MATHEMATICAL MODELING OF BLOOD FLOW IN THE STENOSED ARTERY

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Abstract

In this paper, an analytical study of effects of blood flow in the blood vessels has been investigated. The viscosity of the blood is assumed to be varying radially with hematocrit throughout the region of the artery. The blood flow is assumed to be Newtonian fluid, incompressible, laminar, and steady. In order to model this problem, Navier-Stokes equations was used to derive the governing equations that represent this problem. We simplified the governing equations by linearizing the equations and solved by finite different method. We observed that the size of the blood vessel influenced the blood flow. A little change on the cross-sectional value makes a very significant change on the blood flow rate.

Keywords: Blood flow, Newtonian, Navier-Stokes, Artery, Density, Finite difference method.

1.0 Introduction

In the human circulatory system, artery delivers oxygenated rich blood with nutrients from the heart to each cell of the body. The study of the behaviour of blood flow in the blood vessels provides an understanding of the connection between flow and the development of diseases such as atherosclerosis, aneurysms and thrombosis. The abnormal elongation of arterial thickness is the first step in the formation of atherosclerosis disease. The accumulation of substances in the artery along the wall is known as stenosis, the presence of which changes the flow behavior and hemodynamic conditions of the artery. Coronary artery disease is caused by atherosclerosis that occurred due to stenosis which formed by fatty substances, cholesterol, cellular waste products, and smooth muscle cells accumulation on the arterial wall Akhbar *et al.* (2019). Stenosis is a localize plaque that cause the narrowing on the vessel wall and causing an alternation in the flow structure which consequently reduced the fluid flow passing to the other organs and tissues Chinyoka and Makinde (2014). As the plaque tends to rupture, an individual may suffered to the risk of cardiovascular disease such as heart attack and stroke.

The study of blood flow in the human pulmonary system is consequential for human health as a result of the fact that it deals with measuring the blood pressure and finding the flow via the blood vessel. Many researchers have studied the blood flow in the arteries and veins. Taura *et al.* (2019) worked on physiological fluid by presenting a model for possible pathways of interaction between the cardiovascular and respiratory control systems. He presents a model equation that explains the Frank-Starling mechanism, which plays an important role in the maintenance of the stability of the distribution of blood in the system. As hemodynamics is directly related to overall human health, recently it has gained a serious attention of researchers, physiologists and clinical persons to study the blood flow through arteries. A body of work in this context of arterial blood flow in the presence of stenosis has been reported by David (2020) in his paper.

The bioheat transport phenomena in hemodynamics influence more on the growth of atherogenetic processes however offer a significant insight in fruitful experimental and theoretical investigation. The understanding of the perturbation of the temperature distribution as a function of the vessel diameter is critically important to the development of appropriate models of bioheat transport. In physiological situations, the temperature distributions perturbed when the diameter of the blood vessel is large Rabby *et al.* (2014).

According to previous study, it is interesting to note that the localized cooling regions are present within heated tissues during hyperthermia treatment when the vessel blood is large in size Santabrata and Subir (2005). The normal temperature of the human blood is about 37°C. Thus, irreversible ill effects will occur in the proteins of blood which is the cause of death after such high fever Srinivasacharya *et al.* (2017). Moreover, hypothermia or hyperthermia is widely used for many purposes such as open heart surgeries and cancer treatment, the temperature is substantially important. Temperature magnitude in hyperthermia treatment are important by raising the temperature of cancerous tissue above a therapeutic value 42°C, while maintaining the surrounding normal tissue at sub lethal temperature value. In the past, there has been a number of studies to examine heat transfer in blood vessels. Jahangiri *et al.* (2015) investigate the influence heating protocol on temperature distribution in a single channel vessel and tumor tissue considering hyperthermia treatment. They concluded that large vessel has effect on the heat transfer characteristics in tissues receiving hyperthermia treatment. The effect of heat transfer considering stenosed artery under assumption of the optically thin fluid has been presented by Sharma *et al*. (2016). The heat transfer coefficient considerably effected when the channel size, shape and cross section of channel, fluid properties, and fluid flow arrangement are varying Zain *et al.* (2017). The influence of Reynold number on heat transfer in catheterizes multiple stenosis artery and in nano fluid mini channel has been investigated by Zaman *et al*. (2016). They concluded that there would be a significant influence of Reynolds number on heat transfer enhancement along the geometry and fluid properties. Jahangiri *et al.* (2015) studied mathematical model of the heat and mass transfer through bifurcated arteries with stenosis in mother and daughter artery. They reported that at the throat of the constricted daughter artery, in particular, give appreciable influences on temperature profile.

2.0 Materials and Methods

We have adopted Taura *et al*. (2019) local arterial flow model. This includes the assumptions that the arterial vessel is incompressible material with circular section and without longitudinal movements. Therefore, blood is considered as an incompressible Newtonian fluid and the flow is axially symmetric. The model approach is to use the two-dimensional Navier-Stokes equations and continuity equation for a Newtonian and incompressible fluid in cylindrical coordinate (r,z,t):

$$\frac{\partial u}{\partial t}+w\frac{\partial u}{\partial r}+u\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+v\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}\right) \tag{3.1}$$

$$\frac{\partial w}{\partial t}+w\frac{\partial w}{\partial r}+u\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+v\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}+\frac{w}{r^2}\right) \tag{3.2}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rw)+\frac{\partial u}{\partial z}=0 \tag{3.3}$$

To make our work easier, we now introduce a new variable η to be the radial coordinate, which is define as:

$$\eta=\frac{r}{R(z,t)} \tag{3.4}$$

We assume that P is independent of the radial coordinate η, then the pressure P is uniformly distributed within the cross section $[P=P(z,t)]$

Therefore,

$$\frac{\partial^2 u}{\partial z^2}\le 1, \qquad \frac{\partial^2 w}{\partial z^2}\le 1, \qquad \frac{\partial P}{\partial r}\le 1,$$

We can now use simple algebra to change the variable as

$$\frac{\partial u(r,z,t)}{\partial t}=\frac{\partial u(\eta,t)}{\partial t}.\frac{\partial \eta}{\partial t}+\frac{\partial u(\eta,t)}{\partial t}.\frac{\partial t}{\partial t}$$

$$=-\frac{\eta}{R}\frac{\partial u(\eta,t)}{\partial t}.\frac{\partial R}{\partial t}+\frac{\partial u(\eta,t)}{\partial t}$$

Equations (3.1), (3.2) and (3.3) can be re-written in the new coordinate (η,z,t) as:

$$\frac{\partial u}{\partial t}+\frac{1}{R}\left[\eta\left(\frac{\partial R}{\partial z}+\frac{\partial R}{\partial t}\right)-w\right]\frac{\partial u}{\partial \eta}+u\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial z}+\frac{v}{R^2}\left(\frac{\partial^2 u}{\partial \eta^2}+\frac{1}{\eta}\frac{\partial u}{\partial \eta}\right) \tag{3.5}$$

$$\frac{\partial w}{\partial t}+\frac{1}{R}\left[\eta\left(\frac{\partial R}{\partial z}+\frac{\partial R}{\partial t}\right)-w\right]\frac{\partial w}{\partial \eta}+u\frac{\partial w}{\partial z}=\frac{v}{R^2}\left(\frac{\partial^2 w}{\partial \eta^2}+\frac{1}{\eta}\frac{\partial w}{\partial \eta}+\frac{w}{\eta^2}\right) \tag{3.6}$$

$$\frac{1}{R}\frac{\partial w}{\partial \eta}+\frac{w}{\eta R}+\frac{\partial u}{\partial z}-\frac{\eta}{R}\frac{\partial R}{\partial z}\frac{\partial u}{\partial \eta}=0 \tag{3.7}$$

We now assumed that the velocity profile in the axial direction $u(\eta,z,t)$ to be expressed in the polynomial form as:

$$u(\eta,z,t)=\sum_{k=1}^{N}q_k\left(\eta^{2k}-1\right) \tag{3.8}$$

Also, the velocity profile in the radial direction is

$$w(\eta,z,t)=\frac{\partial R}{\partial z}\eta w+\frac{\partial R}{\partial t}\eta-\frac{\partial R}{\partial t}\frac{1}{N}\eta\sum_{k=1}^{N}\frac{1}{k}\left(\eta^{2k}-1\right) \tag{3.9}$$

Let $N=1$, simplifying equations (3.8) and (3.9) gives

$$u(\eta,z,t)=q(z,t)\left(\eta^2-1\right) \tag{3.10}$$

$$w(\eta,z,t)=\frac{\partial R}{\partial z}\eta w+\frac{\partial R}{\partial t}\eta\left(\eta^2-1\right) \tag{3.11}$$

Substituting equations (3.10) and (3.11) into equations (3.5) and (3.7) gives the dynamic equations of $q(z,t)$ and $R(z,t)$ given below:

$$\frac{\partial q}{\partial t}-\frac{4q}{R}\frac{\partial R}{\partial t}-\frac{2q^2}{R}\frac{\partial R}{\partial z}+\frac{4v}{R^2}q+\frac{1}{\rho}\frac{\partial P}{\partial z}=0 \qquad (3.12)$$

$$2R\frac{\partial R}{\partial t}+\frac{R^2}{2}\frac{\partial q}{\partial z}+q\frac{\partial R}{\partial z}=0 \qquad (3.13)$$

We now define the cross-sectional area $S(z,t)$ and the flow $Q(z,t)$ as:

$$S=\pi R^2, \qquad Q=\iint_s u\partial\eta = R\pi R^2$$

Expressing equations (3.12) and (3.13) in terms of $Q(z,t)$ and $S(z,t)$ from the above definitions gives:

$$\frac{\partial Q}{\partial t}-\frac{3Q}{S}\frac{\partial S}{\partial t}-\frac{2Q^2}{S^2}\frac{\partial S}{\partial z}+\frac{4\pi v}{S}Q+\frac{S}{2\rho}\frac{\partial P}{\partial z}=0 \qquad (3.14)$$

$$\frac{\partial S}{\partial t}+\frac{\partial Q}{\partial z}=0 \qquad (3.15)$$

The systems of equations (3.14) and (3.15) are the governing equation. We used the Finite difference method to solve the non-linear partial differential equations (3.14) and (3.15) by discretizing the equations using the following first order accuracy difference formula:

$$\frac{\partial Q_i}{\partial z}=\frac{Q_i-Q_{i-1}}{\Delta z} \qquad (3.16)$$

And

$$\frac{\partial S_i}{\partial z}=\frac{S_i-S_i}{\Delta z} \qquad (3.17)$$

Therefore,

Equations (3.14) and (3.15) becomes,

$$\frac{\partial Q_i}{\partial t}-\frac{3Q_i}{S_i}\frac{Q_i-Q_{i-1}}{\Delta z}-\frac{2Q_i^2}{S^2}\frac{S_{i+1}-S_i}{\Delta z}+\frac{4\pi v}{S_i}Q_i+\frac{S_i}{2\rho}\frac{\partial P}{\partial z}=0 \qquad (3.18)$$

$$\frac{\partial S_i}{\partial t}=\frac{Q_i-Q_{i-1}}{\Delta z} \qquad (3.19)$$

Where $i=1,2,3,\ldots M$, the pressure gradient $\frac{\partial P}{\partial z}$ is constant all through.

We can simplify governing equations by linearizing equation (3.18) as follows:

$$\frac{\partial Q_i}{\partial t}-\frac{4\pi v}{S_0}Q_i+\frac{S_0}{2\rho}\frac{\partial \rho}{\partial z}+\frac{S_i}{2\rho}\frac{\partial P}{\partial z}=0 \qquad (3.20)$$

The difference equations (3.19) – (3.20) can be written in the form of $\frac{\partial y}{\partial t}=f(y)$

Where,

$$y = (Q_1, Q_2, ..., Q_M, S_1, S_2, ...S_M) \quad (3.21)$$

And

$$f(y) = \begin{bmatrix} -\left(\frac{4\pi v}{S_0} y(1) + \frac{S_0}{2\rho}\frac{\partial P}{\partial z} + \frac{y(M+1)}{2\rho}\frac{\partial P}{\partial z}\right) - \\ \left(\frac{4\pi v}{S_0} y(2) + \frac{S_0}{2\rho}\frac{\partial P}{\partial z} + \frac{y(2+N)}{2\rho}\frac{\partial P}{\partial z}\right) - \\ \left(\frac{4\pi v}{S_0} y(N-1) + \frac{S_0}{2\rho}\frac{\partial P}{\partial z} + \frac{y(2N-1)}{2\rho}\frac{\partial P}{\partial z}\right) - \\ \left(\frac{4\pi v}{S_0} y(N) + \frac{S_0}{2\rho}\frac{\partial P}{\partial z} + \frac{y(2N)}{2\rho}\frac{\partial P}{\partial z}\right) - \\ \left(\frac{y(1) - Q_0}{\Delta z} - \frac{y(2) - y(0)}{\Delta z}\right) - \\ \left(\frac{y(N-1) - y(N-2)}{\Delta z} - \frac{y(N) - y(N-1)}{\Delta z}\right) \end{bmatrix} \quad (3.22)$$

The easiest way to solve (3.19) – (3.22) is by using MatLab buid in function ODE45 embedded with Runge-Kutta method. The necessary parameters for varying are: $Q_0, S_0, \frac{\partial P}{\partial z}, v, \rho$. The other values in normal condition can be obtained from the past works in this field. For example:

Initial value of Q and $Q_0 = 1 - 5.4$ liter/minute Taura *et al.* (2019)

Initial value of S and $S_0 = 1.5 - 2.0cm^3$ Taura *et al.* (2019)

$\frac{\partial P}{\partial z} = 100 - 40mmHg$ Sharma *et al*. (2016)

$v = 0.035cm^2 / s$ David (2020)

$\rho = 1.05g / cm^3$ David (2020)

3.0 Results and Discussion

For us to simulate how the cross-sectional area of the artery affects the blood flow in the artery, the values of the parameters mentioned above are considered as: $\rho = 1.05g / cm^3, v = 0.035cm^2 / \sec, Q_0 = 16.7cm^3 / \sec$ and $S_0 = 1.5cm^2$. We chose the length of the artery model $L = 15cm$ and the number of nodes of the system $N = 3$ and time $t = 0.2\sec$.

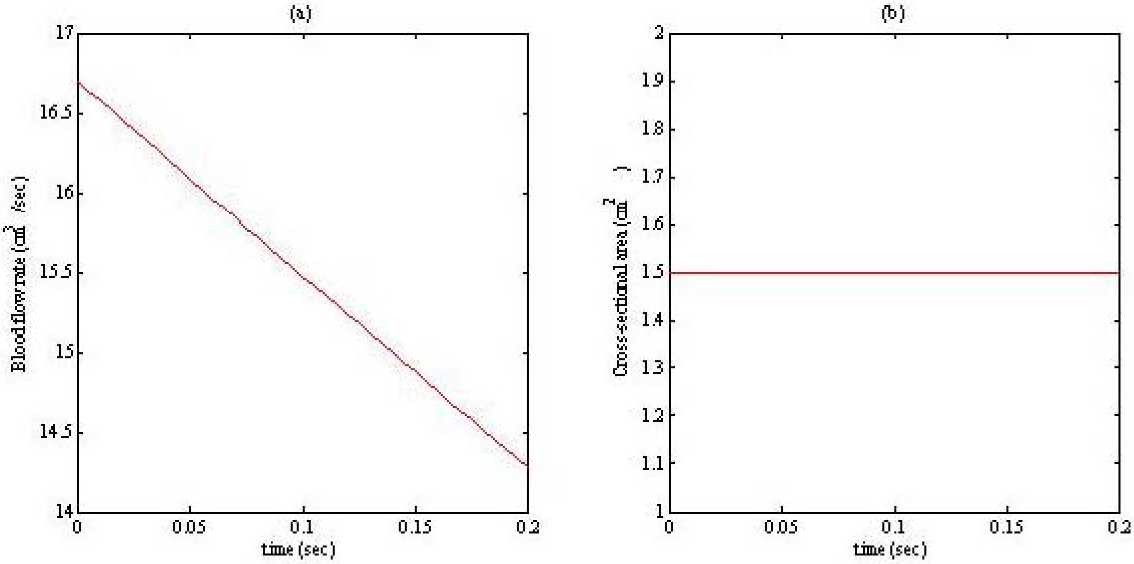

Figure 1: This shows the blood flow rate and cross-sectional areas for each node.

It is observed that the results for Q_1, Q_2, and Q_3 are almost the same as depicted in Figure 1(a). Similarly, the values of S_1, S_2, and S_3 in Figure 1(b) are very close and it is almost a constant. This shows that the values of the blood flow rate and the cross-sectional area are almost the same throughout in the small section of arteries. This could be due to the absence of viscoelastic effect in the model. Now since there is not much difference in the blood flow rate between the sections, we will consider only one section which is S_2 to make the comparison of the different values of the cross-sectional area. As we can see, the value for the blood flow is decreasing from its initial value.

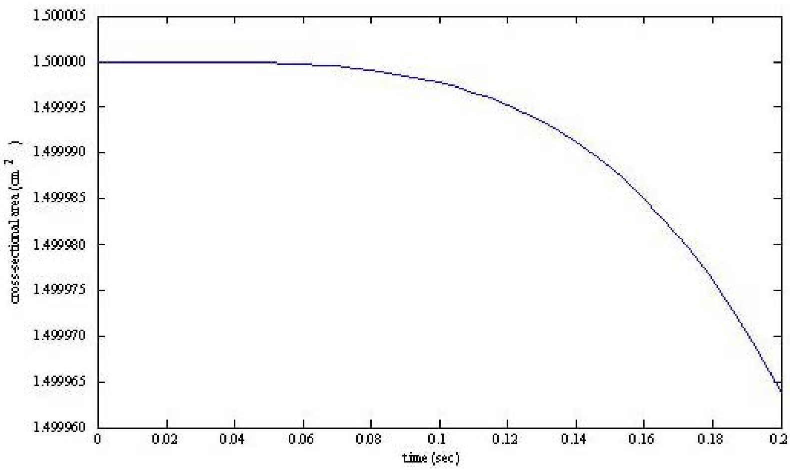

Figure 2: The cross-sectional area against time

This shows that without changing the value of the pressure gradient and the cross-sectional area of the arteries, the blood flow rate through the arteries is decreasing significantly as time increasing. It also shows that the blood flow is linearly decreasing. We assume this condition is valid in the diastole condition only. Next, we compare this result with smaller cross-sectional area.

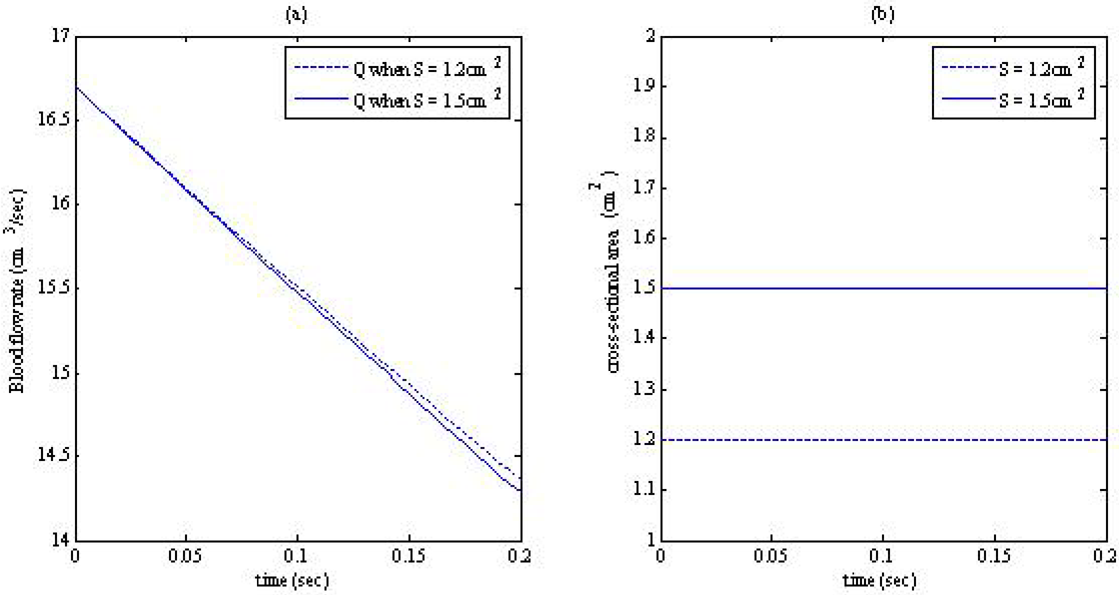

Figure 3: Comparison graph for blood flow with different value of cross-sectional area.

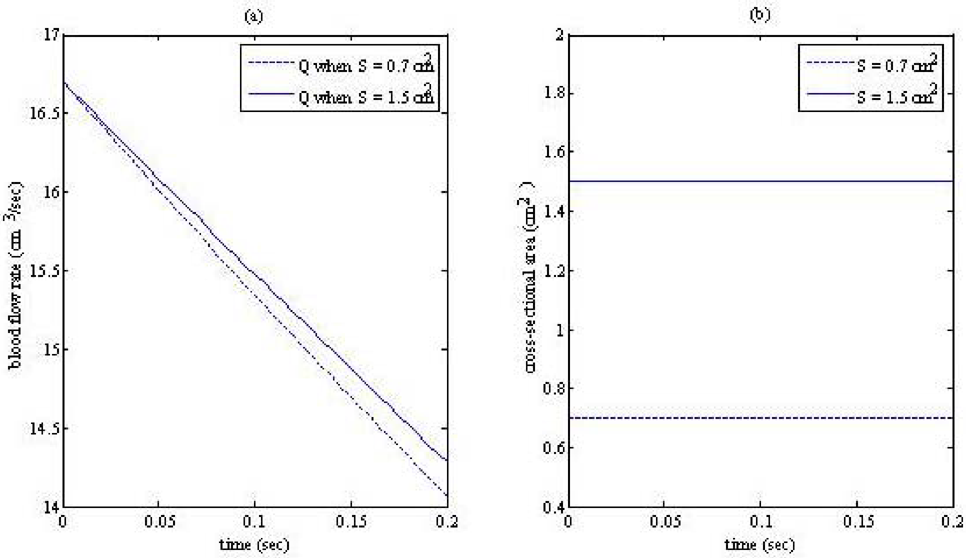

Figure 4: Comparison of *Q* at normal cross-sectional area and much smaller cross-sectional area.

As shown in Figure 5 above, when the value of cross-sectional area is below $0.8cm^2$, the blood flow rate decreases faster than the normal rate. Figure 6 shows the value of blood flow rate when the cross-sectional area is in range of between $0.1cm^2 - 0.8cm^2$. Clearly, if the cross-sectional area continues to decrease below $0.8cm^2$, the blood flow rate also decreased drastically. From this observation, we can say that this condition occur because the cross-sectional area is too small for the blood to get through it. This is also a dangerous condition for human.

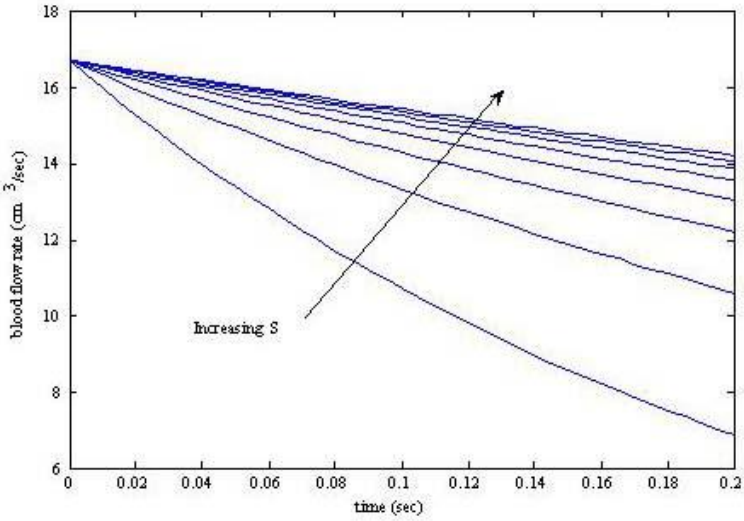

Figure 5: when cross-sectional area is in range between $0.1cm^2 - 0.8cm^2$

From the results obtained, we can conclude that cross-sectional area plays an important part in order for the blood to flow smoothly through the blood vessel. A small change in the value for the cross sectional area may affect the amount of blood flow rate through the arteries which also may affect the blood pressure. In other words, smaller cross-sectional area from normal size may contribute to hypertension or high blood pressure. When a large amount of fluid flows through a small vessel, it may cause the pressure in the vessel to increase.

References

Akhbar Zaman, Nasir Ali, Muhammad Bilal Sajid, and Tasawar Hayat. "Effects of unsteadiness and non-Newtonian rheology on blood flow through a tapered time-variant stenotic artery." *AIP advances*5, no. 3 (2019): 037129.

Chinyoka, T., and O. D. Makinde. "Computational dynamics of arterial blood flow in the presence of magnetic field and thermal radiation therapy." *Advances in Mathematical Physics*2014 (2014).

David Chato. "Heat transfer to blood vessels." Journal of biomechanical engineering 102, no. 2 (2020): 110-118.

Jahangiri, Mehdi, Mohsen Saghafian, and Mahmood Reza Sadeghi. "Numerical simulation of hemodynamic parameters of turbulent and pulsatile blood flow in flexible artery with single and double stenoses." *Journal of Mechanical Science and Technology* 29, no. 8 (2015): 3549-3560.

Rabby, Mir Golam, Sumaia Parveen Shupti, and Md Molla. "Pulsatile non-newtonian laminar blood flows through arterial double stenoses." *Journal of Fluids* 2014 (2014).

Santabrata Chakravarty and Subir Sen. "Dynamic response of heat and mass transfer in blood flow through stenosed bifurcated arteries." *Korea-Australia Rheology Journal* 17, no. 2 (2005): 47-62.

Srinivasacharya, Darbhasayanam, and Gade Madhava Rao. "Micropolar fluid flow through a stenosed bifurcated artery." *Nonlinear Analysis-Modelling and Control* 22, no. 2 (2017): 147-159.

Sharma, Mukesh Kumar, Vinay Nasha, and P. R. Sharma. "A Study for Analyzing the Effect of Overlapping Stenosis and Dilatation on Non-Newtonian Blood Flow in an Inclined Artery." *Journal of Biomedical Science and Engineering* 9, no. 12 (2016): 576.

Taura, L. S. (2019). The use of a continuity equation of fluid mechanics to reduce the abnormality of the cardiovascular system: A control mechanics of the human heart. *Journal of Biophysics and Structural Biology Vol. 4(1), pp. 1-12, March 2012. Available online at* http://www.academicjournals.org/jbsb. DOI: 10.5897/JBSB11.010. ISSN-2141-2200 ©2012.

Zain, Norliza Mohd, and Zuhaila Ismail. "Modelling of Newtonian blood flow through a bifurcated artery with the presence of an overlapping stenosis." *Malaysian Journal of Fundamental and Applied Sciences* 13, no. 4-1 (2017): 304-309.

Zaman, A., N. Ali, O. Anwar Bég, and M. Sajid. "Heat and mass transfer to blood flowing through a tapered overlapping stenosed artery." *International Journal of Heat and Mass*