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A Block Hybrid Scheme for the Solution of First Order Ordinary Differential Equations

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Abstract

A new hybrid linear multistep method (LMM) is considered for the solution of first order initial value problems (IVPs). The new hybrid method is an extension of LMM by the inclusion of extra intermediate off-step points in between the usual grid points in the numerical schemes. A detailed analysis of the method such as the local truncation error and order, consistency and zero stability are investigated and presented. The method as compared with other recently derived methods provide approximation of high accuracy to solution of IVPs in ordinary differential equations.

Keywords: Hybrid linear multistep method, Off-step points, Local truncation error, Consistency, First order initial value problems, Zero-stable.

A BLOCK HYBRID SCHEME FOR THE SOLUTION OF FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract
A new hybrid linear multistep method (LMM) is considered for the solution of first order initial value problems (IVPs). The new hybrid method is an extension of LMM by the inclusion of extra intermediate off-step points in between the usual grid points in the numerical schemes. A detailed analysis of the method such as the local truncation error and order, consistency and zero stability are investigated and presented. The method as compared with other recently derived methods provide approximation of high accuracy to solution of IVPs in ordinary differential equations.

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1. Introduction

Differential equations appear frequently in mathematical models that attempt to describe real life situations. Many natural laws and hypotheses can be translated via mathematical language into equations involving derivatives. These derivatives appear in physics as velocities and accelerations, in geometry as slopes, in biology as rates of growth of populations, in psychology as rates of learning, in chemistry as reaction rates, in economics as rates of change of the cost of living, and in finance as rates of growth of investments. First order differential equations are very useful in many applications [1].

Various approaches like interpolation, numerical integration and Taylor series expansion can be used for the derivation of linear multistep methods (LMMs). LMMs are some of the methods for solving ordinary differential equations (ODEs) and are generally defined as

$$\sum_{j=0}^{k} \varphi_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \tag{1}$$

where φ_j and β_j are uniquely determined and $\varphi_0 + \beta_0 \neq 0, \varphi_k = 1$ [2].

Discrete schemes are generated from equation (1) and applied to solve first order ordinary differential equations.

In literature, [3] constructed continuous schemes which were evaluated at both step and off-step points to obtain hybrid formulae, the methods are effective in treating stiff and highly oscillatory systems of initial value problems (IVPs) in ODEs. Others like [4], [5], [6], [7], [8] and [9] derived multistep numerical methods for solving IVPs of various orders.

Some hybrid Adams type block linear multistep methods are derived in [10] by using power series expansion, the idea of multistep collocation method (MCM) was adopted in the derivation of the schemes to obtain the continuous form which were evaluated at some off grid points and grid points to form block method. The schemes were consistent, zero-stable, and hence convergent.

Also, [11] derived linear multistep methods via interpolation and collocation approach using Laguerre polynomials as basis functions. The discrete form of the continuous methods are evaluated and applied to solve first-order ordinary differential equations. The results by the proposed methods are comparable with the Adams-Moulton methods of the same order. [12] developed a two-step block method of hybrid type for the direct solution of general first-order initial-value problems of the form $y' = f(x, y)$ where all the formulas in the method are obtained from a continuous approximation derived via interpolation and collocation at different points, the method is A-stable which makes it appropriate for solving stiff problems while in [13], a Zero – stable continuous hybrid linear multistep method is proposed for the numerical solution of initial value problems of first order ordinary differential equations. The numerical method was obtained with additional schemes from the same continuous scheme derived via interpolation and collocation procedure. The main scheme together with the additional schemes are then applied in a block form as simultaneous integrators over non – overlapping intervals. The method was found to be consistent, zero– stable, convergent, and accurate.

In this paper, we adopt the use of power series as the basic function in deriving the proposed scheme for the solution of IVPs of first order differential equations.

2. Derivation of the Scheme

Consider the IVP

$$y' = f(x, y); y(x_0) = y_0 \tag{2}$$

where f satisfies the Lipschitz condition of the existence and uniqueness of the solution. The main idea here is to derive the hybrid scheme of the form:

$$\sum_{j=0}^{k} \varphi_j y_{n+j} = h\left(\sum_{j=0}^{k} \beta_j f_{n+j} + \beta_v f_{n+v}\right) \tag{3}$$

where $\varphi_j, \beta_j, \beta_v$ are unknown coefficients and h is the step size. Note that $\varphi_k = 1, \beta_k = 1, \varphi_0$ and β_0 are both non-zero. To generate the hybrid scheme for a k-step method, we assume the approximate solution $y(x)$ of the form:

$$y(x) = \sum_{j=0}^{r+s-1} \varphi_j x^j \tag{4}$$

such that $x \in [x_n, x_{n+k}]$ and x_j are polynomial basis function of degree $r + s - 1$ where r and s are the numbers of interpolations and distinct collocation points respectively chosen to satisfy $1 \leq s < k,\ r > 0$ and the integer $k = 3$ denotes the step number of the scheme.

The approximation is thus constructed by imposing the following conditions:

$$y(x) = \sum_{j=0}^{3} \varphi_j x_{n+s}^j = y_{n+s}, \qquad s = 3, j \neq 0,1,2 \tag{5}$$

$$\left.\begin{array}{l} y'(x) = \sum_{j=0}^{10} j\varphi_j x_{n+\frac{i}{7}}^{j-1} = f_{n+\frac{i}{7}} \\ i = 0,2,4,6,7,8,10,12,14,21 \end{array}\right\} \tag{6}$$

Equations (5) and (6) result to a $(r+s)$ system of equations which can be evaluated for solution through matrix inversion algorithm with a view of obtaining values for φ_j which are then substituted into $\sum_{j=0}^{r+s-1} \varphi_j x^j$ to give

$$\left. \begin{array}{l} y(x) = y_{n+3} + h \sum_{j=0}^{3} \beta_j f_{n+j} + h \sum_{\tau=1}^{6} \beta_{\frac{i}{7}} f_{n+\frac{i}{7}} \\ \qquad j = 0,1,2,3\,, i = 2\tau \end{array} \right\} \tag{7}$$

Assuming $y_{n+j} = Y(x_{n+j})$ to be the numerical approximation to the analytical solution $y(x_{n+j})$ and evaluating equation (7) at points $x = \left\{x_{n+\frac{i}{7}}, i = 0,2,4,6,7,8,10,12,14\right\}$ we obtain the following nine discrete equations.

$$\begin{aligned} y_n = y_{n+3} &- \frac{595727}{409600}\square f_n + \frac{125915643}{7782400}\square f_{n+\frac{2}{7}} - \frac{734770827}{6963200}\square f_{n+\frac{4}{7}} \\ &+ \frac{282965053}{409600}\square f_{n+\frac{6}{7}} - \frac{134397}{100}\square f_{n+1} + \frac{1007908587}{1064960}\square f_{n+\frac{8}{7}} \\ &- \frac{1242236583}{4505600}\square f_{n+\frac{10}{7}} + \frac{7171787}{81920}\square f_{n+\frac{12}{7}} - \frac{7026849}{409600}\square f_{n+2} \\ &- \frac{811647}{4618900}\square f_{n+3} \end{aligned} \tag{8}$$

$$\begin{aligned} y_{n+\frac{2}{7}} = y_{n+3} &- \frac{109563560287}{79659417600}\square f_n + \frac{134845840861}{8128512000}\square f_{n+\frac{2}{7}} - \frac{2935540945931}{27636940800}\square f_{n+\frac{4}{7}} \\ &+ \frac{1126998167741}{1625702400}\square f_{n+\frac{6}{7}} - \frac{131039185313}{97240500}\square f_{n+1} \\ &+ \frac{4008842753131}{4226826240}\square f_{n+\frac{8}{7}} - \frac{4937048204327}{17882726400}\square f_{n+\frac{10}{7}} \\ &+ \frac{712223806291}{8128512000}\square f_{n+\frac{12}{7}} - \frac{1367246632273}{79659417600}\square f_{n+2} \\ &- \frac{102558789}{583683100}\square f_{n+3} \end{aligned} \tag{9}$$

$$\begin{aligned} &y_{n+\frac{4}{7}} \\ &= y_{n+3} - \frac{109697223263}{79659417600}\square f_n + \frac{2577091073639}{154441728000}\square f_{n+\frac{2}{7}} - \frac{6886393235}{65028096}\square f_{n+\frac{4}{7}} \\ &+ \frac{1126159967677}{1625702400}\square f_{n+\frac{6}{7}} - \frac{130971544993}{97240500}\square f_{n+1} + \frac{801464366767}{845365248}\square f_{n+\frac{8}{7}} \\ &- \frac{448723259093}{1625702400}\square f_{n+\frac{10}{7}} + \frac{26375150633}{301056000}\square f_{n+\frac{12}{7}} - \frac{273429344989}{15931883520}\square f_{n+2} \\ &- \frac{844062461}{4803680700}\square f_{n+3} \end{aligned} \tag{10}$$

$$y_{n+\frac{6}{7}}=y_{n+3}-\frac{54163975}{39337984}hf_n+\frac{254473155}{15253504}hf_{n+\frac{2}{7}}-\frac{1443870675}{13647872}hf_{n+\frac{4}{7}}+\frac{556413365}{802816}hf_{n+\frac{6}{7}}$$
$$-\frac{12938085}{9604}hf_{n+1}+\frac{761218875}{802816}hf_{n+\frac{8}{7}}-\frac{2437656975}{8830976}hf_{n+\frac{10}{7}}$$
$$+\frac{70336095}{802816}hf_{n+\frac{12}{7}}-\frac{675146025}{39337984}hf_{n+2}$$
$$-\frac{5995795}{34123012}hf_{n+3} \qquad (11)$$

$$y_{n+1}=y_{n+3}-\frac{713783}{518400}\Box f_n+\frac{410803897}{24624000}\Box f_{n+\frac{2}{7}}-\frac{932358721}{8812800}\Box f_{n+\frac{4}{7}}$$
$$+\frac{44915507}{64800}\Box f_{n+\frac{6}{7}}-\frac{13638997}{10125}\Box f_{n+1}+\frac{1277982671}{1347840}\Box f_{n+\frac{8}{7}}$$
$$-\frac{787026191}{2851200}\Box f_{n+\frac{10}{7}}+\frac{8410703}{96000}\Box f_{n+\frac{12}{7}}-\frac{2224283}{129600}\Box f_{n+2}$$
$$-\frac{16434749}{93532725}\Box f_{n+3} \qquad (12)$$

$$y_{n+\frac{8}{7}}=y_{n+3}-\frac{109682190943}{79659417600}hf_n+\frac{2576546010727}{154441728000}hf_{n+\frac{2}{7}}-\frac{2923851751691}{27636940800}hf_{n+\frac{4}{7}}$$
$$+\frac{225364087897}{325140480}hf_{n+\frac{6}{7}}-\frac{130980212129}{97240500}hf_{n+1}+\frac{308310319447}{325140480}hf_{n+\frac{8}{7}}$$
$$-\frac{4936264196903}{17882726400}hf_{n+\frac{10}{7}}+\frac{712153241683}{8128512000}hf_{n+\frac{12}{7}}-\frac{1367170842193}{79659417600}hf_{n+2}$$
$$-\frac{5995795}{34123012}hf_{n+3} \qquad (13)$$

$$y_{n+\frac{10}{7}}$$
$$=y_{n+3}-\frac{109688078687}{79659417600}\Box f_n+\frac{515340915067}{30888345600}\Box f_{n+\frac{2}{7}}-\frac{172004290139}{6963200}\Box f_{n+\frac{4}{7}}$$
$$+\frac{1126942855357}{1625702400}\Box f_{n+\frac{6}{7}}-\frac{26199998957}{19448100}\Box f_{n+1}+\frac{4009357341903}{4226826240}\Box f_{n+\frac{8}{7}}$$
$$-\frac{448567718357}{1625702400}\Box f_{n+\frac{10}{7}}+\frac{1054975889}{12042240}\Box f_{n+\frac{12}{7}}-\frac{1367137580369}{79659417600}\Box f_{n+2}$$
$$-\frac{844062461}{4803680700}\Box f_{n+3} \qquad (14)$$

$$y_{n+\frac{12}{7}}$$
$$=y_{n+3}-\frac{1353950127}{983449600}\Box f_n+\frac{1673970381}{100352000}\Box f_{n+\frac{2}{7}}-\frac{36092317851}{341196800}\Box f_{n+\frac{4}{7}}$$
$$+\frac{13909458861}{20070400}\Box f_{n+\frac{6}{7}}-\frac{1616827473}{1200500}\Box f_{n+1}+\frac{49481229051}{52183040}\Box f_{n+\frac{8}{7}}$$
$$-\frac{60854846967}{220774400}\Box f_{n+\frac{10}{7}}+\frac{8802093411}{100352000}\Box f_{n+\frac{12}{7}}-\frac{16880901633}{79659417600}\Box f_{n+2}$$
$$-\frac{102558789}{583683100}\Box f_{n+3} \qquad (15)$$

$$y_{n+2} = y_{n+3} - \frac{45712063}{33177600}\square f_n + \frac{52619141911}{3151872000}\square f_{n+\frac{2}{7}} - \frac{2388598835}{22560768}\square f_{n+\frac{4}{7}} + \frac{23015440973}{33177600}\square f_{n+\frac{6}{7}} - \frac{54601793}{40500}\square f_{n+1} + \frac{16377660383}{17252352}\square f_{n+\frac{8}{7}} - \frac{100749626327}{364953600}\square f_{n+\frac{10}{7}} + \frac{14608305859}{165888000}\square f_{n+\frac{12}{7}} - \frac{113326589}{6635520}\square f_{n+2} - \frac{811647}{4618900}\square f_{n+3} \quad (16)$$

3. Analysis of the Derived Scheme

3.1 Order of the block

In this section, the local truncation error and order, consistency and zero stability of the generated scheme are discussed. Following [14] and [2], the local truncation error associated with (1) is defined as the linear difference operator L as.

$$L[y(x),h] = \sum_{j=0}^{k}\left(\varphi_j y(x_{n+j}) - h\beta_j y(x_{n+j})\right) \quad (17)$$

Assuming that $y(x)$ is sufficiently differentiable (17) can be expanded as a Taylor series about the point x to obtain the expression,

$$L[y(x),h] = c_0 y(x_n) + c_1 h y'(x_n) + \cdots + c_q h^q y^q(x_n) + \cdots \quad (18)$$

where the constant coefficient $c_q, q = 0,1,\ldots$ are given as

$$\left.\begin{aligned} C_0 &= \sum_{j=0}^{k}\varphi_j \\ C_1 &= \sum_{j=0}^{k} j\varphi_j \\ &\vdots \\ C_q &= \frac{1}{q!}\sum_{j=0}^{k} j^q\varphi_j - \frac{1}{(q-1)!}\sum_{j=0}^{k} j^{q-1}\beta_j \end{aligned}\right\} \quad (19)$$

Definition 1: A linear multistep method is said to be of order q if $C_0 = C_1 = \ldots = C_q = 0, C_{q+1} \neq 0$.Therefore, $C_{q+1}$is the error constant and $C_{q+1}\square^{q+1}y^{q+1}(x_n)$ is the principal local truncation error at point x_n. Thus, the local truncation error (LTE) of order q can be written as

$$\text{LTE} = C_{q+1}\square^{q+1}y^{q+1}(x_n) + O(h^{q+2}) \quad (20)$$

It is established from our calculation that the generated hybrid linear multistep scheme has order q and relatively small error constant C_{11} as shown in Table 1.

3.2 Consistency

Definition 2: A linear multistep method (1) is consistent if (i) the order $q \geq 1$, (ii)$\sum_{j=0}^{k}\varphi_j = 0$, (iii) $\sum_{j=0}^{k} j\varphi_j = \sum_{j=0}^{k}\beta_j$.

Applying this conditions, schemes (8-16) were found to be consistent.

Table 1: Order and Error Constant

Scheme number	Order	Error Constant
8	11	$\frac{140137}{57977427200}$
9	11	$\frac{408240690679}{169132620289248000}$
10	11	$\frac{2858195138753}{1183928342024736000}$
11	11	$\frac{94093655}{38977064758016}$
12	11	$\frac{42513551}{17610643512000}$
13	11	$\frac{2858095220929}{1183928342024736000}$
14	11	$\frac{51966708167}{21525969854995200}$
15	11	$\frac{1680124653}{696019013536000}$
16	11	$\frac{170133079}{70442574048000}$

3.3 Zero Stability

Definition 3: A linear multistep method of the form (1) is said to be zero stable if no root of the first characteristic polynomial ρ(r) has modulus greater than one and if every root with modulus one is simple.

Also, applying this definition to schemes (8-16) they were found to be zero stable.

3.4 Consistency and Convergence

The hybrid scheme is consistent since each of the method has order $q > 1$. The convergence of the hybrid scheme can be established according to [15] since convergence = consistency + zero stability.

4. Numerical examples

In this section, all calculations are carried out with the aid of MAPLE software. The results are presented in tabular form where y_e is the exact solution, y_n is the numerical solution, and $err = |y_e - y_n|$ is the absolute error Their performance is compared with the exact solution and with other methods in cited literatures.

Example 1: Consider the nonlinear problem given by

$$y' = -10(y-1)^2, y(0) = 2$$

The exact solution is

$$y_e = \frac{2(5x+1)}{10x+1}$$

This problem has been studied earlier by [12] with $h = \frac{1}{100}$ and has also appeared in [16] and [17]. Table 2 gives the results of the exact and numerical solution with the absolute error while Table 3 consists of absolute errors with respect to exact solution at different points in comparison to the errors in the cited literature.

Table 2: Results for problem 1

x	y_e	y_n	Error
0.01	1.9090909090909090909090909091	1.909090909090613807529763	2.95283379328e-13
0.02	1.833333333333333333333333	1.833333333333195875223718	1.37458109615e-13
0.03	1.769230769230769230769231	1.769230769230610863012999	1.58367756232e-13
0.04	1.714285714285714285714286	1.714285714285593153576831	1.21132137455e-13
0.05	1.666666666666666666666667	1.666666666666539157898707	1.27508767960e-13
0.06	1.625000000000000000000000	1.624999999999884383034073	1.15616965927e-13
0.07	1.588235294117647058823529	1.588235294117545967815784	1.01091007745e-13
0.08	1.555555555555555555555556	1.555555555555464504688541	9.1050867015e-14
0.09	1.526315789473684210526316	1.526315789473602819862854	8.1390663462e-14
0.10	1.500000000000000000000000	1.499999999999926292427642	7.3707572358e-14

Table 3: Comparison of absolute errors for problem 1

x	Error in [16]	Error in [17]	Error in [12]	Computed Error
0.01	3.414671e-6	5.222834e-8	4.220821e-9	2.95283379328e-13
0.02	2.749635e-6	8.727145e-8	7.093324e-9	1.37458109615e-13
0.03	1.342943e-5	1.069875e-8	7.147587e-9	1.58367756232e-13
0.04	9.090648e-5	8.987150e-8	7.114519e-9	1.21132137455e-13
0.05	7.969685e-5	4.712423e-8	6.547679e-9	1.27508767960e-13
0.06	6.994886e-5	1.808182e-8	6.062538e-9	1.15616965927e-13
0.07	6.270048e-5	1.602002e-8	5.498647-9	1.01091007745e-13
0.08	6.017101e-5	1.429167e-8	5.019162e-9	9.1050867015e-14
0.09	5.411308 e-5	1.283029e-8	4.557381e-9	8.1390663462e-14
0.10	4.880978e-5	1.159479e-8	4.160552e-9	7.3707572358e-14

Example 2: Consider the IVP.

$$y' = xe^{3x} - 2y, y(0) = 0; 0 \leq x \leq 1, h = 0.1$$

The exact solution is

$$y_e = \frac{e^{-2x}}{25}\left((5x-1)e^{5x} + 1\right)$$

This IVP has appeared in [11], Table 4 gives the results of the exact and numerical solution with the absolute error while Table 5 consists of absolute errors with respect to exact solution at different points in comparison to the errors in the cited literature.

Table 4: Results for problem 1

x	y_e	y_n	Error
0.1	0.005752053971603554552900476	0.005752053971599212267122531	4.342285777945e-15
0.2	0.026812801841435153393671330	0.026812801841425572029777320	9.5813638940le-15
0.3	0.071144527666916251562108330	0.071144527666900050581140880	1.620098096745e-14
0.4	0.150777835474175606635868800	0.150777835474150763238434800	2.48433974340e-14
0.5	0.283616521867177956022528800	0.283616521867141582219944400	3.63738025844e-14
0.6	0.496019565629575738660587100	0.496019565629523770564280900	5.19680963062e-14
0.7	0.826480869814502499348837500	0.826480869814429266422567100	7.32329262704e-14
0.8	1.330857026396823675047574000	1.330857026396778414607960000	4.52604396l4e-14
0.9	2.089774397011155882023154000	2.089774397011060237193530000	9.56448296 24e-14
1.0	3.219099319039541595673248000	3.219099319039491346224326000	5.02494489 22e-14

Table 5: Comparison of absolute errors for problem 1

x	Error in 5-step ADM Scheme [11]	Error in 5-step Optimal Order Scheme [11]	Computed Error
0.1	1.0000e-14	1.0000e-14	4.342285777945e-15
0.2	1.9000e-14	1.9000e-14	9.58136389401e-15
0.3	2.7000e-14	2.7000e-14	1.620098096745e-14
0.4	3.5000e-14	3.5000e-14	2.48433974340e-14
0.5	4.8820e-12	3.3390e-12	3.63738025844e-14
0.6	1.1006e-11	4.6020e-12	5.19680963062e-14
0.7	1.8145e-11	8.9640e-12	7.32329262704e-14
0.8	2.6405e-11	1.1243e-12	4.5260439614e-14
0.9	3.5241e-11	1.6274e-11	9.5644829624e-14
1.0	4.4817e-11	1.9169e-11	5.0249448922e-14

5. Discussion of Result

A block hybrid scheme for the direct solution of first order IVP of ODEs was derived. Table 2 and Table 4 show results of the proposed hybrid scheme for problems 1 and 2 respectively while Table 3 and Table 5 show comparison of errors with other methods that appeared in literature.

6. Conclusion

This scheme is derived through collocation and interpolation technique using power series as basis function. Nine corresponding discrete schemes are applied to solve two IVPs of first order ODEs, numerical results obtained demonstrate better approximation than the other two in comparison, and hence, the new scheme is highly accurate and performs better than the results in literatures cited.

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