad 2.3$$

We then have the first order necessary conditions for optimality as:

$$\frac{\partial z}{\partial t} = \frac{\partial H}{\partial \lambda} = \frac{\partial^2 z}{\partial x^2} + u = f(z,u) \quad 2.4$$

$$\frac{\partial \lambda}{\partial t} = \frac{\partial H}{\partial z} = -\left(\frac{\partial f}{\partial z}\right)^{\lambda^T} = \frac{\partial g}{\partial z} = -2z(x,t) \quad 2.5$$

$$\frac{\partial H}{\partial u} = 0 \ or \left(\frac{\partial f}{\partial u}\right)^{\lambda^T} + \frac{\partial g}{\partial u} = 0 \quad 2.6$$

where

$H = g(z,u) + \lambda^T f(z,u)$ 2.7

Equation (2.5a) gives $\lambda + 2u = 0$ or $\lambda = 2u$ 2.8

By virtue of (2.4) and (2.5a), we have

$$\frac{\partial \lambda}{\partial t} = -2\frac{\partial u}{\partial t} = -2z$$

$$\Rightarrow z(x,t) = \frac{\partial}{\partial t}u(x,t) \quad 2.9$$

Equation (2.6) is here of physical significance under the conditions for optimality which expresses the relationship between the temperature and the heat source at any point of the unit conducting rod of our diffusion model.

$z(x,t)$ is the optimal state of temperature at any point x of the rod while $u(x,t)$ is the optimal control rate of flow of heat through the rod at any position x.

Assuming that equation (2.6) admits the Fourier solution proposed by Ibiejugba (1980) and Duchateau and Zachmann(1986).

$$z(x,t) = \sum_{i=1}^{\infty} \propto_i(t) \sin \pi i x \quad 2.10$$

$$u(x,t) = \sum_{i=1}^{\infty} u_i(t) \sin \pi i x \quad 2.11$$

We then have our new solution

$$z(x,t) = \frac{\partial}{\partial t}\sum_{i=1}^{\infty} u_i(t) \sin \pi i x$$

$$= \sum_{i=1}^{\infty} u_{it}(t) \sin \pi i x \quad 2.12$$

Hence, it immediately follows that

$\propto_i(t) = u_{it}(t)$ 2.13

and

$$z_t(x,t) = \sum_{i=1}^{\infty} u_{itt}(t) \sin \pi i x \quad 2.14$$

$$z_{xx} = \sum_{i=1}^{\infty} i^2(-\pi^2) u_{it} \sin \pi i x \quad 2.15$$

$$z(x,0) \sum_{i=1}^{\infty} u_{it}(0) \sin \pi i x \quad 2.16$$

3. Finite Element Formulation

The stepwise solution procedure is as follows.

STEP 1: Solution continuum discretization.

We discretize the domain $(0 \le t \le 1)$ with elements of equal length.

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STEP 2: Interpolation model.
Consider a one-dimensional rod of length l. Let the nodes be donated by i and j and the nodal values of the field variable u by u_i and u_j. Let us assume our interpolation model for each element as:

$$u(t) = \propto_1 + \propto_2 (t) \quad 3.1$$

That is, we assume that the interpolation model is linear where $\propto_1$ and $\propto_2$ are the unknown coefficients. Using the nodal conditions

$$\left.\begin{array}{l} u(t) = u_i \quad at \quad t = t_i \\ u(t) = u_j \quad at \quad t = t_j \end{array}\right\} \quad 3.2$$

From equation (3.1), we obtain

$$u_i = \propto_1 + \propto_2 t_i \quad 3.3$$

$$u_j = \propto_1 + \propto_2 t_i \quad 3.4$$

Solving (3.3) and (3.4) simultaneously, we have

$$\left.\begin{array}{l} \propto_1 = \dfrac{u_i t_j - u_j t_i}{l} \\ \propto_2 = \dfrac{u_j - u_i}{l} \end{array}\right\} \quad 3.5$$

where t_i and t_j denote the global co-ordinate of nodes i and j respectively.
Substituting solution (3.5) into (3.1), we have

$$u(t) = \frac{(u_i t_j - u_j t_i)}{l} + \frac{(u_j - u_i)t}{l}$$

$$= \frac{(t_j - t)u_i}{l} + \frac{(t - t_i)u_j}{l}$$

$$= \emptyset_i(t)u_i + \emptyset_j(t)u_j$$

$$= [\emptyset(t)]\vec{u}^{(e)} \quad 3.6$$

where $\emptyset_i(t) = \frac{(t_j - t)}{l}$, $\emptyset_j(t) = \frac{(t - t_i)}{l}$ and $\vec{u}^{(e)} = \begin{Bmatrix} u_i \\ u_j \end{Bmatrix}$

STEP 3: Element characteristic matrices and vectors. The necessary conditions for optimality lead to functional

$$J = \frac{1}{2}\int \left[\left(\frac{\partial u_n}{\partial t}\right)^2 + 2\pi^2 n^2 u_n - u_n^2\right] dt \quad 3.7$$

The element characteristic matrices and vectors can be obtained by expressing the functional J in matrix form. Evaluating the functional J over the element e, we obtain

$$J^{(e)} = \frac{1}{2}\int_0^{(e)} \left[-\left(\frac{\partial u_n}{\partial t \partial}\right)^2 + 2\pi^2 n^2 u_n - u_n^2\right] dt \quad 3.8$$

The functional J is expressed as the sum of E elemental quantities $J^{(e)}$ as

$$J = \sum_{e=1}^{E} J^{(e)} \quad 3.9$$

substituting (3.6) in (3.8), we obtain

$$J^{(e)} = \frac{1}{2}\int_{t_i}^{t_j} \left[\vec{u}_n^{(e)T} \left[\frac{\partial \emptyset}{\partial t}\right]^T \left[\frac{\partial \emptyset}{\partial t}\right] \vec{u}_n^{(e)} - \vec{u}_n^{(e)} [\emptyset]^T [\emptyset] \vec{u}_n^{(e)} + 2\pi^2 n^2 [\emptyset]^T \vec{u}_n^{(e)}\right] dt \quad 3.10$$

Using the equivalent minimization problem, the conditions for minimum are

$$\frac{\partial J}{\partial u_i} = \frac{\partial}{\partial u_i} \sum_{e=1}^{E} J^{(e)}$$

$$= \sum_{e=1}^{E} \frac{\partial J^{(e)}}{\partial u_i} = 0 \quad 3.11$$

$$i = 1,2, \ldots \ldots \ldots m$$

where E is the number of elements and m is the number of nodal degrees of freedom.
Equation (3.10) can also be expressed as;

$$\sum_{e=1}^{E} \frac{\partial J^{(e)}}{\partial u_i} = 0$$

$$i.e \sum_{e=1}^{E} \int_{t_i}^{t_j} \left[-\left[\frac{\partial \emptyset}{\partial t}\right]^T \left[\frac{\partial \emptyset}{\partial t}\right] u_n^{(e)} - [\emptyset]^T [\emptyset] \vec{u}_n^{(e)} + \pi^2 n^2 [\emptyset]^T \vec{u}_n^{(e)}\right] dt = \vec{0}$$

$$\Rightarrow \sum_{e=1}^{E} [k^{(e)}] \vec{u}_n^{(e)} = \sum_{e=1}^{E} \vec{p}^{(e)} \quad 3.12$$

Note: in the assembling, J has been replaced by the sum of element functional $J^{(e)}$,
e=1, 2…m.
Looking at (3.10), this can also be stated in linear matrix as;

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ISSN 1596-9770

$$\sum_{e=1}^{m}\frac{\partial J^{(e)}}{\partial u_i}=\sum_{e=1}^{m}\left[k^{(e)}\vec{u}_n^{(e)}-p^{(e)}\right]=0 \qquad 3.13$$

This is necessary to form our governing finite element equations, where element characteristic matrix, $[k^{(e)}]$ is given as;

$$[k^{(e)}]=\int_{t_i}^{t_j}\left[\left[\frac{\partial\emptyset}{\partial t}\right]^T\left[\frac{\partial\emptyset}{\partial t}\right]-[\emptyset]^T[\emptyset]\right]dt \qquad 3.14$$

The element characteristics vector, $\vec{p}^{(e)}$ is given as;

$$\vec{p}^{(e)}=\int_{t_i}^{t_j}\pi^2n^2[\emptyset]^T dt \qquad 3.15$$

and the assembled nodal temperature vector, $\vec{u}^{(e)}$ is given as;

$$\vec{u}^{(e)}=\begin{Bmatrix}u_1\\u_2\\.\\.\\.\\u_m\end{Bmatrix} \qquad 3.16$$

Substituting $[\emptyset(t)]=[\emptyset_i(t)\quad \emptyset_j(t)]=\left[\frac{t_j-t}{l^{(e)}}\quad \frac{t-t_i}{l^{(e)}}\right]$ into equations (3.13) and (3.14), we have;

$$[k^{(e)}]=\int_{t_i}^{t_j}\left[\begin{Bmatrix}\frac{-1}{l^{(e)}}\\ \frac{1}{l^{(e)}}\end{Bmatrix}\begin{Bmatrix}\frac{-1}{l^{(e)}} & \frac{1}{l^{(e)}}\end{Bmatrix}+\begin{Bmatrix}\frac{t_j-t}{l^{(e)}}\\ \frac{t-t_i}{l^{(e)}}\end{Bmatrix}\begin{Bmatrix}\frac{t_j-t}{l^{(e)}} & \frac{t-t_i}{l^{(e)}}\end{Bmatrix}\right]dt$$

$$=\frac{1}{l^{(e)}}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}+\frac{1}{6l^{2(e)}}\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix} \qquad 3.17$$

$$\vec{p}^{(e)}=\pi^2n^2\int_{t_i}^{t_j}\begin{Bmatrix}\frac{t_j-t}{l^{(e)}}\\ \frac{t-t_i}{l^{(e)}}\end{Bmatrix}dt=\frac{\pi^2n^2}{2}\begin{Bmatrix}t_j-t_i\\t_j-t_i\end{Bmatrix} \qquad 3.18$$

STEP 4: Assembly of element matrices and vectors and derivation of governing equations.

We assemble the element characteristics matrices and vectors and obtain the overall equations as;

$$[k]\vec{u}=\vec{p} \qquad 3.19$$

STEP 5: We can now solve the system of equations (3.18) after incorporating the boundary conditions:

 ISSN 1596-9770

4. Result and Discussion

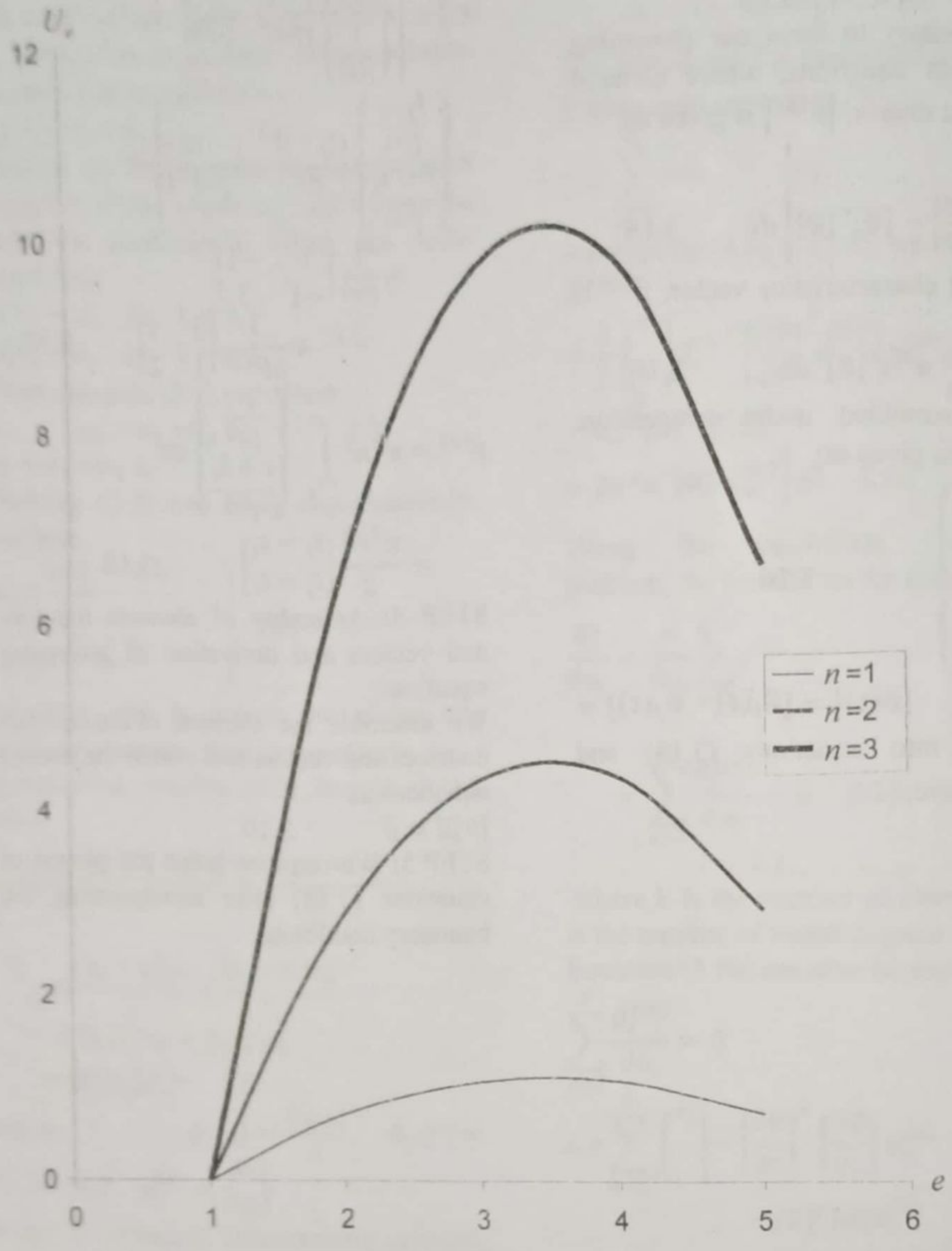

Fig.1: Optimal control profile U_e against number of element e for various values of n when E=5.

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ISSN 1596-9770

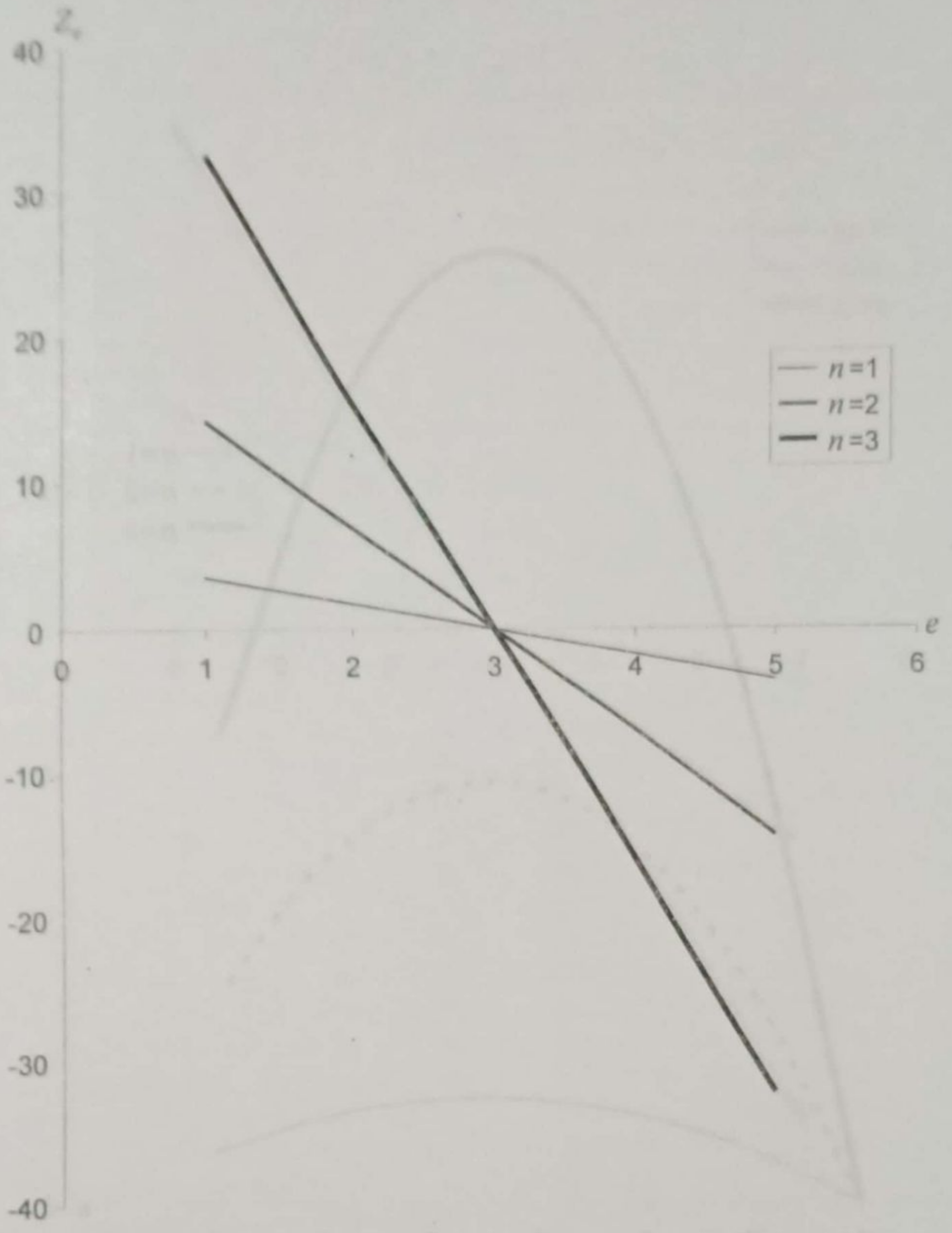

Fig.2: Optimal state profile Z_e against number of element e for various values of n when E=5.

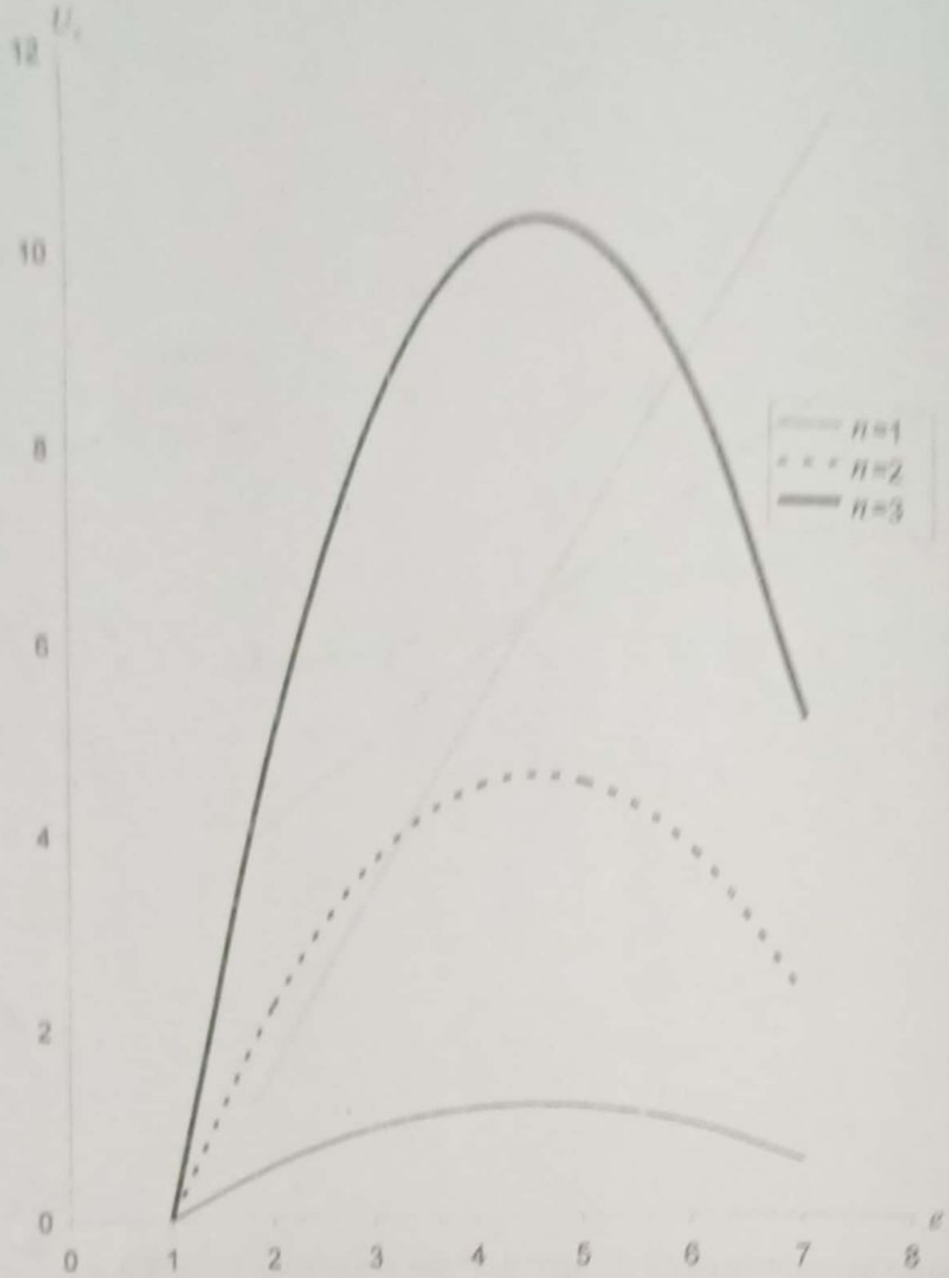

Fig.3: Optimal Control profile U_e against number of element e for various values of n when E=7.

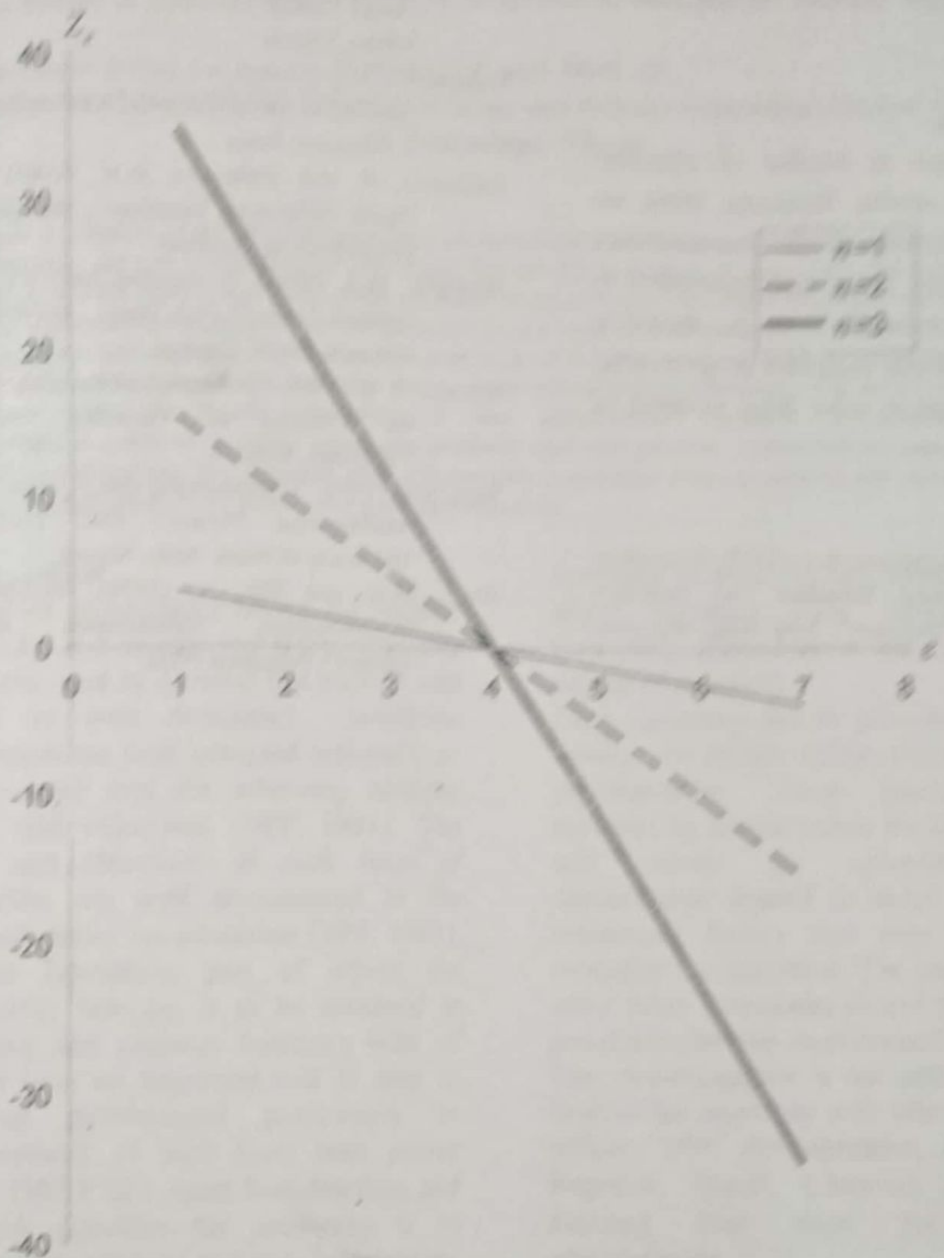

Fig.4: Optimal State profile Z_e against number of element e for various values of n when E=7.

Figures 1 and 3 show the optimal control profile U_e against the number of element, e for various values of n when E = 5 and E = 7 respectively. Generally, the nodal temperatures show cooling process from the left side of the rod to the right side

Figures 2 and 4 show the optimal state profile Z_e against the number of element, e for various values of n when E = 5 and E =

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ISSN 1596-9770

7 respectively. It was discovered that the output for each number of element is symmetric.

5. Conclusion

Increase in number of elements yields better results. However, there are more computations as the characteristics matrix becomes larger and impossible to be solved manually; it is therefore recommended that computer programming should be used to solve large systems of finite elements.

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