

# Mathematical Models of Real Geometrical Factors in Restricted Blood Vessels for the Analysis of *CAD* (Coronary Artery Diseases) Using Legendre, Boubaker and Bessel Polynomials

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**Abstract** Most cardiovascular emergencies are directly caused by coronary artery disease. Coronary arteries can become clogged or occluded, leading to damage to the heart muscle supplied by the artery. Modern cardiovascular medicine can certainly be improved by meticulous analysis of geometrical factors closely associated with the degenerative disease that results in narrowing of the coronary arteries. There are, however, inherent difficulties in developing this type of mathematical models to completely describe the real or ideal geometries that are very critical in plaque formation and thickening of the vessel wall. Neither the mathematical models of the blood vessels with atherosclerosis generated by the heart and blood flow or the NMR/MRI data to construct them are available. In this study, a mathematical formulation for the geometrical factors that are very critical for the understanding of coronary artery disease is presented. Based on the Bloch NMR flow equations, we derive analytical expressions to describe in detail the NMR transverse magnetizations and signals as a function of some NMR flow and geometrical parameters which are invaluable for the analysis of blood

flow in restricted blood vessels. The procedure would apply to the situations in which the geometry of the fatty deposits, (plaque) on the interior walls of the coronary arteries is spherical. The boundary conditions are introduced based on Bessel, Boubaker and Legendre polynomials.

**Keywords** Bloch NMR flow equations · Atherosclerosis · Coronary artery disease · Bessel polynomials · Boubaker polynomials · Legendre polynomials

## Introduction

Atherosclerosis is a degenerative disease that results in narrowing of the coronary arteries. This is caused by fatty deposits, most notably cholesterol, on the interior walls of the coronary arteries Fig. 1. When the walls become narrowed or occluded, they reduce the blood flow to the heart muscle. If the artery remains open to some degree, the reduced blood flow is noticed when the heart is under stress during periods of rapid heartbeat. The resulting pain is called angina. When the artery is completely closed or occluded, a section of the heart muscle can no longer get oxygenated blood, and begins to die. This is called a heart attack [1].

Coronary artery disease (CAD) causes changes in both structure and function of the blood vessels. Atherosclerotic processes cause an abnormal deposition of lipids in the vessel wall, leukocyte infiltration and vascular inflammation, plaque formation and thickening of the vessel wall. These changes lead to a narrowing of the lumen (i.e., stenosis), which restricts blood flow. There are also subtle, yet functionally important changes that can occur before

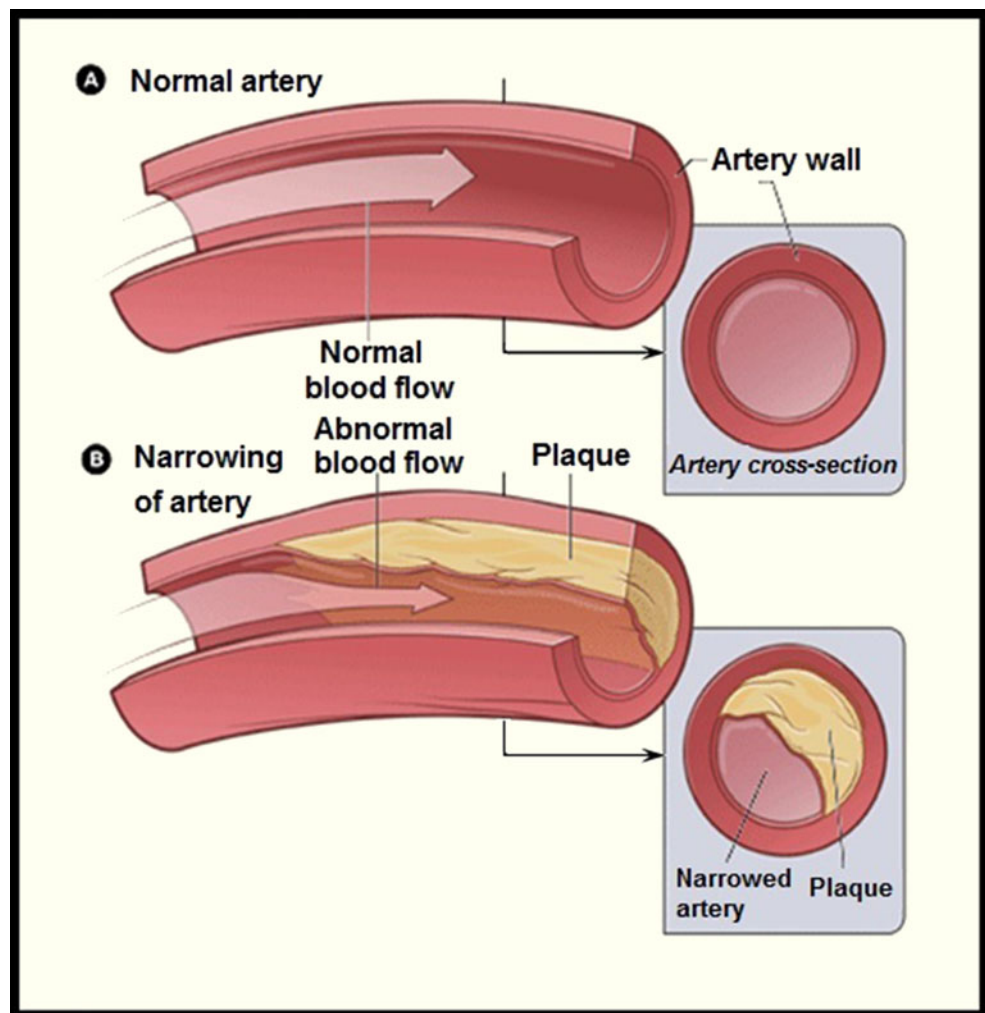
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**Fig. 1** A typical blood vessel. (a) shows a normal artery with normal blood flow. (b) shows an artery with plaque buildup [1]



overt changes in structure are observed. Early in the disease process, the endothelial cells that line the coronary arteries become dysfunctional. Because the endothelium produces important substances such as nitric oxide and prostacyclin that are required for normal coronary function, endothelial dysfunction can lead to coronary vasospasm, impaired relaxation, and formation of blood clots that can partially or completely occlude the vessel.

When coronary artery disease restricts blood flow to the myocardium (ischemia) there is an imbalance between oxygen supply and oxygen demand. When the oxygen supply is insufficient to meet the oxygen demand (reduced oxygen supply/demand ratio), the myocardium becomes hypoxic. This is often associated with chest pain (angina) and other clinical symptoms. Severe ischemia can lead to anoxia and infarction of the tissue. Furthermore, acute or chronic ischemia caused by CAD can impair cardiac mechanical and electrical activities leading to heart failure and arrhythmias. In the presence of coronary artery disease, coronary blood flow may be reduced. This will increase oxygen extraction from the coronary blood and decrease the

venous oxygen content. This leads to tissue hypoxia and angina. As stated in our earlier study [2], tests to accurately diagnose coronary artery disease [3–19] are currently the focus of intense clinical research, their incremental value is not yet proven. Accordingly, their use requires competent mathematical tools for clinical assessment of potential risks and benefits in individual patients.

This study is based on the previous efforts to solve the Bloch NMR flow equations analytically [2, 20–27]. Calculations are based on the resolution of a mathematical model in spherical polar coordinates representing the NMR transverse magnetizations and signals. Boundary conditions are involved in the resolution algorithm at different stages. We consider a function  $M_y$  which describes the NMR transverse magnetizations and signals at a given location  $(x, y, z)$  in the blood vessel. This function changes depending on the degenerative disease that results in narrowing of the blood vessel. The boundary conditions are introduced based on Bessel and Legendre polynomials. This model can allow us to apply the detail mathematical tools presented to obtain qualitative information about the development, the growth

and the maturity of coronary artery disease (CAD) at any point in terms of physical and geometrical factors affecting the individual patient. Within a distributed model the MRI Physics of blood flow in restricted blood vessels is treated using a differential equation derived from the Bloch NMR flow equations [20]. For this type of model casting the governing equations in terms of physical parameters is more straightforward.

We have introduced [2, 20] the transverse magnetization  $M_y$ , as the stream function such that the flow in the region is steady and  $f + \vartheta$  is consequently a function of  $M_y$  alone. But at  $X=0$  (as approached from the region  $x>0$ ) we have [20]:

$$M_y = V_1 y \text{ and } f + \vartheta = \frac{h_1(f_o + \beta y)}{h} = \frac{h_1 \left( f_o + \frac{\alpha}{V_1} M_y \right)}{h} \tag{1a}$$

This must be the relationship between  $f + \vartheta$  and  $M_y$  that is valid over the whole of the region  $x>0$ . Hence in that region we have [20]:

$$\nabla^2 M_y = -\vartheta = f_o \left( \frac{h - h_1}{h} \right) + \beta y - \rho^2 M_y \tag{1b}$$

where:  $\rho^2 = \beta \frac{h_1}{V_1 h}$

Our choice of conditions has led to a linear equation for  $M_y$ . A form of solution which contains the linearity in  $M_y$  has been presented [18] where

$$M_y = (y - a)\psi(x) + \frac{[f_o((h - h_1)/h) + \beta y]}{\rho^2} \tag{1c}$$

Here,  $a$  is a constant and  $\psi(x)$  satisfies Eq. 1a with

$$\rho^2 = \frac{\gamma^2 B_1^2}{V^2} \text{ and } x(y + a) = 1 \tag{1d}$$

Equation 1b can be written in three dimensions as

$$\nabla^2 M_y + \frac{\alpha}{V} M_y = 0 \tag{1e}$$

where  $V$  is the blood flow velocity and  $\alpha$  is a constant to be determined.

**Mathematical formulation of the flow model**

In spherical coordinate Eq. 1c is written as:

$$\frac{1}{r^2} \left( \frac{\partial}{\partial r} \left( r^2 \frac{\partial M_y}{\partial r} \right) + \frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \left( \sin \phi \frac{\partial M_y}{\partial \phi} \right) + \frac{1}{\sin^2 \phi} \frac{\partial^2 M_y}{\partial \theta^2} \right) + a^2 M_y = 0 \tag{2}$$

$$\frac{\partial^2 M_y}{\partial r^2} + \frac{2}{r} \frac{\partial M_y}{\partial r} + \frac{1}{r^2} \frac{\partial^2 M_y}{\partial \phi^2} + \frac{\cot \phi}{r^2} \frac{\partial M_y}{\partial \phi} + \frac{1}{r^2 \sin^2 \phi} \frac{\partial^2 M_y}{\partial \theta^2} + a^2 M_y = 0 \tag{3}$$

By separation of variables, we write:

$$M_y(r, \theta, \phi) = F(r, \theta)\Theta(\phi) \left\{ \begin{aligned} M_y(r, \theta, \phi) &= F(r, \theta)\Theta(\phi) \\ \frac{\partial M_y}{\partial r} &= \Theta \frac{\partial F}{\partial r}, \frac{\partial M_y}{\partial r^2} = \Theta \frac{\partial^2 F}{\partial r^2}, \frac{\partial M_y}{\partial \phi} = \Theta \frac{\partial F}{\partial \phi}, \frac{\partial^2 M_y}{\partial \phi^2} = \Theta \frac{\partial^2 F}{\partial \phi^2}, \frac{\partial^2 M_y}{\partial \theta^2} = F \frac{d^2 \Theta}{d\theta^2} \end{aligned} \right. \tag{3a}$$

Equation 3 becomes:

$$\Theta \frac{\partial^2 F}{\partial r^2} + \frac{2}{r} \Theta \frac{\partial F}{\partial r} = \frac{\Theta}{r^2} \frac{\partial^2 F}{\partial \phi^2} + \Theta \frac{\cot \phi}{r^2} \frac{\partial F}{\partial \phi} + \frac{F}{r^2 \sin^2 \phi} \frac{\partial^2 \Theta}{\partial \theta^2} + a^2 F \Theta = 0 \tag{3b}$$

Multiplying Eq. 3b by  $\frac{1}{F\Theta}$  gives:

$$\frac{1}{F} \frac{\partial^2 F}{\partial r^2} + \frac{2}{rF} \frac{\partial F}{\partial r} = \frac{1}{r^2 F} \frac{\partial^2 F}{\partial \phi^2} + \Theta \frac{\cot \phi}{r^2} \frac{1}{F} \frac{\partial F}{\partial \phi} + \frac{F}{r^2 \sin^2 \phi} \frac{1}{\Theta} \frac{\partial^2 \Theta}{\partial \theta^2} + a^2 = 0 \tag{3c}$$

Multiplying Eq. 3c by  $r^2 \sin^2 \phi$ , gives:

$$\frac{r^2 \sin^2 \phi}{F} \frac{\partial^2 F}{\partial r^2} + \frac{2r^2 \sin^2 \phi}{rF} \frac{\partial F}{\partial r} = \frac{r^2 \sin^2 \phi}{r^2 F} \frac{\partial^2 F}{\partial \phi^2} + \frac{r^2 \sin^2 \phi \cot \phi}{r^2} \frac{\partial F}{\partial \phi} + \frac{1}{\Theta} \frac{d^2 \Theta}{d\theta^2} + a^2 r^2 \sin^2 \phi = 0 \tag{4}$$

Both sides of Eq. 4 must be equal to a constant  $k^2$  such that we have two distinct equations:

$$\frac{\partial^2 F}{\partial r^2} + \frac{2}{r} \frac{\partial F}{\partial r} = \frac{1}{r^2} \frac{\partial^2 F}{\partial \phi^2} + \frac{\cot \phi}{r^2} \frac{\partial F}{\partial \phi} + a^2 F - \frac{k^2 F}{r^2 \sin^2 \phi} = 0 \tag{5}$$

$$-\frac{1}{\Theta} \frac{d^2 \Theta}{d\theta^2} = k^2 \tag{6}$$

The solution to Eq. 6 is:

$$\Theta(\phi) = a_3 e^{ik\theta} + a_4 e^{-ik\theta} \text{ or } \Theta(\phi) = A_3 \cos k\theta + A_4 \sin k\theta \tag{7}$$

(where  $A_3 = a_3 + a_4, a_3 = i(a_3 - a_4)$ )

A solution to Eq. 5 is found by making the assumption  $F(r, \phi) = R(r)\Phi(\phi)$ :

$$\frac{\partial F}{\partial r} = \Phi \frac{dR}{dr}, \frac{\partial^2 F}{\partial r^2} = \Phi \frac{d^2 R}{dr^2}, \frac{\partial F}{\partial \phi} = R \frac{d\Phi}{d\phi}, \frac{\partial^2 F}{\partial \phi^2} = R \frac{d^2 \Phi}{d\phi^2} \tag{7a}$$

Equation 5 becomes:

$$\Phi \frac{d^2 R}{dr^2} + \frac{2\Phi}{r} \frac{dR}{dr} + \frac{R}{r^2} \frac{d^2 \Phi}{d\phi^2} + R \frac{\cot \phi}{r^2} \frac{d\Phi}{d\phi} + a^2 R \Phi - \frac{k^2}{r^2 \sin^2 \phi} R \Phi = 0 \tag{7b}$$

If we multiply Eq. 7b by  $\frac{1}{R\Phi}$ , we have:

$$\frac{r^2}{R} \frac{d^2 R}{dr^2} + \frac{2r}{R} \frac{dR}{dr} + \frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2} + \frac{\cot \phi}{r^2} \cdot \frac{1}{\Phi} \frac{d\Phi}{d\phi} + a^2 r^2 - \frac{k^2}{\sin^2 \phi} = 0 \tag{7c}$$

Multiplying Eq. 7c by  $r^2$  gives:

$$\frac{r^2}{R} \frac{d^2 R}{dr^2} + \frac{2r}{R} \frac{dR}{dr} + a^2 r^2 = -\frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2} + \frac{\cot \phi}{\phi} \frac{d\Phi}{d\phi} + \frac{k^2}{\sin^2 \phi} \tag{8}$$

Both sides of Eq. 8 must be a constant (say  $l(l + 1)$ ), because they are independent of each other,

$$\frac{r^2}{R} \frac{d^2 R}{dr^2} + \frac{2r}{R} \frac{dR}{dr} + a^2 r^2 = -\frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2} + \frac{\cot \phi}{\phi} \frac{d\Phi}{d\phi} + \frac{k^2}{\sin^2 \phi} = l(l + 1) \tag{8a}$$

Equation 8a gives the following equation:

$$r^2 \frac{d^2 R}{dr^2} + 2r \frac{dR}{dr} + (a^2 r^2 - l(l + 1))R = 0 \tag{9}$$

and:

$$\frac{d^2 \Phi}{d\phi^2} + \cot \phi \frac{d\Phi}{d\phi} + \left( l(l + 1) - \frac{k^2}{\sin^2 \phi} \right) \Phi = 0 \tag{10}$$

Equation 9 is the spherical Bessel differential equation. We shall transform this equation into an ordinary Bessel differential equation if we write;

$$x = \frac{\sqrt{\alpha}}{\sqrt{V}} r, \tag{10a}$$

$$r \frac{dR}{dr} = x \frac{\sqrt{V}}{\sqrt{\alpha}} \cdot \frac{dR}{d\left(\frac{\sqrt{V}}{\sqrt{\alpha}} r\right)} = \frac{\sqrt{V}}{\sqrt{\alpha}} \cdot \frac{\sqrt{\alpha}}{\sqrt{V}} \cdot x \frac{dR}{dx} = x \frac{dR}{dx} \tag{10b}$$

$$r^2 \frac{d^2 R}{dr^2} = x^2 \cdot \frac{V}{\alpha} \cdot \frac{d}{d\left(\sqrt{\frac{V}{\alpha}} x\right)} \left( \frac{dR}{d\left(\sqrt{\frac{V}{\alpha}} x\right)} \right) = x^2 \cdot \frac{V}{\alpha} \cdot \frac{\alpha}{V} \frac{d^2 R}{dx^2} = x^2 \frac{d^2 R}{dx^2} \tag{10c}$$

From Eq. 10a–10c we have:

$$x^2 \frac{d^2 R}{dx^2} + 2x \frac{dR}{dx} + (x^2 - l(l + 1))R = 0 \tag{11}$$

$$R = S(x)x^{-\frac{1}{2}} \tag{11a}$$

$$\frac{dR}{dx} = x^{-1/2} \frac{dS}{dx} - \frac{1}{2} x^{-3/2} \frac{dS}{dx} - \frac{1}{2} x^{-3/2} \frac{dS}{dx} + \frac{3}{4} x^{-5/2} S \tag{11b}$$

$$= x^{-1/2} \frac{d^2 S}{dx^2} - x^{-3/2} \frac{dS}{dx} + \frac{3}{4} x^{-5/2} S \tag{11c}$$

Equation 11 gives

$$x^{3/2} \frac{d^2 S}{dx^2} + x^{1/2} \frac{dS}{dx} - \frac{1}{4} x^{-1/2} S - l(l + 1)x^{-1/2} S + x^{3/2} S = 0 \tag{11d}$$

Multiplying Eq 11d by  $x^{1/2}$  gives:

$$x^2 \frac{d^2 S}{dx^2} + x \frac{dS}{dx} + \left\{ x^2 - l\left(l + \frac{1}{2}\right)^2 \right\} S = 0 \tag{12}$$

Equation 12 is a Bessel differential equation of order  $(l + \frac{1}{2})$  with a general solution:

$$S(x) = C_3 J_{l+1/2}(x) + C_4 Y_{l+1/2}(x)$$

From Eq. 11a, we can write

$$R = x^{-1/2} S(x) =^{-1/2} C_3 J_{l+1/2}(x) + x^{-1/2} C_4 Y_{l+1/2}(x) \tag{13}$$

$$R(r) = C_3 \left(\frac{\alpha}{V} r^2\right)^{-1/4} J_{l+1/2}\left(\sqrt{\frac{\alpha}{V}} r\right) + C_4 \left(\frac{\alpha}{V} r^2\right)^{-1/4} Y_{l+1/2}\left(\sqrt{\frac{\alpha}{V}} r\right) \tag{14}$$

where  $J_{l+1/2}\left(\frac{\alpha}{V} r\right)$  and  $Y_{l+1/2}\left(\frac{\alpha}{V} r\right)$  are the Bessel polynomials of the 1st and 2nd kind respectively with the order  $l + \frac{1}{2}$ . Since  $Y_{l+1/2}\left(\sqrt{\frac{\alpha}{V}} r\right)$  is singular at  $r=0$ , then within the region containing  $r=0$  axis, we have  $C_4=0$ . Hence it follows that

$$R(r) = C_3 \left(\frac{\alpha}{V} r^2\right)^{-1/4} J_{l+1/2}\left(\sqrt{\frac{\alpha}{V}} r\right) \tag{15}$$

Equation 10 can be written as:

$$\sin \phi \frac{d^2 \Phi}{d\phi^2} + \cos \phi \frac{d\Phi}{d\phi} + \sin \phi \left\{ l(l+1) - \frac{k^2}{\sin^2 \phi} \right\} \Phi = 0 \tag{16}$$

If we write  $\mu = \cos \phi$ , Eq. 16 becomes

$$\frac{d\Phi}{d\phi} = \frac{d\Phi}{d\mu} \cdot \frac{d\mu}{d\phi} = -\sin \phi \frac{d\Phi}{d\mu} \tag{17}$$

It follows that:

$$(1 - \mu^2) \frac{d^2 \Phi}{d\mu^2} - 2\mu \frac{d\Phi}{d\mu} + \left\{ l(l+1) - \frac{k^2}{\sin^2 \phi} \right\} \Phi = 0 \tag{18}$$

Equation 18 is the Legendre associated differential equation. The solutions are called the associated Legendre polynomials and are given as follows

$$\Phi(\mu) = D_3 P_l^k(\mu) + D_4 Q_l^k(\mu) \tag{19}$$

where:

$$\begin{cases} P_l^k(\mu) = (1 - \mu^2)^{k/2} \frac{d^k}{d\mu^k} P_l(\mu) \\ Q_l^k(\mu) = (1 - \mu^2)^{k/2} \frac{d^k}{d\mu^k} Q_l(\mu) \end{cases} \tag{20}$$

$P_l(\mu)$  and  $Q_l(\mu)$  are the Legendre polynomials of the 1st and 2nd kind respectively. Since we require that the solution be finite on the polar axis of the plague, we write that  $D_4 = 0$ . Therefore, we may write

$$\Phi(\mu) = D_3 P_l^k(\mu) \tag{21}$$

$$\Phi(\phi) = D_3 P_l^k(\cos \phi) \tag{22}$$

**Boundary conditions for the analysis of NMR transverse magnetizations and signals**

Based on Fig. 1, when the blood vessel becomes totally blocked, we expect so much random motion and hence, random orientation of the spins of the blood particles, even in the presence of strong static magnetic field  $B_0$ .

Furthermore, if some particle tunnel through the plague defined by the point  $(h, 2\pi, \phi)$ , we have a very negligible contribution to the NMR signal from most of the spins. Therefore, we may write

$$M_y(h, 2\pi, \phi) \approx 0 \tag{23}$$

This means that at the point where the height of the plague  $h_1$  equals the diameter  $h$ , of the blood vessel, and at  $\theta = 2\pi$ , the transverse magnetization becomes very small so that:

$$R(h)\Theta(2\pi)\Phi(\phi) = 0 \tag{24}$$

Equation 24 implies that  $R(h) = 0$  or  $\Theta(2\pi) = 0$

Since  $k$  is an integer,  $A_3$  in Eq. 7 must be equal to zero for the expression  $\Theta(2\pi) = 0$  to hold. Then, Eq. 7 becomes:

$$\Theta(\theta) = A_4 \sin k\theta \tag{25}$$

For  $R(h) = 0$ , we must have:

$$C_3(\beta^2)^{-1/4} J_{l+1/2}(\beta) = 0 \text{ where } \beta = \sqrt{\frac{\alpha}{V}} h \tag{26}$$

$$J_{l+1/2}(\beta) = 0 \tag{27}$$

so that:

$$\beta = \beta_{l,n} |_{n=1,2,3}; \sqrt{\frac{\alpha}{V}} = \sqrt{\frac{\beta_{l,n}}{h}} \tag{28}$$

By superposition or summing over all  $l, n$  and  $k$ , we obtain

$$M_y(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} A_{k,l,n} \left( \frac{\beta_{l,n}^2}{h^2} r \right)^{-1/2} J_{l+1/2} \left( \frac{\beta_{l,n}}{h} r \right) \{ P_l^k \cos \phi \} \{ \sin k\phi \} \tag{29}$$

where  $A_{k,l,n} = C_3 D_3 A_4$

However, at  $\theta = \frac{\pi}{2k}$ , we have some signal  $M_b(r, \phi)$  so that:

$$M_y \left( r, \frac{\pi}{2k}, \phi \right) = M_b(r, \phi) \tag{30}$$

This implies that:

$$M_b(r, \phi) = \sum_{l=0}^{\infty} \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} A_{k,l,n} \left( \frac{\beta_{l,n}}{h} r \right)^{-1/2} J_{l+1/2} \left( \frac{\beta_{l,n}}{h} r \right) \{ P_l^k \cos \phi \} \tag{31}$$

If we multiply all through by  $r^{-1/2}$ , we have

$$M_b(r, \phi) = \sum_{l=0}^{\infty} \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} A_{k,l,n} \frac{\sqrt{h}}{\sqrt{\beta_{l,n}}} J_{l+1/2} \left( \frac{\beta_{l,n}}{h} r \right) P_l^k \{ \cos \phi \} \tag{32}$$

Since  $C_3, D_3$ , and  $A_4$  are arbitrary constant, the expression of  $A_{k,l,n}$  has to match the intrinsic boundary conditions regardless values of  $n, l$  and  $k$ . For this purpose, this expression is set as:

$$A_{k,l,n} = -\frac{1}{8} \left( B_{4k} \left( \varpi_k \frac{r}{h} \right) \times B_{4l} \left( \varpi_l \frac{r}{h} \right) \times B_{4n} \left( \varpi_n \frac{r}{h} \right) \right) \tag{33}$$

Whith  $B_{4q} |_{q \in \{k,l,n\}}$  the 4q-Boubaker polynomial [28–35] and  $\varpi_q |_{q \in \{k,l,n\}}$   $B_{4q}$  first positive root [29–31].

When involved in an expansion, the  $4q$ -Boubaker polynomial are the unique polynomial set [33] that verifies, simultaneously and conjointly the conditions:

$$\begin{cases} \left. \sum_{q=1}^N B_{4q} \left( \frac{r}{h} \times \varpi_q \right) \right|_{r=0} = -2N \neq 0; \\ \left. \sum_{q=1}^N B_{4q} \left( \frac{r}{h} \times \varpi_q \right) \right|_{r=1} = 0; \end{cases} \tag{34}$$

$$\begin{cases} \left. \sum_{q=1}^N \frac{dB_{4q} \left( \frac{r}{h} \times \varpi_q \right)}{dr} \right|_{r=0} = 0 \\ \left. \sum_{q=1}^N \frac{dB_{4q} \left( \frac{r}{h} \times \varpi_q \right)}{dr} \right|_{r=h} = \sum_{q=1}^N H_q \end{cases} \tag{35}$$

where:

$$H_q = \left. \frac{dB_{4q}(x)}{dx} \right|_{x=\beta_q} = \left( \frac{4\beta_q \left[ 2 - \beta_q^2 \right] \times \sum_{j=1}^q B_{4j}^2(\beta_q)}{B_{4(q+1)}(\beta_q)} + 4\beta_q^3 \right)$$

$$\begin{cases} \left. \sum_{q=1}^N \frac{d^2 B_{4q} \left( \frac{r}{h} \times \varpi_q \right)}{dr^2} \right|_{x=0} = \frac{8}{3} (N(N^2 - 1)) \\ \left. \sum_{q=1}^N \frac{d^2 B_{4q} \left( \frac{r}{h} \times \varpi_q \right)}{dr^2} \right|_{x=1} = \sum_{q=1}^N G_q \end{cases} \tag{36}$$

with:

$$G_q = \left. \frac{d^2 B_{4q}(x)}{dx^2} \right|_{x=\beta_q} = \frac{3\beta_q (4q\beta_q^2 + 12q - 2)H_q - 8q(24q^2\beta_q^2 + 8q^2 - 3q + 4)}{(\beta_q^2 - 1)(12q\beta_q^2 + 4q - 2)}$$

Therefore, the final NMR transverse magnetizations and signal is given analytically as

$$M_y(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} A_{k,l,n} \left( \frac{\beta_{l,n}}{h} r \right)^{-1/2} J_{l+1/2} \left( \frac{\beta_{l,n}}{h} r \right) P_l^k \{ \cos \phi \} (\sin k\phi) \tag{37}$$

with  $A_{k,l,n}$  given by (Eq. 37)

### Conclusion

We have presented a mathematical formulation for blood flow problems involving an inner boundary within which the formation of plague or stenosis acting as obstacles to the blood flow is non uniform. The motion of the blood particle is governed by a differential equation derived from the Bloch NMR flow equations. We constructed a sphere in which the plague can be quantitatively described whether partial or absolute at any given point. Based on our earlier studies [2, 20, 22], we solved the blood flow problems in spherical coordinate by method of separation of variables using the Bessel and Legendre polynomial functions. Equation 10a transforms the spherical Bessel equation into ordinary Bessel equation from which the NMR transverse magnetization and signals were properly derived as functions of Bessel and Legendre polynomials based on the boundary conditions imposed by the flow problems. Dimensionally, it is exciting to note from Eqs. (10a, 11, 12, 26, 28) that, 1) the constant  $\alpha$  defines the velocity on the stenosis in a blood vessel when the diameter available for blood flow reduces from  $h$  to  $\beta$  or  $\beta_{in}$ . 2) the constants  $\beta$  and  $\beta_{in}$  characterize the diameter of the obstructed blood vessel,  $\beta = \beta_{in} = h - h_1$  where  $h_1$  is the height of stenosis. 3) the velocity ratio  $\alpha/V$  determines the severity of plague in the blood vessel. This is a very significant parameter of the flow analysis. For example, the value  $\alpha/V=1$  describes the blood flow in normal artery as shown in Fig. 1a, where  $\beta = \beta_{in} = h$  and  $h_1 = 0$ . On the other extreme situation when  $\alpha/V=0$ , the signal as recorded in Eq. 29 is completely lost and the patient is either dead or in a very serious health condition because  $h = h_1$ , the blood vessel is totally blocked. Based on the detailed analytical procedure presented in this study (Fig. 2), the geometrical factors closely associated with the degenerative disease can be well understood and properly assessed if Eq. 29 is translated to magnetic resonance image by means of an appropriate imaging

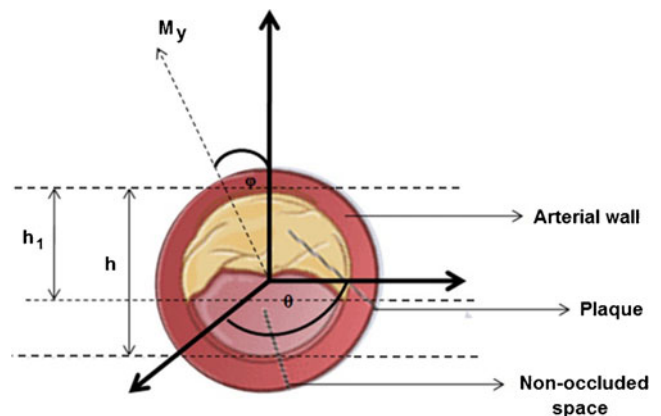


Fig. 2 Geometrical consideration blood vessel with atherosclerosis disease

sequence. Accurate knowledge of the velocity ratio  $\alpha/V$  can enhance our present understanding on how to restore normal coronary perfusion, or if that is not possible, to reduce the oxygen demand by the heart (i.e., normalize the oxygen supply/demand ratio) so as to minimize myocardial hypoxia. We would be able to accurately understand the severe CAD in which one or more coronary arteries is very stenotic, so that patient may undergo the right coronary artery bypass grafts or be treated with the appropriate drugs that reduce the myocardial oxygen demand by decreasing heart rate, contractility, after load or preload.

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