# Eleventh Order Hybrid Block Method for the Solution of Nonlinear First Order Initial Value Problems

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**Abstract**

In this paper, the development of three-step eleventh order hybrid block method for the solution of nonlinear first order initial value problems using power series approximation is discussed. The interpolation and collocation method was adopted in the development of the continuous linear method. The result was evaluated at selected grid points to give a discrete block which eventually gave simultaneous solutions at both grid and off grid points. The three-step block method is consistent, zero stable and therefore convergent. Experimental results confirmed the superiority of the new scheme over an existing method.

Keywords: block method, consistent, convergent, grid points, interpolation, zero stable

## 1 INTRODUCTION

Mathematics is an area of study that has experienced explosion of knowledge which has led to specialized areas; the subject is growing ever larger and permeating into other fields of sciences faster than before (Fatunla, 1987). The mathematical formulation of problems like the reactions of chemicals and vibration of a membrane which arise in real world gives rise to differential equations which are mostly nonlinear; in fact, in many aspects of human endeavour, it is necessary to build a mathematical model to represent certain problems. These mathematical models involve the search for an unknown function that satisfies an equation in which the derivatives of the unknown function play a dominant role.

Differential equations occur in natural sciences, engineering, physics, chemistry, biology, and social sciences; also, the application of mathematics to bio information is rapidly expanding (William and Richard, 2001). Numerical analysis is the study of algorithms that use numerical approximation for the problems of mathematical analysis. The numerical method for solving ordinary differential equations (ODEs) is the most powerful technique ever developed in continuous time dynamics; these are developed since most of the differential equations cannot be solved analytically (Badmus, 2013).

Aboiyar, *et al.* (2015) derived continuous linear multistep methods for solving first-order initial value problems (IVPs) of ordinary differential equations (Odes) with step number k=3 using Hermite polynomials as basis functions. Adams-Bashforth, Adams-Moulton and optimal order methods are derived through collocation and interpolation technique. The result obtained by the optimal order method compared favourably with those of the standard existing methods of Adams-Bashforth and Adams-Moulton.

Ramos, (2017) developed a two-step block method of hybrid type for the direct solution of general first-order initial-value problems of the form $y' = f(x, y)$ where all the formulas in the method are obtained from a continuous approximation derived via interpolation and collocation at different points. The method is A-stable, which makes it appropriate for solving stiff problems

An optimized one-step hybrid block method for the numerical solution of first-order initial value problems was treated by (Bothayna and Muhammed, 2019). The method takes into consideration three hybrid points which are chosen appropriately to optimize the local truncation errors of the main formulas for the block. The method is zero-stable and consistent with fifth algebraic order.

Kamoh, *et al.* (2021) presented a collocation approach for solving initial value problem of ordinary differential equations (ODEs) of the first order. This approach consists of reducing the problem to a set of linear multi-step algebraic equations by approximating the ODE with a shifted Legendre polynomial basis function to determine the unknown constants. The proposed method is simple and efficient; it approximates the solutions very closely to the closed form solutions.

## 2 Statement of the Problem

Consider the first order ordinary differential equation of the form

$$y' = f(x,y), y\left(x_0\right) = y_0 \quad (1)$$

In this work some k-step hybrid block linear multistep methods of the form:

$$\sum_{j}^{k} \alpha_j y_{n+i} = h\sum_{j}^{k} \beta_j f_{n+i} + h\beta_v f_{n+v} \quad (2)$$

$\alpha_0$ and $\beta_0$ are both not zero, $v \in \{0,1,\ldots k\}$

and a hybrid block linear multistep method is developed for solving nonlinear initial value problems in ordinary differential equations of the form (1).

## 3 Derivation of the Schemes

The derivation of the hybrid block method for direct solution of nonlinear first order initial value problem in ordinary differential equations of the form (1) where $x_0$ is the initial point,

$y_0$ is the solution at $x_0$ and $f$is continuous within interval of the integration. A power series of a single variable $x$ in the form:

$$P(x)=\sum_{j=0}^{r+s-1}\alpha_j x^j \tag{3}$$

is used as the basis or trial function, to produce the approximate solution

$$y(x)=\sum_{j=0}^{r+s-1}\alpha_j x^j, \quad \alpha_j \in R, \quad y \in C^m(a,b) \subset P(x) \tag{4}$$

where

i. $\alpha_j s$ are unknown coefficients to be determined,

ii. $r$is the number of interpolations for$1 \leq r \leq k$, and

iii. $s$the number of distinct collocation points with $s > 0$

with the derivative given as

$$y'(x) = \sum_{j=0}^{r+s-1} j\alpha_j x^{j-1} \tag{5}$$

Substituting (5) into (1) gives

$$f(x,y) = \sum_{j=0}^{r+s-1} j\alpha_j x^{j-1} \tag{6}$$

Interpolating (4) and collocating (6) at given step points gives a system of nonlinear equation of the form

$$AX = B \tag{7}$$

which is solved to obtain the values of parameters $\alpha_j s, j = 0,1,2,\ldots,(r+s-1)$ which are then substituted into (4) which yields the new continuous method expressed in the form

$$y(x) = \sum_{j=0}^{r+s+1} \alpha_j(x) y_{n+j} + h\left(\sum_{j=0}^{r+s+1} \beta_j(x) f_{n+j}\right) \tag{8}$$

where $\alpha_j$s and $\beta_j$s are function of $x$.

**Hybrid block method with six off step points at collocation**

The general form of the power series is:

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \ldots \tag{9}$$

If $x_n$ is a particular point, the series could be written as

$$y_n = a_0 + a_1 x_n + a_2 x_n^2 + a_3 x_n^3 + a_4 x_n^4 + a_5 x_n^5 + \ldots \tag{10}$$

Taking $x_n$ as the origin or point of initialization up to another point, say $x_m$, then some steps called the mesh points denoted by $h$ in between $n$ and $m$ is taken into consideration at a certain number of steps $h$=1, 2, 3 ….

Collocating at points $x_n, x_{n+\frac{1}{3}}, x_{n+\frac{2}{3}}, x_{n+1}, x_{n+\frac{4}{3}}, x_{n+\frac{5}{3}}, x_{n+2}, x_{n+\frac{7}{3}}, x_{n+\frac{8}{3}}, x_{n+3}$ and interpolating at point $x_{n+2}$ gives.

$$y(x) = \sum_{j=0}^{10} \alpha_j x^j \tag{11}$$

and the derivative

$$y'(x) = \sum_{j=0}^{10} j\alpha_j x^{j-1} \tag{12}$$

Putting (11) and (12) in matrix form we have

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2h & h & h & h & h & h & h & h & h & h & h \\
(2h)^2 & 0 & 2h\left(\frac{1}{3}h\right) & 2h\left(\frac{2}{3}h\right) & 2h(h) & 2h\left(\frac{4}{3}h\right) & 2h\left(\frac{5}{3}h\right) & 2h(2h) & 2h\left(\frac{7}{3}h\right) & 2h\left(\frac{8}{3}h\right) & 2h(3h) \\
(2h)^3 & 0 & 3h\left(\frac{1}{3}h\right)^2 & 3h\left(\frac{2}{3}h\right)^2 & 3h(h)^2 & 3h\left(\frac{4}{3}h\right)^2 & 3h\left(\frac{5}{3}h\right)^2 & 3h(2h)^2 & 3h\left(\frac{7}{3}h\right)^2 & 3h\left(\frac{8}{3}h\right)^2 & 3h(3h)^2 \\
(2h)^4 & 0 & 4h\left(\frac{1}{3}h\right)^3 & 4h\left(\frac{2}{3}h\right)^3 & 4h(h)^3 & 4h\left(\frac{4}{3}h\right)^3 & 4h\left(\frac{5}{3}h\right)^3 & 4h(2h)^3 & 4h\left(\frac{7}{3}h\right)^3 & 4h\left(\frac{8}{3}h\right)^3 & 4h(3h)^3 \\
(2h)^5 & 0 & 5h\left(\frac{1}{3}h\right)^4 & 5h\left(\frac{2}{3}h\right)^4 & 5h(h)^4 & 5h\left(\frac{4}{3}h\right)^4 & 5h\left(\frac{5}{3}h\right)^4 & 5h(2h)^4 & 5h\left(\frac{7}{3}h\right)^4 & 5h\left(\frac{8}{3}h\right)^4 & 5h(3h)^4 \\
(2h)^6 & 0 & 6h\left(\frac{1}{3}h\right)^5 & 6h\left(\frac{2}{3}h\right)^5 & 6h(h)^5 & 6h\left(\frac{4}{3}h\right)^5 & 6h\left(\frac{5}{3}h\right)^5 & 6h(2h)^5 & 6h\left(\frac{7}{3}h\right)^5 & 6h\left(\frac{8}{3}h\right)^5 & 6h(3h)^5 \\
(2h)^7 & 0 & 7h\left(\frac{1}{3}h\right)^6 & 7h\left(\frac{2}{3}h\right)^6 & 7h(h)^6 & 7h\left(\frac{4}{3}h\right)^6 & 7h\left(\frac{5}{3}h\right)^6 & 7h(2h)^6 & 7h\left(\frac{7}{3}h\right)^6 & 7h\left(\frac{8}{3}h\right)^6 & 7h(3h)^6 \\
(2h)^8 & 0 & 8h\left(\frac{1}{3}h\right)^7 & 8h\left(\frac{2}{3}h\right)^7 & 8h(h)^7 & 8h\left(\frac{4}{3}h\right)^7 & 8h\left(\frac{5}{3}h\right)^7 & 8h(2h)^7 & 8h\left(\frac{7}{3}h\right)^7 & 8h\left(\frac{8}{3}h\right)^7 & 8h(3h)^7 \\
(2h)^9 & 0 & 9h\left(\frac{1}{3}h\right)^8 & 9h\left(\frac{2}{3}h\right)^8 & 9h(h)^8 & 9h\left(\frac{4}{3}h\right)^8 & 9h\left(\frac{5}{3}h\right)^8 & 9h(2h)^8 & 9h\left(\frac{7}{3}h\right)^8 & 9h\left(\frac{8}{3}h\right)^8 & 9h(3h)^8 \\
(2h)^{10} & 0 & 10h\left(\frac{1}{3}h\right)^9 & 10h\left(\frac{2}{3}h\right)^9 & 10h(h)^9 & 10h\left(\frac{4}{3}h\right)^9 & 10h\left(\frac{5}{3}h\right)^9 & 10h(2h)^9 & 10h\left(\frac{7}{3}h\right)^9 & 10h\left(\frac{8}{3}h\right)^9 & 10h(3h)^9
\end{pmatrix} \qquad (13)$$

where

$$B = \left[ y_{n+2}, f_n, f_{n+\frac{1}{3}}, f_{n+\frac{2}{3}}, f_{n+1}, f_{n+\frac{4}{3}}, f_{n+\frac{5}{3}}, f_{n+2}, f_{n+\frac{7}{3}}, f_{n+\frac{8}{3}}, f_{n+3} \right]^T$$

$$X = \left[ \alpha_2, \beta_0, \beta_{\frac{1}{3}}, \beta_{\frac{2}{3}}, \beta_1, \beta_{\frac{4}{3}}, \beta_{\frac{5}{3}}, \beta_2, \beta_{\frac{7}{3}}, \beta_{\frac{8}{3}}, \beta_3 \right]^T \quad (14)$$

Taking the inverse of (14),

$$X = A^{-1}B \quad (15)$$

Solving (15) by Gaussian elimination method using maple 18, the coefficient of the continuous schemes $\alpha_j s$ and $\beta_j s$ were obtained.

The approximate solution is given by

$$y(x) = \alpha_0(x)y_{n+2} + h\left( \begin{array}{l} \beta_n(x)f_n + \beta_{\frac{1}{3}}(x)f_{n+\frac{1}{3}} + \beta_{\frac{2}{3}}(x)f_{n+\frac{2}{3}} + \beta_1(x)f_{n+1} + \beta_{\frac{4}{3}}(x)f_{n+\frac{4}{3}} \\ +\beta_{\frac{5}{3}}(x)f_{n+\frac{5}{3}} + \beta_2(x)f_{n+2} + \beta_{\frac{7}{3}}(x)f_{n+\frac{7}{3}} + +\beta_{\frac{8}{3}}(x)f_{n+\frac{8}{3}} + \beta_3(x)f_{n+3} \end{array} \right) \quad (16)$$

Substituting the $\alpha_j$ and $\beta_j$ into (16) and evaluating at points $x_n, x_{n+\frac{1}{3}}, x_{n+\frac{2}{3}}, x_{n+1}, x_{n+\frac{4}{3}}, x_{n+\frac{5}{3}}, x_{n+\frac{7}{3}}, x_{n+\frac{8}{3}}, x_{n+3}$ gives the following nine discrete schemes that form the block method.

$$y_n = y_{n+2} - \frac{7}{75}hf_n - \frac{771}{1400}hf_{n+\frac{1}{3}} + \frac{51}{700}hf_{n+\frac{2}{3}} - \frac{199}{210}hf_{n+1} + \frac{249}{700}hf_{n+\frac{4}{3}} - \frac{633}{700}hf_{n+\frac{5}{3}} + \frac{299}{2100}hf_{n+2} - \frac{33}{350}hf_{n+\frac{7}{3}} + \frac{3}{140}hf_{n+\frac{8}{3}} - \frac{3}{1400}hf_{n+3} \quad (17)$$

$$y_{n+\frac{1}{3}} = y_{n+2} + \frac{25}{10752}hf_n - \frac{101635}{870912}hf_{n+\frac{1}{3}} - \frac{48425}{108864}hf_{n+\frac{2}{3}} - \frac{11575}{54432}hf_{n+1} + \frac{190775}{435456}hf_{n+\frac{4}{3}} - \frac{123575}{435456}hf_{n+\frac{5}{3}} - \frac{10735}{54432}hf_{n+2} - \frac{3175}{108864}hf_{n+\frac{7}{3}} - \frac{4675}{870912}hf_{n+\frac{8}{3}} + \frac{425}{870912}hf_{n+3} \qquad (18)$$

$$y_{n+\frac{2}{3}} = y_{n+2} - \frac{13}{42525}hf_n + \frac{32}{6075}hf_{n+\frac{1}{3}} - \frac{5494}{42525}hf_{n+\frac{2}{3}} - \frac{17632}{42525}hf_{n+1} - \frac{2174}{8505}hf_{n+\frac{4}{3}} - \frac{17632}{42525}hf_{n+\frac{5}{3}} - \frac{5494}{42525}hf_{n+2} - \frac{32}{6075}hf_{n+\frac{7}{3}} + \frac{13}{42525}hf_{n+\frac{8}{3}} \qquad (19)$$

$$y_{n+1} = y_{n+2} + \frac{7}{38400}hf_n - \frac{201}{89600}hf_{n+\frac{1}{3}} + \frac{33}{22400}hf_{n+\frac{2}{3}} - \frac{2647}{16800}hf_{n+1} + \frac{15909}{44800}hf_{n+\frac{4}{3}} - \frac{15909}{44800}hf_{n+\frac{5}{3}} + \frac{2647}{16800}hf_{n+2} + \frac{33}{2240}hf_{n+\frac{7}{3}} - \frac{201}{89600}hf_{n+\frac{8}{3}} + \frac{7}{38400}hf_{n+3} \qquad (20)$$

$$y_{n+\frac{4}{3}} = y_{n+2} + \frac{23}{340200}hf_{n+\frac{1}{3}} - \frac{167}{170100}hf_{n+\frac{2}{3}} - \frac{701}{85050}hf_{n+1} - \frac{23189}{34020}hf_{n+\frac{4}{3}} - \frac{13903}{34020}hf_{n+\frac{5}{3}} - \frac{23189}{170100}hf_{n+2} + \frac{701}{85050}hf_{n+\frac{7}{3}} - \frac{167}{170100}hf_{n+\frac{8}{3}} + \frac{23}{340200}hf_{n+3} \qquad (21)$$

$$y_{n+\frac{5}{3}} = y_{n+2} + \frac{2497}{21772800}hf_n - \frac{27467}{21772800}hf_{n+\frac{1}{3}} + \frac{17663}{2721600}hf_{n+\frac{2}{3}} - \frac{5779}{272160}hf_{n+1} + \frac{583037}{10886400}hf_{n+\frac{4}{3}} - \frac{2381791}{10886400}hf_{n+\frac{5}{3}} - \frac{225623}{1360800}hf_{n+2} + \frac{42767}{2721600}hf_{n+\frac{7}{3}} - \frac{10063}{4354560}hf_{n+\frac{8}{3}} + \frac{7}{38400}hf_{n+3} \qquad (22)$$

$$y_{n+\frac{7}{3}} = y_{n+2} + \frac{7}{38400}hf_n - \frac{42187}{21772800}hf_{n+\frac{1}{3}} + \frac{25759}{2721600}hf_{n+\frac{2}{3}} - \frac{38599}{1360800}hf_{n+1} + \frac{25759}{2721600}hf_{n+\frac{4}{3}} - \frac{1083167}{10886400}hf_{n+\frac{5}{3}} + \frac{349817}{1360800}hf_{n+2} + \frac{391711}{2721600}hf_{n+\frac{7}{3}} - \frac{163531}{21772800}hf_{n+\frac{8}{3}} + \frac{425}{870912}hf_{n+3} \quad (23)$$

$$y_{n+\frac{8}{3}} = y_{n+2} - \frac{13}{42525}hf_n + \frac{1063}{340200}hf_{n+\frac{1}{3}} - \frac{491}{34020}hf_{n+\frac{2}{3}} + \frac{3373}{85050}hf_{n+1} + \frac{12133}{170100}hf_{n+\frac{4}{3}} + \frac{14117}{170100}hf_{n+\frac{5}{3}} + \frac{9371}{170100}hf_{n+2} + \frac{7817}{17010}hf_{n+\frac{7}{3}} + \frac{19469}{170100}hf_{n+\frac{8}{3}} - \frac{3}{1400}hf_{n+3} \quad (24)$$

$$y_{n+3} = y_{n+2} + \frac{25}{10752}hf_n - \frac{303}{12800}hf_{n+\frac{1}{3}} + \frac{1221}{11200}hf_{n+\frac{2}{3}} - \frac{5039}{16800}hf_{n+1} + \frac{24603}{44800}hf_{n+\frac{4}{3}} - \frac{6369}{8960}hf_{n+\frac{5}{3}} + \frac{13273}{16800}hf_{n+2} - \frac{93}{1600}hf_{n+\frac{7}{3}} + \frac{49143}{89600}hf_{n+\frac{8}{3}} + \frac{1197}{12800}hf_{n+3} \quad (25)$$

## 4 Basic Properties of the Schemes

### 4.1 Order of the Block

Following Fudziah et al. (2020), Thus, the local truncation error (LTE) of order $p$ can be written as

$$\text{LTE} = C_{p+1}h^{p+1}y^{p+1}(x_n) + O\left(h^{p+2}\right) \quad (26)$$

From our calculation, the order of the method is 11 and has relatively small error constant given as:

$$C_{11} = \begin{bmatrix} -\frac{11899}{349192166400}, -\frac{179}{5456127600}, -\frac{18665}{3394147857408}, \frac{62}{82864937925}, \\ \frac{263}{1325839006800}, \frac{90817}{84853696435200}, \frac{171137}{84853696435200}, \frac{5609}{1325839006800} \end{bmatrix} \quad (27)$$

## `4.2 The Stability of the Block Hybrid Method with Six off-grid Points

$$\left(\begin{array}{l} y_{n+3} = y_{n+2} + \frac{25}{10752}hf_n - \frac{303}{12800}hf_{n+\frac{1}{3}} + \frac{1221}{11200}hf_{n+\frac{2}{3}} - \frac{5039}{16800}hf_{n+1} + \\ \frac{24603}{44800}hf_{n+\frac{4}{3}} - \frac{6369}{8960}hf_{n+\frac{5}{3}} + \frac{13273}{16800}hf_{n+2} - \frac{93}{1600}hf_{n+\frac{7}{3}} + \frac{49143}{89600}hf_{n+\frac{8}{3}} \\ + \frac{1197}{12800}hf_{n+3} \end{array}\right) \quad (28)$$

$$\left(\begin{array}{l} \rho(r) = r^3 - r^2 \\ \rho(r) = 1,0,0 \end{array}\right) \quad (29)$$

### 4.3 Consistency

$$\left.\begin{array}{l} \sum \alpha = 1 - 1 = 0 \\ \rho(r) = r^3 - r^2 \\ \frac{\rho(r)}{dr} = 3r^2 - 2r \\ = 1 \end{array}\right\} \quad (30)$$

$$\left.\begin{array}{l} \sigma(1) = \frac{25}{10752} - \frac{303}{12800} + \frac{1221}{11200} - \frac{5039}{16800} + \frac{24603}{44800} - \frac{6369}{8960} + \\ \frac{13273}{16800} - \frac{93}{1600} + \frac{49143}{89600} + \frac{1197}{12800} \end{array}\right\} \quad (31)$$

## 5 Numerical Examples

**Consider two nonlinear problems:**

**Problem 1:**

$$\left.\begin{array}{r} y' = 10(y-1)^2; y(0) = 2 \\ y(x) = 1 + \frac{1}{1+10x} \end{array}\right\}$$

**Problem 2:**

$$y' = \frac{-y^3}{2}; y(0) = 1$$

$$y(x) = \frac{1}{\sqrt{1+x}}$$

Table 1: Comparison of results for Problem 1

| $x$ | $y_e(x)$ | $y_c(x)$ | Error | Error Kamoh *et al.* (2021) |
|---|---|---|---|---|
| **0.01** | 1.9090909090909090909 | 1.9090909090883875693 | 2.5215216e-12 | 1.7533e-10 |
| **0.02** | 1.8333333333333333333 | 1.8333333333313054109 | 2.0279224e-12 | 2.3200e-10 |
| **0.03** | 1.7692307692307692308 | 1.7692307692266878581 | 4.0813727e-12 | 2.4115e-10 |
| **0.04** | 1.7142857142857142857 | 1.7142857142820396358 | 3.6746499e-12 | 2.3140e-10 |
| **0.05** | 1.6666666666666666667 | 1.6666666666634712705 | 3.1953962e-12 | 2.1484e-10 |
| **0.06** | 1.6250000000000000000 | 1.6249999999970446828 | 2.9553172e-12 | 1.9660e-10 |
| **0.07** | 1.5882352941176470588 | 1.5882352941150129362 | 2.6341226e-12 | 1.7887e-10 |
| **0.08** | 1.5555555555555555556 | 1.5555555555532065746 | 2.3489810e-12 | 1.6250e-10 |
| **0.09** | 1.5263157894736842105 | 1.5263157894715604936 | 2.1237169e-12 | 1.4773e-10 |
| 0.10 | 1.5000000000000000000 | 1.4999999999980809096 | 1.9190904e-12 | 1.3457e-10 |

Table 2: Comparison of results for Problem 2

| $x$ | $y_e(x)$ | $y_c(x)$ | Error |
|---|---|---|---|
| **0.2** | 0.91287092917527685577 | 0.91287092917487886036 | 3.9799541e-13 |
| **0.4** | 0.84515425472851657752 | 0.84515425472778839334 | 7.2818418e-13 |
| **0.6** | 0.79056941504209483300 | 0.79056941504147010303 | 6.2472997e-13 |
| **0.8** | 0.74535599249992989883 | 0.74535599249940306961 | 5.2682922e-13 |
| 1.0 | 0.70710678118654752440 | 0.70710678118609391550 | 4.5360890e-13 |

## CONCLUSION AND DISCUSSION

This research is centred on the derivation of eleventh order hybrid block method with six off step points. The schemes were implemented on two nonlinear initial value problems, the approximate solutions gotten through these schemes were compared with the exact solutions in which a better accuracy was observed. It is shown that the hybrid block method derived for the direct solution of nonlinear first order initial value problems in ODEs is of order eleven and gives very low error terms. The consistency and zero stability of the new method guarantee its convergence. Based on the result obtained in Table 1 there is improvement on the convergence rate of the scheme. The new method is highly accurate and performs better than the literature cited.

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