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Lagrangian-Dual and Sensitivity Analysis In Portfolio Selection

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ABSTRACT

In this paper, we consider the Lagrangian dual problem in portfolio optimization problems. The Lagrangian dual can be used to solve integer programming in which knapsack problem is one of them. We modelled Knapsack problem as a portfolio problem which consists of health care and oil and gas sector from 2010-2014. We used Lagrangian duality to solve the problem and the Lagrangian multiplier as the sensitivity coefficients.

Keywords: Lagrangian-Dual, Sensitivity Analysis, Knapsack problem, Portfolio Selection

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1. INTRODUCTION

The Knapsack problem can be defined as a set of items, each with a weight(w) and a profit(p), determine the number(n) of each item to include in a collection(j) so that the total

weight is less than or equal to a given limit and the total profit(p) is as large as possible.

Mathematically it can be represented as follows:

$$\max_x \sum_{j=1}^{n} p_j x_j \quad (1)$$

$$s.t \ \sum_{j=1}^{n} w_j x_j \leq C \quad (2)$$

$$x_j = 0 \ or \ 1, j = 1, \ldots, n \quad (3)$$

The difficulty of the problem is caused by the integrality requirement of equation (3). The knapsack problem has received wide attention from the operations research community, because of its uses in many practical problems. Applications include resource allocation in distributed systems, capital budgeting and cutting stock problems [5,6,7].The knapsack problem have been applied in areas such as information technologies [1], resource constrained project scheduling [2], auditing [3] and health care [4].

The Lagrangian dual can be used to find dual bounds of the integer programming in which knapsack problem is among them. M. Fisher [8] used Lagrangian Relaxation Method to Solving Integer Programming Problems.Mayank Verma and R.R.K. Sharma [9] proposed a two-level, multi-item, multi-period-capacitated dynamic lot-sizing problem with inclusions of backorders and setup times, is solved using a novel procedure. In their paper they used a single-constraint continuous knapsack problem and the reduced problem is solved using bounded variable linear programs (BVLPs).

In terms of computational time, their developed procedure is efficient than the CPLEX solver of GAMS. In this study, we shall be modelling Knapsack problem to a portfolio selection of health care and oil and gas sector.

2. MATERIALS AND METHODS

In this section, we shall considered a function of three variables and derive a lagrangian dual from the figure 3.8 below. We shall also be using the same as the sensitivity analysis.

Proof: We shall provide the proof of Lagrangian dual If we consider $z = f(x, y)$ where z is a function of two independent variables x and increases in x and y will produce a combined increase in z.

Given $z = f(x, y)$ then

$$z + \delta x = f(x + \delta x, y + \delta y)$$

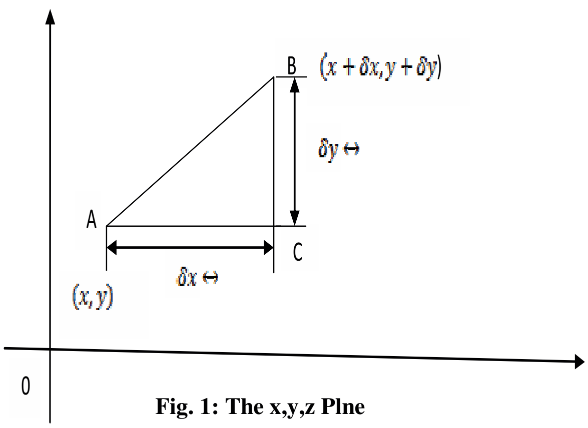

Fig. 1: The x,y,z Plne

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For C: $f(x+\delta x, y) = f(x,y) + \delta x f'_x(x,y) + \frac{(\delta x)^2}{2!} f''_{xx}(x,y) + \cdots$ (4)

where $f'_x(x,y)$ denotes $\frac{\partial}{\partial x} f(x,y)$; $f''_{xx}(x,y)$ denotes $\frac{\partial^2}{\partial x^2} f(x,y)$

From B to C: $(x+\delta x)$ is constant; y changes to $(y+\delta y)$

∴

$$f(x+\delta x, y+\delta y) = f(x+\delta x, y) + \delta y f'_y(x+\delta x, y) + \frac{\delta y^2}{2!} f''_{yy}(x+\delta x, y) + \frac{\delta y^2}{2!} f''_{yy}(x+\delta x, y) + \ldots \quad (5)$$

differentiate (4) with respect to y, we have

$$f'_y(x+\delta x, y) = f'_y(x,y) + \delta x f'_{yx}(x,y) + \frac{(\delta x)^2}{2!} f'''_{yxx}(x,y) + \ldots \quad (6)$$

$$f''_{yy}(x+\delta x, y) = f''_{yy}(x,y) + \delta x f'''_{yyx}(x,y) + \frac{(\delta x)^2}{2!} f^{iv}_{yyxx}(x,y) + \ldots \quad (7)$$

substituting equations (4), (6) and (7) to (5)
it now becomes

$$f(x+\delta x, y+\delta y) = f(x,y) + \delta x f'_x(x,y) + \frac{(\delta x)^2}{2!} f''_{xx}(x,y) + \ldots$$
$$+ \delta y \{ f'_y(x,y) + \delta x f''_{yx}(x,y) + \frac{(\delta x)^2}{2!} f'''_{yxx}(x,y) + \ldots \}$$
$$+ \frac{(\delta y)^2}{2!} \{ f''_{yy}(x,y) + \delta x f'''_{yyx}(x,y) + \frac{(\delta x)^2}{2!} f^{iv}_{yyxx}(x,y) + \ldots \}$$
$$+ \ldots \quad (8)$$

We rearranging the terms by collecting together all the first derivatives, and then all the

second derivatives, and so on, we get

$$f(x+\delta x, y+\delta y) = f(x,y) + \{ \delta x f'_x(x,y) + \delta y f'_y(x,y) \}$$
$$+ \frac{1}{2!} \{ (\delta x)^2 f''_{xx}(x,y) + 2\delta x \delta y f''_{yx}(x,y) + (\delta y)^2 f''_{yy}(x,y) \} + \ldots \quad (9)$$

Then, if $z = f(x,y)$, then $z + \delta z = f(x+\delta x, y+\delta y)$

Substituting $z + \delta z = f(x+\delta x, y+\delta y)$ to equation (9) ,we have

$$z + \delta z = z + \left\{ \delta x \frac{\partial z}{\partial x} + \delta y \frac{\partial z}{\partial y} \right\} + \frac{1}{2!} \left\{ (\delta x)^2 \frac{\partial^2 z}{\partial x^2} + 2\delta x \delta y \frac{\partial^2 z}{\partial y \partial x} \; (\delta y)^2 \frac{\partial^2 z}{\partial y^2} \right\} + \ldots \quad (10)$$

Subtracting z from each side

$$\delta z = \frac{\partial z}{\partial x} \delta x + \frac{\partial z}{\partial y} \delta y + \frac{1}{2!} \left\{ \frac{\partial^2 z}{\partial x^2} (\delta x)^2 + 2 \frac{\partial^2 z}{\partial y \partial x} (\delta x \delta y) + \frac{\partial^2 z}{\partial y^2} (\delta y)^2 \right\} + \ldots \quad (11)$$

since δx and δy are small, the expression in the brackets is of the next order of smallness and can be discarded. Therefore, we arrive at the result below

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if $z = f(x, y)$, then $\delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y$ (12)

for z to have a stationary value, then

$$0 = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y \quad (13)$$

We find δy from equation (13)

$$\frac{\partial f}{\partial y}\delta y = -\frac{\partial f}{\partial x}\delta x \quad (14)$$

$$\therefore \delta y = -\frac{\frac{\partial f}{\partial x}\delta x}{\frac{\partial f}{\partial y}} \quad (15)$$

Substituting equation (15) to (12) and rearrange it, we have

$$\delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial f}{\partial x}[-\frac{\frac{\partial f}{\partial x}\delta x}{\frac{\partial f}{\partial y}}]\delta x \quad (16)$$

$$\delta z = \frac{\partial z}{\partial x}\delta x - \frac{\partial f}{\partial x}[\frac{\frac{\partial f}{\partial x}\delta x}{\frac{\partial f}{\partial y}}]\delta x \quad (17)$$

Now we define λ as the value of $[\frac{\frac{\partial f}{\partial x}\delta x}{\frac{\partial f}{\partial y}}]$ as the stationary point of the constrained functions.

(17) can be written as

$$\delta z = [\frac{\partial z}{\partial x} - \lambda\frac{\partial f}{\partial x}]\delta x \quad (18)$$

Dividing through by δx

$$\frac{\delta z}{\delta x} = [\frac{\partial z}{\partial x} - \lambda\frac{\partial f}{\partial x}] \quad (19)$$

Then if $\delta x \to 0$, we have

$$\frac{dz}{dx} = \frac{\partial(z-\lambda f)}{\partial x} \quad (20)$$

if $\frac{dz}{dx} = 0$ and $L = z - \lambda f$

$\frac{\partial(z-\lambda f)}{\partial x} = 0$ or $\frac{\partial L}{\partial x} = 0$

The equations reduced to $L = z - \lambda f$ (21)

where,

$L =$ Lagragian function

z = objective function

λ = lagragian multiplier or sensitivity coefficients

$f =$ constraints

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Substituiting knapsack problem (1-3) into lagragian function, we have

$$L(X,\lambda) = \sum_{j=1}^{n} p_j x_j - \lambda[C - \sum_{j=1}^{n} w_j x_j] \quad (22)$$

where:

$$z = \sum_{j=1}^{n} p_j x_j \quad (23)$$

$$f = C - \sum_{j=1}^{n} w_j x_j \quad (24)$$

$p =$ profit

$w =$ weight

$n =$ number of items from which to choose

$x = \{0,1\}$

$j =$ 1 to n

$X = x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}$

Dual function:

$$D(\lambda) = \max_{X \epsilon \{0,1\}^n} \sum_{j=1}^{n} p_j x_j - \lambda[C - \sum_{j=1}^{n} w_j x_j] \quad (25)$$

Thus the Lagragian dual:

$$\min_{\lambda \geq 0} D(\lambda) = \min_{\lambda \geq 0} (\max_{X \epsilon \{0,1\}^n} \sum_{j=1}^{n} p_j x_j - \lambda[C - \sum_{j=1}^{n} w_j x_j]) \quad (26)$$

In this study, we shall be using λ as the sensitivity coefficient. Equations(22) can be rearrange as follows

$$\lambda = \frac{z}{f} \quad (27)$$

Equations (27) say that at maximum point the ratio of z to f is the same for every x_i where i = 1,2,3,..., and moreover it equals λ. The numerator z gives the marginal benefit of each x_i to the function z to be maximized. Moreover, λ gives the approximate change in z due to a one unit change in x_i.

Similarly, the denominators have a marginal cost interpretation, namely, f gives the marginal cost of using x_i, in other words, the approximate change in f due to a unit change in x_i. We therefore summarize λ as the common benefit-cost ratio(sensitivity coefficients) for all the $x's$. i.e

$$\lambda = \frac{z}{f}$$

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3. RESULTS AND DISCUSION

able 1 below shows the expected profit of both health care and oil and gas return which include 18 companies which are Total(TOT), Eterna(ETE), Conoil(CON), Japaul(JAP), Mobil(MOB), Mrs(MRS), Forte(FOR), Oando(OAN), Ekocarp(EKO), Evansmedical(EVA), Fidson(FID), Glaxosmithkline(GLA), May & Baker(MAY), Morison(MOR), Neimeth(NEI), Nigerian-German(NIG), Pharma-Deko(PHA) and Union Diagnostic(UNI) from 2010-2014. The results were obtained from the Nigeria Stock Exchange (NSE). In order to obtain an estimate for an expected return on our investments or expected profit, we will solve for the mean of our data points from the return. We will also calculate the earning per share price which was extracted from the annual reports of each companies. Table 1 shows the Expected profit (EP) and Earning per share (EPS)

Table 1: Expected Profit and Earning Per Share of listed Stocks

Stocks	Mean(Expected Profits) (EP)	Earning Per Share (EPS)
TOT*	0.0036	11.796
ETE*	0.0440	0.706
CON	-0.0260	298.4
JAP*	0.0706	1.596
MOB	-0.0380	1227.2
MRS	0.0220	3.188
FOR	-0.1551	4.89
OAN*	0.1427	2.64
EKO*	0.0333	31.35
EVA*	0.0333	0.33
FID	-0.0483	24.5
GLA	-0.0692	2.88
MAY*	0.0713	0.22
MOR*	0.1568	16.58
NEI*	0.0384	9
NIG*	0.0628	138.8
PHA*	0.0612	186.4
UNI*	0.0203	2.3

As shown in Table 1 stocks with * by their names have positive figures as their mean whiles those with the * have negative mean. A share with a positive mean indicates that share is expected to yield positive returns (profits) while those with negative means those shares will decline in value over a period. With the main objective of this work being to maximize the return on our investments with the help of knapsack concept, we will now consider stocks with positive expected returns. These profitable stocks are shown in the table below with their respective earning per share.

Table 2: Profitable shares with their respective prices.

Variables	Stocks	Mean(Expected Profits) (EP)	Earning Per Share (EPS)
x_1	TOT	0.0036	11.796
x_2	ETE	0.0440	0.706
x_3	JAP	0.0706	1.596
x_4	MRS	0.0220	3.188
x_5	OAN	0.1427	2.64
x_6	EKO	0.0333	31.35
x_7	EVA	0.0333	0.33
x_8	MAY	0.0713	0.22
x_9	MOR	0.1568	16.58
x_{10}	NEI	0.0384	9
x_{11}	NIG	0.0628	1.38
x_{12}	PHA	0.0612	18.64
x_{13}	UNI	0.0203	2.3

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Table 2 captures profitable shares as well as their respective prices. The various expected profits for the individual shares will form the coefficients of the objective function of our problem whiles their corresponding earning per share prices will be the coefficients of the constraint.

3.1 Formulation of the capacity
There are millions of shares of these stocks which are traded daily on the stock exchange. The amount of money you need to invest on the Nigerian Stock Exchange (NSE) depends on the price of shares you select. Shares are usually traded in batches or round lots of 100. Where the price of a particular stock is high, an investor can contact a broker to buy fever than 100 shares or what is commonly referred to as odd lots. In this study, we shall adopt the ₦ 10.00 per share as the amount to be invested.

3.2 Objective Function
The objective function which seeks to maximize the Return of Investment (R), will be equated to the summation of the expected returns of the various individuals shares. The coefficients of the objective function are derived from the expected profits indicated in the table.

Thus is given by:

$$R = 0.0036x_1 + 0.044x_2 + 0.0706x_3 + 0.022x_4 + 0.1427x_5 + 0.033x_6 + 0.0333x_7 + 0.0713x_8 + 0.1568x_9 + 0.0384x_{10} + 0.0628x_{11} + 0.0612x_{12} + 0.0203x_{13} \quad (28)$$

3.3 Constraint
The constraint consist of the summation of the individual earning per share which is on the financial statement of each stock is considered. These coefficients are indicated in the table as the earning per share.

Thus is given by:

$$11.796x_1 + 0.706x_2 + 1.596x_3 + 3.188x_4 + 2.64x_5 + 31.35x_6 + 0.33x_7 + 0.22x_8 + 16.58x_9 + 9x_{10} + 1.38x_{11} + 18.64x_{12} + 2.3x_{13} \leq 10 \quad (29)$$

3.4 Lagrangian Dual Solution
The numerical results below are the lagrangian dual in equation (26) and the objective function(28), constraints (29) were used to solve the dual.

$$L(\underline{x}, \lambda) = 0.0036x_1 + 0.044x_2 + 0.0706x_3 + 0.022x_4 + 0.1427x_5 + 0.033x_6 + 0.0333x_7 + 0.0713x_8 + 0.1568x_9 + 0.0384x_{10} + 0.0628x_{11} + 0.0612x_{12} + 0.0203x_{13} + \ldots \lambda[10 - (11.796x_1 + 0.706x_2 + 1.596x_3 + 3.188x_4 + 2.64x_5 + 31.35x_6 + 0.33x_7 + 0.22x_8 + 16.58x_9 + 9x_{10} + 1.38x_{11} + 18.64x_{12} + 2.3x_{13})]$$

$$L(\underline{x}, \lambda) = (0.0036-11.\ \lambda)x_1 + (0.044-0.706\lambda)x_2 + (0.0706-1.596\lambda)x_3 + (0.022-3.188\lambda)x_4 + (0.1427-2.64\lambda)x_5 + (0.033-31.35\lambda)x_6 + (0.0333-0.33\lambda)x_7 + (0.0713-0.22\lambda)x_8 + (0.1568-16.58\lambda)x_9 + (0.0384-9\lambda)x_{10} + (0.0628-1.38\lambda)x_{11} + (0.0612-18.64\lambda)x_{12} + (0.0203-2.3\lambda)x_{13} + 10\lambda$$

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Table 3: Optimal Solution of the Lagragian subproblem

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	$D(\lambda)$	λ
1	1	1	1	1	1	1	1	1	1	1	1	1	0.76-89.72λ	[0.00030534,∞]
0	1	1	0	1	0	1	1	0	0	1	0	1	0.445+0.828λ	[0.06232295,0.00030534]
0	0	1	1	1	1	1	1	1	1	1	1	1	0.7124-77.224λ	[0.04423559,0.06232295]
0	0	0	1	1	1	1	1	1	1	1	1	1	0.6418-75.628λ	[0.00690088,0.04423559]
0	0	0	0	1	1	1	1	1	1	1	1	1	0.6198-72.44λ	[0.05405303,0.00690088]
0	0	0	0	0	1	1	1	1	1	1	1	1	0.4771-69.8λ	[0.00105263,0.05405303]
0	0	0	0	0	0	1	1	1	1	1	1	1	0.4441-38.45λ	[0.10090909,0.00105263]
0	0	0	0	0	0	0	1	1	1	1	1	1	0.4108-38.12λ	[0.32409091,0.10090909]
0	0	0	0	0	0	0	0	1	1	1	1	1	0.3395-37.9λ	[0.00945718,0.32409091]
0	0	0	0	0	0	0	0	0	1	1	1	1	0.1827-21.32λ	[0.0042667,0.00945718]
0	0	0	0	0	0	0	0	0	0	1	1	1	0.1443-21.32λ	[0.04550725,0.0042667]
0	0	0	0	0	0	0	0	0	0	0	1	1	0.0815-10.94λ	[0.00328326,0.04550725]
0	0	0	0	0	0	0	0	0	0	0	0	1	0.0203-7.7λ	[0.00882609,0.00328326]
0	0	0	0	0	0	0	0	0	0	0	0	0	10λ	[∞,0.00882609]

From Table 3, the optimal solution of the dual is given as

$D(\lambda) =$ 0.445+0.828λ, where λ = 0.00030534

When we substitute λ = 0.00030534 to the dual $D(\lambda) =$ 0.445+0.828λ.

λ indicates that if the Earning per share constraints were increase by one unit, the value

λ = 0.00030534 estimates the increase of the total expected profit.

It also shows that there will be a positive increase in the profit if the value of capacity increase, from the result indicates a small change. The results also show that there is no much gain in the Nigeria Stock Exchange from 2010 to 2014.

4. Future Work

Future work is to apply Lagrangian dual to other integer programming problems which includes facility location problem, capital budgeting, set covering problem, fixed charge problem and Job scheduling problem.

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