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Improvements of Successive Overrelaxation Iterative (SOR) Method for *L*-Matrices

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ABSTRACT
In order to accelerate the convergence of the classical iterative schemes, such as Jacobi, Gauss-Seidel and SOR, for the solution of the linear system $Ax = b$, several classes of preconditioners have been introduced. The present work proposes a preconditioner of the class $I + S$ applied to the classical SOR iterative matrix for linear systems with an irreducible L-matrix as the coefficient matrix. Convergence conditions for the preconditioned system are derived and analysed using standard procedures available in literature. Comparison of results for various spectral radii obtained from numerical experiments proved to be in agreement with the theorems advanced.

Keywords: linear system, L-matrix, iteration, spectral radius, convergence

INTRODUCTION

Many physical phenomena that involve two or more variables are modeled as partial differential equations. The approximation of partial derivatives by finite differences, in most cases, are eventually turned into linear system of algebraic equations of the form

$$Ax = b \quad (1)$$

where $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is a known nonsingular matrix, $b \in \mathbb{R}^{n\times 1}$ is a known vector and $x \in \mathbb{R}^{n\times 1}$ is the unknown vector. An iterative method for solving the system (1) consists of a process where the coefficient matrix A is split into the form $A = M - N$, where M is nonsingular, and the system (1) is converted into an equivalent system of the form

$$x = M^{-1}Nx + M^{-1}b \quad (2)$$

The sequence of solution vectors is obtained from (2) through the general linear iteration formula

$$x^{(k+1)} = M^{-1}Nx^{(k)} + M^{-1}b\,, \quad k = 0,1,2,\cdots \quad (3)$$

where $M^{-1}N$ is referred to as iteration matrix of the method. In this paper we assume, without loss of generality, that $A = I - L - U$, where I is the identity matrix, $-L$ and $-U$ the strictly lower and strictly upper triangular matrices of A, respectively. By the foregoing splitting, the iteration matrix of the classical SOR method is defined by

$$T_{SOR} = (I - \omega L)^{-1}\{(1-\omega)I + \omega U\} \quad (4)$$

where $M^{-1} = (I - \omega L)^{-1}$ and $N = \{(1-\omega)I + \omega U\}$. A necessary and sufficient condition for convergence and stability of the method requires that the spectral radius of the iterative method be less than 1, and the method converges faster when the spectral radius is near 0 than when it is near 1. Preconditioning, therefore, is the technique of reducing the spectral radius of the corresponding iterative matrix in order to speed up the convergence of a classical iterative method. It involves the manipulation of system (1) thus,

$$PAx = Pb \quad (5)$$

Or more compactly,

$$A'x = b' \quad (6)$$

where $A' = PA$, $b' = Pb$ and P is a nonsingular matrix known as the preconditioner.

Since the introduction of SOR method by Young (1950), a great many researchers have written about iterative methods for solving linear systems. These include, Varga (1957), Varga (1959), Young and Edison (1970), Young (1972), Varga (1981) and Young (1987). Modifications and generalizations of the basic iterative methods such as the SOR have also been done in many articles which include, to mention but just a few, Hadjidimos (1978), Martins (1988), Hadjidimos (2000) and Youssef (2012), Bamigbola and Ibrahim (2014), Muleta and Gofe (2018), Vatti *et al.* (2018). Of recent, there has been a lot of research involving preconditioners of the class $I + S$ used in accelerating the convergence of basic iterative methods. In this class abound advances by Milaszewicz (1987), Gunawardena *et al.* (1991), Kohno *et al.* (1997), Li and Sun (2000), Hadjidimos *et al.* (2003), Ndanusa and Adeboye (2012), Mayaki and Ndanusa (2019)

MATERIALS AND METHODS

The Preconditioned SOR Iterative Methods

Assume the coefficient matrix A in (1) is a nonsingular $L-$matrix and the linear system (1) is transformed into the preconditioned system (5), where $P = (I + S)$, I being the identity matrix and S a nonnegative matrix known as the preconditioner. Then a new preconditioner S is proposed thus.

$$S = \begin{cases} -a_{ij}\,, & (i,j) = (1,2),(2,1),(n-1,n),(n,n-1) \\ 0\,, & \text{otherwise} \end{cases}$$

The overrelaxation parameter ω is applied to the preconditioned system (6) thus

$$\omega A'x = \omega b' \quad (7)$$

A regular splitting of the coefficient matrix $\omega A'$ results in

$$\omega A' = \omega(D' - L' - U') \quad (8)$$

where D', $-L'$ and $-U'$ are the diagonal, the strictly lower triangular and the strictly upper triangular parts of matrix A'.

$$\omega A' = \omega(I + D_* - L' - U')$$

where $I + D_*$ is a splitting of D' into the sum of identity matrix I and a diagonal matrix D_*.

$$= I - \omega L' + \omega SL + \omega D_* - I + \omega I - \omega U' - \omega SL$$
$$= [I - \omega(L' - SL - D_*)] - \{(1-\omega)I + \omega(U' + SL)\}$$

Thus

$$\omega A' = [I - \omega(L' - SL - D_*)] - \{(1-\omega)I + \omega(U' + SL)\}$$

is a regular splitting of $\omega A' = M - N$, where $M = [I - \omega(L' - SL - D_*)]$ and $N = \{(1-\omega)I + \omega(U' + SL)\}$. Hence, the preconditioned SOR scheme is defined as

$$x^{(k+1)} = [I - \omega(L' - SL - D_*)]^{-1}\{(1-\omega)I + \omega(U' + SL)\}x^{(k)} + [I - \omega(L' - SL - D_*)]^{-1}\omega b'$$

Or more compactly,

$$x^{(k+1)} = T_1 x^{(k)} + c$$

where the preconditioned SOR iteration matrix is $T_1 = M^{-1}N = [I - \omega(L' - SL - D_*)]^{-1}\{(1-\omega)I + \omega(U' + SL)\}$ and $c = M^{-1}\omega b' = [I - \omega(L' - SL - D_*)]^{-1}\omega b'$.

Similarly, from equation (7)

$$\omega A' = \omega(D' - L' - U')$$
$$= \omega D' - \omega L' - \omega U'$$
$$= D' - \omega L' - (1-\omega)D' - \omega U'$$

Therefore,

$$\omega A' = (D' - \omega L') - [(1-\omega)D' + \omega U']$$

is another splitting of the preconditioned coefficient matrix $\omega A' = M - N$, where

$$\left.\begin{aligned} M &= (D' - \omega L') \\ N &= [(1-\omega)D' + \omega U'] \end{aligned}\right\} \quad (9)$$

And on substituting equations (9) into equation (3) we obtain our second preconditioned SOR iterative scheme as

$$x^{(k+1)} = (D' - \omega L')^{-1}[(1-\omega)D' + \omega U']x^{(k)} + (D' - \omega L')^{-1}\omega b'$$

That is,

$$x^{(k+1)} = T_2 x^{(k)} + c$$

where the preconditioned SOR iteration matrix $T_2 = M^{-1}N = (D' - \omega L')^{-1}[(1-\omega)D' + \omega U']$ and $c = M^{-1}\omega b' = (D' - \omega L')^{-1}\omega b'$.

The entries of the coefficient matrix $A' = (a'_{ij})$ of the preconditioned linear system (7) is further examined as follows.

$$\left.\begin{aligned}
a'_{ii} &= 1 - a_{12}a_{21}, && i = 1,2 \\
a'_{ii} &= 1, && i = 3,4,5,\cdots,n-2 \\
a'_{ii} &= 1 - a_{n-1,n}a_{n,n-1}, && i = n-1,n \\
a'_{ij} &= 0, && (i,j) = (1,2),(2,1),(n-1,n),(n,n-1) \\
a'_{ij} &= a_{ij}(i \neq j), && i = 3,4,5,\cdots,n-2 \\
a'_{1j} &= a_{1j} - a_{12}a_{2j}, && j = 3,4,5,\cdots,n \\
a'_{2j} &= a_{2j} - a_{21}a_{1j}, && j = 3,4,5,\cdots,n \\
a'_{n-1,j} &= a_{n-1,j} - a_{n-1,n}a_{nj}, && j = 1,2,3,\cdots,n-2 \\
a'_{n,j} &= a_{n,j} - a_{n,n-1}a_{n-1,j}, && j = 2,3,\cdots,n-2
\end{aligned}\right\} \quad (10)$$

From (10), all the off-diagonal entries of A' (represented by the last 6 lines) satisfy the $L-$matrix requirement. The diagonal entries (represented by the first 3 lines) must be greater than 0. Thus we must have

$$1 - a_{12}a_{21} > 0, \text{ which implies } 1 > a_{12}a_{21}$$
$$1 - a_{n-1,n}a_{n,n-1} > 0, \text{ which implies } 1 > a_{n-1,n}a_{n,n-1}$$

But $a_{12}a_{21} \geq 0$ and $a_{n-1,n}a_{n,n-1} \geq 0$. Hence,

$$0 \leq a_{12}a_{21} \leq 1, \text{ and}$$
$$0 \leq a_{n-1,n}a_{n,n-1} \leq 1.$$

Convergence Analysis

The following lemmas are needed in order to prove the main theorems.

Lemma 1 (Varga (1981))

Let $A \geq 0$ be an irreducible $n \times n$ matrix. Then,

i. A has a positive real eigenvalue equal to its spectral radius.
ii. To $\rho(A)$ there corresponds an eigen vector $x > 0$.
iii. $\rho(A)$ increases when any entry of A increases.
iv. $\rho(A)$ is a simple eigenvalue of A.

Lemma 2 (Varga (1981))

i. Let A be a nonnegative matrix. Then If $\alpha x \leq Ax$ for some nonnegative vector $x, x \neq 0$, then $\alpha \leq \rho(A)$.
ii. If $Ax \leq \beta x$ for some positive vector x, then $\rho(A) \leq \beta$. Moreover, if A is irreducible and if $0 \neq \alpha x \leq Ax \leq \beta x$ for some nonnegative vector x, then $\alpha \leq \rho(A) \leq \beta$ and x is a positive vector.

Lemma 3 (Li and Sun (2000))

Let $A = M - N$ be an $M-$splitting of A. Then the splitting is convergent, i.e., $\rho(M^{-1}N < 1)$, if and only if A is a nonsingular $M-$matrix.

Theorem 1

Let $T_{SOR} = (I - \omega L)^{-1}\{(1-\omega)I + \omega U\}$ be the SOR iteration matrix while $T_1 = [I - \omega(L' - SL - D_*)]^{-1}\{(1-\omega)I + \omega(U' + SL)\}$ and $T_2 = (D' - \omega L')^{-1}\{(1-\omega)D' + \omega U'\}$ be the preconditioned SOR iteration matrices. If $A \in \mathbb{R}^{n\times n}$ is an irreducible $L-$matrix with $0 \leq a_{12}a_{21} < 1$, $0 \leq a_{n-1,n}a_{n,n-1} < 1$ and $0 < \omega < 1$, then T_{SOR}, T_1 and T_2 are nonnegative and irreducible matrices.

Proof:

When $\omega = 0$, T_{SOR}, T_1 and T_2 become the identity matrix I. When $\omega < 0$ and $\omega > 1$, negative entries are introduced to these three matrices. Thus, for T_{SOR}, T_1 and T_2 to be nonnegative, the only allowable range of values of ω is $0 < \omega < 1$.

For $0 < \omega < 1$, and by binomial expansion, $(I - \omega L)^{-1} = I + \omega L + \omega^2 L^2 + \cdots + \omega^{n-1}L^{n-1} \geq 0$, since $L \geq 0$. More so, $(1-\omega)I + \omega U \geq 0$, since $U \geq 0$. Thus $T_{SOR} = (I - \omega L)^{-1}[(1-\omega)I + \omega U] \geq 0$. Hence, T_{SOR} is a nonnegative matrix.

For $0 < \omega < 1$,

$$T_{SOR} = [I + \omega L + \omega^2 L^2 + \cdots + \omega^{n-1}L^{n-1}][(1-\omega)I + \omega U]$$
$$= (1-\omega)I + \omega(1-\omega)L + \omega U + \omega^2 LU + \omega^2(1-\omega)L^2 + \omega^3 L^2 U + \cdots$$
$$= (1-\omega)I + \omega(1-\omega)L + \omega U + \text{nonnegative terms}$$

Since $A = I - L - U$ is irreducible, so also is the matrix $(1-\omega)I + \omega(1-\omega)L + \omega U$ since the coefficients of I, L and U are not zero and less than 1 in absolute value. Hence, T_{SOR} is an irreducible matrix.

The preconditioned iteration matrix T_1 is defined by

$$T_1 = [I - \omega(L' - SL - D_*)]^{-1}\{(1-\omega)I + \omega(U' + SL)\}$$

Since $U' \geq 0$, $SL \geq 0$, for $0 < \omega < 1$, $(1-\omega)I + \omega(U' + SL) \geq 0$ and $[I - \omega(L' - SL - D_*)]^{-1} = I + \omega(L' - SL - D_*) + \omega^2(L' - SL - D_*)^2 + \cdots + \omega^{n-1}(L' - SL - D_*)^{n-1} \geq 0$, since $(L' - SL - D_*) \geq 0$.Consequently, $T_1 = [I - \omega(L' - SL - D_*)]^{-1}\{(1-\omega)I + \omega(U' + SL)\} \geq 0$. Hence, T_1 is a nonnegative matrix.

Suppose the matrix $A = I - L - U$ is an irreducible matrix; then the preconditioned matrix A' is

$$A' = PA = (I + S)A = (I - L_s - U_s)(I - L - U)$$
$$= I - L - U - L_s + L_sL + L_sU - U_s + U_sL + U_sU$$
$$= I - L - U - L_s + L_sL - (L_sU)_L - (L_sU)_U - U_s - (U_sL)_L - (U_sL)_U + U_sU$$
$$= I - L - L_s + L_sL - (L_sU)_L - (U_sL)_L - U - U_s + U_sU - (U_sL)_U - (L_sU)_U$$
$$= I - (L + L_s - L_sL + (L_sU)_L + (U_sL)_L) - (U + U_s - U_sU + (U_sL)_U + (L_sU)_U)$$
$$= I - L' - U'$$

where $L' = L + L_s - L_sL + (L_sU)_L + (U_sL)_L$, $U' = U + U_s - U_sU + (U_sL)_U + (L_sU)_U$ and $-(Q)_L$ and $-(Q)_U$ denote the strictly lower and strictly upper parts of the matrix Q respectively. Since A is irreducible, it is obvious that $A' = I - L' - U'$ is irreducible, as it inherits the nonzero structure of the irreducible matrix A'.

Now,

$$T_1 = [I - \omega(L' - SL - D_*)]^{-1}\{(1-\omega)I + \omega(U' + SL)\}$$
$$= (I + \omega(L' - SL - D_*) + \omega^2(L' - SL - D_*)^2 + \cdots + \omega^{n-1}(L' - SL - D_*)^{n-1})\{(1-\omega)I + \omega U' + \omega SL\}$$
$$= (1-\omega)I + \omega U' + \omega SL + \omega(1-\omega)(L' - SL - D_*) + \omega^2(L' - SL - D_*)U' + \omega^2(L' - SL - D_*)SL + \omega^2(1-\omega)(L' - SL - D_*)^2 + \cdots$$
$$= (1-\omega)I + \omega(1-\omega)L' + \omega U' + \omega SL + \omega(1-\omega)(-SL - D_*) + \omega^2(L' - SL - D_*)U' + \omega^2(L' - SL - D_*)SL + \omega^2(1-\omega)(L' - SL - D_*)^2 + \cdots$$
$$= (1-\omega)I + \omega(1-\omega)L' + \omega U' + \text{nonnegative terms}$$

Since $A' = I - L' - U'$ is irreducible, it implies, for $0 < \omega < 1$, the matrix $(1-\omega)I + \omega(1-\omega)L' + \omega U'$ is also irreducible, because the coefficients of I, L' and $U'$are different from zero and less than one in absolute value. Therefore, the matrix $T_1 = [I - \omega(L' - SL - D_*)]^{-1}\{(1-\omega)I + \omega(U' + SL)\}$ is irreducible. Hence T_1 is a nonnegative and irreducible matrix.

Similarly,

$$T_2 = (D' - \omega L')^{-1}[(1-\omega)D' + \omega U']$$
$$= [D'(I - \omega D'^{-1}L')]^{-1}[(1-\omega)D' + \omega U']$$
$$= [D'(I - \omega D'^{-1}L')]^{-1}[(1-\omega)D' + \omega U']$$
$$= \left(I - \omega D'^{-1}L'\right)^{-1}D'^{-1}[(1-\omega)D' + \omega U']$$
$$= \left(I - \omega D'^{-1}L'\right)^{-1}\left[(1-\omega)I + \omega D'^{-1}U'\right]$$
$$= [I + \omega D'^{-1}L' + \omega^2(D'^{-1}L')^2 + \cdots + \omega^{n-1}(D'^{-1}L')^{n-1}]\left[(1-\omega)I + \omega D'^{-1}U'\right]$$
$$= (1-\omega)I + \omega(1-\omega)D'^{-1}L' + \omega D'^{-1}U' + \text{nonnegative terms}$$

Using similar arguments it is conclusive that $T_2 = (D' - \omega L')^{-1}[(1-\omega)D' + \omega U']$ is a nonnegative and irreducible matrix.

Theorem 2

Let $T_{SOR} = (I - \omega L)^{-1}\{(1-\omega)I + \omega U\}$ and $T_1 = [I - \omega(L' - SL - D_*)]^{-1}\{(1-\omega)I + \omega(U' + SL)\}$ be the SOR and preconditioned SOR iteration matrices respectively. If $0 < \omega < 1$ and if $A \in \mathbb{R}^{n\times n}$ is an irreducible L −matrix with $0 \le a_{12}a_{21} < 1$, $0 \le a_{n-1,n}a_{n,n-1} < 1$, then

(i) $\rho(T_1) < \rho(T_{SOR})$, if $\rho(T_{SOR}) < 1$
(ii) $\rho(T_1) = \rho(T_{SOR})$, if $\rho(T_{SOR}) = 1$
(iii) $\rho(T_1) > \rho(T_{SOR})$, if $\rho(T_{SOR}) > 1$

Proof:

T_{SOR} and T_1 are nonnegative and irreducible matrices (Theorem 1). Let $\rho(T_{SOR}) = \mu$; then there exists a positive vector $x = (x_1, x_2, \cdots, x_n)^T$, such that

$$T_{SOR}x = \mu x$$

That is,

$$(I - \omega L)^{-1}\{(1-\omega)I + \omega U\}x = \mu x$$

$$(1-\omega)I + \omega U = \mu(I - \omega L) \quad (11)$$

Therefore, for this $x > 0$

$$T_1x - \mu x = [I - \omega(L' - SL - D_*)]^{-1}\{(1-\omega)I + \omega(U' + SL)\}x - \mu x$$
$$= [I - \omega(L' - SL - D_*)]^{-1}\{(1-\omega)I + \omega(U' + SL)\}x - \mu[I - \omega(L' - SL - D_*)]^{-1}[I - \omega(L' - SL - D_*)]x$$
$$= [I - \omega(L' - SL - D_*)]^{-1}\{(1-\omega)I + \omega(U' + SL) - \mu[I - \omega(L' - SL - D_*)]\}x$$
$$= [I - \omega(L' - SL - D_*)]^{-1}\{(1-\omega-\mu)I + \omega U' + \mu\omega L' - \mu\omega SL + \omega SL - \mu\omega D_*\}x$$
$$= [I - \omega(L' - SL - D_*)]^{-1}\{(1-\omega-\mu)I + \omega(U + U_S + U_*) + \mu\omega(L + L_S + L_*) + \omega(\mu-1)(-SL) - \mu\omega D_*\}x$$
$$[I - \omega(L' - SL - D_*)]^{-1}\{(1-\omega-\mu)I + \omega U + \mu\omega L + -\mu\omega D_* + \omega D_* + \omega(\mu-1)(-SL) - \omega D_* + \omega L_* + \omega U_* + \mu\omega L_* - \omega L_* + \omega U_S + \mu\omega L_S\}x$$

From (11),

$$\omega U + \mu\omega L = -(1-\omega-\mu)I$$
$$T_1x - \mu x = [I - \omega(L' - SL - D_*)]^{-1}\{(1-\omega-\mu)I - (1-\omega-\mu)I + \omega(\mu-1)(-D_*) + \omega(\mu-1)(-SL) + \omega(\mu-1)L_* - \omega(D_* - L_* - U_*) + \omega U_S + \mu\omega L_S\}x$$
$$= [I - \omega(L' - SL - D_*)]^{-1}\{\omega(\mu-1)(-D_* - SL + L_*) - \omega(D_* - L_* - U_*) + \omega U_S + \mu\omega L_S\}x$$
$$= [I - \omega(L' - SL - D_*)]^{-1}\{\omega(\mu-1)(-D_* - SL + L_*) - \omega(-SL - SU) + \mu\omega L_S - \omega L_S + \omega L_S + \omega U_S\}x$$
$$= [I - \omega(L' - SL - D_*)]^{-1}\{\omega(\mu-1)(-D_* - SL + L_*) + \omega(\mu-1)L_S + \omega SL + \omega SU + \omega(L_S + U_S)\}x$$
$$= [I - \omega(L' - SL - D_*)]^{-1}\{\omega(\mu-1)(-D_* - SL + L_* + L_S) + \omega SL + \omega SU + \omega(-S)\}x$$
$$= [I - \omega(L' - SL - D_*)]^{-1}\{\omega(\mu-1)(-D_* - SL + L_* + L_S) + S - \omega S + \omega SU - S + \omega SL\}x$$
$$= [I - \omega(L' - SL - D_*)]^{-1}\{\omega(\mu-1)(-D_* - SL + L_* + L_S) + S[(1-\omega)I + \omega U] - S(I - \omega L)\}x$$

From (11),

$$(1-\omega)I + \omega U = \mu(I - \omega L)$$

$$T_1x-\mu x=[I-\omega(L'-SL-D_*)]^{-1}\{\omega(\mu-1)(-D_*-SL+L_*+L_S)+\mu S(I-\omega L)-S(I-\omega L)\}x$$
$$=[I-\omega(L'-SL-D_*)]^{-1}\{\omega(\mu-1)(-D_*-SL+L_*+L_S)+(\mu-1)S(I-\omega L)\}x$$
$$=(\mu-1)[I-\omega(L'-SL-D_*)]^{-1}\{\omega(-D_*-SL+L_*+L_S)+S(I-\omega L)\}x$$

From (11),

$$(I-\omega L)=\frac{(1-\omega)I+\omega U}{\mu}$$
$$T_1x-\mu x=(\mu-1)[I-\omega(L'-SL-D_*)]^{-1}\{\omega(-D_*-SL+L_*+L_S)+S[\frac{(1-\omega)I+\omega U}{\mu}]\}x$$
$$=\frac{(\mu-1)}{\mu}[I-\omega(L'-SL-D_*)]^{-1}\{\mu\omega(-D_*-SL+L_*+L_S)+(1-\omega)S+\omega SU\}x$$

Let $F=G_1x$, where $G_1=[I-\omega(L'-SL-D_*)]^{-1}\{\mu\omega(-D_*-SL+L_*+L_S)+(1-\omega)S+\omega SU\}$. Then $G_1=[I-\omega(L'-SL-D_*)]^{-1}\{\mu\omega(-D_*-SL+L_*+L_S)+(1-\omega)S+\omega SU\}\geq 0$, because $\mu\omega(-D_*-SL+L_*)\geq 0$, $\mu\omega L_S+(1-\omega)S\geq 0$ and $\omega SU\geq 0$. Also, $[I-\omega(L'-SL-D_*)]^{-1}=I+\omega(L'-SL-D_*)+\omega^2(L'-SL-D_*)^2+\cdots+\omega^{n-1}(L'-SL-D_*)^{n-1}\geq 0$, since $(L'-SL-D_*)\geq 0$. Therefore, $G_1=[I-\omega(L'-SL-D_*)]^{-1}\{\mu\omega(-D_*-SL+L_*+L_S)+(1-\omega)S+\omega SU\}\geq 0$. Consequently, $F=[I-\omega(L'-SL-D_*)]^{-1}\{\mu\omega(-D_*-SL+L_*+L_S)+(1-\omega)S+\omega SU\}x\geq 0$, since $x>0$.

(i) If $\mu<1$, then $T_1x-\mu x\leq 0$ but not equal to 0. Therefore, $T_1x\leq \mu x$. From Lemma 2, we have $\rho(T_1)<\mu=\rho(T_{SOR})$.

(ii) If $\mu=1$, then $T_1x-\mu x=0$. Therefore, $T_1x=\mu x$. From Lemma 2, we have $\rho(T_1)=\mu=\rho(T_{SOR})$.

(iii) If $\mu>1$, then $T_1x-\mu x\geq 0$ but not equal to 0. Therefore, $T_1x\geq \mu x$. From Lemma 2, we have $\rho(T_1)>\mu=\rho(T_{SOR})$.

Theorem 3

Let $T_{SOR}=(I-\omega L)^{-1}\{(1-\omega)I+\omega U\}$ and $T_2=(D'-\omega L')^{-1}[(1-\omega)D'+\omega U']$ and be the SOR and preconditioned SOR iteration matrices respectively. If $0<\omega<1$ is and $A\in\mathbb{R}^{n\times n}$ is an irreducible L −matrix with $0\leq a_{12}a_{21}<1$, $0\leq a_{n-1,n}a_{n,n-1}<1$, then

i. $\rho(T_2)<\rho(T_{SOR})$, if $\rho(T_{SOR})<1$;
ii. $\rho(T_2)=\rho(T_{SOR})$, if $\rho(T_{SOR})=1$;
iii. $\rho(T_2)>\rho(T_{SOR})$, if $\rho(T_{SOR})>1$.

Proof:

From Theorem 1, T_{SOR} and T_2 are nonnegative and irreducible matrices. Suppose $\rho(T_{SOR})=\mu$, then there exists a positive vector $x=(x_1,x_2,\cdots,x_n)^T$, such that

$$T_{SOR}x=\mu x$$

That is,

$$(I-\omega L)^{-1}\{(1-\omega)I+\omega U\}x=\mu x$$
$$(1-\omega)I+\omega U=\mu(I-\omega L) \qquad (12)$$

Therefore, for this $x>0$,

$$T_2x-\mu x=(D'-\omega L')^{-1}\{(1-\omega)D'+\omega U'\}x-\mu x$$
$$=(D'-\omega L')^{-1}\{(1-\omega)D'+\omega U'\}x-(D'-\omega L')^{-1}(D'-\omega L')\mu x$$
$$=(D'-\omega L')^{-1}\{(1-\omega)D'+\omega U'-\mu(D'-\omega L')\}x$$
$$=(D'-\omega L')^{-1}\{(1-\omega-\mu)D'+\mu\omega L'+\omega U'\}x$$
$$=(D'-\omega L')^{-1}\{(1-\omega-\mu)(I+D_*)+\mu\omega(L+L_S+L_*)+\omega(U+U_S+U_*)\}x$$
$$=(D'-\omega L')^{-1}\{(1-\omega-\mu)D_*+\mu\omega L_*+\mu\omega L_S+\mu\omega L+\omega U_S+\omega U_*+(1-\omega-\mu)I+\omega U)\}x$$

From equation (12)

$$(1-\omega-\mu)I+\omega U=-\mu\omega L$$
$$T_2x-\mu x=(D'-\omega L')^{-1}\{(1-\omega-\mu)D_*+\mu\omega L_*+\mu\omega L_S+\omega U_S+\omega U_*)\}x$$
$$=(D'-\omega L')^{-1}\{(\mu-1)(-D_*)+\mu\omega L_*-\omega L_*+\mu\omega L_S-\omega D_*+\omega L_*+\omega U_*+\omega U_S)\}x$$
$$=(D'-\omega L')^{-1}\{(\mu-1)(-D_*)+(\mu-1)\omega L_*-\omega(D_*-L_*-U_*)+\mu\omega L_S+\omega U_S\}x$$
$$=(D'-\omega L')^{-1}\{(\mu-1)(-D_*+\omega L_*)-\omega(-(SL+SU))+\mu\omega L_S-\omega L_S+\omega L_S+\omega U_S\}x$$
$$=(D'-\omega L')^{-1}\{(\mu-1)(-D_*+\omega L_*)+\omega SL+\omega SU+(\mu-1)\omega L_S+\omega(L_S+U_S)\}x$$
$$=(D'-\omega L')^{-1}\{(\mu-1)(-D_*+\omega L_*+\omega L_S)+\omega SL+\omega SU-\omega S\}x$$
$$=(D'-\omega L')^{-1}\{(\mu-1)(-D_*+\omega L_*+\omega L_S)-S+\omega SL+S-\omega S+\omega SU\}x$$
$$=(D'-\omega L')^{-1}\{(\mu-1)(-D_*+\omega L_*+\omega L_S)+(1-\omega)S+\omega SU-S(I-\omega L)\}x$$
$$=(D'-\omega L')^{-1}\{(\mu-1)(-D_*+\omega L_*+\omega L_S)+S[(1-\omega)I+\omega U]-S(I-\omega L)\}x$$

From equation (12)

$$(1-\omega)I+\omega U=\mu(I-\omega L)$$
$$T_2x-\mu x=(D'-\omega L')^{-1}\{(\mu-1)(-D_*+\omega L_*+\omega L_S)+\mu S(I-\omega L)-S(I-\omega L)\}x$$
$$=(D'-\omega L')^{-1}\{(\mu-1)(-D_*+\omega L_*+\omega L_S)+(\mu-1)S(I-\omega L)\}x$$

From equation (12)

$$(I-\omega L)=\frac{(1-\omega)I+\omega U}{\mu}$$
$$=(\mu-1)(D'-\omega L')^{-1}\left\{-D_*+\omega L_*+\omega L_S+\frac{(1-\omega)S+\omega SU}{\mu}\right\}x$$
$$=\frac{(\mu-1)}{\mu}(D'-\omega L')^{-1}\{-\mu D_*+\mu\omega L_*+\mu\omega L_S+(1-\omega)S+\omega SU\}x$$

Let $F=G_2x$, with $G=(D'-\omega L')^{-1}\{-\mu D_*+\mu\omega L_*+\mu\omega L_S+(1-\omega)S+\omega SU\}$. It is clear that $-\mu D_*+\mu\omega L_*+\mu\omega L_S+(1-\omega)S+\omega SU\geq 0$, since $\omega SU\geq 0$, $-\mu D_*\geq 0$, $\mu\omega L_*\geq 0$ and $\mu\omega L_S+(1-\omega)S\geq 0$. Since D' is a nonsingular matrix, we let $D'-\omega L'$ be a splitting of some matrix J, i.e., $J=D'-\omega L'$. Also, D' is an M −matrix and $\omega L'\geq 0$. Thus, $J=D'-\omega L'$ is an M −splitting. Now, $\omega {D'}^{-1}L'$ is a strictly lower triangular matrix, and by implication its eigenvalues lie on its main diagonal; in this case they are all zeros. Therefore, $\rho\left(\omega {D'}^{-1}L'\right)=0$. since $\rho\left(\omega {D'}^{-1}L'\right)<1$, $J=D'-\omega L'$ is a convergent splitting. By the foregoing, $J=D'-\omega L'$ is an M −splitting and $\rho\left(\omega {D'}^{-1}L'\right)<1$, we employ Lemma 3.3 to establish that J is an M −matrix. Since J is an M −matrix, by definition, $J^{-1}=(D'-\omega L')^{-1}\geq 0$. Thus, $G_2\geq 0$ and $F\geq 0$.

(i) If $\mu < 1$, then $T_2x - \mu x \leq 0$ but not equal to 0. Therefore, $T_2x \leq \mu x$. From Lemma 2, we have $\rho(T_2) < \mu = \rho(T_{SOR})$.

(ii) If $\mu = 1$, then $T_2x - \mu x = 0$. Therefore, $T_2x = \mu x$. From Lemma 2, we have $\rho(T_2) = \mu = \rho(T_{SOR})$.

(iii) If $\mu > 1$, then $T_2x - \mu x \geq 0$ but not equal to 0. Therefore, $T_2x \geq \mu x$. From Lemma 2, we have $\rho(T_2) > \mu = \rho(T_{SOR})$.

If in T_{SOR}, T_1 and T_2 the relaxation parameter $\omega = 1$, the iteration matrices of the Gauss-Seidel method results in each case. Therefore, the following corollaries are direct implications of Theorems 1 and 2.

Corollary 1 Let $T_{GS} = (I - L)^{-1}U$ be the Gauss-Seidel iteration matrix and $T_{GS1} = [I - (L' - SL - D_*)]^{-1}(U' + SL)$ be the preconditioned Gauss-Seidel iteration matrix. If $A \in \mathbb{R}^{n\times n}$ is an irreducible L −matrix with $0 \leq a_{12}a_{21} < 1$, $0 \leq a_{n-1,n}a_{n,n-1} < 1$, then

i. $\rho(T_{GS1}) < \rho(T_{GS})$, if $\rho(T_{GS}) < 1$;

ii. $\rho(T_{GS1}) = \rho(T_{GS})$, if $\rho(T_{GS}) = 1$;

iii. $\rho(T_{GS1}) > \rho(T_{GS})$, if $\rho(T_{GS}) > 1$.

Corollary 2 Let $T_{GS} = (I - L)^{-1}U$ be the Gauss-Seidel iteration matrix and $T_{GS2} = (D' - L')^{-1}U'$ be the preconditioned Gauss-Seidel iteration matrix. If $A \in \mathbb{R}^{n\times n}$ is an irreducible L −matrix with $0 \leq a_{12}a_{21} < 1$, $0 \leq a_{n-1,n}a_{n,n-1} < 1$, then

i. $\rho(T_{GS2}) < \rho(T_{GS})$, if $\rho(T_{GS}) < 1$;

ii. $\rho(T_{GS2}) = \rho(T_{GS})$, if $\rho(T_{GS}) = 1$;

iii. $\rho(T_{GS2}) > \rho(T_{GS})$, if $\rho(T_{GS}) > 1$.

RESULTS AND DISCUSSION

In order to illustrate the results in the previous section, we present sample matrices and determine the corresponding spectral radii of the iteration matrices of the preconditioned methods T_1 and T_2, as well as those of the methods of the classical SOR, Gunawardena *et al.* (1991) and Ndanusa and Adeboye (2012). The results are given in Tables I and II. In the Tables, $T_G = [I - \omega(L + SL)]^{-1}\{(1 - \omega)I + \omega(U - S + SU)\}$ denote the iteration matrix of the method of Gunawardena *et al.* (1991) and $T_{NA} = (I - \omega\bar{L})^{-1}\{(1 - \omega)I + \omega(\bar{U} - D_1)\}$ denote iteration matrix of the method of Ndanusa and Adeboye (2012).

Problem 1

Let

$$A = \begin{pmatrix} 1 & -0.297 & -0.244 & 0 \\ -0.305 & 1 & 0 & -0.203 \\ -0.256 & 0 & 1 & -0.254 \\ 0 & -0.263 & -0.237 & 1 \end{pmatrix}$$

Problem 2

Let

$$A = \begin{pmatrix} 1 & -0.23661 & 0 & -0.25833 & -0.05480 \\ -0.15730 & 1 & -0.01535 & -0.31542 & -0.12652 \\ 0 & -0.18436 & 1 & -0.01523 & 0 \\ -0.12589 & -0.00357 & -0.12354 & 1 & -0.13654 \\ -0.21365 & -0.01489 & -0.13940 & -0.04890 & 1 \end{pmatrix}$$

Table 1: Comparison of results for Problem 1

ω	T_{SOR}	T_1	T_2	T_G	T_{NA}
0.1	0.9502804790	0.9372884197	0.9320577508	0.9402372379	0.9351153898
0.2	0.8977415361	0.8725081471	0.8616461629	0.8770106635	0.8679416975
0.3	0.8419865987	0.8054951896	0.7884547669	0.8098650194	0.7982368471
0.4	0.7825060928	0.7360514251	0.7120821997	0.7382207198	0.7256999962
0.5	0.7186219444	0.6639313773	0.6319885298	0.6613117719	0.6499460086
0.6	0.6493889513	0.5888214416	0.5474069636	0.5780721630	0.5704624297
0.7	0.5734015584	0.5103053888	0.4571607539	0.4869025333	0.4865309881
0.8	0.4883586106	0.4278029316	0.3592125666	0.3850991994	0.3970677198
0.9	0.3898409564	0.3404492464	0.2491800449	0.2669680004	0.3002417195

Table 2: Comparison of results for Problem 2

ω	T_{SOR}	T_1	T_2	T_G	T_{NA}
0.1	0.9433814720	0.9348244213	0.9334375820	0.9354069300	0.9306207143
0.2	0.8842186340	0.8671088946	0.8639973840	0.8676225507	0.8585524626
0.3	0.8222178790	0.7966021659	0.7913264100	0.7962564586	0.7835036459
0.4	0.7570107410	0.7229997252	0.7149727624	0.7208128415	0.7051088312
0.5	0.6881194420	0.6459240253	0.6343347202	0.6406373744	0.6228938605
0.6	0.6148968014	0.5648933772	0.5485668314	0.5548198004	0.5362138206
0.7	0.5364118257	0.4792699254	0.4563852284	0.4619924830	0.4441306819
0.8	0.4512020588	0.3881659704	0.3555863470	0.3598337933	0.3451317516
0.9	0.3566223781	0.2902570756	0.2414370780	0.2434052324	0.2362969098

From Tables 1 and 2, it was observed that the preconditioned iterative methods T_1 and T_2 gave better results than the classical SOR method, thereby confirming the effectiveness of the preconditioned iterative methods. In comparison to existing preconditioned methods T_G and T_{NA}, the preconditioned method T_2 performed better than both T_G and T_{NA} for Problem 1. For the same Problem, T_1 gave better results than T_G for the first two values of the relaxation parameter ω, even as T_1 lags behind T_{NA} for all values of ω. In Table 2, T_2 exhibited more accurate results than T_G, although it did not match the performance of T_{NA}. And lastly, T_{NA} displayed better results than T_1 for all values of ω, even as T_1 performed better than T_G for the first three values of ω.

CONCLUSION

We introduced a new preconditioner of the type $I + S$ and went further to develop the preconditioned iterative techniques T_1 and T_2 for the SOR iterative method for solving linear systems with L −matrix. Theoretical convergence analysis carried out established that the two techniques are convergent. Further numerical experiments confirmed the results of theoretical analysis. The new preconditioned methods are shown to compare favourably with similar methods in literature.

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