# EXTENDED BLOCK HYBRID BACKWARD DIFFERENTIATION FORMULA FOR SECOND ORDER FUZZY DIFFERENTIAL EQUATIONS USING LEGENDRE POLYNOMIAL AS BASIS FUNCTION

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**Abstract.**
*In this paper, we developed an implicit continuous four-step Extended Block Hybrid Backward Differentiation Formulae (EBHBDF) for the direct solution of Fuzzy Differential Equations (FDEs). For this purpose, the Legendre polynomial was employed as the basis function for the development of schemes in a collocation and interpolation techniques. in this regard and the results are satisfied the convex triangular fuzzy number. We also compare the numerical results with the exact solution, and it shows that the proposed method is good approximation for the analytic solution of the given second order Fuzzy Differential Equations*

## Introduction

The study of Fuzzy Differential e\Equations (FDEs) appears as a natural way to model the propagation of uncertainty in a dynamical environment. FDEs play an important role for modeling physical and engineering problems since they mimic the real situation to handle the system under uncertainty. Though, it is difficult to obtain the exact solution of FDEs due to the complexity of arithmetic in Fuzzy. The concept of Fuzzy set theory was first developed by Zadeh (1965) and there is need for efficient numerical technique to handle the corresponding FDEs. In recent years, the theory of FDEs has attracted wide spread attention and had been rapidly growing. It was massively studied by several researchers (Oregan, Lakshmikantham, & Nieto, 2003; Nieto, 2006). Chang and Zadeh (1972) first introduced the concept of Fuzzy derivative, followed by Duois and Prade (1982) who defined and used extension principle in their approach. Bede (2008) described the exact solutions of FDEs. Buckley and Feuring (2001) used two analytical methods to solve nth order linear differential equations with Fuzzy initial conditions, the first method Fuzzified the crisp solution to obtain a Fuzzy function and then check if it satisfied the differential equations and the second is the reverse of the first method. Ahmada, Hasan, and Baets (2013) studied the analytical and numerical based solution for Fuzzy differential equations FDEs. Oregan *et al*. (2003) obtained the exact solution of Fuzzy first-order boundary value problems. In all of the above attempts, the FDEs are converted into coupled or uncoupled system of differential equations depending on the sign of the coefficients. Much recently, Tapaswini and Chakraverty (2014) developed a new analytical method based on Fuzzy centre which solve with respects to the sign of the coefficients.

In the last few years, second-order fuzzy differential equations have been studied by Abbasbandy and Viranloo (2002), Abbasbandy, Viranloo, L´opez-Pouso, and Nieto (2004), Allahviranlo, Ahmady, and Ahmady (2007), Allahviranlo, Ahmady, and Ahmady (2008), Wang and Guo (2011) and Rabiei, Ismail, Ahmadian and Salahshour (2013), Fookand and Ibrahim (2017). In the work of Allahviranlo *et al*. (2008), the authors obtained the approximate solution of nth-order linear differential equations with fuzzy initial conditions by using the collocation method. Wang and Guo (2011) have developed numerical methods for addressing second-order fuzzy differential equation by Adomian decomposition methods. Rabiei *et al.* (2013) have developed the fuzzy improved Runge-Kutta Nystrom (FIRKN) method for solving second-order fuzzy differential equations. Meanwhile Fookand and Ibrahim (2017) proposed block backward differentiation formula method for solving second order fuzzy initial value problems. Jameel et al (2017) developed numerical solution of second-order nonlinear two-point fuzzy boundary value problems (TPFBVP) by combining the finite difference method with Newton's method. In this paper, we construct an Extended Block Hybrid Backward Differentiation Formula (EBHBDF) method capable of solving both Initial and boundary value problem of linear and non-linear type of second order FDEs with small errors and less computation

**Preliminaries**

The definitions reviewed in this section are required in our work.

The link between the crisp and fuzzy domains represented by the r-level set (or r-cut set) of a fuzzy set $\tilde{A}$, denoted by $[\tilde{A}]_r$ which is the crisp set of all $x \in X$ such that $\mu_{\tilde{A}} \geq r$ i.e., $[\tilde{A}] = \{x \in X \mid \mu_{\tilde{A}} > r, r \in [0.1]\}$

**Definition 2.2**

One of the important tools that uses to fuzzify the crisp models are fuzzy numbers which are subsets of the real numbers set and represents vague values. Fuzzy numbers are linked to degrees of membership which state how true it is to say if something belongs or not to a determined set. A fuzzy number $\mu$ is called a triangular fuzzy number (Dubois and Prade, 1982) is defined by three numbers $\alpha < \beta < \gamma$ where the graph of $\mu(x)$ is a triangle with the base on the interval$[\alpha, \beta]$ and its membership function has the following form (Figure 1)

$$\mu(x,\alpha,\beta,\gamma) = \begin{cases} 0 & if\ x < \alpha \\ \dfrac{x-\alpha}{\beta-\alpha}, & if\ \alpha \leq x \leq \beta \\ \dfrac{\gamma - x}{\gamma - \beta}, & if\ \beta \leq x \leq \gamma \\ 0, & if\ x > \gamma \end{cases}$$

$$[\mu(x)]_r = [\alpha + r(\beta - \alpha), \gamma - r(\gamma - \beta)], r \in [0,1].$$

And its r-level is:

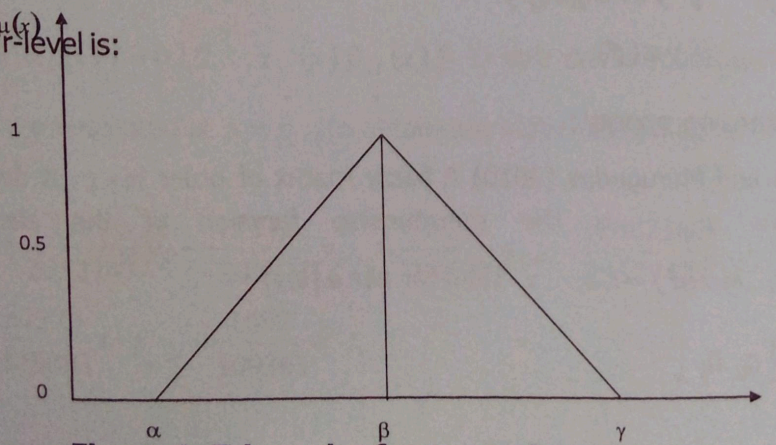



**Figure 1: Triangular fuzzy number**

In this paper, the class of all fuzzy subsets of $R$ will be denoted by $\tilde{E}$ and satisfy the following properties (Dubois and Prade, 1982, Mansouri and Ahmady, 2012)

1. $\mu(x)$ is normal, i.e., $\exists t_0 \in R$ with $\mu(t_0)=1$,

2. $\mu(x)$ is convex fuzzy set, i.e., $\mu(\lambda x+(1-\lambda)y)\geq \min\{\mu(x),\mu(y)\}\forall x,y\in \mathbb{R},\lambda[0,1]$,

3. $\mu$ upper semi-continuous on $R$

4. $\{x\in R:\mu(x)>0\}$ is compact.

Where $\tilde{E}$ is the space of fuzzy numbers and $R$ is a proper subset of $\tilde{E}$.

Define the r-level set $x\in \mathbb{R},[\mu]_r=\{x\backslash \mu(x)\geq r\},0\leq r\leq 1$, where $[\mu]_0=\{x\backslash \mu(x)>0\}$ is compact Ghanbari (2009) which is a closed bounded interval and denoted by $[\mu]_r=(\underline{\mu}(x),\overline{\mu}(x))$. In the parametric form (Dubois and Prade, 1982) which is represented by an ordered pair of function $(\underline{\mu}(x;r),\overline{\mu}(x;r)),r\in[0,1]$ that satisfies the following conditions:

1. $\underline{\mu}(x;r)$ Is bounded left continuous non-decreasing function over $[0,1]$.

2. $\overline{\mu}(x;r)$ is bounded left continuous non-increasing function over $[0,1]$.

3. $\underline{\mu}(x;r)\leq\overline{\mu}(x;r)$. A crisp number r is simply represented by $\underline{\mu}(r)=\overline{\mu}(r)=r$.

**Definition 2.3** **Fard (2009)** A mapping $\tilde{f}:T\rightarrow E$ for some interval $T\subseteq\tilde{E}$ is called a fuzzy process or fuzzy function with crisp variable, and we denote r-level set by:

$$[\tilde{f}(x;r)]_r=[\underline{f}(x;r),\overline{f}(x;r)],x\in T,r\in[0,1]$$

Where $\tilde{E}$ be the set of all upper semicontinuous normal convex fuzzy numbers.

**Definition 2.4** Zadeh(2005) Each function $f:X\rightarrow Y$ induces another function $\tilde{f}:F(X)\rightarrow F(Y)$ defined for each fuzzy interval U in X by:

$$\tilde{f}(U)(y)=\begin{cases}Sup_{x\in f^{-1}(y)}U(x), & if\ y\in range(f)\\ 0, & if\ y\notin(f)\end{cases}$$

This is called the Zadeh's extension principle.

**Definition 2.5** Sriram and Murugadas (2010) A fuzzy matrix of order $m\times s$ is defined $[\tilde{A}]=[\tilde{a}_{ij},\mu_{\tilde{a}_{ij}}]$ as, where $\mu_{\tilde{a}_{ij}}$ is the membership function of the element $\tilde{a}_{ij}\ in[\tilde{A}],\forall\tilde{a}_{ij}\in\hat{E}\ for\ i=1,2,...m\ and\ j=1,2,...,s$. Thus for all $r\in[0,1]$

$$[\tilde{A}]_r=[\underline{A},\overline{A}]_r, and[\tilde{a}_{ij}]_r=[\underline{a}_{ij},\overline{a}_{ij}]_r.$$

## vation of the Method

is section, we construct the main method and additional methods derived from its second vative which are combined to form the four-step Extended Block Backward Differentiation nula (EBBDF) on the interval from $x_n$ to $x_n + kh$ where $h$ be the chosen step-length. We ume that the exact solution $y(x)$ on the interval $[x_n, x_{n+4}]$ is locally represented by $Y(x)$ en by

$$(x) = \sum_{j=0}^{p+q-1} b_j \varphi_j(x)$$

are unknown coefficients to be determined, and $\varphi_j(x)$ are Legendre polynomial basis nction of degree $p+q-1$ such that the number of interpolation points and the number of stinct collocation points $q$ are respectively chosen to satisfy $p>0$ and $q>0$. The proposed ass of methods is thus constructed by specifying the following parameters:

$$_j(x) = x^j_{n+i}, j = 0,...k, p = 5, q = 2, k = 4$$

3y imposing the following conditions

$$\sum_{j=0}^{6} b_j x^j_{n+i} = y_{n+i}, i = 0,...4 \quad (2)$$

$$\sum_{j=0}^{6} j(j-1) b_j x^{j-2}_{n+i} = f_{n+i}, i = 0,...4 \quad (3)$$

Assuming that $y_{n+i} = Y(x_n + ih)$, denote the numerical approximation to the exact solution $y(x_{n+i})$, $f_{n+i} = Y''(x_n + ih, y_{n+i})$, denote the approximation to $y''(x_{n+i})$ and $n$ is the grid index. It should be noted that equation (2) and (3) lead to a system of seven equations which must be solved to obtain the coefficients $b_j$, $j = 0,1...,6$. The main method is then obtained by substituting the values of $b_j$ into equation (2). After some algebraic computation, the method yields the expression in the form (4).

$$Y(x) = \sum_{j=0}^{4} \alpha_j(x) + \alpha_{\frac{15}{4}}(x) y_{\frac{15}{4}} + h^2\left(\beta_{\frac{15}{4}}(x) f_{\frac{15}{4}} + \beta_4(x) f_4\right) \quad (4)$$

Where $\alpha_j(x)$ $j = 0,1,2....,\alpha_{\frac{15}{4}}(x), \beta_{\frac{15}{4}}(x), \beta_4(x)$ are continuous coefficients. The continuous form in (4) are evaluated at $x = x_{n+4}$ to obtain the main method as

$$y_{n+4} = \frac{2797}{2952747} y_n - \frac{11432}{1202971} y_{n+1} + \frac{41430}{765527} y_{n+2} - \frac{1268216}{2952747} y_{n+3} + \frac{314654720}{227361519} y_{n+\frac{15}{2}} + \frac{37376}{328083} h^2 f_{n+\frac{15}{2}} - \frac{1692}{109361} h^2 f_{n+4} \quad (5)$$

$$y_{n+4}=\frac{2797}{2952747}y_n-\frac{11432}{1202971}y_{n+1}+\frac{41430}{765527}y_{n+2}-\frac{1268216}{2952747}y_{n+3}+\frac{314654720}{227361519}y_{n-\frac{15}{2}}+$$
$$\frac{37376}{328083}h^2 f_{n+\frac{15}{2}}-\frac{1692}{109361}h^2 f_{n-4} \qquad (5)$$

Differentiating (4) twice to obtain the additional method at $x=x_{n+1}, x=x_{n+2}, x=x_{n+3}$ as

$$y_{n+3}=\frac{19697337}{32100299}y_{n+2}-\frac{2908143}{32100299}y_{n+1}-\frac{14399488}{32100299}y_{n+\frac{15}{4}}+\frac{4904669}{32100299}y_n+$$
$$\frac{1}{32100299}h^2\left(9559872 f_{n+\frac{15}{4}}-5026329 f_{n+4}-6889743 f_{n+1}\right) \qquad (6)$$

$$y_{n+2}=\frac{184320}{2521662}y_{n+\frac{15}{4}}-\frac{96881}{2521662}y_{n+3}+\frac{1445682}{2521662}y_{n+1}-\frac{77154}{2521662}y_n+$$
$$\frac{1}{2521662}h^2\left(-305536 f_{n+\frac{15}{4}}+173327 f_{n+4}-1202971 f_{n+2}\right) \qquad (7)$$

$$y_{n+1}=\frac{44619776}{4828761}y_{n+\frac{15}{4}}+\frac{96881}{4828761}y_{n+3}+\frac{41892741}{4828761}y_{n+1}+\frac{409871}{4828761}y_n+$$
$$\frac{1}{4828761}h^2\left(1315776 f_{n+\frac{15}{4}}-1735965 f_{n+4}-25262391 f_{n+3}\right) \qquad (8)$$

The first derivative formula is also obtained by differentiating the continuous form in equation (4) once as follows

$$z_n=-\frac{1}{252623910h}\left(\begin{array}{l}420477750h^2 f_{n+4}-802771200h^2 f_{n+\frac{15}{4}}+647551751y_n-\\ 1745131500y_{n+1}+2733179625y_{n+2}-2879530500y_{n+3}+\\ 1243930624y_{n+\frac{15}{4}}\end{array}\right) \qquad (9)$$

$$6820845570hz_{n+1}=7219392512y_{n+\frac{15}{4}}-17750608645y_{n+3}$$
$$+21483497970y_{n+2}-9967046355y_{n+1}-985235482y_n+$$
$$h\left(-4151790720 f_{n+\frac{15}{4}}-2105548830 f_{n+4}\right)$$

$$z_{n+3}=\frac{1}{252623910h}\left(\begin{array}{l}23222430h^2 f_{n+4}-56327040h^2 f_{n+\frac{15}{4}}-1650726y_n+\\ 17439975y_{n+1}-114633090y_{n+2}-125047615y_{n+3}+\\ 223891456y_{n+\frac{15}{4}}\end{array}\right)$$

$$z_{n+2}=-\frac{1}{974406510h}\left(\begin{array}{l}130540410h^2 f_{n+4}-273208320h^2 f_{n+\frac{15}{4}}-21997129y_n+\\ 305930520y_{n+1}+874173465y_{n+2}-1752289000y_{n+3}+\\ 594182144y_{n+\frac{15}{4}}\end{array}\right)$$

$$z_{n+4} = \frac{1}{6820845570h}\left(\begin{array}{l} 88503030h^2 f_{n+4} + 3476309760h^2 f_{n+\frac{15}{4}} + 24233363y_n - \\ 305930520y_{n+1} + 1411179165y_{n+2} - 11611207300y_{n+3} + \\ 10420477952y_{n+\frac{15}{4}} \end{array}\right)$$

$$z_{n+\frac{7}{2}} = -\frac{1}{27283382280h}\left(\begin{array}{l} 1132420905h^2 f_{n+4} - 2361355920h^2 f_{n+\frac{15}{4}} - 70213297y_n + \\ 719018370y_{n+1} - 4301529705y_{n+2} + 44129411480y_{n+3} - \\ 40476686848y_{n+\frac{15}{4}} \end{array}\right)$$

## Numerical Examples and Discussion of Results

In this section, the efficiency and accuracy of the EBHBDF method formulated in above is tested on fuzzy system. The self-starting method is implemented efficiently by combining the methods as simultaneous numerical integrator for IVP's for example, the method presented in (5) - (9) are combined to obtain the initial conditions at $x_{n+4}$, $n(\text{mod}\,4) \neq 0$

and $0 \leq n \leq N$ using computed values $y(x_{n+4})$ over sub-interval $[x_0, x_4]$

In this section, we solved the fuzzy differential equations to show the accuracy of the method proposed in the above. The results of the exact solutions and numerical solutions are presented in the tables and figures. A comparison of the numerical solutions and exact solution is carried out to obtain the errors. Let the exact solution $Y(t,r) = |\underline{Y}(t,r), \overline{Y}(t,r)|$, the absolute error formula, considered in tables 1 – 2 is as follows:

The error, $\in$ is defined as the maximum error through the whole interval of integration.

Maximum Error $= \in$

$\underline{\in} = |\underline{y} - \underline{Y}|, \quad \overline{\in} = |\overline{y} - \overline{Y}|$

The notation used in the tables and figures are the following:

$h$: *step size*

$r$: *fuzzy numbers with fuzz bounded r – level interval*

$\underline{Y}$: *lower bounded exact solution*

$\overline{Y}$: *upper bounded exact solution*

$\underline{y}$: *lower bounded numerical solution*

$\overline{y}$: *upper bounded numerical solution*

**Problem1:** We consider the following fuzzy linear initial value problem.

$$y'' = -y, \quad x \geq 0$$

$$y(0) = 0, y'(0) = [0.9 + 0.1r, 1.1 - 0.1r]$$

Exact solution at $x = 1$

$$Y(x,r) = [(0.9 + 0.1r)\sin(x), \ (1.1 - 0.1r)\sin x]$$

**Problem 2:** We consider the following fuzzy linear initial value problem

$$y'' = -y + x, \qquad x \geq 0$$

$$y'(0) = [1.8 + 0.2r, 2.2 - 0.2r]$$

Exact solution at $x = 1$

$$y_1 = \left(\frac{4}{5} + \frac{1}{5}r\right)\sin x + \left(\frac{9}{10} + \frac{1}{10}r\right)\cos(x) + x$$

$$y_2 = \left(\frac{6}{5} - \frac{1}{5}r\sin(x)\right) + \left(\frac{11}{10} - \frac{1}{10}r\right)\cos x + x$$

**Problem 3:** We consider a second-order Fuzzy linear differential equation with positive coefficients, subject to Fuzzy boundary conditions.

$$y'' + \bar{y} + t = 0$$

$\bar{y}(0) = \bar{y}(1) = [0.1r - 0.1, 0.1 - 0.1r]$ Exact solutions: First condition;
$Y[t,r] = -t + (0.1r - 0.1)\cos(t) + (1.13376 + 0.0546302r)\sin(t)$

Second condition; $\tilde{Y}[t,r] = -t + (0.1 - 0.1r)\cos(t) + (1.24303 - 0.0546302\,r)\sin(t)$

**Problem 4**

$$y''(x,r) = \frac{-[y'(x,r)]^2}{y(x,r)}, \quad x \in [0,1], \quad y(0,r) = [0.9 + 0.1r,\ 1.1 - 0.1r]$$
$$y(1,r) = [1.9 + 0.1r,\ 2.1 - 0.1r]$$

Using the Maple 2015 software package to obtained the exact solution of Problem 4 as follows

$$\underline{Y}(x,r) = \sqrt{1.4 + 0.1r}\sqrt{\frac{0.1(9.0 + 1.0r)^2}{14.0 + 1.0r} + 2x}$$

$$\bar{Y}(x,r) = \sqrt{1.6 - 0.1r}\sqrt{\frac{-0.1(-11.0 + 1.0r)^2}{-16.0 + 1.0r} + 2x}$$

Also we can represent the exact solution of Problem 4 for all $r \in [0,1]$ and $x \in [0,1]$ in figure 4

**Table 1: Error at t = 1 in solving problem 1**

| | | BDF | | BBDF | | EBHBDF | |
|---|---|---|---|---|---|---|---|
| $h$ | $r$ | $\underline{\varepsilon}$ | $\bar{\varepsilon}$ | $\underline{\varepsilon}$ | $\bar{\varepsilon}$ | $\underline{\varepsilon}$ | $\bar{\varepsilon}$ |
| $10^{-1}$ | 0 | 3.09591e-05 | 3.78389e-05 | 5.40487e-05 | 6.60595e-05 | 2.80947e-07 | 3.4337e-07 |
| | 0.2 | 3.16471e-05 | 3.71510e-05 | 5.52498e-05 | 6.48584e-05 | 2.8719e-07 | 3.3713e-07 |
| | 0.4 | 3.23351e-05 | 3.64630e-05 | 5.64509e-05 | 6.36573e-05 | 2.93433e-07 | 3.3089e-07 |
| | 0.6 | 3.30231e-05 | 3.57750e-05 | 5.76519e-05 | 6.24563e-05 | 2.99676e-07 | 3.2464e-07 |
| | 0.8 | 3.37111e-05 | 3.50870e-05 | 5.88530e-05 | 6.12552e-05 | 3.0592e-07 | 3.1840e-07 |
| | 1.0 | 3.43990e-05 | 3.43990e-05 | 6.00541e-05 | 6.00541e-05 | 3.12163e-07 | 3.1216e-07 |
| **Execution Time** | | 1.26s | | 0.6s | | 0.52s | |

| $h$ | $r$ | BDF $\underline{\varepsilon}$ | BDF $\overline{\varepsilon}$ | BBDF $\underline{\varepsilon}$ | BBDF $\overline{\varepsilon}$ | EBHBDF $\underline{\varepsilon}$ | EBHBDF $\overline{\varepsilon}$ |
|---|---|---|---|---|---|---|---|
| $10^{-2}$ | 0 | 3.14945e-08 | 3.84933e-08 | 6.851e-08 | 8.373e-08 | 6.12e-11 | 7.06e-11 |
| | 0.2 | 3.21944e-08 | 3.77934e-08 | 7.003e-08 | 8.221e-08 | 6.02e-11 | 7.14e-11 |
| | 0.4 | 3.28943e-08 | 3.70935e-08 | 7.155e-08 | 8.069e-08 | 6.25e-11 | 6.97e-11 |
| | 0.6 | 3.35941e-08 | 3.63937e-08 | 7.307e-08 | 7.916e-08 | 6.36e-11 | 6.79e-11 |
| | 0.8 | 3.42940e-08 | 3.56938e-08 | 7.459e-08 | 7.764e-08 | 6.45e-11 | 6.54e-11 |
| | 1.0 | 3.49939e-08 | 3.49939e-08 | 7.612e-08 | 7.611e-08 | 6.48e-11 | 6.48e-11 |

**Table 2: Error at t = 1 in solving problem 2**

| $h$ | $r$ | BDF $\underline{\varepsilon}$ | BDF $\overline{\varepsilon}$ | BBDF $\underline{\varepsilon}$ | BBDF $\overline{\varepsilon}$ | EBHBDF $\underline{\varepsilon}$ | EBHBDF $\overline{\varepsilon}$ |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 1.708944e-05 | 2.85313e-05 | 2.25608e-05 | 4.09196e-05 | 1.09762e-07 | 2.0352e-07 |
| | 0.2 | 1.823363e-05 | 2.73871e-05 | 2.43967e-05 | 3.90838e-05 | 1.19138e-07 | 1.9414e-07 |
| | 0.4 | 1.937782e-05 | 2.62430e-05 | 2.62326e-05 | 3.72479e-05 | 1.28514e-07 | 1.8477e-07 |
| | 0.6 | 2.052201e-05 | 2.50988e-05 | 2.80684e-05 | 3.54120e-05 | 1.3789e-07 | 1.7539e-07 |
| | 0.8 | 2.166619e-05 | 2.39546e-05 | 2.99043e-05 | 3.35761e-05 | 1.47266e-07 | 1.6601e-07 |
| | 1.0 | 2.281038e-05 | 2.28104e-05 | 3.17402e-05 | 3.17402e-05 | 1.56643e-07 | 1.5664e-07 |
| **Execution Time** | | 1.26s | | 0.6s | | 0.57s | |

| $h$ | $r$ | BDF $\underline{\varepsilon}$ | BDF $\overline{\varepsilon}$ | BBDF $\underline{\varepsilon}$ | BBDF $\overline{\varepsilon}$ | EBHBDF $\underline{\varepsilon}$ | EBHBDF $\overline{\varepsilon}$ |
|---|---|---|---|---|---|---|---|
| 2 | 0 | 1.67951e-08 | 2.83038e-08 | 3.56459e-08 | 6.04823e-08 | 2.93e-10 | 3.51e-10 |
| | 0.2 | 1.79460e-08 | 2.71529e-08 | 3.81297e-08 | 5.79987e-08 | 2.95e-10 | 3.42e-10 |
| | 0.4 | 1.90969e-08 | 2.60021e-08 | 4.06131e-08 | 5.55149e-08 | 3.04e-10 | 3.42e-10 |
| | 0.6 | 2.02477e-08 | 2.48512e-08 | 4.30966e-08 | 5.30309e-08 | 3.08e-10 | 3.33e-10 |
| | 0.8 | 2.13986e-08 | 2.37003e-08 | 4.55803e-08 | 5.05478e-08 | 3.18e-10 | 3.23e-10 |
| | 1.0 | 2.25495e-08 | 2.25495e-08 | 4.80643e-08 | 4.80643e-08 | 3.26e-10 | 3.26e-10 |

**Table 3: Solution of Problem 3 at x=1/12**

| r | $\underline{Y}$ | $\underline{y}$ | $\overline{Y}$ | $\overline{y}$ |
|---|---|---|---|---|
| 0 | -0.08861562589 | -0.08861521274 | 0.1106903314 | 01197852491 |
| 0.1 | -0.07819560308 | -0.07819518960 | 0.1002703086 | 0.1093652261 |
| 0.2 | -0.06777558028 | -0.06777516657 | 0.08985028585 | 0.09894520301 |
| 0.3 | -0.05735555747 | -0.05735514341 | 0.07943026304 | 0.08852517997 |
| 0.4 | -0.04693553467 | -0.04693512029 | 0.04817019462 | 0.07810515682 |
| 0.5 | -0.03651551186 | -0.03651509720 | 0.03775017182 | 0.06768513374 |
| 0.6 | -0.02609548906 | -0.02609507415 | 0.02733014901 | 0.05726511070 |
| 0.7 | -0.01567546625 | -0.01567505104 | 0.01691012621 | 0.04684508756 |
| 0.8 | -0.00525544345 | -0.005255027944 | 0.00649010340 | 0.03642506446 |
| 0.9 | 0.005164579351 | 0.005164995189 | 0.008146109966 | 0.02600504135 |
| 1 | 0.015584602156 | 0.01558501824 | 0.018641544080 | 0.018641543730 |

**Table 4: Difference approximate solution $\underline{y}$ (xr) at h=1/20 for Problem 4**

| $r$ | $\underline{y}(0,r)$ | $\underline{y}(0.2,r)$ | $\underline{y}(0.4,r)$ | $\underline{y}(0.6,r)$ | $\underline{y}(0.8,r)$ | $\underline{y}(1.0,r)$ |
|---|---|---|---|---|---|---|
| 0 | 0.900000000 | 1.170466311 | 1.389241611 | 1.577971655 | 1.746424118 | 1.90000000 |
| 0.25 | 0.925000000 | 1.193992139 | 1.412663429 | 1.601755296 | 1.770768885 | 1.92500000 |
| 0.5 | 0.950000000 | 1.217576679 | 1.436138431 | 1.625575424 | 1.795131656 | 1.95000000 |
| 0.75 | 0.9750000000 | 1.241216591 | 1.459664055 | 1.649430460 | 1.819511708 | 1.97500000 |
| 1 | 1.0000000000 | 1.264908775 | 1.483237895 | 1.673318913 | 1.843908357 | 2.00000000 |

**Table 5: Difference approximate solution $\bar{y}(x,r)$ at h=1/20 for Problem 4**

| r | $\bar{y}(0,r)$ | $\bar{y}(0.2,r)$ | $\underline{y}(0.4,r)$ | $\underline{y}(0.62,r)$ | $\underline{y}(0.8,r)$ | $\underline{y}(1.0,r)$ |
|---|---|---|---|---|---|---|
| 0 | 1.0900000000 | 1.359189815 | 1.577021613 | 1.768218943 | 1.940669873 | 2.09900000 |
| 0.25 | 1.0750000000 | 1.336271154 | 1.554226771 | 1.745171056 | 1.917191552 | 2.07500000 |
| 0.5 | 1.0500000000 | 1.312438641 | 1.530521316 | 1.721190509 | 1.892748880 | 2.05000000 |
| 0.75 | 1.0250000000 | 1.288650350 | 1.506857690 | 1.697239374 | 1.868320953 | 2.02500000 |
| 1 | 1.0000000000 | 1.264908775 | 1.483237895 | 1.673318913 | 1.843908357 | 2.00000000 |

Table 6: Accuracy of Numerical solution of Problem 4 at h = 1/120 and r = 0.75

| **x** | $\left[\underline{E}_{\frac{1}{20}}\right]_{0.75}$ | $\left[\overline{E}_{\frac{1}{20}}\right]_{0.75}$ |
|---|---|---|
| **0** | 0 | 0 |
| **0.2** | 2.57E-06 | 2.05E-06 |
| **0.4** | 2E-06 | 1.63E-06 |
| **0.6** | 1.26E-06 | 1.03E-06 |
| **0.8** | 5.88E-07 | 4.87E-07 |
| **1** | 0 | 0 |

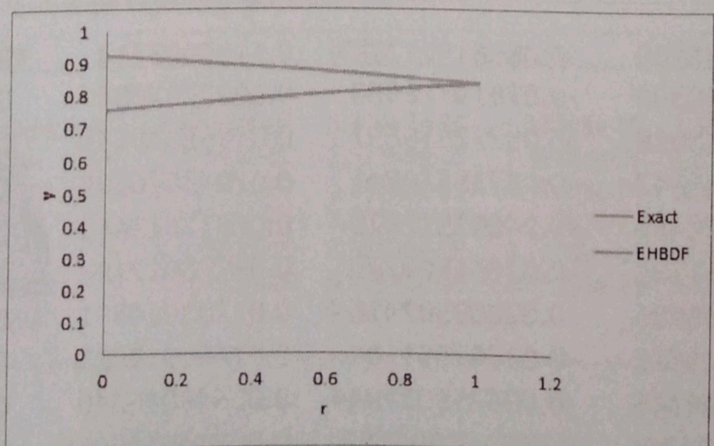



**Figure 2: The exact solution and the approximate solution in Table 1 with h=0.1**

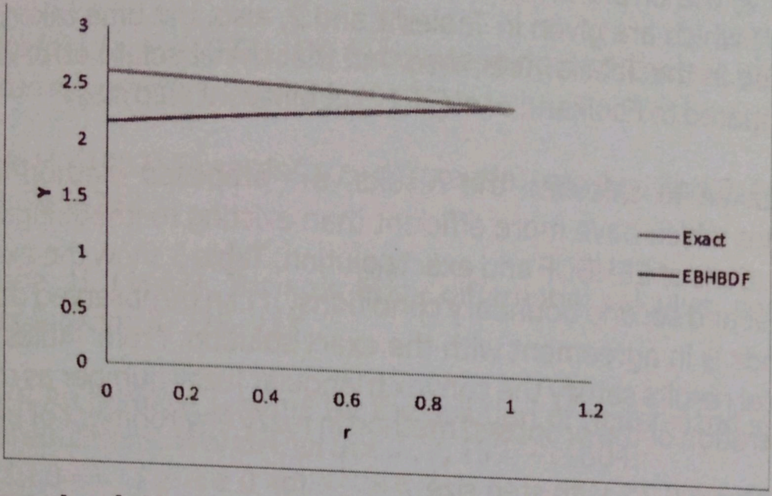
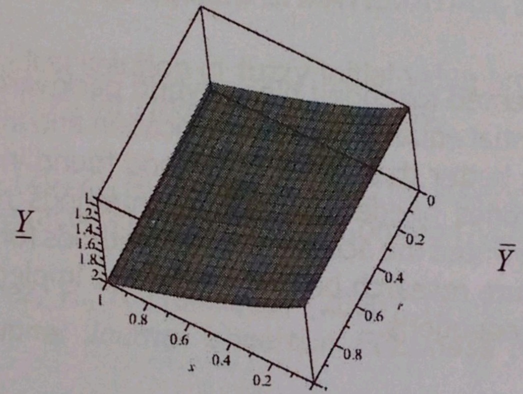



**Figure 3: The exact solution and the approximate solution in Table 2 with h=0.1**



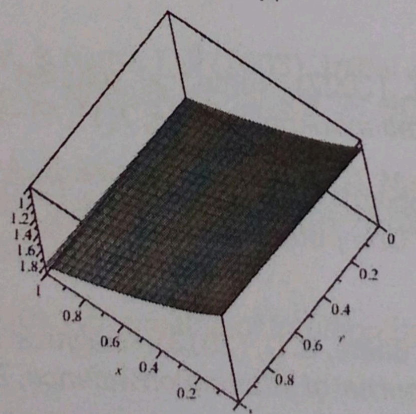



**Figure 4: Exact analytical solution of problem 4 for all $r \in [0,1]$ and $x \in [0,1]$**

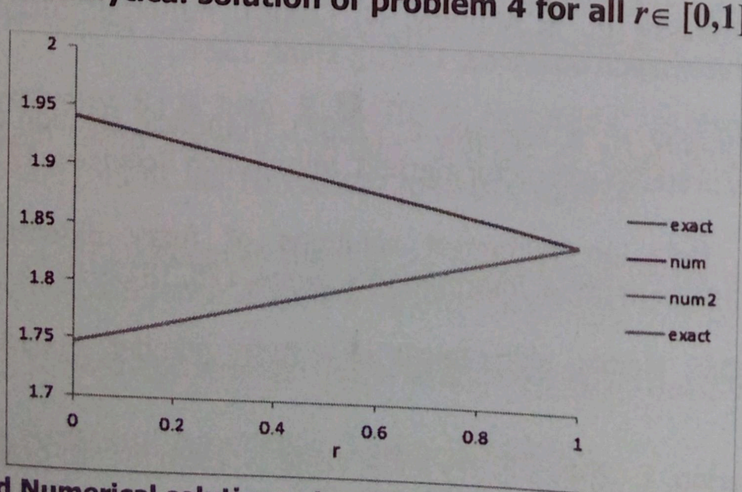



**Figure 5: Exact and Numerical solutions at x=0.8 and for all r in Problem 4 when h = 1/120**

For problems 1 and 2, the errors of EBHBD are compared with BDF and BBDF proposed by Foolkand *et al* (2017) which are given in Tables 1 and 2, also, the time taken for the proposed method are presented in the Tables. It is observed that the absolute error of the proposed is very small when compared to Foolkand *et al* (2017) at different step size.

However, for time taken to calculate the results, the proposed method in this paper has significant advantages which have more efficient than existing method. Figures 2 and 3 show the approximate solutions of EBHBDF and exact solution. Table 3 show the exact and numerical solutions with the first and second boundary conditions. It can be observed that the behavior of the proposed methods is in agreement with the exact solution. From Tables 4 and 5, one can see that the numerical results satisfy the convex triangular fuzzy number as mentioned in Sect. 2. Also for more illustration of the proposed method in fuzzy environment of problem 4, we solved this problem at $r = 0.75$ with step size $h = \frac{1}{20}$ for $0 \le x_i \le 1,\ i = 0,1,2, \quad n$ as shown in Table 6

## Conclusion

In this study, we have presented extended block hybrid backward differentiation formula for the solution of fuzzy differential equations using collocation and interpolation techniques. The method proposed performs better than existing method found in the literature. The method avoids complicated subroutines needed for existing methods requiring starting values or predictors. We have demonstrated the accuracy of the methods for fuzzy differential problems. It is recommended that future research be focused on the implementation of the method to parabolic partial differential equations.

## References


Abbasbandy, S., & Viranloo, T. (2002) Numerical solution of fuzzy differential equation. *Mathematical & Computational Applications*, 7(1), 41–52.

Abbasbandy, S., Viranloo, T., L. opez-Pouso, O., & Nieto, J. (2004). Numerical methods for fuzzy differential inclusions. *Computers & Mathematics with Applications*, 48(10-11), 1633–1641.

Ahmada, M. Z., Hasan, M. K., & Baets, B. D. (2013). Analytical and numerical solutions of fuzzy differential equations, *Journal of Information Science*, 236, 156–167.

Allahviranloo, T., Ahmady, E., & Ahmady, N. (2008). nth-order fuzzy linear differential equations. *Information Sciences*, 178(5), 1309–1324.

Allahviranloo, T., Ahmady, N., & Ahmady, E. (2007). Numerical solution of fuzzy differential equations by predictor-corrector method. *Information Sciences*, 177(7), 1633-1647.

Bede, B. (2008). Note on numerical solutions of fuzzy differential equations by predictorcorrector method. *Information Science*, 178, 1917–1922.

Bodjanova, S. (2006). Median alpha-levels of a fuzzy number. *Fuzzy Set Syst*, 157(7), 879–891.

Buckley, J. J., & Feuring, T. (2001). Fuzzy initial value problem for nth-order linear differential equations, *Fuzzy Sets System*, 121, 247–255.

Chang, S. L., & Zadeh, L. A. (1972). On fuzzy mapping and control. *IEEE Transactions on System Man cybernetics*, 2, 30-34.

Dubois D., & Prade H. (1982) Towards fuzzy differential calculus. Part 3: Differentiation. *Fuzzy Set Syst* 8, 225–233.

Dubois, D., & Prade, H. (1982). Towards fuzzy differential calculus, part 3: Differentiation. *Fuzzy Sets System*, 8, 225–233.

Fard, O. S. (2009). An iterative scheme for the solution of generalized system of linear fuzzy differential equations. *World Appl Sci. J.*, 7, 1597–11604.

Fookand, T. K., & Ibrahim, Z. B. (2017). Block backward differentiation formulas for solving second order fuzzy differential equations. *MATEMATIKA*, 33( 2), 215–226.

Ghanbari, M. (2009). Numerical solution of fuzzy initial value problems under generalization differentiability by H. P. M. *Int J Ind Math* 1(1), 19–39.

Mansouri, S., & Ahmady, N. (2012). Numerical method for solving Nth-order fuzzy differential equation by using characterization theorem. *Commun Numer Anal*, 1–12.

Nieto, J. J., Rodriguez-Lopez, R., & Franco, D. (2006). Linear first-order fuzzy differential equations, *International Journal Uncertain Fuzziness Knowledge.-Based System*, 14,687-709.

Oregan, D., Lakshmikantham, V., & Nieto, J. J. (2003). Initial and boundary value problems for fuzzy differential equations. *Nonlinear Analysis*, 54, 405–415.

Rabiei, F., Ismail, F., Ahmadian, A., & Salahshour, S. (2013). Numerical solution of second-order fuzzy differential equation using improved runge-kuttanystrom method. *Mathematical Problems in Engineering*, 1–10.

Sriram, S., & Murugadas, P. (2010). On semiring of intuitionistic fuzzy matrices. *Appl Math Sci*, 4(23),1099–1105.

Tapaswini, S., & Chakraverty, S.(2014). New analytical method for solving n-th order fuzzy differential equations. Ann. *Fuzzy Mathematics Information*, 8, 231–244.

Tapaswini, S., Chakraverty, S., & Juan, S. N.(2017). Numerical Solution of Fuzzy boundary value problem using Galerkin method, *Journal of India Academy of Sciences, Sandhana*, 42(1), 45-61.doi:10.1007/s12046-016-0578-4.

Wang, L., & Guo, S. (2011). Adomian method for second-order fuzzy differential equation, World Academic of Science. *Engineering and Technology*, 5, 4–23.

Zadeh, L. A. (2005). Toward a generalized theory of uncertainty. *Inf Sci*, 172(2), 1–40.

Zadeh, L. A. (1965). Fuzzy sets, *Information Control*, 8. 338-353.