Pulse Vaccination Strategy in a SVEIRS Epidemic Model with Two-Time Delay and Saturated Incidence

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Abstract Finding the best way to vaccinate people against infectious disease is an important issue for health workers. In this study a compartmental two-time delay SVEIRS mathematical model with pulse vaccination and saturated incidence was formulated to examine the dynamics of infectious disease in a population. The existence of the disease free periodic solution was established and the compact form was derived. From our study, it was discovered that short pulse vaccination or long latent period or long immune period will guarantee eradication of the disease in the population. Lastly, the conditions for the incurability of the disease were examined.

Keywords SVEIRS, Incurability, Permanence, Global Attractivity, Disease-Free Periodic Solution, Immunity

1. Introduction

To control infectious diseases, vaccination is always the approach recommended by health workers and epidemiologists. Controlling and eliminating diseases has been studied by many authors, see [1] and [2]. Constant vaccination strategy or approach has been the major method proposed by many WHO workers and consultants, but Agur [3] started the study of pulse vaccination strategy as an alternative to constant vaccination strategy. Since then a lot of researchers had worked on the theoretical and practical advantages of pulse vaccination strategy, see [3-7]. The application of pulse vaccination strategy to epidemiological models was restricted to SIS and SIR models, but recently researchers have been studying more complex classifications [6] and [8-10]. Pulse vaccination is the repeated application of vaccine across age cohorts.

Meng et al [11] and Jin et al [12] studied an SIR model with some people failing to obtain immunity after first dose but gained immunity after later doses.

As we know immunity to infectious diseases after being vaccinated against them might not be life long, so in this study we assume that the latent and immunity (not permanent) period are constants. Delayed models (ordinary delay or two-time delay) with nonlinear or saturated incidence have also been proposed by some researchers, see [10-16], so a two-time delay model was replicated in this study but of SVEIRS type. Precisely, we consider a saturated incidence of the form $\beta I/(1+\alpha S)$ and the two-time delay (latent and immunity) for this model.

The focus of this work is to determine the conditions for which the disease will be eradicated or otherwise become incurable in a population.

2. Model Formulation

In this section we are going to formulate the model by considering vaccination as an epidemiological class.

In this regard, we use the following notations and assumptions for subsequent development: Let $S(t)$, $V(t)$, $E(t)$, $I(t)$ and $R(t)$ denote respectively the susceptible, vaccinated, exposed, infectious and recovered partitions of the population, and $N(t)$ be the total population. We assume that new individuals enter into the susceptible class of the population at a constant recruitment rate A and death will occur to them at the rate $\mu S(t)$ (where μ is the per capital natural mortality rate). With the assumption of saturated incidence $\beta I(t)/(1+\alpha S(t))$ the interaction between susceptible individuals and infectious individuals can the represented by $\beta I(t)/(1+\alpha S(t))$, where β is the adequate contact rate that can lead to infection and α is the saturated incidence parameter. If $\beta\upsilon$ is the adequate contact of an infective and a vaccinated individual per unit time, then the interaction of infectives and vaccinated is describe by $\beta\upsilon V(t)I(t)/(1+\alpha S(t))$.

Death will occur to the vaccinated class, exposed class, infectious class and recovered class at the rate, $\mu V(t)$, $\mu E(t)$, $\mu I(t)$ and $\mu R(t)$ respectively and there will be an additional disease induced death of infectious individuals at the rate $\varphi I(t)$, where φ is the death rate due to infection. We let γ be the recovery parameter of infectious

individuals and ρ is the average rate for vaccinated individuals to obtain immunity and move to recovered class, τ is the average immune period of the population and ω, the latent period of the disease.

Note: Parameters $A, \beta, \mu, \gamma, \omega, \tau, \alpha, \varphi, \upsilon, \rho$ are positive constant.

So, using the formulation above we have the following system as the SVEIRS model:

$$S'(t) = A - \frac{\beta S(t)I(t)}{1+\alpha S(t)}$$

$$-\mu S(t) + \rho I(t-\tau)e^{-\mu\tau}$$

$$V'(t) = -\left(\gamma+\mu+\frac{\upsilon\beta I(t)}{1+\alpha S(t)}\right)V(t)$$

$$E'(t) = \frac{\beta S(t)I(t)}{1+\alpha S(t)} + \frac{\upsilon\beta I(t)}{1+\alpha S(t)}$$

$$-\frac{\beta S(t-\omega)I(t-\omega)e^{-\mu\omega}}{1+\alpha S(t-\omega)} \tag{1}$$

$$-\frac{\upsilon\beta V(t-\omega)I(t-\omega)e^{-\mu\omega}}{1+\alpha S(t-\omega)} - \mu E(t)$$

$$I'(t) = \frac{\beta S(t-\omega)I(t-\omega)e^{-\mu\omega}}{1+\alpha S(t-\omega)}$$

$$+\frac{\upsilon\beta V(t-\omega)I(t-\omega)e^{-\mu\omega}}{1+\alpha S(t-\omega)}$$

$$-\frac{\beta S(t-\omega-\tau)I(t-\omega-\tau)e^{-(\omega+\tau)\mu}e^{-\varphi\tau}}{1+\alpha S(t-\omega-\tau)}$$

$$-(\mu+\varphi+\rho)I(t)$$

$$R'(t) = \gamma V(t)$$

$$+\rho I(t)\frac{\beta S(t-\omega-\tau)I(t-\omega-\tau)e^{-(\omega+\tau)\mu}e^{-\varphi\tau}}{1+\alpha S(t-\omega-\tau)}$$

$$-\mu R(t) - \rho I(t-\tau)e^{-\mu\tau}$$

Now, we are going to incorporate pulse vaccination strategy into our model, this will lead us to the following impulsive system:

$$\left.\begin{array}{l} S'(t) = A - \dfrac{\beta S(t)I(t)}{1+\alpha S(t)} \\[2mm] -\mu S(t) + \rho I(t-\tau)e^{-\mu\tau} \\[2mm] V'(t) = -\left(\gamma+\mu+\dfrac{\upsilon\beta I(t)}{1+\alpha S(t)}\right)V(t) \\[2mm] E'(t) = \dfrac{\beta S(t)I(t)}{1+\alpha S(t)} + \dfrac{\upsilon\beta I(t)}{1+\alpha S(t)} \\[2mm] -\dfrac{\beta S(t-\omega)I(t-\omega)e^{-\mu\omega}}{1+\alpha S(t-\omega)} \\[2mm] -\dfrac{\upsilon\beta V(t-\omega)I(t-\omega)e^{-\mu\omega}}{1+\alpha S(t-\omega)} - \mu E(t) \\[2mm] I'(t) = \dfrac{\beta S(t-\omega)I(t-\omega)e^{-\mu\omega}}{1+\alpha S(t-\omega)} \\[2mm] +\dfrac{\upsilon\beta V(t-\omega)I(t-\omega)e^{-\mu\omega}}{1+\alpha S(t-\omega)} \\[2mm] -\dfrac{\beta S(t-\omega-\tau)I(t-\omega-\tau)e^{-(\omega+\tau)\mu}e^{-\varphi\tau}}{1+\alpha S(t-\omega-\tau)} \\[2mm] -(\mu+\varphi+\rho)I(t) \\[2mm] R'(t) = \gamma V(t) + \rho I(t) \\[2mm] +\dfrac{\beta S(t-\omega-\tau)I(t-\omega-\tau)e^{-(\omega+\tau)\mu}e^{-\varphi\tau}}{1+\alpha S(t-\omega-\tau)} \\[2mm] -\mu R(t) - \rho I(t-\tau)e^{-\mu\tau} \end{array}\right\}, t \neq \kappa T, \kappa \in \mathbb{Z}^+ \tag{2}$$

$$\left.\begin{array}{l} S(t^+) = (1-\theta)S(t^-) \\[1mm] V(t^+) = V(t^-) + \theta S(t^-) \\[1mm] E(t^+) = E(t^-) \\[1mm] I(t^+) = I(t^-) \\[1mm] R(t^+) = R(t^-) \end{array}\right\} t = \kappa T, \kappa \in \mathbb{Z}^+$$

$\mathbb{Z}^+ = \{0,1,2,...\}$, T is the period of pulsing and

$$N(t) = S(t) + V(t) + E(t) + I(t) + R(t) \forall t \geq 0 \tag{3}$$

where $N(t)$ is the total population at time t

.

3. Model Description and Preliminaries

By system (2)and equation (3) we have;

$$N'(t) = A - \mu N(t) - \varphi I(t) \qquad (4)$$

(by summing from the first to the fifth equation of (2)), therefore the total population is not constant over time. It follows from (1) that

$$A - (\mu + \varphi)N(t) \le N'(t) = A - \mu N(t) - \varphi I(t)$$
$$\le A - \mu N$$

$$\Rightarrow \frac{A}{\mu + \varphi} \le \liminf_{t\to\infty} N(t) \le \limsup_{t\to\infty} N(t) \le \frac{A}{\mu} \qquad (5).$$

Following the approach of Gao et al [10,17] and Song et al [14], we can see from (2) that the first, second and fourth equations of the system does not contain the variables $E(t)$ and $R(t)$, so we are going to focus on the following equivalent system of (2).

$$\begin{cases} \left. \begin{aligned} & \frac{dS(t)}{dt} = A - \frac{\beta S(t)I(t)}{1+\alpha S(t)} \\ & \quad - \mu S(t) + \rho I(t-\tau)e^{-\mu\tau} \\ & \frac{dV(t)}{dt} = -\left(\gamma + \mu + \frac{\upsilon\beta I(t)}{1+\alpha S(t)}\right)V(t) \\ & \frac{dI(t)}{dt} = \frac{\beta S(t-\omega)I(t-\omega)e^{-\mu\omega}}{1+\alpha S(t-\omega)} \\ & \quad + \frac{\upsilon\beta V(t-\omega)I(t-\omega)e^{-\mu\omega}}{1+\alpha S(t-\omega)} \\ & \quad - \frac{\beta S(t-\omega-\tau)I(t-\omega-\tau)e^{-(\omega+\tau)\mu}e^{-\varphi\tau}}{1+\alpha S(t-\omega-\tau)} \\ & \quad - (\mu+\varphi+\rho)I(t) \\ & \frac{dN(t)}{dt} = A - \mu N(t) - (\varphi+\rho)I(t) \end{aligned} \right\}, t \ne \kappa T, \kappa \in \mathbb{Z}^+ \\ \left. \begin{aligned} & S(t^+) = (1-\theta)S(t^-) \\ & V(t^+) = V(t^-) + \theta S(t^-) \\ & I(t^+) = I(t^-) \\ & N(t^+) = N(t^-) \end{aligned} \right\} t = \kappa T, \kappa \in \mathbb{Z}^+ \end{cases} \qquad (6)$$

The initial conditions for (6)are of the form

$$S(\xi) = \phi_1(\xi); V(\xi) = \phi_2(\xi); I(\xi) = \phi_3(\xi);$$
$$N(\xi) = \phi_4(\xi); -l \le \xi \le 0 \qquad (7).$$

We have,
$\phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T$ such that $C_+ = C([-l,0], \mathfrak{R}_+^4)$ and C_+ is the space of continuous functions on $[-l,0]$ with uniform norm; where

$$l = \max\{\tau, \omega\} \text{ moreso, } \mathfrak{R}_+ = [0,\infty);$$

$$\mathfrak{R}_+^4 = \{J \in \mathfrak{R}^4 : J \ge 0\}$$

The solution of (6) is a piecewise continuous function $J : \mathfrak{R}_+ \to \mathfrak{R}_+^4, J(t)$ has a point of discontinuity at $-\kappa\tau(\kappa \in J^+)$ of the first kind and continuous on $(kT, (k+1)T], k \in J^+$ and $J(kT^+) = \lim_{t\to T^+} J(t)$ exists.

Since ϕ is smooth then the solution of (6) exists and is unique (using definition of smooth function).
We assume that
$\phi_i(0) > 0 \forall i = 1,2,3,4$ and since $S'(t)|_{s=0} > 0$
and $I'(t) = 0$ where ever $I(t) = 0 \forall t \ne \kappa T, \kappa \in \mathbb{Z}^+$.
Also, using the fact that

$$S(\kappa T^+) = (1-\theta)S(\kappa T^-), V(\kappa T^+) = V(\kappa T^-) + \theta S(\kappa T^-),$$
$$I(\kappa T^+) = I(\kappa T^-), N(\kappa T^+) = N(\kappa T^-)$$

for $t \ne \kappa T, \kappa \in \mathbb{Z}^+$.
Now from (6), since $N(t) = S(t) + V(t) + I(t)$ then we have the following lemma.

Lemma 1
Suppose $J(t)$ is a solution of (6) with initial condition (7) then,

$$S(t) \le \frac{A}{\mu}; V(t) \le \frac{A}{\mu}, I \le \frac{A}{\mu} \forall t \text{ large enough.}$$

Proof
Since $N(t^+) = N(t^-) = N(t) \forall t \ge 0$ then $N(t)$ is continuous on $t \in [0, +\infty]$ and by (4)

$$\frac{dN(t)}{dt} = A - \mu N(t) - \varphi I(t). \text{ We have}$$

$$\frac{dN(t)}{dt} = A - \mu N(t) - \varphi I(t) \le A - \mu N(t)$$

$$\Rightarrow \liminf_{t\to\infty} N(t) \le \limsup_{t\to\infty} N(t) \le \frac{A}{\mu}.$$

So $N(t)$ is uniformly bounded. If there exist a positive integer κ_1, by definition of $N(t)$ we have that,

$$S(t) \le \frac{A}{\mu}, V(t) \le \frac{A}{\mu}, I(t) \le \frac{A}{\mu} \forall t \ge \kappa_1 T$$

More so, biologically we assume that
$\phi_i(0) > 0 \forall i = 1,2,3,4$ then the system (6) is positively invariant in the closed set:

$$\Theta = \{(S,V,I,N) \in \mathfrak{R}_+^4 |; 0 \le S+V+I \le \frac{A}{\mu}, N \le \frac{A}{\mu}\}.$$

Definition 1 Uniform Persistence
System (6) is said to be uniformly persistent if there is a $\lambda > 0$
(independent of the initial conditions) such that every solution $(S(t), V(t), I(t), N(t))$ of system (6) with initial conditions (7) satisfies

$$\liminf_{t\to\infty} S(t) \geq \lambda, \liminf_{t\to\infty} V(t) \geq \lambda,$$
$$\liminf_{t\to\infty} I(t) \geq \lambda, \liminf_{t\to\infty} N(t) \geq \lambda$$

Definition 2 Permanence

System(6) is said to be permanent in Θ if there exists a compact region $\Theta_0 \in \text{int } \Theta$ such that every solution of system (6) with initial conditions (7) will eventually enter and remain in region Θ_0.

Lemma 2 [10]

Consider the following impulsive system

$$\begin{cases} u'(t) = a - bu(t); & t \neq \kappa T \\ u(t^+) = (1-\theta)u(t^-); & t = \kappa T \end{cases} \quad (8)$$

where $a > 0, b > 0, 0 < \theta < 1$. Then there exists a unique positive periodic solution of system (8) given by

$$\tilde{u}_{a,b}(t) = \frac{a}{b} + (u^*_{a,b} - \frac{a}{b})e^{-b(t-\kappa T)}; \kappa T < t \leq (\kappa+1)T$$

which is asymptotically stable, where

$$u^*_{a,b} = \frac{a(1-\theta)(1-e^{-bT})}{b(1-(1-\theta)e^{-bT})}$$

Lemma 3 [18]

Let us consider the following impulsive differential inequalities:

$$w'(t) \leq (\geq)P(t)w(t) + g(t); \quad t \neq t_k$$
$$w(t_k^+) \leq (\geq)d_k w(t_k) + b_k; \quad t = t_k; k \in N$$

where $P(t), g(t) \in C[R_+, R]; d_k \geq 0$ and b_k are constants.

Assume,

i)the sequence $\{t_k\}$ satisfies $0 \leq t_0 < t_1 < t_2 < ....$, with $\lim_{t\to\infty} t_k = \infty$;

ii) $w \in PC'[R_+, R]$ and $w(t)$ is left continuous at $t_k, k \in N$. Then

$$w(t) \leq (\geq)w(t_0) \prod_{t_0 < t_k < t} d_k \exp\left(\int_{t_0}^t P(s)ds\right) + \sum_{t_0 < t_k < t} \left(\prod_{t_0 < t_k < t} d_k \exp\left(\int_{t_0}^t P(s)ds\right)\right)b_k$$
$$+ \int_{t_0}^t \prod_{s < t_k < t} d_k \exp\left(\int_s^t P(\theta)d\theta\right)q(s)ds; \quad t \geq t_0$$

Lemma 4 [9] and [19]

Consider the following equation,

$$\dot{x}(t) = a_1 x(t-\omega) - a_2 x(t)$$

where $a_1, a_2, \omega > 0; x(t) > 0$ for $-\omega \leq t \leq 0$. We have;

(i)If $a_1 < a_2$; then $\lim_{t\to+\infty} x(t) = 0$

(ii)If $a_1 > a_2$; then $\lim_{t\to+\infty} x(t) = +\infty$

3.1. Model Analysis

3.1.1. Global Attractivity of Disease-free Periodic Solution

From (5) and the fact that

$$\frac{dS(t)}{dt} = A - \frac{\beta S(t)I(t)}{1+\alpha S(t)} - \mu S(t) + \rho I(t-\tau)e^{-\mu\tau}$$

we have

$$\frac{dS(t)}{dt} \geq A - \left(\mu + \frac{A\beta}{\mu + \alpha A}\right)S(t)$$

Now we show that there exist a disease-free periodic solution for system (6), that is the infectious individuals are absent from the population (i.e. $I(t) = 0 \forall t \geq 0$).

So if $I(t) = 0 \forall t \geq 0$ then (6) simplifies to

$$\begin{cases} \dfrac{dS(t)}{dt} = A - \mu S(t) \\ \dfrac{dV(t)}{dt} = -(\gamma + \mu)V(t) \\ \dfrac{dN(t)}{dt} = A - \mu N(t) \end{cases} \quad t \neq \kappa T, \kappa \in \mathbb{Z}^+$$

$$\begin{cases} S(t^+) = (1-\theta)S(t^-) \\ V(t^+) = V(t^-) + \theta S(t^-) \\ N(t^+) = N(t^-) \end{cases} \quad t = \kappa T, \kappa \in \mathbb{Z}^+$$

(9)

which is the disease-free system whose unique solution exists [9,20] and derived subsequently.

For $I(t) = 0 \forall t \geq 0$ we have

$$\lim_{t\to\infty} N(t) = \frac{A}{\mu}$$

Therefore using the fact that $N(t) = S(t) + V(t) + I(t)$ we have the following equations

$$V(t) = \frac{A}{\mu} - S(t) \quad (10)$$

and

$$\begin{cases} \dfrac{dS(t)}{dt} = A - \mu S(t) & t \neq \kappa\tau \\ S(t^+) = (1-\theta)S(t^-) & t = \kappa\tau \end{cases} \quad (11).$$

Solving (11) between pulses and using the discrete dynamical system determined by the stroboscopic map and the integral of the system, we have

$$\tilde{S}_{A,\mu}(t) = \frac{A}{\mu} + \left(S^*_{A,\mu} - \frac{A}{\mu}\right)e^{-\mu(t-\kappa T)}; \quad \kappa T < t \leq (\kappa+1)T$$

$$\Rightarrow S^* = \frac{A(1-\theta)(1-e^{-\mu T})}{\mu(1-(1-\theta)e^{-\mu T})}$$

$$\equiv \frac{A}{\mu}\left[1 - \frac{\theta e^{-\mu(t-\kappa T)}}{1-(1-\theta)e^{-\mu T}}\right]; \quad \kappa T < t \leq (\kappa+1)T, \kappa \in \mathbb{Z}^+$$

(12)

By (10) and (12) we have

$$V^*(t) = \frac{A}{\mu}\left[\frac{\theta(1-e^{-\mu T})e^{-(\mu+\gamma)(t-\kappa T)}}{(1-e^{-(\mu+\gamma)T})(1-(1-\theta)e^{-\mu T})}\right]$$

(13),

with $\kappa T < t \le (\kappa+1)T$; $\kappa \in \mathbb{Z}^+$ and

$$\left.\begin{array}{l}S^*(0^+) = S^*(\kappa T^+) \\ \quad = \frac{A}{\mu}\left[1 - \frac{\theta}{1-(1-\theta)e^{-\mu T}}\right] \\ V^*(0^+) = V(\kappa T^+) \\ \quad = \frac{A}{\mu}\left[\frac{\theta(1-e^{-\mu T})}{(1-e^{-(\mu+\gamma)T})(1-(1-\theta)e^{-\mu T})}\right]\end{array}\right\} ; t = \kappa T$$

By Lemma 2, (13) is GAS and the solution of (9) is

$$\left.\begin{array}{l}S(t) = \frac{A}{\mu} + \left[\frac{A}{\mu}\left(1 - \frac{\theta}{1-(1-\theta)e^{-\mu T}}\right) - \frac{A}{\mu}\right]e^{-\mu(t-\kappa T)} \\ \quad = S(0^+) - \frac{A}{\mu}\left[1 - \frac{\theta}{1-(1-\theta)e^{-\mu T}}\right] \\ V(t) = V(0^+)e^{-(\mu+\gamma)(t-\kappa T)}\end{array}\right\}$$

$$\kappa T < t \le (k+1)T; \quad \kappa \in \mathbb{Z}^+$$

So, the disease-free periodic solution of (6) is;

$$(S^*(t), \frac{A}{\mu} - S^*(t), 0, \frac{A}{\mu}) .$$

Next we determine the global attractivity condition of the disease-free periodic solution $(S^*(t), (A/\mu) - S^*(t), 0, A/\mu)$ of system (6).

First we consider the following lemma;

Lemma 5

System (9) has a unique positive solution

$$(S^*(t), V^*(t), \frac{A}{\mu}),$$

meaning that the system (6) has a disease-free periodic solution

$$(S^*(t), V^*(t), 0, \frac{A}{\mu}) \forall\ t \in (\kappa T, (\kappa+1)T], \quad \kappa \in \mathbb{Z}^+,$$

so for any solution $(S(t), V(t), I(t), N(t))$ of system (6), then $S(t) \to S^*(t), V(t) \to V^*(t)$ and

$$N(t) \to \frac{A}{\mu} \ as\ t \to \infty.$$

Proof

The proof follows from Lemma 2.

Now, we set the basic reproduction number R^* to

$$R^* = \frac{A\beta e^{-\mu\omega}}{\mu+\varphi}\left(\frac{r_1 r_2}{\mu(r_2+\theta)+\alpha r_1 r_2} + \frac{\theta r_1 r_2 r_3}{\mu(r_2+\theta)(1-r_3)+\alpha\theta r_1 r_2 r_3}\right)$$

where $r_1 = \mu + \rho e^{-\mu\tau}, r_2 = e^{\mu T} - 1, r_3 = e^{-(\mu+\gamma)T}$

then we state and prove the following theorem.

Theorem 1

If $R^* < 1$, then the disease-free periodic solution $(S^*(t), V^*(t), 0, \frac{A}{\mu})$ of system (6) is globally attractive.

Proof

Let $(S(t), V(t), I(t), N(t))$ be a solution of system (6) with initial conditions (7). Since $R^* < 1$, we can choose an $\varepsilon > 0$ sufficiently small such that;

$$\beta e^{-\mu\omega}\left(\begin{array}{c}\frac{Ar_1 r_2}{\mu(r_2+\theta)+\alpha r_1 r_2} \\ + \frac{A\theta r_1 r_2 r_3}{\mu(r_2+\theta)(1-r_3)+\alpha\theta r_1 r_2 r_3} \\ +\varepsilon\end{array}\right) - (\mu+\varphi) < 0$$

$$\Rightarrow \beta e^{-\mu\omega}\left(\begin{array}{c}\frac{Ar_1 r_2}{\mu(r_2+\theta)+\alpha r_1 r_2} \\ + \frac{A\theta r_1 r_2 r_3}{\mu(r_2+\theta)(1-r_3)+\alpha\theta r_1 r_2 r_3} + \varepsilon\end{array}\right) < (\mu+\varphi)\ (14)$$

From the first and fifth equations of system (6) it follows that for $\kappa \ge \kappa_1$;

$$S'(t) \le A - \mu S(t) + \rho e^{-\mu\tau}; \quad t \ne \kappa T, \quad \kappa \in \mathbb{Z}^+$$

$$S(t) = (1-\theta)S(t); \quad\quad\quad t = \kappa T, \quad \kappa \in \mathbb{Z}^+$$

By Lemma 3, we have

$$S(t) \le S(\kappa_1 T^+) \prod_{\kappa_1 < \kappa T < t} (1-\theta)\exp\int_{\kappa_1 T}^{t} -\mu ds +$$

$$\int_{\kappa_1 T}^{t} \prod_{s<\kappa T<t}(1-\theta)\exp\int_{s}^{t} -\mu d\vartheta(A+\rho e^{-\mu\tau})ds$$

$$= S_1 + S_2$$

where

$$S_1 = S(\kappa_1 T^+)\prod_{\kappa_1 < \kappa T < t}(1-\theta)\exp\int_{\kappa_1 T}^{t} -\mu d(T\varsigma)$$

$$= S(\kappa_1 T^+)(1-\theta)^{[t/T]}e^{-\mu(t-\kappa T)}$$

and

$$S_2 = \frac{A+\rho e^{-\mu\tau}}{\mu}\left[\frac{(1-\theta)^{[t/T]-\kappa_1+1}(e^{\mu T}-1)e^{\kappa_1\mu T}}{-(1-\theta)(e^{\mu T}-1)e^{\mu T[t/T]}e^{\mu T}} - e^{\mu T[t/T]}\right]$$

so if we set

$$q(t) = e^{-\mu t}\begin{bmatrix} S(\kappa_1 T^+)(1-\theta)^{[{t}/{T}]}e^{\kappa_1 \mu T} \\[2mm] -\dfrac{A+\rho e^{-\mu\tau}}{\mu}\dfrac{(1-\theta)^{[{t}/{T}]-\kappa_1+1}(e^{\mu T}-1)e^{\kappa_1 \mu T}}{e^{\mu T}-1+\theta} \end{bmatrix}$$

$$q(t) \le e^{-\mu t}S(\kappa_1 T^+)(1-\theta)^{[{t}/{T}]}e^{\kappa_1 \mu T}$$

$$= e^{-\mu t}S(\kappa_1 T^+)(1-\theta)^{[{t}/{T}]+1}e^{\kappa_1 \mu T} < e^{-\mu t}S(\kappa_1 T^+)e^{\kappa_1 \mu T}$$

and

$$S(t) \le S_1 + S_2 = q(t) + \frac{A+\rho e^{-\mu\tau}}{\mu}\left[1 - \frac{\theta e^{\mu T([{t}/{T}]+1-{t}/{T})}}{e^{\mu T}-1+\theta}\right]$$

From these, we have

$$S(t) \le e^{-\mu t}S(\kappa_1 T^+)e^{\kappa_1 \mu T} + \frac{A+\rho e^{-\mu\tau}}{\mu}\left[1 - \frac{\theta}{e^{\mu T}-1+\theta}\right]$$

$$\Rightarrow \limsup_{t\to\infty} S(t) \le \frac{A+\rho e^{-\mu\tau}}{\mu}\left[1 - \frac{\theta}{e^{\mu T}-1+\theta}\right]$$

So if there exist a positive integer $\kappa_2 \ge \kappa_1$ and an arbitrarily small positive constant ε such that $t \ge \kappa_2 T$, we have

$$S(t) \le \frac{A+\rho e^{-\mu\tau}}{\mu}\left[1 - \frac{\theta}{e^{\mu T}-1+\theta}\right] + \varepsilon = \Psi_1 \quad (15)$$

Again from the second and sixth equations of (6), we have

$$\forall \kappa \ge \kappa_1$$
$$V'(t) \le -(\gamma+\mu)V(t), \qquad t \ne \kappa T, \quad \kappa \in \mathbb{Z}^+$$
$$V(t) = V(t) + \theta S(t) \qquad t = \kappa T, \quad \kappa \in \mathbb{Z}^+$$

We consider the following comparison impulsive differential system

$$y'(t) = -(\gamma+\mu)y; \qquad t \ne \kappa T \qquad \kappa \in \mathbb{Z}^+$$
$$y(t) = y + \theta S; \qquad t = \kappa T \qquad \kappa \in \mathbb{Z}^+ \quad (16).$$

Solving the equation between pulses and using discrete dynamical system approach, we have

$$y^*(t) = \frac{\theta e^{-(\mu+\gamma)T}(A+\rho e^{-\mu\tau})}{\mu(1-e^{-(\mu+\gamma)T})}\left[1 - \frac{\theta}{e^{\mu T}-1+\theta}\right]; \kappa T < t \le (\kappa+1)T \ \ \kappa \in \mathbb{Z}^+$$

$$y^*(0^+) = \frac{\theta(A+\rho e^{-\mu\tau})}{\mu(1-e^{-(\mu+\gamma)T})}\left[1 - \frac{\theta}{e^{\mu T}-1+\theta}\right]; t = \kappa T, \qquad \kappa \in \mathbb{Z}^+$$

which is GAS by Lemma 2.

Let $S(t), V(t), I(t), N(t)$ be the solution of system (6) with initial condition (7) and

$S(0^+) = S_0 > 0, V(0^+) = V_0 > 0$, $y(t)$ be the solution of system (16) with initial condition $y(0^+) = V_0$

By comparison theorem for impulsive differential equation [20], there exists an integer $\kappa > 0$ such that

$$V(t) < y(t) < y^*(t) + \varepsilon_0; \qquad \kappa T < t \le (\kappa+1)T, \quad \kappa \in \mathbb{Z}^+$$

$$\Rightarrow V(t) < y^* + \varepsilon_0 \le \frac{\theta e^{-(\mu+\gamma)T}(A+\rho e^{-\mu\tau})}{\mu(1-e^{-(\mu+\gamma)T})}\left[1 - \frac{\theta}{e^{\mu T}-1+\theta}\right] + \varepsilon_0 = \Psi_2 \quad (17).$$

From (15), (17) and third equation of (6), $\forall t > \kappa_2 T + \omega$ we have,

$$I' \le \left(\frac{\beta e^{-\mu\omega}\Psi_1 I(t-\omega)}{1+\alpha\Psi_1}\right) + \left(\frac{\upsilon\beta e^{-\mu\omega}\Psi_2}{1+\alpha\Psi_1}\right) - (\mu+\varphi+\rho)I(t) \quad (18).$$

Let us consider the following impulsive equation as the comparison equation

$$x' = \left(\frac{\beta e^{-\mu\omega}\Psi_1 x(t-\omega)}{1+\alpha\Psi_1}\right) + \left(\frac{\beta e^{-\mu\omega}\Psi_2 x(t-\omega)}{1+\alpha\Psi_1}\right) - (\mu+\varphi+\rho)x(t).$$

From (14), we have

$$\left(\frac{\beta e^{-\mu\omega}\Psi_1}{1+\alpha\Psi_1}\right) + \left(\frac{\beta e^{-\mu\omega}\Psi_2}{1+\alpha\Psi_1}\right) < (\mu+\varphi+\rho).$$

And so by Lemma 5 we have,

$$\lim_{t\to\infty} x(t) = 0$$

Since $I(s) = x(s) = \phi_3 > 0 \ \forall \ s \in [-l, 0]$, then by comparison theorem of differential equations and the fact that the solution of system (6) are non-negative with $I(t) \ge 0$ then we have $I(t) \to 0$ as $t \to \infty$.

Next we assume that $0 < I(t) < \varepsilon \ \ \forall \ t \ge 0$. From (6) we have

$$S'(t) \ge A - \mu S - \beta\varepsilon$$
$$V'(t) \ge -\upsilon\beta\varepsilon - (\gamma+\mu)V(t)$$

If we have

$$\begin{cases} \left.\begin{aligned} b_1'(t) &= A - \mu b_1 - \beta\varepsilon \\ c_1'(t) &= -\upsilon\beta\varepsilon - (\gamma+\mu)c_1'(t) \end{aligned}\right\} t \ne \kappa T; \quad \kappa \in \mathbb{Z}^+ \\[3mm] \left.\begin{aligned} b_1(t^+) &= (1-\theta)b_1 \\ c_1(t^+) &= c_1 + \theta b_1 \end{aligned}\right\} \qquad t = \kappa T; \quad \kappa \in \mathbb{Z}^+ \quad (19) \\[3mm] b_1(0^+) = S(0^+) \\ c_1(0^+) = V(0^+) \end{cases}$$

and assuming that $b_1(t) \to S^*(t),\ c_1(t) \to V^*(t)$ and

$$N \to \frac{A}{\mu} \text{ as } \varepsilon \to 0$$

where $(\tilde{b}_1(t), \tilde{c}_1(t))$ is a unique positive periodic solution of (19).

From (25) we have

$$\tilde{b}_1 = \frac{A - \beta \varepsilon}{\mu}\left(1 - \frac{\theta e^{-\mu(t - \kappa T)}}{1 - (1 - \theta)e^{-\mu T}}\right)$$

$$\tilde{c}_1 = -\frac{\upsilon \beta \varepsilon}{\mu + \gamma}$$

$$+ \frac{A - \beta \varepsilon}{\mu}\left[\frac{\left(\frac{\theta}{1 - e^{-(\mu + \gamma)T}}\right)}{\left(1 - \frac{\theta e^{-\mu T}}{1 - (1 - \theta)e^{-\mu T}}\right)}\right]e^{-(\mu + \gamma)(t - \kappa T)}$$

$$\forall \kappa T < t \le (\kappa + 1)T$$

Again, using the comparison theorem of impulsive system [11], we have the following:

If there exist $\varepsilon_1 > 0\ \forall T_1 > 0$ then

$$\begin{cases} S(t) > \tilde{b}_1 - \varepsilon_1 \\ V(t) > \tilde{c}_1 - \varepsilon_1 \end{cases} \quad \forall t > T \qquad (20).$$

From (6) we have

$$S'(t) \le A - \mu S(t) + \rho \varepsilon e^{-\mu \tau}$$

$$V'(t) \le -\gamma V(t) - \mu V(t)$$

So, if we have
$\tilde{b}_2, \tilde{c}_2$ as the unique positive periodic solution of

$$\begin{cases} b_2'(t) = A - \mu b_2(t) - \rho \varepsilon e^{-\mu \tau} \\ c_2'(t) = -(\gamma + \mu)c_2(t) \end{cases} t \ne \kappa T; \quad \kappa \in \mathbb{Z}^+$$
$$\begin{cases} b_2(t^+) = (1 - \theta)b_2 \\ c_2(t^+) = c_2 + \theta b_2 \end{cases} \quad t = \kappa T; \quad \kappa \in \mathbb{Z}^+\ (21).$$
$$b_2(0^+) = S(0^+)$$
$$c_2(0^+) = V(0^+)$$

Then

$$S(t) \le \tilde{b}_2(t), V(t) \le \tilde{c}_2(t),$$

and $\tilde{b}_2(t) \to S^*(t),\ \tilde{c}_2(t) \to V^*(t)$ as $\varepsilon \to 0$.

So, from (21) we have

$$\tilde{b}_2 = \frac{A + \rho e^{-\mu \tau}}{\mu}\left(1 - \frac{\theta e^{-\mu(t - \kappa T)}}{1 - (1 - \theta)e^{-\mu T}}\right)$$

$$\tilde{c}_2 = \frac{\theta e^{-(\mu + \gamma)(t - \kappa T)}}{1 - e^{-(\mu + \gamma)T}}\left(1 - \frac{\theta e^{-\mu T}}{1 - (1 - \theta)e^{-\mu T}}\right)$$

$$\forall \kappa T < t \le (\kappa + 1)T$$

Now, using the comparison theorem of impulsive system, as presented in [21], we have the following; for $\varepsilon_1 > 0\ \exists\ T_2 > 0$ such that

$$\begin{cases} S(t) < \tilde{b}_2(t) + \varepsilon_1 \\ V(t) < \tilde{c}_2(t) + \varepsilon_1 \end{cases} \quad \forall t > T_2 \qquad (22).$$

If we let $\varepsilon \to 0$, then from (20) and (22) we have

$$\begin{cases} S^*(t) - \varepsilon_1 < S(t) < S^*(t) + \varepsilon_1 \\ V^*(t) - \varepsilon_1 < V(t) < V^*(t) + \varepsilon_1 \end{cases} \text{ as } t \to \infty$$

$$\Rightarrow S(t) \to S^*(t), V(t) \to V^*(t) \text{ as } t \to \infty.$$

From the above and using the fact that

$$\lim_{t \to \infty} N(t) = \frac{A}{\mu}, \text{ the proof is complete.}$$

Next, let

$$\tau_* = \frac{1}{\mu}\ln\left[1 + \frac{\theta(\mu + \varphi + \rho)\mu}{A\beta e^{-\mu \omega} - \mu(\mu + \varphi + \rho)}\right]$$

$$\theta_* = \frac{(e^{\mu \tau} - 1)(A\beta e^{-\mu \omega} - \mu(\mu + \varphi + \rho))}{\mu(\mu + \varphi + \rho)}$$

and

$$\omega_* = -\frac{1}{\mu}\ln\left[\frac{\mu(\mu + \varphi + \rho)(1 - (1 - \theta)e^{-\mu \tau})}{A\beta(1 - e^{-\mu \tau})}\right]$$

Corollary 1

If $\theta > \theta_*$ or $\omega > \omega_*$, the infection-free periodic solution

$\left(S^*(t), \frac{A}{\mu} - S^*(t), 0, \frac{A}{\mu}\right)$ is globally attractive.

Corollary 2

For system (6) with initial condition (7) we have the infection-free periodic solution

$\left(S^*(t), \frac{A}{\mu} - S^*(t), 0, \frac{A}{\mu}\right)$ is globally attractive provided that;

(i) $A\beta e^{-\mu \omega} > \mu(\mu + \varphi + \rho)$ and $\tau < \tau_*$

(ii) $A\beta(1 - e^{-\mu \tau}) > \mu(\mu + \varphi + \rho)(1 - (1 - \theta)e^{-\mu \tau})$
    and $\omega > \omega_*$.

3.1.2. Incurability or Permanence of the Disease

The system (6) is said to be permanent if there are positive constants $q_i, Q_i, i = 1, 2, 3, 4$ and a finite time T_0 such that for all solution $(S(t), V(t), I(t), N(t))$ with initial values,

$$S(0^+) > 0, V(0^+) > 0, I(0^+) > 0 \text{ and } N(0^+) > 0;$$

$$q_1 \leq S(t) \leq Q_1, q_2 \leq V(t) \leq Q_2, q_3 \leq I(t) \leq Q_3;$$

$$q_4 \leq N(t) \leq Q_4 \text{ holds } \forall \ t \geq T_0$$

Let,

$$R_* = \frac{A\beta e^{-\mu\omega}}{(1+\alpha)(\mu+\varphi+\rho)} \left(\frac{(1-e^{-(\mu+\varphi+\rho)\tau})}{+\frac{\upsilon\theta e^{-(\upsilon\beta+\mu+\gamma)T}}{1-e^{-(\upsilon\beta+\mu+\gamma)T}}} \right) \left(\frac{(1-\theta)(1-e^{-\mu T})}{1-(1-\theta)e^{-\mu T}} \right)$$

and

$$I_* = \frac{\mu}{\beta}(R_* - 1)$$

Theorem 2

If $R_* > 1$, then the disease is permanently incurable, i.e. there exist a positive constant q_2 such that $I(t) \geq q_2$ for a large enough t.

Proof

Suppose $\aleph(t) = (S(t), V(t), I(t), N(t))$ is any nonnegative solution of (6) with initial conditions (7) and so from the third equation of (7), we have the following equivalent equation:

$$I'(t) = \frac{\beta e^{-\mu\omega} S(t)I(t)}{1+\alpha S(t)} + \frac{\upsilon\beta e^{-\mu\omega} V(t)I(t)}{1+\alpha S(t)}$$

$$+\beta e^{-(\omega+\tau)\mu} e^{-\varphi\tau} \left[\frac{S(t)I(t)}{1+\alpha S(t)} - \frac{S(t-\omega-\tau)I(t-\omega-\tau)}{1+\alpha S(t-\omega-\tau)} \right]$$

$$-\frac{\beta e^{-(\omega+\tau)\mu} e^{-\varphi\tau} S(t)I(t)}{1+\alpha S(t)} - (\mu+\varphi+\rho)I(t)$$

$$-\beta e^{-\mu\omega} \left(\frac{S(t)I(t)}{1+\alpha S(t)} - \frac{S(t-\omega)I(t-\omega)}{1+\alpha S(t)} \right)$$

$$-\upsilon\beta e^{-\mu\omega} \left(\frac{V(t)I(t)}{1+\alpha S(t)} - \frac{V(t-\omega)I(t-\omega)}{1+\alpha S(t)} \right)$$

$$= \frac{\beta e^{-\mu\omega} S(t)I(t)}{1+\alpha S(t)} + \frac{\upsilon\beta e^{-\mu\omega} V(t)I(t)}{1+\alpha S(t)}$$

$$-\frac{\beta e^{-(\omega+\tau)\mu} e^{-\varphi\tau} S(t)I(t)}{1+\alpha S(t)}$$

$$-(\mu+\varphi+\rho)I(t)$$

$$-\beta e^{-\mu\omega} \left[\frac{d}{dt} \int_{t-\omega}^{t} \frac{S(\theta)I(\theta)}{1+\alpha S(\theta)} d\theta + \upsilon \frac{d}{dt} \int_{t-\omega}^{t} \frac{V(\theta)I(\theta)}{1+\alpha S(\theta)} d\theta \right] \quad (23)$$

$$+\beta e^{-(\omega+\tau)\mu} e^{-\varphi\tau} \frac{d}{dt} \int_{t-\omega-\tau}^{t} \frac{S(\theta)I(\theta)}{1+\alpha S(\theta)} d\theta$$

Next, assume a positive solution $(S(t), V(t), I(t), N(t))$ of system (6) and set

$$D(t) = I(t) + \beta e^{-\mu\omega} \left[\int_{t-\omega}^{t} \frac{S(\theta)I(\theta)}{1+\alpha S(\theta)} d\theta + \upsilon \int_{t-\omega}^{t} \frac{V(\theta)I(\theta)}{1+\alpha S(\theta)} d\theta \right]$$

$$-\beta e^{-(\omega+\tau)\mu} e^{-\varphi\tau} \frac{d}{dt} \int_{t-\omega-\tau}^{t} \frac{S(\theta)I(\theta)}{1+\alpha S(\theta)} d\theta$$

From (23), we calculate the derivative of $R(t)$ evaluated along the solution of (6) as

$$D'(t) = \frac{\beta e^{-\mu\omega} S(t)I(t)}{1+\alpha S(t)} + \frac{\upsilon\beta e^{-\mu\omega} V(t)I(t)}{1+\alpha S(t)}$$

$$-\frac{\beta e^{-(\omega+\tau)\mu} e^{-\varphi\tau} S(t)I(t)}{1+\alpha S(t)} - (\mu+\varphi+\rho)I(t)$$

$$= (\mu+\varphi+\rho)I(t) \left[\beta e^{-\mu\omega} \left(\frac{\frac{S(t)}{(\mu+\varphi+\rho)(1+\alpha S(t))}}{+\frac{\upsilon V(t)}{(\mu+\varphi+\rho)(1+\alpha S(t))}} - \frac{e^{-(\mu+\varphi)\tau} S(t)}{(\mu+\varphi+\rho)(1+\alpha S(t))} \right) - 1 \right]$$

$$\Rightarrow$$

$$D'(t) = (\mu+\varphi+\rho)I(t) \left[\beta e^{-\mu\omega} \left(\frac{(1-e^{-(\mu+\varphi)\tau})}{(\mu+\varphi+\rho)(1+\alpha S(t))} + \frac{\upsilon V(t)}{(\mu+\varphi+\rho)(1+\alpha S(t))} \right) - 1 \right] \quad (24).$$

Since $R_* > 1$, we can easily see that $I_* > 0$ and there exist a positive constant $2\varepsilon_1$ small enough such that;

$$\frac{\beta e^{-\mu\omega}}{(\mu+\varphi+\rho)}\left(\frac{A_1(1-e^{-(\mu+\varphi+\rho)\tau})}{1+\alpha A_1}+\frac{\upsilon A_2}{1+\alpha A_1}\right)$$

$$> \frac{\beta e^{-\mu\omega}}{(\mu+\varphi+\rho)}\left(\frac{A_1(1-e^{-(\mu+\varphi+\rho)\tau})}{1+\alpha}+\frac{\upsilon A_2}{1+\alpha}\right)>1 \quad (25),$$

where

$$A_1 = \frac{A}{\beta I_* + \mu}\left(\frac{(1-\theta)(1-e^{-(\beta I_*+\mu)T})}{1-(1-\theta)e^{-(\beta I_*+\mu)T}}\right)-\varepsilon_1$$

$$A_2 = \frac{A}{\beta I_* + \mu}\left(\frac{\theta e^{-(\upsilon\beta I_*+\mu+\gamma)T}}{1-e^{-(\upsilon\beta I_*+\mu+\gamma)T}}\right)\left(\frac{(1-\theta)(1-e^{-(\beta I_*+\mu)T})}{1-(1-\theta)e^{-(\beta I_*+\mu)T}}\right)-\varepsilon_1$$

Now, we assume that for any $t_0 > 0$ it is not possible that $I(t) < I_* \quad \forall \ t \geq t_0$.

By contradiction, suppose that this is not valid, then there exist a $t_0 > 0$ such that $I(t) < I_* \quad \forall \ t \geq t_0$.

From system (6) we have

$$\begin{cases} S'(t) \geq A - \beta S(t)I_* - \mu S(t) \\ V'(t) \geq -(\gamma+\mu)V(t)-\upsilon\beta V(t)I_* \end{cases} \begin{matrix} t \neq \kappa T; \kappa \in \mathbb{Z}^+ \end{matrix}$$

$$\begin{cases} S(t^+) = (1-\theta)S(t) \\ V(t^+) = V(t)+\theta S(t) \end{cases} \quad t = \kappa T; \kappa \in \mathbb{Z}^+$$

Consider the following comparison impulsive system for $t \geq t_0$;

$$q'(t) = A - (\mu+\beta I_*)q(t) \quad t \neq \kappa T; \quad \kappa \in \mathbb{Z}^+$$
$$q(t) = (1-\theta)q(t) \quad t = \kappa T; \quad \kappa \in \mathbb{Z}^+ \quad (26).$$

By Lemma 2, we have

$$\tilde{q}(t) = \frac{A}{(\mu+\beta I_*)}+\left(q^*-\frac{A}{(\mu+\beta I_*)}\right)e^{-(\mu+\beta I_*)(t-\kappa T)}$$

as the unique positive periodic solution of (26) which is globally asymptotically stable with

$$q^* = \frac{A}{(\mu+\beta I_*)}\left(\frac{(1-\theta)(1-e^{-(\mu+\beta I_*)T})}{1-(1-\theta)e^{-(\mu+\beta I_*)T}}\right)$$

By comparison theorem of impulsive differential equation, there is $T_1 \geq (t_0+\omega+\tau)$ such that

$$S(t) > \frac{A}{\beta I_*+\mu}\left(\frac{(1-\theta)(1-e^{-(\beta I_*+\mu)T})}{1-(1-\theta)e^{-(\beta I_*+\mu)T}}\right)-\varepsilon_1 = A_1$$

$$V(t) > \frac{A}{\beta I_*+\mu}\left(\frac{\theta e^{-(\upsilon\beta I_*+\mu+\gamma)T}}{1-e^{-(\upsilon\beta I_*+\mu+\gamma)T}}\right)\left(\frac{(1-\theta)(1-e^{-(\beta I_*+\mu)T})}{1-(1-\theta)e^{-(\beta I_*+\mu)T}}\right)-\varepsilon_1 = A_2$$

$$\Rightarrow$$
$$S(t) > A_1 \quad (27).$$
$$V(t) > A_2$$

From (24) and (27) we have that

$$R'(t) > (\mu+\varphi)I(t)\left[\beta e^{-\mu\omega}\left(\frac{(1-e^{-(\mu+\varphi)\tau})A_1}{(\mu+\varphi)(1+\alpha)}+\frac{\upsilon A_2}{(\mu+\varphi)(1+\alpha)}\right)-1\right] \ for \ t \geq t_1$$
$$(28).$$

Set

$$I_l = \min_{t \in [t_1+\omega, t_1+\omega+\tau]} I(t)$$

We want to show that $I(t) \geq I_* \quad \forall t \geq t_1$.

Suppose the contrary, there is a $t_2 \geq t_1+\omega+\tau$ such that $I(t_2) = I_l$ and $I(t) \geq I_l$ for $t \in [t_1,t_2]$.

From (6), (25) and (28) we see that

$$I'(t_2) \geq \left[\beta e^{-\mu\omega}\left(\frac{(1-e^{-(\mu+\varphi+\rho)\tau})A_1}{(\mu+\varphi+\rho)(1+\alpha)}+\frac{\upsilon A_2}{(\mu+\varphi+\rho)(1+\alpha)}\right) -(\mu+\varphi+\rho)\right]I_l = (\mu+\varphi+\rho)I_l(R_*-1)>0$$

which is a contradiction. Hence the claim is true. From (28), we have that

$$R'(t) > (\mu+\varphi)I_l(R_*-1) > 0$$
$$\Rightarrow R(t) \to \infty \text{ as } t \to +\infty$$

This contradicts

$$R(t) \leq \frac{A}{\mu}+\beta e^{-\mu\omega}\frac{A^2}{\mu^2}\left[(1-e^{(\mu+\varphi)\tau})+\upsilon\right]$$

Also for t large enough we have two other cases to examine:

$(i) I(t) \geq I_*$

$(ii) I(t)$ oscillate about I_* for t large enough

$$m = \min\left\{\frac{I_*}{2}, m_2\right\}$$
Set

We hope to show that $I(t) \geq m_2$. For the first case $(I(t) \geq I_*)$ the conclusion is obvious.

For the second case, if there exist t^* (large enough) such that

$$I(t^*) = I_*, S(t) > A_1 \text{ and } V(t) > A_2 \text{ for } t \geq t^* \quad (29).$$

$I(t)$ is uniformly continuous because it is not affected by impulse and it is bounded.

Therefore there exist a $\vartheta(0 < \vartheta < \tau)$ which is independent of the choice of t^*

such that $I(t) > \dfrac{I^*}{2} \, \forall t^* \leq t \leq t^* \vartheta$

From the condition that

$$\frac{A_1^* e^{-\mu\omega - (\mu+\varphi+\rho)\tau}}{1+\alpha} + \frac{\upsilon A_2^* e^{-\mu\omega}}{1+\alpha} > 1$$

then there exist $\varepsilon > 0$ sufficiently small such that

$$\frac{(A_1^* - \varepsilon) e^{-\mu\omega - (\mu+\varphi+\rho)\tau}}{1+\alpha} + \frac{\upsilon(A_2^* - \varepsilon) e^{-\mu\omega}}{1+\alpha} > 1$$

$$\Rightarrow \frac{A_1 e^{-\mu\omega - (\mu+\varphi+\rho)\tau}}{1+\alpha} + \frac{\upsilon A_2 e^{-\mu\omega}}{1+\alpha} > 1 \tag{30}$$

From (6), (29) and (30) we have

$$I(t) \geq \left(\frac{e^{-(\mu+\varphi+\rho)\tau} A_1}{1+\alpha A_1} + \frac{\upsilon A_2}{1+\alpha A_1} \right) \beta e^{-\mu\omega} \frac{I_*}{2} \vartheta \geq I_* \vartheta$$

$$\doteq m_2, \quad t \leq t^* + \omega \leq t^* + \omega + \tau$$

We can continue with the same argument for

$$t \leq t^* + \omega + \kappa\tau \leq t^* + \omega + (\kappa+1)\tau.$$

Therefore, we conclude that with the choice of m_2 which is independent of the nonnegative solution of (6) we have $I(t) \geq m_2$.

Hence the proof of Theorem 2.

4. Conclusions

Theorem 1 states the global attractiveness of periodic solution

$$\left(S^*(t), \frac{A}{\mu} - S^*(t), 0, \frac{A}{\mu} \right)$$

of system (6) with initial condition (7) in domain Θ for the case $R^* < 1$, i.e the disease will be eradicated from the population. Corollary 1 and corollary 2 implied that the disease will be eradicated if the pulse vaccination rate is larger than θ_*, the latent period is longer than ω_* and the infectivity period is shorter than τ_*. In this work, we have studied the delayed SVEIRS epidemic model with pulse vaccination and saturated incidence. We have been able to determine the conditions for which the disease will be eradicated in the population through the use of pulse vaccination strategy. Also we found that a smaller pulse vaccination rate or a shorter latent period of the disease or a shorter immunity period could lead to the disease been permanent in the population. Furthermore, the inclusion of saturated incidence in term of susceptible, allowed us to study the dynamics of the diseases with respect to population

with very high proportion of it prone to being infected.

Lastly, with this new model, optimal control theory can be employed to determine the cost benefit of maximizing the recovered individuals and minimizing the infected and susceptible individuals by adopting the framework proposed in [22].

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