Ornstein-Uhlenbeck Operator for Correlated Random Variables

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***Abstract*—We investigate Ornstein-Uhlenbeck operator which serves as an important tool with application in many fields, including sensitivity analysis involving Lévy processes using Malliavin calculus. Some processes with multivariate random variables have feature of correlation among the random variables. Hence, there is need to obtain the Ornstein-Uhlenbeck operator for such phenomenon. This paper was therefore designed to derive the Ornstein-Uhlenbeck operator for correlated multivariate random variables.**

***Index Terms*—Gaussian random variables, Skorohod integral, Ornstein-Uhlenbeck operator.**

I. Introduction

The Ornstein-Uhlenbeck (O-U) operator is employed in sensitivity analysis of financial instruments using Malliavin calculus, and has applications in other fields, namely, geometry, functional calculus and analysis, etc. Bally et al. [1] applied the operator in providing numerical algorithm for sensitivity computation in a model driven by a Lévy process. Bavouzet and Messaoud [2], Bavouzet et al. [3], Bally & Clement [4] and Udoye et al. [5] used the operator in the sensitivity analysis in a jump-type market model, while Udoye and Ekhaguere [6] applied the operator in deriving the greeks *delta* and *gamma* of an interest rate derivative driven by a variance gamma Lévy process.

Chang and Feng [7] studied the operator with quadratic potentials. Otten [8] applied the operator as a basis for proving exponential decay of revolving waves. Metafune [9] studied L^p-spectrum of the operator. Chen and Liu [10] derived complex form of the operators and semigroups. Cappa [11] studied the operator in convex domains of Banach spaces. Cerrai and Lunardi [12] proved Schauder estimates for stationary and evolution equations under the operator in a separable Banach space, endowed with a centred Gaussian measure. Feo et al. [13] discussed Gaussian symmetrization method and regularity approximations for solutions to nonlocal equation of fractional powers of the O-U operator while Casarino [14] considered an O-U semigroup on $\mathbb{R}^n$ with covariance matrix. Wei et al. [15] discussed the problem in the parameter estimation for the squared radial O-U process under an α-stable noises for distinct observation.

A good model of a financial derivative with correlated random variables should be able to consider the feature of correlation in order to avoid wrong forecasting and hedging by a risk manager. The O-U operator for correlated random variables is an important tool to be considered. Di Bernardino et al. [16] discussed some phenomena in both physical and biological sciences that can be understood mathematically by taking into account statistical properties of level crossings of random Gaussian processes and emphasized that some of them require consideration of correlated level crossings emerging from multiple correlated processes. Zubeldia and Mandjes [17] considered acyclic network of single-server queues along with heterogeneous processing rates and presumed that each queue is fed by the superposition of a large number of i.i.d. Gaussian processes having stationary increments and positive drifts that can be correlated across different queues. Rau et al. [18] applied hierarchical logistic Gaussian processes to deduce redshift distributions of galaxies samples, by means of their cross-correlations through spatially overlapping spectroscopic samples. Perger et al. [19] discussed auto-correlations functions of astrophysical processes and correlated time variations in the structure of Gaussian processes. Hong et al. [20] discussed estimation of models with multivariate, multimode and nonlinear processes involving correlated noises. Hence, in this paper, we derive the O-U operator for correlated Gaussian random variables.

The rest of the paper is organized as follows: Section 2 discusses important tools needed in the work, Section 3 derives the results, and concludes the work.

II. Mathematical Foundation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For $p, n \geq 1$, define $C^p(\mathbb{R}^n)$ as functions $F : \mathbb{R}^n \to \mathbb{R}$ that are p times continuously differentiable. Let $X_1, ..., X_n$ denote a sequence of random variables and let $S_{(n,p)}$ be a set of simple functionals such that $F = \widehat{F}(X_1, ..., X_n) \in S$ while $P_{(n,p)}$ is the space of simple processes $U_i = u_i(X_1, ..., X_n)$ of length n, where $u_i \in C^p(\mathbb{R}^n)$, $i = 1, ..., n$.

Definition II.1. The *O-U operator* $L : S_{(n,2)} \to S_{(n,0)}$ on F is defined as

$$LF = -\sum_{i=1}^{n}[(\partial_{ii}^2 F)(X_1, ..., X_n) + \phi_i(x_i)(\partial_i F)(X_1, ..., X_n)],$$

where

$$\phi_i(x_i) = \partial_{x_i} \ln[f(\mathbf{x})] = \frac{f_i'(\mathbf{x})}{f(\mathbf{x})}, \quad f(\mathbf{x}) \neq 0; 1 \leq i \leq n$$

otherwise, $\phi_i(\mathbf{x}) = 0$, where f_i is the density function of the random variable $X_i, i = 1, ..., n$.

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Definition II.2. Let Z_i and Z_j be non-zero correlated Gaussian random variables. The correlation coefficient between Z_i and Z_j is given by

$$\rho_{Z_iZ_j} = \frac{\text{Cov}(Z_i,Z_j)}{\sqrt{\text{Var}(Z_i)}\sqrt{\text{Var}(Z_j)}} = \frac{\sigma_{ij}}{\sigma_i\sigma_j},\ -1 \le \rho_{Z_iZ_j} \le 1$$

where $\text{Var}(Z_i) = \mathbb{E}[(Z_i - \mathbb{E}(Z_i))^2]$, $i = 1,...,n$, is the variance of Z_i and $\text{Cov}(Z_i,Z_j) = \mathbb{E}[(Z_i - \mathbb{E}(Z_i))(Z_j - \mathbb{E}(Z_j))]$, $i = 1,...,n;\ j = 1,...,n$, is the covariance between Z_i and Z_j.

In what follows, $\mathbf{\Sigma}$ is an $n \times n$ covariance matrix of a multivariate Gaussian random vector $\mathbf{Z}$.

Definition II.3. The probability density function of a multivariate Gaussian random vector $\mathbf{Z} \sim \mathcal{N}(\mu, \mathbf{\Sigma})$ is given by

$$f(\mathbf{z}) = \frac{1}{\sqrt{(2\pi)^n \det(\mathbf{\Sigma})}} \exp\left(-\frac{1}{2}(\mathbf{z}-\mu)^T\mathbf{\Sigma}^{-1}(\mathbf{z}-\mu)\right)$$

where $\mu \in \mathbb{R}^n$ is a vector denoting $\mathbb{E}[\mathbf{Z}]$ (expectation of Z_i, $i = 1,...,n$); $\mathbf{\Sigma} \in \mathbb{R}^{n\times n}$, $\mathbf{Z} = [Z_1, Z_2, ..., Z_n]^T \in \mathbb{R}^n$ is a Gaussian random vector, ^{T} denotes transpose, $\det(\mathbf{\Sigma})$ denotes the determinant of $\mathbf{\Sigma}$.

In the next section, we derive the O-U operator for correlated random variables.

III. The O-U Operator for Correlated Gaussian Random Variables

The derivation of O-U operator for correlated Gaussian random variables is as follows:

Theorem III.1. *Let σ_{ii} be the diagonal entries of the inverse of the covariance matrix $\mathbf{\Sigma}$ and let σ_{ij} be other entries, $i,j = 1,...,n$. The density function f of n-dimensional correlated Gaussian random variables $Z_1,...,Z_n$ satisfies the following:*

1) $$\ln f(\mathbf{z}) = \mathbb{K} - \frac{1}{2}\left[\sum_{i=1}^{n}(z_i-\mu_i)^2\sigma_{ii} + 2\sum_{\substack{i,j=1\\ i<j}}^{n}(z_i-\mu_i)(z_j-\mu_j)\sigma_{ij}\right]$$

   where
   $$\mathbb{K} = -\frac{n}{2}\ln(2\pi) - \frac{1}{2}\ln(\det(\mathbf{\Sigma})) \qquad (1)$$
   is a constant.

2) $$\partial_{z_i} \ln f(z_1, z_2, ..., z_n) = -\left[(z_i-\mu_i)\sigma_{ii} + \sum_{j\neq i}(z_j-\mu_j)\sigma_{ij}\right]$$

   where $\mathbf{z} = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$.

Proof: In general, from the density function

$$f(\mathbf{z}) = \frac{1}{\sqrt{(2\pi)^n \det(\mathbf{\Sigma})}} \cdot \exp\left(-\frac{1}{2}(\mathbf{z}-\mu)^T\mathbf{\Sigma}^{-1}(\mathbf{z}-\mu)\right),$$

$$\begin{aligned}\ln f(\mathbf{z}) &= -\frac{n}{2}\ln(2\pi) - \frac{1}{2}\ln(\det(\mathbf{\Sigma})) \\ &\quad -\frac{1}{2}[(\mathbf{z}-\mu)^T\mathbf{\Sigma}^{-1}(\mathbf{z}-\mu)] \\ &= \mathbb{K} - \frac{1}{2}[(\mathbf{z}-\mu)^T\mathbf{\Sigma}^{-1}(\mathbf{z}-\mu)],\end{aligned}$$

where $\mathbb{K} = -\frac{n}{2}\ln(2\pi) - \frac{1}{2}\ln(\det(\mathbf{\Sigma}))$ is given by equation (1).

Let c_{ij} denote the elements on the position ith row and jth column of the covariance matrix.
$\mathbf{\Sigma}$ is the covariance matrix given by

$$\mathbf{\Sigma} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1n} \\ c_{21} & c_{22} & c_{23} & \cdots & c_{2n} \\ c_{31} & c_{32} & c_{33} & \cdots & c_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{n1} & c_{n2} & c_{n3} & \cdots & c_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbb{E}(Z_1-\mu_1)^2 & \cdots & \mathbb{E}(Z_1-\mu_1)(Z_n-\mu_n) \\ \mathbb{E}(Z_2-\mu_2)(Z_1-\mu_1) & \cdots & \mathbb{E}(Z_2-\mu_2)(Z_n-\mu_n) \\ \vdots & \vdots & \vdots \\ \mathbb{E}(Z_n-\mu_n)(Z_1-\mu_1) & \cdots & \mathbb{E}(Z_n-\mu_n)^2 \end{bmatrix},$$

$$(\mathbf{z}-\mu) = \begin{bmatrix} Z_1 - \mathbb{E}[Z_1] \\ Z_2 - \mathbb{E}[Z_2] \\ Z_3 - \mathbb{E}[Z_3] \\ \vdots \\ Z_n - \mathbb{E}[Z_n] \end{bmatrix}, \qquad \mu_i = \mathbb{E}[Z_i].$$

Let

$$\mathbf{\Sigma^{-1}} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \cdots & \sigma_{2n} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \cdots & \sigma_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \sigma_{n3} & \cdots & \sigma_{nn} \end{bmatrix}$$

be the inverse of the $n \times n$ covariance matrix.

The result is trivial for $n = 1$.
Let $n = 2$, then;

$$\begin{bmatrix} z_1-\mu_1 & z_2-\mu_2 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} z_1-\mu_1 \\ z_2-\mu_2 \end{bmatrix}$$

$$= \begin{bmatrix} z_1-\mu_1 & z_2-\mu_2 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \begin{bmatrix} z_1-\mu_1 \\ z_2-\mu_2 \end{bmatrix}$$

$$= \begin{bmatrix} (z_1-\mu_1)\sigma_{11} + (z_2-\mu_2)\sigma_{12} \\ (z_1-\mu_1)\sigma_{12} + (z_2-\mu_2)\sigma_{22} \end{bmatrix}^T \begin{bmatrix} z_1-\mu_1 \\ z_2-\mu_2 \end{bmatrix}$$

$$= (z_1-\mu_1)^2\sigma_{11} + 2(z_1-\mu_1)(z_2-\mu_2)\sigma_{12} + (z_2-\mu_2)^2\sigma_{22}.$$

Therefore,

$$\ln f(z_1, z_2) = \mathbb{K} - \frac{1}{2}[(z_1-\mu_1)^2\sigma_{11} + 2(z_1-\mu_1)(z_2-\mu_2)\sigma_{12} + (z_2-\mu_2)^2\sigma_{22}].$$

Thus, for $n = 2$;

$$\frac{\partial \ln f(z_1, z_2)}{\partial z_1} = -\frac{1}{2}[2(z_1-\mu_1)\sigma_{11} + 2(z_2-\mu_2)\sigma_{12}] = -[(z_1-\mu_1)\sigma_{11} + (z_2-\mu_2)\sigma_{12}].$$

$$\frac{\partial \ln f(z_1, z_2)}{\partial z_2} = -\frac{1}{2}[2(z_1-\mu_1)\sigma_{12} + 2(z_2-\mu_2)\sigma_{22}] = -[(z_2-\mu_2)\sigma_{22} + (z_1-\mu_1)\sigma_{12}].$$

Hence, the result holds for $n = 2$.

Assume that the result holds for $n = k$,
$\ln f(z_1, z_2, z_3, ..., z_k) =$

$$\mathbb{K} - \frac{1}{2}\left[\sum_{i=1}^{k}(z_i-\mu_i)^2\sigma_{ii} + 2\sum_{i\neq j}(z_i-\mu_i)(z_j-\mu_j)\sigma_{ij}\right].$$

$$\frac{\partial \ln f(z_1,...,z_k)}{\partial z_1} = -\frac{1}{2}\left[\frac{\partial}{\partial z_1}((z_1-\mu_1)^2\sigma_{11} + (z_2-\mu_2)^2\sigma_{22} + \cdots + (z_k-\mu_k)^2\sigma_{kk}) + 2\frac{\partial}{\partial z_1}((z_1-\mu_1)(z_2-\mu_2)\sigma_{12} + (z_1-\mu_1)(z_3-\mu_3)\sigma_{13} + \cdots + (z_1-\mu_1)(z_k-\mu_k)\sigma_{1k}) + 0\right]$$

$$= -\frac{1}{2}[(2(z_1-\mu_1)\sigma_{11} + 0 + \cdots + 0) + 2((z_2-\mu_2)\sigma_{12} + (z_3-\mu_3)\sigma_{13} + \cdots + (z_k-\mu_k)\sigma_{1k})]$$

$$= -[(z_1-\mu_1)\sigma_{11} + \textstyle\sum_{1\neq j}(z_j-\mu_j)\sigma_{1j}].$$

$$\frac{\partial \ln f(z_1,...,z_k)}{\partial z_2} = -\frac{1}{2}\left[\frac{\partial}{\partial z_2}((z_1-\mu_1)^2\sigma_{11} + (z_2-\mu_2)^2\sigma_{22} + \cdots + (z_k-\mu_k)^2\sigma_{kk}) + 2\frac{\partial}{\partial z_2}((z_2-\mu_2)\cdot(z_3-\mu_3)\sigma_{23} + (z_2-\mu_2)(z_4-\mu_4)\sigma_{24} + \cdots + (z_2-\mu_2)(z_k-\mu_k)\sigma_{2k}) + 0\right]$$

$$= -\frac{1}{2}[(2(z_2-\mu_2)\sigma_{22} + 0 + \cdots + 0) + 2((z_3-\mu_3)\sigma_{23} + (z_4-\mu_4)\sigma_{24} + \cdots + (z_k-\mu_k)\sigma_{2k})]$$

$$= -[(z_2-\mu_2)\sigma_{22} + \sum_{2\neq j}(z_j-\mu_j)\sigma_{2j}].$$

$$\vdots$$

$$\frac{\partial \ln f(z_1,...,z_k)}{\partial z_k} = -\frac{1}{2}\left[\frac{\partial}{\partial z_k}((z_1-\mu_1)^2\sigma_{11} + (z_2-\mu_2)^2\sigma_{22} + \cdots + (z_k-\mu_k)^2\sigma_{kk}) + 2\frac{\partial}{\partial z_k}((z_1-\mu_1)(z_2-\mu_2)\sigma_{12} + (z_1-\mu_1)(z_3-\mu_3)\sigma_{13} + \cdots + (z_1-\mu_1)(z_k-\mu_k)\sigma_{1k} + (z_2-\mu_2)(z_3-\mu_3)\sigma_{23} + (z_2-\mu_2)(z_4-\mu_4)\sigma_{24} + \cdots + (z_2-\mu_2)(z_k-\mu_k)\sigma_{2k} + (z_3-\mu_3)(z_4-\mu_4)\sigma_{34} + (z_3-\mu_3)(z_5-\mu_5)\sigma_{35} + \cdots + (z_3-\mu_3)(z_k-\mu_k)\sigma_{3k} + (z_4-\mu_4)(z_5-\mu_5)\sigma_{45} + (z_4-\mu_4)(z_6-\mu_6)\sigma_{46} + \cdots + (z_4-\mu_4)(z_k-\mu_k)\sigma_{4k} \\ \vdots \\ + (z_{k-2}-\mu_{k-2})(z_{k-1}-\mu_{k-1})\sigma_{k-2,k-1} + (z_{k-2}-\mu_{k-2})(z_k-\mu_k)\sigma_{k-2,k} + (z_{k-1}-\mu_{k-1})(z_k-\mu_k)\sigma_{k-1,k})\right].$$

$$\frac{\partial \ln f(z_1,...,z_k)}{\partial z_k} = -[(z_k-\mu_k)\sigma_{kk} + ((z_1-\mu_1)\sigma_{1k} + (z_2-\mu_2)\sigma_{2k} + \cdots + (z_{k-2}-\mu_{k-2})\sigma_{k-2,k} + (z_{k-1}-\mu_{k-1})\sigma_{k-1,k})]$$

$$= -[(z_k-\mu_k)\sigma_{kk} + \sum_{j\neq k}(z_j-\mu_j)\sigma_{kj}],\ j = 1,...,k-1.$$

Since it is true for $n = k$; there is need to show that it is true for $n = k+1$.
$\ln f(z_1, z_2, ..., z_{k+1}) =$

$$\mathbb{K} - \frac{1}{2}\left[\sum_{i=1}^{k+1}(z_i-\mu_i)^2\sigma_{ii} + 2\sum_{i<j}(z_i-\mu_i)(z_j-\mu_j)\sigma_{ij}\right].$$

Assume that the derivative is true for $z_i, i = 1, ..., k$ respectively, we show that it is true for z_{k+1}.

$$\frac{\partial \ln f(z_1,...,z_k,z_{k+1})}{\partial z_{k+1}} = -\frac{1}{2}\left[\frac{\partial}{\partial z_{k+1}}((z_1-\mu_1)^2\sigma_{11} + (z_2-\mu_2)^2\sigma_{22} + \cdots + (z_{k+1}-\mu_{k+1})^2\sigma_{k+1,k+1}) + 2\frac{\partial}{\partial z_{k+1}}((z_1-\mu_1)(z_2-\mu_2)\sigma_{12} + (z_1-\mu_1)(z_3-\mu_3)\sigma_{13} + \cdots + (z_1-\mu_1)(z_k-\mu_k)\sigma_{1k} + (z_1-\mu_1)(z_{k+1}-\mu_{k+1})\sigma_{1,k+1} + (z_2-\mu_2)(z_3-\mu_3)\sigma_{23} + (z_2-\mu_2)(z_4-\mu_4)\sigma_{24} + \cdots + (z_2-\mu_2)(z_k-\mu_k)\sigma_{2k} + (z_2-\mu_2)(z_{k+1}-\mu_{k+1})\sigma_{2,k+1}\right.$$

$$\vdots$$

$$+(z_{k-1}-\mu_{k-1})(z_k-\mu_k)\sigma_{k-1,k}+(z_{k-1}-\mu_{k-1})$$
$$\cdot(z_{k+1}-\mu_{k+1})\sigma_{k-1,k+1}+(z_k-\mu_k)(z_{k+1}-\mu_{k+1})\sigma_{k,k+1})\Big]$$
$$=-\frac{1}{2}[2(z_{k+1}-\mu_{k+1})\sigma_{k+1,k+1}+((z_1-\mu_1)\sigma_{1,k+1}$$
$$+(z_2-\mu_2)\sigma_{2,k+1}+\cdots+(z_{k-1}-\mu_{k-1})\sigma_{k-1,k+1}$$
$$+(z_k-\mu_k)\sigma_{k,k+1})]$$
$$=-\frac{1}{2}\Big[2(z_{k+1}-\mu_{k+1})\sigma_{k+1,k+1}+\sum_{j\neq k+1}(z_j-\mu_j)\sigma_{k+1,j}\Big].$$

Therefore, the result is true for all n. ■

Corollary III.2. *Let σ_{ii} denote the diagonal entries of the inverse of the covariance matrix $\mathbf{\Sigma}$. The density function f of n-dimensional uncorrelated Gaussian random variables $Z_1,...,Z_n$ satisfies the following:*

1) $\ln f(\mathbf{z})=\mathbb{K}-\frac{1}{2}\Big[\sum_{i=1}^{n}(z_i-\mu_i)^2\sigma_{ii}\Big]$ *where*
$$\mathbb{K}=-\frac{n}{2}\ln(2\pi)-\frac{1}{2}\ln(\det(\mathbf{\Sigma}))$$
is a constant.
2) $\partial_{z_i}\ln f(z_1,z_2,...,z_n)=-\Big[(z_i-\mu_i)\sigma_{ii}\Big].$

Theorem III.3. *Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space. Let $F=\widehat{F}(Z_1,...,Z_n)$ be a functional where $\widehat{F}:\mathbb{R}^n\to\mathbb{R}$. Assume that $Z_1,...,Z_n$ is a sequence of correlated Gaussian random variables, then,*

1) *the Skorohod integral operator $\delta:P_{(n,1)}\to S_{(n,0)}$ given for simple process $U\in P_{(n,1)}$ satisfies*
$$\delta(U)(Z_1,...,Z_n)=-\sum_{i=1}^{n}\Big[\partial_i u_i-\Big((z_i-\mu_i)\sigma_{ii}$$
$$+\sum_{j\neq i}(z_j-\mu_j)\sigma_{ij}\Big)u_i\Big](Z_1,...,Z_n)$$
where $U_i(Z_1,...,Z_n)(\omega)=u_i(Z_1(\omega),...,Z_n(\omega))$; $u_i:\mathbb{R}^n\to\mathbb{R}$, $i\in N$, $\omega\in\Omega$.
2) *the Ornstein-Uhlenbeck operator $L:S_{(n,2)}\to S_{(n,0)}$ satisfies*
$$L_iF(Z_1,...,Z_n)=-\Big[\partial_i^2F(Z_1,...,Z_n)$$
$$-\Big((z_i-\mu_i)\sigma_{ii}+\sum_{j\neq i}(z_j-\mu_j)\sigma_{ij}\Big)\partial_iF(Z_1,...,Z_n)\Big].$$

Proof:

1) In general
$\delta_{i,\pi}(U)(Z_1,...,Z_n)$
$$=-[\partial_i(\pi_iu_i)+(\pi_iu_i)\partial_{z_i}\ln f](Z_1,...,Z_n).$$
Since Z_i's are Gaussian random variables, its density function f_i, $i=1,...,n$ is everywhere differentiable on $\mathbb{R}$. Its weight function $\pi_i=1$ and its derivative $\pi_i'=0$.
$\Rightarrow\delta(U)(Z_1,...,Z_n)=$
$$-\sum_{i=1}^{n}[(\pi_i\partial_iu_i+u_i\partial_i\pi)+(\pi_iu_i)\partial_{z_i}\ln f](Z_1,...,Z_n)$$
$$=-\sum_{i=1}^{n}[\partial_iu_i+u_i\partial_{z_i}\ln f](Z_1,...,Z_n).$$
By Theorem III.1, we get
$\delta(U)(Z_1,...,Z_n)$
$$=-\sum_{i=1}^{n}\Big[\partial_iu_i+u_i\partial_{z_i}\ln\Big(\frac{1}{\sqrt{(2\pi)^n\det(\mathbf{\Sigma})}}$$
$$\cdot\exp\Big(-\frac{1}{2}(\mathbf{z}-\mu)^T\mathbf{\Sigma}^{-1}(\mathbf{z}-\mu)\Big)\Big)\Big](Z_1,...,Z_n)$$
$$=-\sum_{i=1}^{n}\Big[\partial_iu_i+u_i\partial_{z_i}\Big(\mathbb{K}-\frac{1}{2}\Big[\sum_{i=1}^{n}(z_i-\mu_i)^2\sigma_{ii}$$
$$+2\sum_{\substack{i,j=1\\i<j}}^{n}(z_i-\mu_i)(z_j-\mu_j)\sigma_{ij}\Big]\Big)\Big]$$
$$=-\sum_{i=1}^{n}\Big[\partial_iu_i-\Big((z_i-\mu_i)\sigma_{ii}+\sum_{j\neq i}(z_j-\mu_j)\sigma_{ij}\Big)u_i\Big]$$
$$\cdot(Z_1,...,Z_n).$$
2) The **Ornstein-Uhlenbeck operator** $L=\partial\delta:S_{(n,2)}\to S_{(n,0)}$ satisfies
$LF(Z_1,...,Z_n)$
$$=-\sum_{i=1}^{n}\partial_i\Big[\partial_iF+F\partial_{z_i}\ln\Big(\frac{1}{\sqrt{(2\pi)^n\det(\mathbf{\Sigma})}}$$
$$\cdot\exp\Big(-\frac{1}{2}(\mathbf{z}-\mu)^T\mathbf{\Sigma}^{-1}(\mathbf{z}-\mu)\Big)\Big)\Big](Z_1,...,Z_n)$$
$$=-\sum_{i=1}^{n}\Big[\partial_i\partial_iF+\partial_iF\partial_{z_i}\Big(\mathbb{K}-\frac{1}{2}\Big[\sum_{i=1}^{n}(z_i-\mu_i)^2\sigma_{ii}$$
$$+2\sum_{\substack{i,j=1\\i<j}}^{n}(z_i-\mu_i)(z_j-\mu_j)\sigma_{ij}\Big]\Big)\Big](Z_1,...,Z_n)$$
$$=-\sum_{i=1}^{n}\Big[\partial_i^2F-\Big((z_i-\mu_i)\sigma_{ii}+\sum_{j\neq i}(z_j-\mu_j)\sigma_{ij}\Big)$$
$$\cdot\partial_iF\Big](Z_1,...,Z_n).$$
■

Corollary III.4. *Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space. Let $F=\widehat{F}(Z_1,...,Z_n)$ be a functional such that $\widehat{F}:\mathbb{R}^n\to\mathbb{R}$. Assume that $Z_1,...,Z_n$ is a sequence of uncorrelated Gaussian random variables. Then,*

1) *the Skorohod integral operator $\delta:P_{(n,1)}\to S_{(n,0)}$ given for simple process $U\in P_{(n,1)}$ satisfies*
$$\delta(U)(Z_1,...,Z_n)=-\sum_{i=1}^{n}\Big[D_iu_i(Z_1,...,Z_n)$$
$$-(z_i-\mu_i)\sigma_{ii}u_i(Z_1,...,Z_n)\Big],$$

where σ_{ii} denote the diagonal elements of the $n \times n$ inverse covariance matrix. $U_i(Z_1, ..., Z_n)(\omega) = u_i(Z_1(\omega), ..., Z_n(\omega))$; $u_i : \mathbb{R}^n \to \mathbb{R}$, $i \in N$, $\omega \in \Omega$.

2) *the Ornstein-Uhlenbeck operator* $L : S_{(n,2)} \to S_{(n,0)}$ *satisfies*

$$L_i F(Z_1, ..., Z_n) = -\Bigg[D_i^2 F(Z_1, ..., Z_n) - (z_i - \mu_i)\sigma_{ii} D_i F(Z_1, ..., Z_n)\Bigg].$$

IV. Conclusion

We have derived the expression of the Ornstein-Uhlenbeck operator for correlated Gaussian random variables. This is to be adopted when computing sensitivities using Malliavin calculus in financial markets and phenomenon involving correlated multivariate random variables. The operator makes it easier to compute the greeks of financial instruments in a given Lévy market involving more than one random variables. The greeks deal with the effect of changes with respect to parameters of a given model. For future research, we suggest its application in a phenomenon with correlated random variables.

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