Application of Chebyschev Polynomial to Economization of Power Series

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Abstract
The truncated power series of the continuous function $f(x)$ will not generally be a good approximate due to the great error it possesses, however the error may be reduced by making n so large so as to include many terms. But the cost of evaluating large number of terms is high; it is often possible to reduce considerably the necessary number of terms without increasing the error significantly, this is the feature that allows economization. In this paper a better power series representations of functions by Chebyschev polynomials to economize Maclaurin series is sought for. The procedure is tested on two functions.

Key words: Power series, Maclaurin series, Chebyschev polynomial, Continuous function

1. Introduction

The idea of deriving an approximate representation of a function (an approximant), $f(x)$, which is specified by $n+1$ parameters that have been derived from the first $n+1$ coefficients of the power series is a well known phenomenon in approximation theory. In the context of this paper the representation is the original power series truncated at the n^{th} order, $f_n(x)$.

Economization is the process of finding an alternative representation for the function containing $n+1$ parameters that possesses the same functional form as the initial approximation but also incorporates information present in the higher orders of the original power series to minimize maximum error of the approximant over a range value of x, this implies that one has an economy of representation. This process is based on the fundamental property of the Chebyschev polynomial.

Power series is an infinite series which does not terminate at a particular power of x. The power series takes the simpler form

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_o + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \qquad 1.1$$

As a result of this lack of termination there is need to specify a domain which includes only those real numbers, x, for which $f(x)$ is finite. Power series are useful in chemistry, physics, engineering and mathematics for a number of reasons.

i. They provide means to formulate alternative representation of transcendental functions such as the exponential, logarithmic and trigonometric functions.

ii. As a direct result of the above reason, they also allow us to investigate how an equation describing some physical property behaves for either small or large values for one of the independent variables.

For example, the Einstein model for the molar capacity at constant volume, C_v, of a solid yields the formula:

$$C_v = 3R\left(\frac{hv}{KT}\right)^2 \left\{\frac{e^{hv/2KT}}{e^{hv/2KT-1}}\right\}^2 \qquad 1.2$$

Substituting the exponential terms by their two-term series approximation it can be shown that at high values of T, C_v tends to the limit 3R

$$e^{hv/2KT} = 1 + \frac{hv}{2KT} + \cdots \qquad 1.3$$

$$\Rightarrow C_v = 3R\left(\frac{hv}{KT}\right)^2 \left\{\frac{1+\frac{hv}{2KT}}{1+\frac{hv}{2KT}-1}\right\} = 3R\left(\frac{hv}{KT}\right)^2 \left\{\frac{1+\frac{hv}{2KT}}{\frac{hv}{2KT}}\right\} \qquad 1.4$$

iii. Power series are used when the formula of association between one property and another is unknown.

2. Approximation by Chebyshev Polynomials, $e_n(x)$

The application of Chebyschev methods to the economization of power series is an occasionally useful technique. The procedure is to convert the polynomial to a linear combination of Chebyshev polynomial; it may be possible to drop some of the last terms without permitting the error to exceed a prescribed limit. The Chebyshev polynomial are useful in numerical work for the interval $-1 \leq x \leq 1$ because

1. $|T_n(x)| \leq 1$ within $-1 \leq x \leq 1$
2. The maxima and minima are of comparable value.
3. The maxima and minima are spread reasonably uniformly over the interval $-1 \leq x \leq 1$
4. All Chebyshev polynomials satisfy a three recurrence relation.

These properties of Chebyshev polynomial produce an approximating polynomial which minimizes error in its application. This is different from the least square approximation where the sum of the squares of errors is minimized whereas the maximum error itself can be quite large. In Chebyshev approximation, the average error can be large but the maximum error is minimized. A truncated series to approximate the functions on the interval [0,1] with a precision of 0.001 is taken into consideration.

The economized polynomial is given by

$$e_n(x) = f(x_{n+1}) - \frac{a_{n+1}}{2^n} T_{n+1}(x) \qquad 2.1$$

where $e_n(x)$ is the economized power series and $f(x_{n+1})$ is represented by Maclaurin series truncated expansion.

The Chebyschev polynomial, $T_{n+1}(x)$ is given by

$$T_n(x) = cos(ncos' x), -1 \leq x \leq 1, \qquad n \geq 0 \qquad 2.2$$

To express $T_{n+1}(x)$ as a polynomial function, recall that

$$cos(n+1)\theta = cos(n\theta + \theta) = \cos n\theta cos\theta - sinn\theta sin\theta \qquad 2.3a$$

$$cos(n-1)\theta = cos(n\theta - \theta) = \cos n\theta cos\theta + sinn\theta sin\theta \qquad 2.3b$$

Adding 2.3a and 2.3b results into

$$cos(n+1)\theta = 2\cos\theta cosn\theta - cos(n-1)\theta \qquad 2.4$$

Therefore,

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \ n \geq 1 \qquad 2.5$$

By definition,

$$T_n(x) = cos(ncos' x) \qquad 2.6$$

Now putting $n = 0$ and $n = 1$

$$\Rightarrow T_0(x) = cos(0) = 1 \qquad 2.7$$

And

$$T_1(x) = cos(cos' x) = x \qquad 2.8$$

Using equation(2.5), (2.7) and (2.8) $T_n(x)$, $n \geq 1$ can be obtained.

3. Numerical Application

This method is tested on the approximant generated from power series of two known functions of different orders.

Example1. Obtain the $4^{th}, 5^{th}$ and 6^{th} order economized series of

$$e^{1/2x}, \quad x\epsilon[-1,1] \qquad 3.1$$

For $n = 4$, $f(x_{n+1}) = f(x_5)$ so $e^{1/2x}$ is expanded up to x^5

$$f_5(x) = 1 + \frac{1}{2}x + \frac{1}{8}x^2 + \frac{1}{48}x^3 + \frac{1}{384}x^4 + \frac{1}{3840}x^5 \qquad 3.2$$

From equation (2.1) $a_{n+1} = \frac{1}{3840}$ and $\frac{1}{n!} = \frac{1}{24}$ 3.3

By Chebyschev polynomial

$$T_5(x) = 16x^5 - 20x^3 + 5x \qquad 3.4$$

Substituting equation (3.2) and (3.4) in (2.1) we have

$e_5(x) = 1 + \frac{1}{2}x + \frac{1}{8}x^2 + \frac{1}{48}x^3 + \frac{1}{384}x^4 + \frac{1}{3840}x^5 - \left(\frac{1}{3840}\right)\left(\frac{1}{24}\right)(16x^5 - 20x^3 + 5x)$

3.5

Likewise, for $n = 5$, we have $e^{1/2x}$ expanded up to x^6 3.6

$f_6(x) = 1 + \frac{1}{2}x + \frac{1}{8}x^2 + \frac{1}{48}x^3 + \frac{1}{384}x^4 + \frac{1}{3840}x^5 + \frac{1}{46080}x^6$ 3.7

$\Rightarrow a_{n+1} = \frac{1}{46080}$ and $\frac{1}{n!} = \frac{1}{120}$ 3.8

$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$ 3.9

$e_6(x) = 1 + \frac{1}{2}x + \frac{1}{8}x^2 + \frac{1}{48}x^3 + \frac{1}{384}x^4 + \frac{1}{3840}x^5 + \frac{1}{46080}x^6 - \left(\frac{1}{46080}\right)\left(\frac{1}{120}\right)(32x^6 - 48x^4 + 18x^2 - 1)$

3.10

Similarly, for $n = 6$, we have $e^{1/2x}$ expanded up to x^7

$f_7(x) = 1 + \frac{1}{2}x + \frac{1}{8}x^2 + \frac{1}{48}x^3 + \frac{1}{384}x^4 + \frac{1}{3840}x^5 + \frac{1}{46080}x^6 + \frac{1}{645120}x^7$ 3.11

$\Rightarrow \quad a_{n+1} = \frac{1}{645120}$ and $\frac{1}{n!} = \frac{1}{720}$

$T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x$

$e_7(x) =$

$1 + \frac{1}{2}x + \frac{1}{8}x^2 + \frac{1}{48}x^3 + \frac{1}{384}x^4 + \frac{1}{3840}x^5 + \frac{1}{46080}x^6 + \frac{1}{645120}x^7 -$

$\left(\frac{1}{645120}\right)\left(\frac{1}{720}\right)(64x^7 - 112x^5 + 56x^3 - 7x)$ 3.12

Table 1: Comparison of economized series with Maclaurin series, $e^{\frac{1}{2}x}$

x	$e^{\frac{1}{2}x}$	Maclaurin of order			Economized of order		
		4	5	6	4	5	6
0.0	1.000000000	1.000000000	1.000000000	1.000000000	1.0000000 00	1.000000000	1.000000181
0.2	1.105170918	1.105170918	1.105170918	1.105170918	1.105161747	1.105170920	1.105170983
0.4	1.221402758	1.221402758	1.221402759	1.221402756	1.221393077	1.221402759	1.221402614
0.6	1.349858808	1.349858808	1.349858805	1.349858762	1.349858573	1.349858804	1.349858626
0.8	1.491824698	1.491824698	1.491824681	1.491824356	1.491829486	1.491824681	1.491824492
1.0	1.648721271	1.648721271	1.648721168	1.648719618	1.648687066	1.648721166	1.648719437
Maximum error		0.000000000	0.000000103	0.000000547	0.000009681	0.000000105	0.000001834

Example 2. Obtain the $3^{rd}, 5^{th} and\ 7^{th}$ order economized series of $cosh(x)$. $-1 \leq x \leq 1$

For $n = 3$, $f(x_{n+1}) = f_4(x)$ expanding $cosh(x)$ up to x^4 we have

$f_5(x) = cosh(x) = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4$ 3.13

By (2.1) $a_{n+1} = \frac{1}{24}$ and $\frac{1}{n!} = \frac{1}{6}$ 3.14

By Chebyschev polynomial

$T_4(x) = 8x^4 - 8x^2 + 1$ 3.15

Substituting equation (3.10) and (3.12) in (2.1) we have

$e_4(x) = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \left(\frac{1}{24}\right)\left(\frac{1}{6}\right)(8x^4 - 8x^2 + 1)$ 3.16

Also, for $n = 5$, $f(x_{n+1}) = f_6(x)$, expanding $cosh(x)$ up to x^6 we have

$$f_5(x) = cosh(x) = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^6 \qquad 3.17$$

$a_{n+1} = \frac{1}{720}$ and $\frac{1}{n!} = \frac{1}{120}$

$$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1 \qquad 3.18$$

And that

$$e_6(x) = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^6 - \left(\frac{1}{720}\right)\left(\frac{1}{120}\right)(32x^6 - 48x^4 + 18x^2 - 1) \qquad 3.19$$

Similarly, for $n = 7$, $f(x_{n+1}) = f_8(x)$ expanding $cosh(x)$ up to x^8 we have

$$f_8(x) = cosh(x) = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^6 + \frac{1}{8!}x^8$$

$a_{n+1} = \frac{1}{40320}$ and $\frac{1}{n!} = \frac{1}{5040}$ 3.20

$$T_8(x) = 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1$$

$$e_8(x) = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^6 + \frac{1}{8!}x^8 -$$
$$\left(\frac{1}{40320}\right)\left(\frac{1}{5040}\right)(128x^8 - 256x^6 + 160x^4 - 32x^2 + 1) \qquad 3.21$$

Table 2: Comparison of economized series with Maclaurin series, $cosh(x)$

x	$cosh(x)$	Maclaurin of order			Economized of order		
		3	5	7	3	5	7
0.0	1.000000000	1.000000000	1.000000000	1.000000000	0.993055556	1.000011574	0.999999995
0.2	1.020066756	1.020066667	1.020066756	1.020066756	1.015255556	1.020070862	1.020066755
0.4	1.081072372	1.081066667	1.081072356	1.081072372	1.081588888	1.081063302	1.081072377
0.6	1.185465218	1.185400000	1.185464800	1.185465217	1.191255556	1.185456094	1.185465217
0.8	1.337434946	1.337066667	1.337430756	1.337434917	1.342922222	1.337439462	1.337434914
1.0	1.543080635	1.541666667	1.543055556	1.543080358	1.534722222	1.543043982	1.543080353
Maximum error		0.001413968	0.000025079	0.000000277	0.008358413	0.000036653	0.000000282

4. Result and Discussion

Table 1 compares the errors of the power series of $e^{\frac{1}{2}x}$

- Observe that the economized polynomial of order five is more accurate than the sixth order of the Maclaurin series.
- Also near $x = 0$ the economized series of order 6 is less accurate.

Table 2 compares the errors of the power series of $cosh(x)$

- The economized polynomial of order five is more accurate than the third order of the Maclaurin series, likewise the economized polynomial of order seven is more accurate than the fifth order of the Maclaurin series.
- Near $x = 0$ the economized series are less accurate.

5. Conclusion

Approximation of power series by Chebyschev polynomial is studied in this paper, it is found out that it is possible to reduce the degree of polynomial without permitting the error to exceed the prescribed limit. Because of the relative ease with which they can be developed, such economized series are frequently used for approximation to functions. These examples gave a modification of the Maclaurin series whose errors are slightly greater than those of the original Maclaurin series.

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