A Note on the Existence of Unique Solution of In-Situ Combustion Oil Shale In Porous Medium

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Abstract

This paper establishes the criteria for the existence of unique solution of the equations governing the in-situ combustion of oil shale and examines the properties of solution. Our proof revealed that velocity, mass and temperature are increasing function of time.
Keywords and phrases: combustion, In-situ, oil recovery, oil shale, porous medium.

1. Introduction

Oil shale gains attention as a potential abundant source of oil whenever the price of crude oil rises. Though, oil shale mining and processing raise a number of environmental concerned such as land use, waste disposal, water use, waste-water management, greenhouse- gas emissions and air pollution. Oil shale is found all over the world, including China, Israel, and Russia. The United States, however, has the most shale resources. However, all the types of kerogen consist mainly of hydrocarbons; smaller amounts of sulphur, oxygen and nitrogen; and a variety of minerals.(Abdelrahman, 2015).

Deposits of oil shale have been found in 27 countries worldwide. Oil shale becomes an important alternative energy due to its huge reserves. It is quite possible to satisfy the future oil requirement. Upon being heated, kerogen in oil shale can be converted to oil and gas. The heating process is called pyrolysis or retorting (Zheng *et al*., 2017).

In situ processes introduce heat to the oil shale which is still embedded in its natural geological formation. One of the in situ methods is the in- situ combustion (ISC). In-situ combustion is simply combustion heating of in-place oil shale within a deposit at the fire front. The main idea in in-situ combustion is burning of a portion of oil shale to produce sufficient heat to retort the remainder. A great portion of the potentially recoverable shale oil resource is in low-grade deposits that may never be recovered by primary mining techniques. In-situ processing presents the opportunity of recovering shale oil from these low-grade deposits without the adverse environmental impacts normally related with mining and above ground processing which comprises three steps: conduction heating, hot gas injection, and in situ combustion. In the step of conduction heating, oil shale is heated, in the second step, hot gas is injected into the oil shale layer and the surrounding cool oil shale would be heated. In-situ combustion is the process whereby hot air is injected into the oil shale layer in order to react the organic component (kerogen). The combustion reaction produces enough heat to propagate with a combustion wave leading to cracking and vaporizations of lighter components. The pyrolysis of oil shale is triggered by the combustion reactions in the absence of extra heat supply. Shale oil is obtained through the pyrolysis process (Zheng *et al*., 2017).

Several works have been done on the in-situ combustion of oil shale. Lapene *et al.* (2007) modeled coupled mass and heat transport in reactive porous medium using homogeneous description at a Darcy-scale. Local non-equilibrium transport of heat was treated with a two field temperature, one for the gas and one for the solid phase.

Olayiwola *et al.* (2011) extended Lapene *et al.*(2007) model to a situation where there is Arrhenius heat generation and chemical reaction. They made additional assumption that the reaction is in steady-state so that time derivatives are zero $\left(\frac{\partial}{\partial t}=0\right)$. They examined the properties of solution of the model and obtained the analytical solution using asymptotic expansion.

In another development, Olayiwola *et al.* (2012) studied coupled heat transport in Arrhenius reactive porous medium using a homogeneous description at the Darcy-scale. They assumed that there is a perfect contact between gas and solid phase. Eigenfunctions expansion technique was used and the outcome showed that the heat transfer increases as Frank-Kamenetskii number increases and scaled thermal conductivity decreases.

Zheng *et al.* (2017) focused on the numerical simulation of in situ combustion of oil shale. Numerical test was used for the stimulation of oil shale and their result showed that varying gas injection rate and oxygen was important in the field test in-situ combustion.

This paper aim at establish the criteria for the existence of unique solution and examine the properties of solution of the equations governing the in-situ combustion of oil shale as enhanced oil recovery technique in a porous medium.

3.0 Model Formulation

Here, we extend Olayiwola *et al.* (2012) model by incorporating combustion front velocity. Then, the equations that describe the in-situ combustion of oil shale are:

The continuity equation:

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u_c)}{\partial x}=0 \qquad (1)$$

The momentum equation:

$$\rho\left(\frac{\partial u_c}{\partial t}+\frac{\partial u_c}{\partial x}\right)=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left(\mu\frac{\partial u_c}{\partial x}\right) \qquad (2)$$

Gas phase energy equation:

$$\left.\begin{array}{l}\rho_g \varepsilon c_g\left(\frac{\partial T_g}{\partial t}-u_e\frac{\partial T_g}{\partial x}\right)+\rho_g \varepsilon c_g u_f\frac{\partial T_g}{\partial x}=\frac{\partial}{\partial x}\left(\kappa_g\frac{\partial T_g}{\partial x}\right)+\Gamma(T_g-T_s)+h(T_e-T_g)+\\ \varepsilon\Delta H A C_f^{\alpha}C_{ox}^{\beta}e^{-\frac{E}{RT_s}}\end{array}\right\} \quad (3)$$

The solid phase energy equation:

$$\rho_s(1-\varepsilon)c_s\left(\frac{\partial T_s}{\partial t}-u_e\frac{\partial T_s}{\partial x}\right)=\frac{\partial}{\partial x}\left(\kappa_s\frac{\partial T_s}{\partial x}\right)-\Gamma(T_g-T_s)+h(T_e-T_s)+(1-\varepsilon)\Delta H A C_f^{\alpha}C_{ox}^{\beta}e^{-\frac{E}{RT_s}}\Bigg\} \quad (4)$$

The oxygen mass balance:

$$\varepsilon\rho_g\left(\frac{\partial C_{ox}}{\partial t}-u_e\frac{\partial C_{ox}}{\partial x}\right)+\varepsilon\rho_g u_f\frac{\partial C_{ox}}{\partial x}=\rho_g\varepsilon\frac{\partial}{\partial x}\left(D_{ox}\frac{\partial C_{ox}}{\partial x}\right)+\varepsilon A C_f^{\alpha}C_{ox}^{\beta}e^{-\frac{E}{RT_s}} \quad (5)$$

The fuel mass balance:

$$(1-\varepsilon)\rho_s\left(\frac{\partial C_f}{\partial t}-u_e\frac{\partial C_f}{\partial x}\right)=(1-\varepsilon)A C_f^{\alpha}C_{ox}^{\beta}e^{-\frac{E}{RT_s}} \quad (6)$$

Darcy's law

$$u_f=-\frac{K}{\mu}\left(\frac{\partial P}{\partial x}-\rho_g\nabla Z\right) \quad (7)$$

Where

A is the frequency, E is the activation energy, α and β are orders of the gaseous reaction, ρ_g is gas density, R is the gas constant, u_e is combustion front velocity, κ_s is thermal conductivity of solid phase, κ_g is thermal conductivity of gas phase, μ is viscosity, t is time, x is position, ε is the porosity, c_s is heat capacity of solid phase, c_g is the heat capacity of gas phase, T_e is the external temperature, T_s is the temperature of solid phase, T_g is the temperature of gas phase, ΔH is heat generation constant, h is heat transfer coefficient, C_{ox} is concentration of oxygen, C_f is fuel concentration, Γ is exchange term between the phases, K is the permeability, D_{ox} is the diffusion of oxygen, u_f is filtration velocity and P is the pressure.

3.1 Coordinate Transformation

Here, we shall neglect the gravitational effect due to the small size in the vertical direction and we let $\rho_g = \rho_s = \rho$. it simple to eliminate the continuity equation (1) by means of streamline function

$$\eta(x,t) = \left(\rho^2\right)^{-\frac{1}{2}} \int_0^x \rho(s,t)ds \tag{8}$$

Then, the coordinate transformation becomes

$$\frac{\partial}{\partial x} \rightarrow \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial}{\partial \eta} \tag{9}$$

$$\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial t} = -u_c \frac{\partial}{\partial \eta} + \frac{\partial}{\partial t} \tag{10}$$

Using the equations (9) and (10) can be simplified as:

$$\rho \frac{\partial u_c}{\partial t} = -\frac{\partial \mathrm{P}}{\partial \eta} + \frac{\partial}{\partial \eta}\left(\mu \frac{\partial u_c}{\partial \eta}\right) \tag{11}$$

$$\left.\begin{aligned} &\rho \varepsilon c_g \left(\frac{\partial \mathrm{T}_g}{\partial t} - 2u_c \frac{\partial \mathrm{T}_g}{\partial \eta}\right) + \rho \varepsilon c_g u_f \frac{\partial \mathrm{T}_g}{\partial \eta} = \frac{\partial}{\partial \eta}\left(\kappa_g \frac{\partial \mathrm{T}_g}{\partial \eta}\right) + \Gamma(\mathrm{T}_g - \mathrm{T}_s) + h(\mathrm{T}_e - \mathrm{T}_g) + \\ &\varepsilon \Delta \mathrm{HAC}_f^\alpha C_{ox}^\beta e^{-\frac{\mathrm{E}}{R\mathrm{T}_s}} \end{aligned}\right\} \tag{12}$$

$$\left.\begin{aligned} &\rho(1-\varepsilon)c_s \left(\frac{\partial \mathrm{T}_s}{\partial t} - 2u_c \frac{\partial \mathrm{T}_s}{\partial \eta}\right) = \frac{\partial}{\partial \eta}\left(\kappa_s \frac{\partial \mathrm{T}_s}{\partial \eta}\right) - \Gamma(\mathrm{T}_g - \mathrm{T}_s) + h(\mathrm{T}_e - \mathrm{T}_s) + \\ &(1-\varepsilon)\Delta \mathrm{HAC}_f^\alpha C_{ox}^\beta e^{-\frac{\mathrm{E}}{R\mathrm{T}_s}} \end{aligned}\right\} \tag{13}$$

$$\rho \varepsilon \left(\frac{\partial C_{ox}}{\partial t} - 2u_c \frac{\partial C_{ox}}{\partial \eta}\right) + \varepsilon \rho u_f \frac{\partial C_{ox}}{\partial \eta} = \rho \varepsilon \frac{\partial}{\partial \eta}\left(D_{ox} \frac{\partial C_{ox}}{\partial \eta}\right) + \varepsilon \mathrm{AC}_f^\alpha C_{ox}^\beta e^{-\frac{\mathrm{E}}{R\mathrm{T}_s}} \tag{14}$$

$$(1-\varepsilon)\rho \left(\frac{\partial C_f}{\partial t} - 2u_c \frac{\partial C_f}{\partial \eta}\right) = (1-\varepsilon)\mathrm{AC}_f^\alpha C_{ox}^\beta e^{-\frac{\mathrm{E}}{R\mathrm{T}_s}} \tag{15}$$

$$\frac{\partial \mathrm{P}}{\partial \eta} = -\frac{\mu}{\mathrm{K}} u_f \tag{16}$$

Here, we assume the porous space of the medium is filled initially with fuel and oxygen. The initial and boundary conditions were formulated as follows:

Initial conditions:

At $t=0$ and $\forall\eta$

$$u_e = u_{inj}\ \ T_g = T_0,\ C_{ox} = 0,\ C_f = C_{f0} \tag{17}$$

Boundary conditions:

$$\left.\begin{aligned}
&\left.\frac{\partial T_s}{\partial\eta}\right|_{\eta=0} = 0,\ \left.\frac{\partial T_s}{\partial\eta}\right|_{\eta=L} = 0\\
&\left.T_g\right|_{\eta=0} = T_{inj},\ \left.\frac{\partial T_g}{\partial\eta}\right|_{\eta=L} = 0\\
&\left.C_{ox}\right|_{\eta=0} = C_{ox0},\ \left.\frac{\partial C_{ox}}{\partial\eta}\right|_{\eta=L} = 0\\
&\left.u_e\right|_{\eta=0} = u_{inj},\ \left.u_e\right|_{\eta=L} = 0\\
&\left.C_f\right|_{\eta=0} = C_{f0}\\
&\left.P\right|_{\eta=L} = P_{outlet}
\end{aligned}\right\} \tag{18}$$

3.2 Existence and Uniqueness of solution

To Scientists and Engineers, the question of existence and uniqueness of solution remain to be a pivot in models and designs. When a problem is formulated we need to examine the solution(s) so as to predict the behavior of such solution(s). We are interested in the existence and uniqueness of solution of the system of equations (11) – (16) satisfying (17) and (18) in order to be able to predict the behavior of the solution. First, in the absence of convection and combustion front and assuming no thermal exchange and no heat transfer between phases. We let $\kappa_s, \kappa_g, D_{ox}$ be constants and we shall follow the approach used by Olayiwola (2015).

Theorem 3.1: let, $D_{ox} = \dfrac{K_g}{\rho c_g} = D_1$ Then there exists a unique solution of problem (11) – (16) satisfy (17) and (18)

Proof: let, $D_{ox} = \dfrac{K_g}{\rho c_g} = D_1,\ \psi = \left(T_g - \dfrac{\Delta H}{c_g}C_{ox}\right)$ and $\varphi = \left(T_s - \dfrac{\Delta H}{c_s}C_f\right)$

Then (19) – (22) become

$$\frac{\partial \psi}{\partial t} = D_1 \frac{\partial^2 \psi}{\partial \eta^2} \tag{19}$$

$$\psi(\eta,0)=T_0, \quad \psi(0,t)=\left(T_{sg}-\frac{\Delta H}{c_g}C_{ast}\right), \quad \psi_\eta(L,t)=0 \tag{20}$$

and

$$\frac{\partial \varphi}{\partial t} = K_1 \frac{\partial^2 T_s}{\partial \eta^2} \tag{21}$$

$$\varphi(\eta,0)=\left(T_0-\frac{\Delta H}{c_s}C_{f0}\right), \quad \left.\frac{\partial T_s}{\partial \eta}\right|_{\eta=0}=0, \quad \left.\frac{\partial T_s}{\partial \eta}\right|_{\eta=L}=0 \tag{22}$$

Using Eigenfunction expansion technique, we obtain the solution of the problem (19) and (20) as

$$\psi(\eta,t)=\sum_{n=1}^{\infty}\frac{4T_0}{(2n-1)\pi}e^{-D_1\left(\frac{(2n-1)\pi}{2L}\right)^2 t}\sin\left(\frac{(2n-1)\pi}{2L}x\right) \tag{23}$$

and by direct integration, we obtain the solution of problem (21) and (22) as

$$\varphi(\eta,t)=\left(T_0-\frac{\Delta H}{c_s}C_{f0}\right) \tag{24}$$

Then, we obtain

$$T_g(\eta,t)=\sum_{n=1}^{\infty}\frac{4T_0}{(2n-1)\pi}e^{-D_1\left(\frac{(2n-1)\pi}{2L}\right)^2 t}\sin\left(\frac{(2n-1)\pi x}{2L}\right)+\frac{\Delta H}{c_g}C_{as}(\eta,t) \tag{25}$$

$$C_{as}(\eta,t)=\frac{c_g}{\Delta H}\left(T_g(\eta,t)-\sum_{n=1}^{\infty}\frac{4T_0}{(2n-1)\pi}e^{-D_1\left(\frac{(2n-1)\pi}{2L}\right)^2 t}\sin\left(\frac{(2n-1)\pi x}{2L}\right)\right) \tag{26}$$

$$T_s(\eta,t)=\left(T_0-\frac{\Delta H}{c_s}C_{f0}\right)+\frac{\Delta H}{c_s}C_f(\eta,t), \tag{27}$$

$$C_f(\eta,t)=\frac{c_s}{\Delta H}\left(\left(T_s(\eta,t)-\left(T_0-\frac{\Delta H}{c_s}C_{f0}\right)\right)\right) \tag{28}$$

Hence, there exists a unique solution of problem (11) – (16). This completes the proof.

We shall now consider an alternative method for the existence of unique solution of the problem.

Before delving into the method, we shall substitute the solutions (26) - (28) into the equations (11) - (16).

Therefore, the equation (11) – (16) in dimensionless form become:

$$\frac{\partial u}{\partial t} = -\frac{\partial P}{\partial \eta} + \frac{1}{Re}\frac{\partial}{\partial \eta}\left(\mu \frac{\partial u}{\partial \eta}\right) \quad (29)$$

$$\left.\begin{array}{l} \frac{\partial \phi}{\partial t} - 2u\frac{\partial \phi}{\partial \eta} + v\frac{\partial \phi}{\partial \eta} = \frac{1}{Pe}\frac{\partial}{\partial \eta}\left(\lambda_s \frac{\partial \phi}{\partial \eta}\right) + \sigma(\phi - \theta) + \gamma(a - \phi) + \delta(b\theta + C_{f0})^{\alpha} \\ \left(c\phi + d\left(1 - \sum_{n=1}^{\infty}\frac{4}{(2n-1)\pi} e^{-\frac{1}{Pem}\left(\frac{(2n-1)\pi}{2}\right)^2 t} \sin\left(\frac{(2n-1)\pi\eta}{2}\right)\right)\right)^{\beta} e^{\frac{\theta}{1+\epsilon\theta}} \end{array}\right\} \quad (30)$$

$$\left.\begin{array}{l} \frac{\partial \theta}{\partial t} - 2u\frac{\partial \theta}{\partial \eta} = \frac{1}{Pe}\frac{\partial}{\partial \eta}\left(\lambda_s \frac{\partial \theta}{\partial \eta}\right) - \sigma_1(\phi - \theta) + \gamma_1(a_1 - \phi) + \delta_1(b_1\theta + C_{f0})^{\alpha} \\ \left(c_1\phi + d_1\left(1 - \sum_{n=1}^{\infty}\frac{4}{(2n-1)\pi} e^{-\frac{1}{Pem}\left(\frac{(2n-1)\pi}{2}\right)^2 t} \sin\left(\frac{(2n-1)\pi\eta}{2}\right)\right)\right)^{\beta} e^{\frac{\theta}{1+\epsilon\theta}} \end{array}\right\} \quad (31)$$

$$\left.\begin{array}{l} \frac{\partial X}{\partial t} - 2u\frac{\partial X}{\partial \eta} + v\frac{\partial X}{\partial \eta} = \frac{1}{Pem}\frac{\partial}{\partial \eta}\left(D \frac{\partial X}{\partial \eta}\right) + \delta_2(b_2\theta + C_{f0})^{\alpha} \\ \left(c_2\phi + d_2\left(1 - \sum_{n=1}^{\infty}\frac{4}{(2n-1)\pi} e^{-\frac{1}{Pem}\left(\frac{(2n-1)\pi}{2}\right)^2 t} \sin\left(\frac{(2n-1)\pi\eta}{2}\right)\right)\right)^{\beta} e^{\frac{\theta}{1+\epsilon\theta}} \end{array}\right\} \quad (32)$$

$$\left.\begin{array}{l} \frac{\partial Y}{\partial t} - 2u\frac{\partial Y}{\partial \eta} = \delta_3(b_3\theta + C_{f0})^{\alpha} \\ \left(c_3\phi + d_3\left(1 - \sum_{n=1}^{\infty}\frac{4}{(2n-1)\pi} e^{-\frac{1}{Pem}\left(\frac{(2n-1)\pi}{2}\right)^2 t} \sin\left(\frac{(2n-1)\pi\eta}{2}\right)\right)\right)^{\beta} e^{\frac{\theta}{1+\epsilon\theta}} \end{array}\right\} \quad (33)$$

$$v = -Da\,Re\,\eta(1-\eta) \quad (34)$$

With initial and boundary conditions

$$\left.\begin{array}{lll}
\phi(\eta,0)=0, & \phi(0,t)=\phi_1 & \left.\dfrac{\partial\phi}{\partial\eta}\right|_{\eta=1}=0 \\
\theta(\eta,0)=0, & \left.\dfrac{\partial\theta}{\partial\eta}\right|_{\eta=0}=0 & \left.\dfrac{\partial\theta}{\partial\eta}\right|_{\eta=1}=0, \\
X(\eta,0)=0, & X(0,t)=1, & \left.\dfrac{\partial X}{\partial\eta}\right|_{\eta=1}=0 \\
Y(\eta,0)=1, & Y(0,t)=1 & \\
\left.P\right|_{\eta=1}=\phi_1 & & \\
u(\eta,0)=1, & \left.u\right|_{\eta=0}=1, & \left.u\right|_{\eta=1}=0
\end{array}\right\} \tag{35}$$

Where

$Re=\dfrac{\rho L u_{\infty}}{\mu_0}=\dfrac{L u_{\infty}}{\upsilon}=$ Reynolds number, $pe=\dfrac{\rho c_g L u_{\infty}}{\kappa_{g0}}=$ Peclet number, $pem=\dfrac{L u_{\infty}}{D_1}=$ Mass transfer peclet number, $\sigma=\dfrac{\Gamma L}{\rho c_g u_{\infty}}=$ Heat exchange coefficient,

$\gamma=\dfrac{hL}{\rho c_g u_{\infty}}$, $\delta=\dfrac{L\Delta HA}{\rho c_g \in T_0 u_{\infty}}e^{-\frac{E}{RT_0}}=$ Frank-kameneskii parameter, $a=\dfrac{T_s-T_0}{\in T_0}$, $b=\dfrac{c_s \in T_0}{\Delta H}$

$pem=\dfrac{L u_{\infty}}{D_1}=$ Mass transfer peclet number, $c=\dfrac{c_g \in T_0}{\Delta H}$, $d=\dfrac{c_g T_0}{\Delta H}$

We let $\dfrac{\partial P}{\partial\eta}=\eta(1-\eta)$ and consider the following asymptotic expansion of temperatures θ and ϕ and concentrations X and Y and velocity u in $\in$

Let

$$\left.\begin{array}{l}
u=u_0+\in u_1+\ldots \\
\theta=\theta_0+\in\theta_1+\ldots \\
X=X_0+\in X_1+\ldots \\
Y=Y_0+\in Y_1+\ldots
\end{array}\right\} \tag{36}$$

and equate the powers of $\in$ in equation (29) – (35), we have

$\in^0$:

$$\frac{\partial u_0}{\partial t}=\frac{1}{\mathrm{Re}}\frac{\partial^2 u_0}{\partial \eta^2}-\eta(1-\eta) \tag{37}$$

$$\left.\begin{aligned}&\frac{\partial \phi_0}{\partial t}=\frac{1}{\mathrm{Pe}}\frac{\partial^2 \phi_0}{\partial \eta^2}+2u_0\frac{\partial \phi_0}{\partial \eta}+Da\,\mathrm{Re}\,\eta(1-\eta)\frac{\partial \phi_0}{\partial \eta}+\sigma(\phi_0-\theta_0)+\gamma(a-\phi_0)+\delta(b\theta_0+C_{f0})^{\alpha}\\&\left(c\phi_0+d\left(1-\sum_{n=1}^{\infty}\frac{4}{(2n-1)\pi}e^{-\frac{1}{\mathrm{Pem}}\left(\frac{(2n-1)\pi}{2}\right)^2 t}\sin\left(\frac{(2n-1)\pi\eta}{2}\right)\right)\right)^{\beta}e^{\theta_0}\end{aligned}\right\}(38)$$

$$\left.\begin{aligned}&\frac{\partial \theta_0}{\partial t}=\frac{1}{\mathrm{Pe}}\frac{\partial^2 \theta_0}{\partial \eta^2}+2u_0\frac{\partial \theta_0}{\partial \eta}+Da\,\mathrm{Re}\,\eta(1-\eta)\frac{\partial \theta_0}{\partial \eta}+\sigma_1(\phi_0-\theta_0)+\gamma_1(a-\phi_0)+\delta_1(b_1\theta_0+C_{f0})^{\alpha}\\&\left(c_1\phi_0+d_1\left(1-\sum_{n=1}^{\infty}\frac{4}{(2n-1)\pi}e^{-\frac{1}{\mathrm{Pem}}\left(\frac{(2n-1)\pi}{2}\right)^2 t}\sin\left(\frac{(2n-1)\pi\eta}{2}\right)\right)\right)^{\beta}e^{\theta_0}\end{aligned}\right\}(39)$$

$$\left.\begin{aligned}&\frac{\partial \mathrm{X}_0}{\partial t}=\frac{1}{Pem}\frac{\partial^2 \mathrm{X}_0}{\partial \eta^2}+2u_0\frac{\partial \mathrm{X}_0}{\partial \eta}+Da\,\mathrm{Re}\,\eta(1-\eta)\frac{\partial \mathrm{X}_0}{\partial \eta}+\delta_2(b_2\theta_0+C_{f0})^{\alpha}\\&\left(c_2\phi_0+d_2\left(1-\sum_{n=1}^{\infty}\frac{4}{(2n-1)\pi}e^{-\frac{1}{\mathrm{Pem}}\left(\frac{(2n-1)\pi}{2}\right)^2 t}\sin\left(\frac{(2n-1)\pi\eta}{2}\right)\right)\right)^{\beta}e^{\theta_0}\end{aligned}\right\}\quad(40)$$

$$\left.\begin{aligned}&\frac{\partial \mathrm{Y}_0}{\partial t}=2u_0\frac{\partial \mathrm{Y}_0}{\partial \eta}+\delta_3(b_3\theta_0+C_{f0})^{\alpha}\\&\left(c_3\phi_0+d_3\left(1-\sum_{n=1}^{\infty}\frac{4}{(2n-1)\pi}e^{-\frac{1}{\mathrm{Pem}}\left(\frac{(2n-1)\pi}{2}\right)^2 t}\sin\left(\frac{(2n-1)\pi\eta}{2}\right)\right)\right)^{\beta}e^{\theta_0}\end{aligned}\right\}\quad(41)$$

$$\left.\begin{array}{l}\phi_0(\eta,0)=0, \quad \phi_0(0,t)=q, \quad \left.\dfrac{\partial\phi_0}{\partial\eta}\right|_{\eta=1}=0\\ \theta_0(\eta,0)=0, \quad \left.\dfrac{\partial\theta_0}{\partial\eta}\right|_{\eta=0}=0 \quad \left.\dfrac{\partial\theta_0}{\partial\eta}\right|_{\eta=1}=0,\\ X_0(\eta,0)=0, \quad X_0(0,t)=1, \quad \left.\dfrac{\partial X_0}{\partial\eta}\right|_{\eta=1}=0\\ Y_0(\eta,0)=1, \quad Y_0(0,t)=1\\ u_0(\eta,0)=1, \ \left.u_0\right|_{\eta=0}=1, \ \left.u_0\right|_{\eta=1}=0\end{array}\right\} \tag{42}$$

ϵ^1:

$$\frac{\partial u_1}{\partial t}=\frac{1}{\mathrm{Re}}\left(\frac{\partial u_0}{\partial\eta}\right)^2+\frac{1}{\mathrm{Re}}\frac{\partial^2 u_0}{\partial\eta^2}+\frac{u_0}{\mathrm{Re}}\frac{\partial^2 u_0}{\partial\eta^2} \tag{43}$$

$$\left.\begin{array}{l}\dfrac{\partial\phi_1}{\partial t}=\dfrac{1}{\mathrm{Pe}}\left(\dfrac{\partial\phi_0}{\partial\eta}\right)^2+\dfrac{1}{\mathrm{Pe}}\dfrac{\partial^2\phi_0}{\partial\eta^2}+\dfrac{\phi_0}{\mathrm{Pe}}\dfrac{\partial^2\phi_0}{\partial\eta^2}+2u_1\dfrac{\partial\phi_1}{\partial\eta}+Da\,\mathrm{Re}\,\eta(1-\eta)\dfrac{\partial\phi_1}{\partial\eta}+\sigma(\phi_1-\theta_1)-\gamma\phi_1+\\ c\delta\beta\phi_1(b\theta_0+C_{f0})^{\alpha}\left(c\phi_0+d\left(1-\displaystyle\sum_{n=1}^{\infty}\frac{4}{(2n-1)\pi}e^{-\frac{1}{\mathrm{Pem}}\left(\frac{(2n-1)\pi}{2}\right)^2 t}\sin\left(\frac{(2n-1)\pi\eta}{2}\right)\right)\right)^{\beta-1}e^{\theta_0}+\\ b\delta\alpha\theta_1(b\theta_0+C_{f0})^{\alpha-1}\\ \left(c\phi_0+d\left(1-\displaystyle\sum_{n=1}^{\infty}\frac{4}{(2n-1)\pi}e^{-\frac{1}{pem}\left(\frac{(2n-1)\pi}{2}\right)^2 t}\sin\left(\frac{(2n-1)\pi\eta}{2}\right)\right)\right)^{\beta}e^{\theta_0}+\theta_1e^{\theta_0}-\theta_0^2e^{\theta_0}\end{array}\right\} \tag{44}$$

$$\left.\begin{array}{l}\dfrac{\partial\theta_1}{\partial t}=\dfrac{1}{\mathrm{Pe}}\left(\dfrac{\partial\theta_0}{\partial\eta}\right)^2+\dfrac{1}{\mathrm{Pe}}\dfrac{\partial^2\theta_0}{\partial\eta^2}+\dfrac{\theta_0}{\mathrm{Pe}}\dfrac{\partial^2\theta_0}{\partial\eta^2}+2u_1\dfrac{\partial\theta_1}{\partial\eta}+\sigma_1(\phi_1-\theta_1)-\gamma\theta_1+c_1\delta_1\beta\phi_1(b\theta_0+C_{f0})^{\alpha}\\ \left(c_1\phi_0+d_1\left(1-\displaystyle\sum_{n=1}^{\infty}\frac{4}{(2n-1)\pi}e^{-\frac{1}{\mathrm{Pem}}\left(\frac{(2n-1)\pi}{2}\right)^2 t}\sin\left(\frac{(2n-1)\pi\eta}{2}\right)\right)\right)^{\beta-1}e^{\theta_0}+b_1\delta_1\alpha\theta_1(b_1\theta_0+C_{f0})^{\alpha-1}\\ \left(c_1\phi_0+d_1\left(1-\displaystyle\sum_{n=1}^{\infty}\frac{4}{(2n-1)\pi}e^{-\frac{1}{pem}\left(\frac{(2n-1)\pi}{2}\right)^2 t}\sin\left(\frac{(2n-1)\pi\eta}{2}\right)\right)\right)^{\beta}e^{\theta_0}+\theta_1e^{\theta_0}-\theta_0^2e^{\theta_0}\end{array}\right\} \tag{45}$$

$$\left.\begin{array}{l}\frac{\partial X_1}{\partial t}=\frac{1}{Pem}\left(\frac{\partial X_0}{\partial\eta}\right)^2+\frac{1}{Pem}\frac{\partial^2 X_0}{\partial\eta^2}+\frac{X_0}{Pe}\frac{\partial^2 X_0}{\partial\eta^2}+2u_1\frac{\partial X_1}{\partial\eta}+Da\,Re\,\eta(1-\eta)\frac{\partial X_1}{\partial\eta}+\\ c_2\delta_2\beta\phi_1\left(b_2\theta_0+C_{f0}\right)^{\alpha}\\ \left(c_2\phi_0+d_2\left(1-\sum_{n=1}^{\infty}\frac{4}{(2n-1)\pi}e^{-\frac{1}{Pem}\left(\frac{(2n-1)\pi}{2}\right)^2 t}\sin\left(\frac{(2n-1)\pi\eta}{2}\right)\right)\right)^{\beta-1}e^{\theta_0}+b_2\delta_2\alpha\theta_1\left(b_2\theta_0+C_{f0}\right)^{\alpha-1}\\ \left(c_2\phi_0+d_2\left(1-\sum_{n=1}^{\infty}\frac{4}{(2n-1)\pi}e^{-\frac{1}{pem}\left(\frac{(2n-1)\pi}{2}\right)^2 t}\sin\left(\frac{(2n-1)\pi\eta}{2}\right)\right)\right)^{\beta}e^{\theta_0}+\theta_1 e^{\theta_0}-\theta_0^2 e^{\theta_0}\end{array}\right\}\quad(46)$$

$$\left.\begin{array}{l}\frac{\partial Y_1}{\partial t}=2u_1\frac{\partial Y_1}{\partial\eta}+c_3\delta_3\beta\phi_1\left(b_3\theta_0+C_{f0}\right)^{\alpha}\\ \left(c_3\phi_0+d_3\left(1-\sum_{n=1}^{\infty}\frac{4}{(2n-1)\pi}e^{-\frac{1}{Pem}\left(\frac{(2n-1)\pi}{2}\right)^2 t}\sin\left(\frac{(2n-1)\pi\eta}{2}\right)\right)\right)^{\beta-1}e^{\theta_0}+b_3\delta_3\alpha\theta_1\left(b_3\theta_0+C_{f0}\right)^{\alpha-1}\\ \left(c_3\phi_0+d_3\left(1-\sum_{n=1}^{\infty}\frac{4}{(2n-1)\pi}e^{-\frac{1}{pem}\left(\frac{(2n-1)\pi}{2}\right)^2 t}\sin\left(\frac{(2n-1)\pi\eta}{2}\right)\right)\right)^{\beta}e^{\theta_0}+\theta_1 e^{\theta_0}-\theta_0^2 e^{\theta_0}\end{array}\right\}\quad(47)$$

$$\left.\begin{array}{lll}\phi_1(\eta,0)=0, & \phi_1(0,t)=0, & \left.\frac{\partial\phi_1}{\partial\eta}\right|_{\eta=1}=0\\ \theta_1(\eta,0)=0, & \left.\frac{\partial\theta_1}{\partial\eta}\right|_{\eta=0}=0 & \left.\frac{\partial\theta_1}{\partial\eta}\right|_{\eta=1}=0,\\ X_1(\eta,0)=0, & X_1(0,t)=0, & \left.\frac{\partial X_1}{\partial\eta}\right|_{\eta=1}=0\\ Y_1(\eta,0)=0, & Y_1(0,t)=0 & \\ u_1(\eta,0)=0, & \left.u_1\right|_{\eta=0}=0, & \left.u_1\right|_{\eta=1}=0\end{array}\right\}\quad(48)$$

This question of existence and uniqueness of solutions to these equations has been addressed by Ayeni (1978) who consider a similar set of equations and showed among other results that existence and uniqueness are somewhat well known. In his work, he studied the following system of parabolic equations.

$$\left.\begin{aligned}\frac{\partial\phi}{\partial t} &= \Delta\phi + f(x,t,\phi,u,v), & x\in R^n, t>0\\ \frac{\partial u}{\partial t} &= \Delta u + g(x,t,\phi,u,v), & x\in R^n, t>0\\ \frac{\partial v}{\partial t} &= \Delta v + h(x,t,\phi,u,v), & x\in R^n, t>0\end{aligned}\right\} \quad (49)$$

$$\phi(x,0)=f_0(x)$$

$$u(x,0)=g_0(x)$$

$$v(x,0)=h_0(x)$$

$$x=(x_1,x_2,...,x_n)$$

(S.1): $f_0(x)$, $g_0(x)$ and $h_0(x)$ are bounded for $x\in R^n$.Each has at most a countable number of discontinuities.

(S.2): f, g, h satisfies the uniform Lipschitz condition

$$|\varphi(x,t,\phi_1,u_1,v_1)-\varphi(x,t,\phi_2,u_2v_2)|\le M(|\phi_1-\phi_2|+|u_1-u_2|+|v_1-v_2|), \qquad (x,t)\in G$$

Where $G=\{(x,t): x\in R^n, 0<t<\tau\}$.

Our proof of existence of unique solution of the system of parabolic equations (37) – (42) will be analogous to his proof.

Theorem 3.2: There exists a unique solution $u_0(\eta,t),\phi_0(\eta,t),\theta_0(\eta,t),X_0(\eta,t)$ and $Y_0(\eta,t)$ of equations (37) – (41) which satisfy (42).

Lemma 3.1 (Ayeni(1978)) :

Let (f_0,g_0,h_0) and (f,g,h) satisfy **(S.1)** and **(S.2)** respectively. Then there exists a solution of problem (49),

Proof of lemma 3.1, see Ayeni (1978)

Proof of theorem (3.1):

We rewrite equations (37) – (42) as

$$\frac{\partial u_0}{\partial t}=\frac{1}{\mathrm{Re}}\frac{\partial^2 u_0}{\partial \eta^2}+f(\eta,t,u_0,\phi_0,\theta_0,\mathrm{X}_0,\mathrm{Y}_0),\quad \eta\in R^n, t>0, \tag{50}$$

$$\frac{\partial \phi_0}{\partial t}=\frac{1}{\mathrm{Pe}}\frac{\partial^2 \phi_0}{\partial \eta^2}+g(\eta,t,u_0,\phi_0,\theta_0,\mathrm{X}_0,\mathrm{Y}_0)\quad \eta\in R^n, t>0, \tag{51}$$

$$\frac{\partial \theta_0}{\partial t}=\frac{1}{\mathrm{Pe}}\frac{\partial^2 \theta_0}{\partial \eta^2}+h(\eta,t,u_0,\phi_0,\theta_0,\mathrm{X}_0,\mathrm{Y}_0)\quad \eta\in R^n, t>0, \tag{52}$$

$$\frac{\partial \mathrm{X}_0}{\partial t}=\frac{1}{Pem}\frac{\partial^2 \mathrm{X}_0}{\partial \eta^2}+\mathrm{Z}(\eta,t,u_0,\phi_0,\theta_0,\mathrm{X}_0,\mathrm{Y}_0)\quad \eta\in R^n, t>0, \tag{53}$$

Where,

$$f(\eta,t,u_0,\phi_0,\theta_0,\mathrm{X}_0,\mathrm{Y}_0)=-\eta(1-\eta) \tag{54}$$

$$\left.\begin{aligned}&g(\eta,t,u_0,\phi_0,\theta_0,\mathrm{X}_0,\mathrm{Y}_0)=\sigma(\phi_0-\theta_0)+\gamma(a-\phi_0)+2u_0\frac{\partial\phi_0}{\partial\eta}+Da\,\mathrm{Re}\,\eta(1-\eta)\frac{\partial\phi_0}{\partial\eta}+\\&\delta(b\phi_0+C_{f0})^{\alpha}\left(c\phi_0+d\left(1-\sum_{n=1}^{\infty}\frac{4}{(2n-1)\pi}e^{-\frac{1}{pem}\left(\frac{(2n-1)}{2}\pi\right)^2 t}\sin\left(\frac{(2n-1)\pi\eta}{2}\right)\right)\right)^{\beta}e^{\theta_0}\end{aligned}\right\} \tag{55}$$

$$\left.\begin{aligned}&h(\eta,t,u_0,\phi_0,\theta_0,\mathrm{X}_0,\mathrm{Y}_0)=\sigma_1(\phi_0-\theta_0)+\gamma_1(a-\theta_0)+2u_0\frac{\partial\theta_0}{\partial\eta}+\delta_1(b_1\theta_0+C_{f0})^{\alpha}\\&\left(c_1\phi_0+d_1\left(1-\sum_{n=1}^{\infty}\frac{4}{(2n-1)\pi}e^{-\frac{1}{pem}\left(\frac{(2n-1)}{2}\pi\right)^2 t}\sin\left(\frac{(2n-1)\pi\eta}{2}\right)\right)\right)^{\beta}e^{\theta_0}\end{aligned}\right\} \tag{56}$$

$$\left.\begin{aligned}&\mathrm{Z}(\eta,t,u_0,\phi_0,\theta_0,\mathrm{X}_0,\mathrm{Y}_0)=2u_0\frac{\partial\mathrm{X}_0}{\partial\eta}+Da\,\mathrm{Re}\,\eta(1-\eta)\frac{\partial\mathrm{X}_0}{\partial\eta}+\delta_2(b_2\phi_0+C_{f0})^{\alpha}\\&\left(c_2\phi_0+d_2\left(1-\sum_{n=1}^{\infty}\frac{4}{(2n-1)\pi}e^{-\frac{1}{pem}\left(\frac{(2n-1)}{2}\pi\right)^2 t}\sin\left(\frac{(2n-1)\pi\eta}{2}\right)\right)\right)^{\beta}e^{\theta_0}\end{aligned}\right\} \tag{57}$$

Ignoring the second term at the right hand side, the fundamental solutions of equations (50) – (53) are (see Toki and Tokis (2007)):

$$F(\eta,t)=\frac{\mathrm{Re}^{\frac{1}{2}}\eta}{2t\sqrt{\pi t}}\exp\left(-\frac{\mathrm{Re}\,\eta^2}{4t}\right) \tag{58}$$

$$G(\eta,t)=\frac{\mathrm{Pe}^{\frac{1}{2}}\eta}{2t\sqrt{\pi t}}\exp\left(-\frac{\mathrm{Pe}\eta^2}{4t}\right) \tag{59}$$

$$H(\eta,t)=\frac{\mathrm{Pe}^{\frac{1}{2}}\eta}{2t\sqrt{\pi t}}\exp\left(-\frac{\mathrm{Pe}\eta^2}{4t}\right) \tag{60}$$

$$J(\eta,t)=\frac{\mathrm{Pem}^{\frac{1}{2}}\eta}{2t\sqrt{\pi t}}\exp\left(-\frac{\mathrm{Pem}\,\eta^2}{4t}\right) \tag{61}$$

Clearly

$$f(\eta,t,u_0,\phi_0,\theta_0,X_0,Y_0)=-\eta(1-\eta),$$

$$\left.\begin{array}{l} g(\eta,t,u_0,\phi_0,\theta_0,X_0,Y_0)=\sigma(\phi_0-\theta_0)+\gamma(a-\phi_0)+2u_0\dfrac{\partial\phi_0}{\partial\eta}+Da\,\mathrm{Re}\eta(1-\eta)\dfrac{\partial\phi_0}{\partial\eta}+\delta(b\phi_0+C_{f0})^{\alpha} \\ \left(c\phi_0+d\left(1-\displaystyle\sum_{n=1}^{\infty}\frac{4}{(2n-1)\pi}e^{-\frac{1}{pem}\left(\frac{(2n-1)}{2}\pi\right)^2 t}\sin\left(\frac{(2n-1)\pi\eta}{2}\right)\right)\right)^{\beta}e^{\theta_0} \end{array}\right\}$$

$$\left.\begin{array}{l} h(\eta,t,u_0,\phi_0,\theta_0,X_0,Y_0)=\sigma_1(\phi_0-\theta_0)+\gamma_1(a-\theta_0)2u_0\dfrac{\partial\theta_0}{\partial\eta}+\delta_1(b_1\theta_0+C_{f0})^{\alpha} \\ \left(c_1\phi_0+d_1\left(1-\displaystyle\sum_{n=1}^{\infty}\frac{4}{(2n-1)\pi}e^{-\frac{1}{pem}\left(\frac{(2n-1)}{2}\pi\right)^2 t}\sin\left(\frac{(2n-1)\pi\eta}{2}\right)\right)\right)^{\beta}e^{\theta_0} \end{array}\right\},$$

$$\left.\begin{array}{l} Z(\eta,t,u_0,\phi_0,\theta_0,X_0,Y_0)=2u_0\dfrac{\partial X_0}{\partial\eta}+Da\,\mathrm{Re}\eta(1-\eta)\dfrac{\partial X_0}{\partial\eta}+\delta_2(b_2\phi_0+C_{f0})^{\alpha} \\ \left(c_2\phi_0+d_2\left(1-\displaystyle\sum_{n=1}^{\infty}\frac{4}{(2n-1)\pi}e^{-\frac{1}{pem}\left(\frac{(2n-1)}{2}\pi\right)^2 t}\sin\left(\frac{(2n-1)\pi\eta}{2}\right)\right)\right)^{\beta}e^{\theta_0} \end{array}\right\}$$

are Lipschitz continuous .Hence by theorem 3.2, the results follow. This completes the proof.

3.3 Properties of Solution

Theorem 3.2: Let $\mathrm{Re}=\mathrm{Pe}=Da=\sigma=\sigma_1=\gamma=\gamma_1=Pem=1$ and $\alpha=\beta=0$ in equation (37) – (41). Then $\frac{\partial u_0}{\partial t}\geq 0$, $\frac{\partial\phi_0}{\partial t}\geq 0$, $\frac{\partial\theta_0}{\partial t}\geq 0$, $\frac{\partial X_0}{\partial t}\geq 0$, $\frac{\partial Y_0}{\partial t}\geq 0$

In the proof, we shall make use of the following lemma of kolodner and Pederson (1966). **Lemma (Kolodner and Pederson (1966))**: Let $u(x,t)=0\left(e^{a|x|^2}\right)$ be a solution on $R^n \times [0,t)$ of the differential inequality $\frac{\partial u}{\partial t}-\Delta u+k(x,t)u \geq 0$ where k is bounded from below if $u(x,t)\geq 0$, then $u(x,t)\geq 0$ for all $(x,t)\in R^n \times [0,t_0)$

Proof of Theorem 3.2: Given,

$$\frac{\partial u_0}{\partial t}-\frac{\partial^2 u_0}{\partial \eta^2}-\eta(1-\eta)=0 \tag{62}$$

$$\frac{\partial \phi_0}{\partial t}-\frac{\partial^2 \phi_0}{\partial \eta^2}-2u_0\frac{\partial \phi_0}{\partial \eta}-\eta(1-\eta)\frac{\partial \phi_0}{\partial \eta}+\theta-\delta e^{\theta_0}=0 \tag{63}$$

$$\frac{\partial \theta_0}{\partial t}-\frac{\partial^2 \theta_0}{\partial \eta^2}-2u_0\frac{\partial \theta_0}{\partial \eta}-\eta(1-\eta)\frac{\partial \theta_0}{\partial \eta}+\theta_0-\delta_1 e^{\theta_0}=0 \tag{64}$$

$$\frac{\partial X_0}{\partial t}-\frac{\partial^2 X_0}{\partial \eta^2}-2u_0\frac{\partial X_0}{\partial \eta}-\eta(1-\eta)\frac{\partial X_0}{\partial \eta}-\delta_2 e^{\theta_0}=0 \tag{65}$$

$$\frac{\partial Y_0}{\partial t}-2u_0\frac{\partial X_0}{\partial \eta}-\delta_3 e^{\theta_0}=0 \tag{66}$$

Differentiating with respect to t, we have

$$\frac{\partial}{\partial t}\left(\frac{\partial u_0}{\partial t}\right)-\frac{\partial^2}{\partial \eta^2}\left(\frac{\partial u_0}{\partial t}\right)=0 \tag{67}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \phi_0}{\partial t}\right)-\frac{\partial^2}{\partial \eta^2}\left(\frac{\partial \phi_0}{\partial t}\right)=(2u_0+\eta(1-\eta))\frac{\partial}{\partial \eta}\left(\frac{\partial \phi_0}{\partial t}\right)+2\frac{\partial \phi_0}{\partial \eta}\frac{\partial u_0}{\partial t}+\left(\delta e^{\theta_0}-1\right)\frac{\partial \theta_0}{\partial t} \tag{68}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \theta_0}{\partial t}\right)-\frac{\partial^2}{\partial \eta^2}\left(\frac{\partial \theta_0}{\partial t}\right)+\left(1-\delta_1 e^{\theta_0}\right)\frac{\partial \theta_0}{\partial t}=(2u_0+\eta(1-\eta))\frac{\partial \theta_0}{\partial \eta}\frac{\partial u_0}{\partial t}+2\frac{\partial \theta_0}{\partial \eta}\frac{\partial u_0}{\partial t} \tag{69}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial X_0}{\partial t}\right)-\frac{\partial^2}{\partial \eta^2}\left(\frac{\partial X_0}{\partial t}\right)=(2u_0+\eta(1-\eta))\frac{\partial}{\partial \eta}\left(\frac{\partial X_0}{\partial t}\right)+2\frac{\partial X_0}{\partial \eta}\frac{\partial u_0}{\partial t}+\delta_2 e^{\theta_0}\frac{\partial \theta_0}{\partial t} \tag{70}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial Y_0}{\partial t}\right)=2u_0\frac{\partial}{\partial \eta}\left(\frac{\partial Y_0}{\partial t}\right)+2\frac{\partial Y_0}{\partial \eta}+\delta_3 e^{\theta_0}\frac{\partial \theta_0}{\partial t}=0 \tag{71}$$

Let

$$m=\frac{\partial u_0}{\partial t},\ n=\frac{\partial \phi_0}{\partial t},\ r=\frac{\partial \theta_0}{\partial t},\ w=\frac{\partial X_0}{\partial t},\ Z=\frac{\partial Y_0}{\partial t}$$

Then

$$\frac{\partial m}{\partial t}-\frac{\partial^2 m}{\partial \eta^2}=0$$

$$\frac{\partial n}{\partial t}-\frac{\partial^2 n}{\partial \eta^2}\geq 0 \text{ Since } \left(2u_0+\eta(1-\eta)\right)\frac{\partial n}{\partial \eta}+2\frac{\partial \phi_0}{\partial \eta}m+\left(\delta e^{\theta_0}-1\right)r\geq 0$$

$$\frac{\partial r}{\partial t}-\frac{\partial^2 r}{\partial \eta^2}+\left(1-\delta_1 e^{\theta_0}\right)r\geq 0 \text{ Since } \left(2u_0+\eta(1-\eta)\right)\frac{\partial r}{\partial \eta}+2\frac{\partial \theta_0}{\partial \eta}n\geq 0$$

$$\frac{\partial w}{\partial t}-\frac{\partial^2 w}{\partial \eta^2}\geq 0 \quad \text{Since } \left(2u_0+\eta(1-\eta)\right)\frac{\partial w}{\partial \eta}+2\frac{\partial X_0}{\partial \eta}n+\delta_2 e^{\theta_0}r\geq 0$$

$$\frac{\partial Z}{\partial t}\geq 0 \text{ Since } \quad 2u_0\frac{\partial Z}{\partial \eta}+2\frac{\partial Y_0}{\partial \eta}m+\delta_3 e^{\theta_0}r\geq 0$$

These can be written as

$$\frac{\partial m}{\partial t}-\frac{\partial^2 m}{\partial \eta^2}+k(\eta,t)m\geq 0$$

$$\frac{\partial n}{\partial t}-\frac{\partial^2 n}{\partial \eta^2}+k_1(\eta,t)n\geq 0$$

$$\frac{\partial r}{\partial t}-\frac{\partial^2 r}{\partial \eta^2}+k_2(\eta,t)r\geq 0$$

$$\frac{\partial w}{\partial t}-\frac{\partial^2 w}{\partial \eta^2}+k_3(\eta,t)w\geq 0$$

$$\frac{\partial Z}{\partial t}-0\frac{\partial^2 Z}{\partial \eta^2}+k_4(\eta,t)Z\geq 0$$

Where

$$k(\eta,t)=0,\ k_1(\eta,t)=0,\ \ k_2(\eta,t)=\left(1-\delta e^{\theta_0}\right),\ \ k_3(\eta,t)=0,\ \ k_4(\eta,t)=0$$

Clearly, k_2 is bounded from below and k, k_1, k_3 and k_4 are bounded everywhere. Hence, by Kolodner and Pederson's lemma, $r(\eta,t)\geq 0$, $m(\eta,t)\geq 0$, $n(\eta,t)\geq 0$, $w(\eta,t)\geq 0$ and $Z(\eta,t)\geq 0$, that is $\frac{\partial u_1}{\partial t}\geq 0, \frac{\partial \phi_1}{\partial t}\geq 0, \frac{\partial \theta_1}{\partial t}\geq 0$, $\frac{\partial X_1}{\partial t}\geq 0$, and $\frac{\partial Y_1}{\partial t}\geq 0$. This completes the proof.

4.0 Conclusion

To examine the properties of solution of the in-situ combustion of oil shale as enhanced oil recovery technique in porous medium, we used an approach by Ayeni (1978) and Kolodner and Pederson (1966).Our result revealed that velocity u ,mass X and Y, temperature θ and ϕ are increasing function of time.

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