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Second Derivatives Single Step Block Hybrid Method for Non-Linear Dynamical System

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Abstract

In this paper, a modified single-step method is proposed to integrate nonlinear dynamical systems resulting to ordinary differential equations. In order to obtain higher order A-stable method, we have used second derivative of the solutions and imposed some special sets of off-grid points in the formulation process of the algorithms. The consistency, convergence and order of accuracy of the algorithms were successfully established and in addition, the method is found to be A-stable. The proposed method which is self-starting were applied as simultaneous numerical integrators on non-overlapping intervals. In order to demonstrate the effectiveness of the proposed algorithms, some nonlinear dynamical systems of IVPs with applications in population growth models, chaos and vibratory theory are considered. and results obtained are compared with those from related schemes and from other methods in the literature.

1. Introduction

Numerical methods for Ordinary Differential Equations (ODEs) are very important tools for scientific computation, as they are widely used for solution of real life problems. More so, nonlinear problems have shown up in large domains as science and technology have progressed. Nonlinear problems cannot be solved using the standard linear technique. Because of this, a new approach to understanding complex systems has been developed over the years. Nonlinear dynamic systems often exhibit population growth and chaos as

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two of their primary complicated dynamic features. Even while nonlinear dynamic systems may be used to examine chaotic or disordered problems and unearth their intricate laws, they are not the primary focus of nonlinear problems [2, 6]. An appropriate nonlinear mathematical model must be constructed to accurately depict the data's underlying law in order to get insight into the system's features. Nonlinear dynamics, on the other hand, are more diversified and dynamic, and they vary based on the prior state in a more complicated manner. As a result, practical engineering has hit an insurmountable obstacle. In general, finding an analytical solution is very impossible when there is a complex chaotic state present. When it comes to describing an unknown system state, individuals tend to focus on methods with high approximation accuracy and ease of use, rather than attempting to solve the precise problem themselves.

Many approaches have been developed to solve the numerical solution of nonlinear dynamic systems throughout the years, and the major ones are as follows: perturbation method [16], averaging method [8], Runge-Kutta method [3], Euler method [5], gradient method [4], linear multistep method and others. These approaches have certain benefits for handling specific systems, but they provide unpleasant results when dealing with issues of generic nonlinear dynamic systems, such as reduced accuracy, complexity and big computation quantities, Runge phenomena, etc. As a result, the issue today is: can we discover an effective approach for studying nonlinear dynamic systems that has both high approximation accuracy and avoids the Runge phenomenon? Good convergence and approximation, stability, are properties of block hybrid linear multistep methods derived through point collocation technique of orthogonal functions [1, 7, 10, 12–14, 17].

Because of this, the goal of this work is to develop a class of one-step block hybrid methods that improve accuracy and zero-stability while also ensuring convergence by using the derivative of the iterative method in our derivation process.

The methods are implemented as block method whereby, there is no requirement for a different strategy for finding starting values. In the implementation process, we obtain initial conditions at x_{n+1}, $n = 0, 1, \ldots, N-1$ using the computed values y_{n+1} over sub-intervals $[x_0, x_1], \ldots, [x_{N-1}, x_N]$. For instance when $n = 0, (y_\eta, y_1)$ are obtained simultaneously over the sub-interval $[x_0, x_1]$, as y_0 is known from the IVP, for $n = 1, (y_{\eta+1}, y_2)$ are also obtained simultaneously over the sub-interval $[x_1, x_2]$, as y_1 is now known from the previous block, and so on. Therefore, the sub-interval $[x_n, x_{n+1}]$ do not over-lap and the solutions obtained in this manner are more accurate than those obtained in the conventional way.

2. Theoretical Procedure of the Method

The proposed one-step second derivative block intra-step point method for the solution of nonlinear dynamic systems of first order ordinary differential equations is of the form:

$$y_{n+1} = y_n + h \sum_{j=0}^{1} \beta_j f_{n+j} + h^2 \sum_{j=0}^{1} \gamma_j g_{n+j} \tag{2.1}$$

and the additional method

$$y_{n+\eta} = y_n + h \sum_{j=0}^{1} \beta_j f_{n+j} + h^2 \sum_{j=0}^{1} \gamma_j g_{n+j} \tag{2.2}$$

where $\beta_1 \neq 0$, $\gamma_1 \neq 0$, β_j, $\beta_{\eta j}$, γ_j, $\gamma_{\eta j}$ are unknown coefficients, ν is the intra-step points. The general approach in the derivation of (2.1) and (2.2) involves the use of continuous collocation approach using a trial function of the form:

$$Y(x) = \sum_{j=0}^{r+2s+1} a_j x^j \tag{2.3}$$

where a_j are unknown coefficients to be determined, r and s are numbers of interpolation and collocation points respectively. We interpolate (2.3) at x_n and collocate its first derivative at x_n and x_{n+1}, and a countable number of intra-step points defined as $x_{n+\eta} = x_n + h\eta$. Here, $\eta in (0,1)$ are points generated from the Bhaskara cosine formula [15]. These lead to a system of equations of the form:

$$\begin{cases} Y(x_n) = y_n, \\ Y'(x_{n+j\nu}) = f_{n+j\nu}, \\ Y''(x_{n+j\nu}) = g_{n+j\nu}, \\ j = 0, 1, \quad \nu = 1, 2, \ldots, m \end{cases} \tag{2.4}$$

which is solved using matrix inversion method to obtain a_j and then substituted into (2.3) to get the continuous scheme of the form:

$$\begin{aligned} y(x) = y_n + h(\beta_0(x) f_n + \beta_\nu(x) f_{n+j\nu}) \\ + h^2(\gamma_0(x) g_n + \gamma_\nu(x) g_{n+j\nu} + \gamma_1(x) g_{n+1}) \end{aligned} \tag{2.5}$$

The continuous scheme (2.5) generated produces the main and additional algorithms which are merged to generate approximations simultaneously. In this paper, we consider two different blocks.

The specification of one-step second derivative block method with 5 intra-points is given as $k = 1$, $m = 5$, $\eta_j = \left(\frac{5}{74}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{69}{74}\right)$, $x \in [x_n, x_{n+1}]$ which results in system of equations

$$Y_\omega = D\Psi_{\omega - n} \tag{2.6}$$

where

$$\begin{aligned} Y_\omega = \Big(& y_n, f_n, f_{n+\frac{5}{74}}, f_{n+\frac{1}{4}}, f_{n+\frac{1}{2}}, f_{n+\frac{3}{4}}, f_{n+\frac{69}{74}}, f_{n+1}, \\ & g_n, g_{n+\frac{5}{74}}, g_{n+\frac{1}{4}}, g_{n+\frac{1}{2}}, g_{n+\frac{3}{4}}, g_{n+\frac{69}{74}}, g_{n+1} \Big)^\top \end{aligned}$$

and

$\Phi_\omega = \left(\alpha_0, \beta_0, \beta_{\frac{5}{74}}, \beta_{\frac{1}{4}}, \beta_{\frac{1}{2}}, \beta_{\frac{3}{4}}, \beta_{\frac{69}{74}}, \beta_1, \gamma_0, \gamma_{\frac{5}{74}}, \gamma_{\frac{1}{4}}, \gamma_{\frac{1}{2}}, \gamma_{\frac{3}{4}}, \gamma_{\frac{69}{74}}, \gamma_1\right)$.

The matrix D for this method is given in Appendix.

Equation (2.6) is solved by matrix inversion technique which yield the continuous coefficients $\alpha_0(x)$, $\beta_j(x), \gamma_j(x)$, which are then substituted into (2.5) to obtain its equivalent continuous scheme:

$$\begin{aligned} y(x) = y_n + h \Big(& \beta_0(x) f_n + \beta_{\frac{5}{74}}(x) f_{n+\frac{5}{74}} + \beta_{\frac{1}{4}}(x) f_{n+\frac{1}{4}} \\ & + \beta_{\frac{1}{2}}(x) f_{n+\frac{1}{2}} + \beta_{\frac{3}{4}}(x) f_{n+\frac{3}{4}} + \beta_{\frac{69}{74}}(x) f_{n+\frac{69}{74}} + \beta_1(x) f_{n+1} \Big) \\ + h^2 \Big(& \gamma_0(x) g_n + \gamma_{\frac{5}{74}}(x) g_{n+\frac{5}{74}} + \gamma_{\frac{1}{4}}(x) g_{n+\frac{1}{4}} + \gamma_{\frac{1}{2}}(x) g_{n+\frac{1}{2}} \\ & + \gamma_{\frac{3}{4}}(x) g_{n+\frac{3}{4}} \gamma_{\frac{69}{74}} g_{n+\frac{69}{74}} + \gamma_1(x) g_{n+1} \Big). \end{aligned} \tag{2.7}$$

Evaluating (2.7) at $x = \frac{5}{74}h, \frac{1}{4}h, \frac{1}{2}h, \frac{3}{4}h, \frac{69}{74}h$ and h gives the following discrete schemes which form the block for the one-step second derivative block intra-points method with $m = 5$ (OSDBM5).

(2.8)
$$
\begin{aligned}
y_{n+\frac{5}{74}} = y_n &+ \frac{22900024067154983684706038\,5\,hf_n}{81971632290700361955935629 44}\\
&+ \frac{3082196653420347171129603501 5\,hf_{n+\frac{5}{74}}}{794803209597695208727781572608}\\
&+ \frac{479906684273373618194618000 0\,hf_{n+1/4}}{790663609705300413921583677632 7}\\
&+ \frac{3450076071236439375\,hf_{n+1/2}}{341027590071575057530 88}\\
&+ \frac{6140574416512977257660000\,hf_{n+3/4}}{790663609705300413921583677632 7}\\
&- \frac{1073156567769432075257796 55\,hf_{n+\frac{69}{74}}}{794803209597695208727781572608}\\
&+ \frac{22770646151768088090844 15\,hf_{n+1}}{81971632290700361955935629 44}\\
&+ \frac{5505892519428027553412 5\,h^2g_n}{2375989341759430781331467 52}\\
&- \frac{2090910219414764265627 5\,h^2g_{n+\frac{5}{74}}}{293040994594261917090447 36}\\
&- \frac{2957002796078266059650 00\,h^2g_{n+1/4}}{2899389841236891873566496801}\\
&- \frac{26786422976951818125\,h^2g_{n+1/2}}{28236130534597543224934 4}\\
&- \frac{8994757674275907860500 0\,h^2g_{n+3/4}}{2899389841236891873566496801}\\
&- \frac{1024953598291893879 25\,h^2g_{n+\frac{69}{74}}}{418629992277517024414924 8}\\
&- \frac{1334898531657665905075\,h^2g_{n+1}}{2375989341759430781331467 52}
\end{aligned}
$$

(2.9)
$$
\begin{aligned}
y_{n+\frac{1}{4}} = y_n &+ \frac{1943212527496001\,hf_n}{34093872933120000}\\
&+ \frac{672766595746510020168338779492501\,hf_{n+\frac{5}{74}}}{687397370462871531872675954688 0000}\\
&+ \frac{47060462768187769\,hf_{n+1/4}}{526168068417351360}\\
&+ \frac{1963790013\,hf_{n+1/2}}{671759728640}\\
&+ \frac{19206102090155\,hf_{n+3/4}}{105233613683470272}
\end{aligned}
$$

$$
\begin{aligned}
-\frac{6417699225046998570497415915 17\, hf_{n+\frac{69}{74}}}{27495894818514861274907038187 5200} \\
&+ \frac{6710626995623\, hf_{n+1}}{1363754917324800} \\
&+ \frac{597715882969\, h^2 g_n}{790582560768000} \\
&+ \frac{4963027059592093628627263\, h^2 g_{n+\frac{5}{74}}}{1369950595619632044834816000} \\
&- \frac{669184831493\, h^2 g_{n+1/4}}{154358069209344} \\
&- \frac{299379159\, h^2 g_{n+1/2}}{150323855360} \\
&- \frac{445505665001\, h^2 g_{n+3/4}}{771790346046720} \\
&- \frac{1197758139000212953365 89\, h^2 g_{n+\frac{69}{74}}}{27399011912392640896696320 0} \\
&- \frac{2241217229\, h^2 g_{n+1}}{22588073164800}
\end{aligned}
$$

$$
\begin{aligned}
(2.10) \qquad y_{n+\frac{1}{2}} = y_n &+ \frac{17996522320261\, hf_n}{213086705832000} \\
&+ \frac{2397161746783962471673661058 71\, hf_{n+\frac{5}{74}}}{268514597837059192137764044 8000} \\
&+ \frac{319896474521248\, hf_{n+1/4}}{1644275213804223} \\
&+ \frac{153451983\, hf_{n+1/2}}{1312030720} \\
&+ \frac{26394146969312\, hf_{n+3/4}}{8221376069021115} \\
&- \frac{1248620965583234972388189079 79\, hf_{n+\frac{69}{74}}}{1342572989185295960688820224 0000} \\
&+ \frac{22213495592711\, hf_{n+1}}{1065433529160000} \\
&+ \frac{1596937613\, h^2 g_n}{1235285251200} \\
&+ \frac{33701731647972683621 57\, h^2 g_{n+\frac{5}{74}}}{53513695141391876751 3600} \\
&+ \frac{1551045736\, h^2 g_{n+1/4}}{430686577035} \\
&- \frac{5197001\, h^2 g_{n+1/2}}{293601280}
\end{aligned}
$$

$$-\frac{1671184136\,h^2 g_{n+3/4}}{602961207849} - \frac{5047051038806095992223\,h^2 g_{n+\frac{69}{74}}}{267568475706959383756800} - \frac{2579379871\,h^2 g_{n+1}}{6176426256000}$$

$$(2.11)\qquad y_{n+\frac{3}{4}} = y_n + \frac{1564933587251\,hf_n}{15589333760000} + \frac{86235065051280332051276556917\,hf_{n+\frac{5}{74}}}{1047702134526553165481902080000} + \frac{15845155729673\,hf_{n+1/4}}{80196321965760} + \frac{155171040579\,hf_{n+1/2}}{671759728640} + \frac{26061104016757\,hf_{n+3/4}}{240588965897280} - \frac{56251752175435417469664759647\,hf_{n+\frac{69}{74}}}{314310640357965949644570624000} + \frac{753115181853\,hf_{n+1}}{15589333760000} + \frac{1747267617\,h^2 g_n}{1084475392000} + \frac{2310809407156927663313\,h^2 g_{n+\frac{5}{74}}}{298288718100383662080000} + \frac{681768631\,h^2 g_{n+1/4}}{117633035520} - \frac{299379159\,h^2 g_{n+1/2}}{150323855360} - \frac{3778927211\,h^2 g_{n+3/4}}{352899106560} - \frac{2857204560657354607459\,h^2 g_{n+\frac{69}{74}}}{626406308010805690368000} - \frac{1034958591\,h^2 g_{n+1}}{1084475392000}$$

$$(2.12)\qquad y_{n+\frac{5}{69}} = y_n + \frac{20221603651679559569621\,hf_n}{192535718271610673960000} + \frac{49850707848181703965480 1\,hf_{n+\frac{5}{74}}}{622280344911608807424000 0} + \frac{119160702951379968501984156 8\,hf_{n+1/4}}{602548094578037200062173203 5}$$

$$+\frac{39868522991012976378507\,hf_{n+1/2}}{170513795035787528765440} + \frac{35638633238676056778197247 04\,hf_{n+3/4}}{1807644283734111600186519610 5} + \frac{769051431703194651320653\,hf_{n+\frac{69}{74}}}{1866841034734826422272000 0} + \frac{148963089941292358633 71\,hf_{n+1}}{192535718271610673960000} + \frac{131291407474186803729\,h^2g_n}{770142873086442695840 00} + \frac{861151080504415783 3\,h^2g_{n+\frac{5}{74}}}{105538929050832076800 0} + \frac{14012917501073299128088\,h^2g_{n+1/4}}{220956396984978804569920 5} - \frac{26786422976951818125\,h^2g_{n+1/2}}{282361305345975432249344} - \frac{613149069216113684775 2\,h^2g_{n+3/4}}{9469559870784805910139 45} - \frac{28171179896528379749\,h^2g_{n+\frac{69}{74}}}{31661678715249623040 00} - \frac{113877534642128422479\,h^2g_{n+1}}{77014287308644269584000}$$

$$y_{n+1} = y_n + \frac{3506128349813\,hf_n}{33294797786250} + \frac{262138373250404721337405181\,hf_{n+\frac{5}{74}}}{327776608687816396652544 0000} + \frac{1625876519575552\,hf_{n+1/4}}{8221376069021115} + \frac{153451983\,hf_{n+1/2}}{656015360} + \frac{1625876519575552\,hf_{n+3/4}}{8221376069021115} + \frac{262138373250404721337405181\,hf_{n+\frac{69}{74}}}{3277766086878163966525440000} + \frac{3506128349813\,hf_{n+1}}{33294797786250} + \frac{330127123\,h^2g_n}{193013320500} + \frac{1069234221816037002 1\,h^2g_{n+\frac{5}{74}}}{1306486697787887616000} \tag{2.13}$$

$$+\frac{19213240832\,h^2 g_{n+1/4}}{3014806039245}$$
$$-\frac{19213240832\,h^2 g_{n+3/4}}{3014806039245}$$
$$-\frac{10692342218160370021\,h^2 g_{n+\frac{69}{74}}}{1306486697787887616000}$$
$$-\frac{330127123\,h^2 g_{n+1}}{193013320500}$$

3. Analysis of the method

3.1. Local truncation error and order. Let the linear operator defined on the method be

$$L[y(x),h]=\sum_{j=0}^{k}\alpha_j\cdot y(x+jh)+h\sum_{j=0}^{k}\beta_j f_{n+j}+h^2\sum_{j=0}^{k}\gamma_j g_{n+j} \tag{3.1}$$

Assuming that $y(x)$ is sufficiently differentiable, we can expand the terms in (3.1) as a Taylor series about the point x to obtain the expression

$$L[y(x),h]=C_0 y(x)+C_1 h y(x)+\dots C_q h^q y(x)+\dots \tag{3.2}$$

where the constant C_q, $q=0,1,\dots$ are given as follows

$$\begin{cases} C_0=\sum_{j=0}^{k}\alpha_j, \\ C_1=\sum_{j=1}^{k} j\alpha_j-\sum_{j=0}^{k} j\beta_j, \\ C_2=\frac{1}{2!}\sum_{j=1}^{k} j^2\alpha_j-\sum_{j=1}^{k} j\beta_j-\sum_{j=0}^{k}\gamma_j, \\ \dots \\ C_q=\frac{1}{q!}\sum_{j=1}^{k} j^q\alpha_j-\frac{1}{(q-1)!}\sum_{j=1}^{k} j^{q-1}\beta_j-\frac{1}{(q-2)!}\sum_{j=1}^{k} j^{q-2}\gamma_j, \end{cases} \tag{3.3}$$

According to Henrici [9], we say the method (2.1) is of order p if $C_0=C_1=\ldots=C_p=0$, $C_{p+1}\neq= 0$. The C_{p+1} is the error constant and $C_{p+1}h^{p+1}y^{(p+1)}(x_n)$ is the principal truncation error at the point x_n. From our analysis, the block methods have the following order and error constants summarized in Table 1 respectively. It is noted from Table 1 that OSDBM5 is of uniform accurate order 14.

Table 1. Order and Error Constants for the Proposed One-step Second Derivative block method with 5 intra-points (OSBDM5)

Methods, Equation	Order, p	Error Constant, C_{p+1}
(8)	14	$\frac{21374359100493024975}{980732750726580551455020713275288243404 8}$
(9)	14	$\frac{4754047589}{1411151632466261606934773 76000}$
(10)	14	$\frac{222373}{440984885145706752167116 80000}$
(11)	14	$\frac{12402993}{645245373784298860052480 00}$
(12)	14	$\frac{25082462978234246925 3}{112109885124903755732222474523634302976 00}$
(13)	14	$\frac{222373}{984341261485952571801600}$

3.2. **Consistency.** The block method OSBDM5 is said to be consistent if the order of the individual block member is greater or equal to one. That is, $p > 1$ Therefore, we can infer from table1 that the methods are consistent. In what follows, the method OSBDM5 can generally be written as a matrix difference equation as follows:

$$A^{(1)}Y_w = A^{(0)}Y_{w-1} + h(B^{(0)}F_{w-1} + B^{(1)}F_w) + h^2(C^{(0)}G_{w-1} + C^{(1)}G_w) \tag{3.4}$$

And the matrices $A^{(0)}, A^{(1)}, B^{(0)}, B^{(1)}, C^{(0)}$ and $C^{(1)}$ and are matrices whose entries are given by the coefficients of the method OSBDM5.

3.3. **Zero stability.** Zero-stability is concerned with the stability of the difference system in the limit as h tends to zero [9]. Thus, as $h \to 0$, the method (3.4) tends to the difference system

$$A^{(1)}Y_w - A^{(0)}Y_{w-1} = 0 \tag{3.5}$$

whose first characteristic polynomial $\rho(\lambda)$ is given by

$$\rho(\lambda) = |\lambda A^{(1)} - A^{(0)}|. \tag{3.6}$$

Definition 3.1. (Zero-stability). The block method (3.5) is said to be zero stable if the roots of the first characteristic polynomial $\rho(\lambda)$ satisfies $|\lambda_j| \leq 1$, $j = 1, 2, \cdots$ and for those roots with $|\lambda_j| = 1$, the multiplicity must not exceed 1 [11].

Therefore, the characteristic polynomials of the methods OSDBM5 and OSDBM6 are respectively given as: $\rho(\lambda) = \lambda^5(\lambda - 1) = 0$, $\lambda = \{0, 0, 0, 0, 0, 1\}$ and $\rho(\lambda) = \lambda^6(\lambda - 1) = 0$, $\lambda = \{0, 0, 0, 0, 0, 0, 1\}$ Therefore, our methods are zero stable since they both satisfy $|\lambda_j| \leq 1$.

3.4. **Convergence.** The necessary and sufficient conditions for one-step second derivative method OSDBM5 to be convergent are that they must be consistent and zero stable [9]. Following this theorem, OSDBM5 are convergent.

3.5. **Region of Absolute stability.** The region of absolute stability is determined by obtaining the stability polynomial of the form:

$$\sigma(z) = (A^{(1)} - zB^{(1)} - z^2C^{(1)})^{-1}(A^{(0)} + zB^{(0)} - z^2C^{(0)}) \tag{3.7}$$

where $z = \lambda h$. The matrix $\sigma(z)$ has eigenvalues $\{0, 0, \ldots, \lambda_k\}$, and the dominant eigenvalue λ_k is a rational function with real coefficient given by

$$\lambda_k = \frac{P(z)}{P(-z)} \tag{3.8}$$

It is clear from the stability functions that for $Re(z) < 0$, $|\lambda_k| \leq 1$. The method is A-stable since their regions of absolute stability contains the left half-plane (Figure 1).

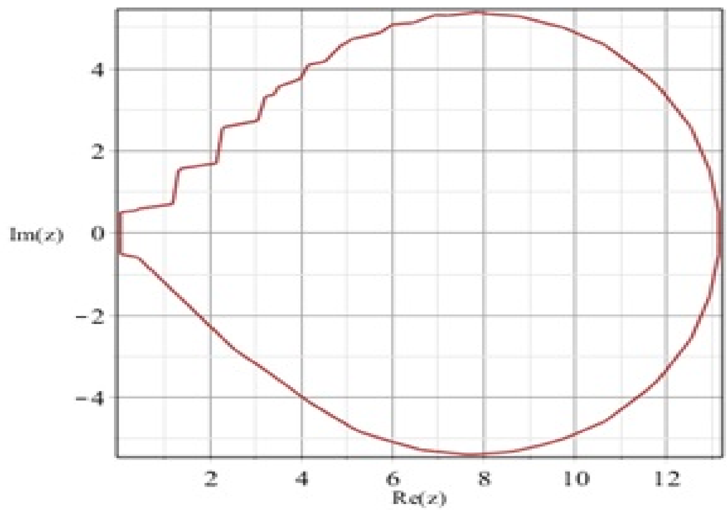

Figure 1: Stability region for OSDBM5

4. Numerical experiments

In this section, we test the performance of the one-step second derivative block methods on some systems of initial value problems on nonlinear dynamic problems. We find the absolute errors of the approximate solution on the partition πN as $|y(x)-y(x_n)|$ and also make comparisons with some existing methods in the literature. For the purpose of comparative analysis of performance of the new methods on the various numerical examples, we use the following notations: OSDBM5 is the new One step second derivative block method with 5 intra-points.

Problem 1. Consider the Mathieu equation expressed as a system of two first order equations:

$$\begin{cases} y_1' = y_2, \\ y_2' = -(\delta + \epsilon \cos 2t) y_1. \end{cases}$$

The problem is solved subject to the initial conditions $x_0 = 0.1$, $y_0 = 0$. In this problem we define

$$\begin{cases} \phi_1(t) = 0, \quad \psi_1(t) = y_{2,r} \\ \phi_2(t) = 0, \quad \psi_2(t) - (\delta + \epsilon \cos 2t) y_{1,r+1}. \end{cases}$$

The simulation of the solution profiles and phase portrait is given in Figure 2.

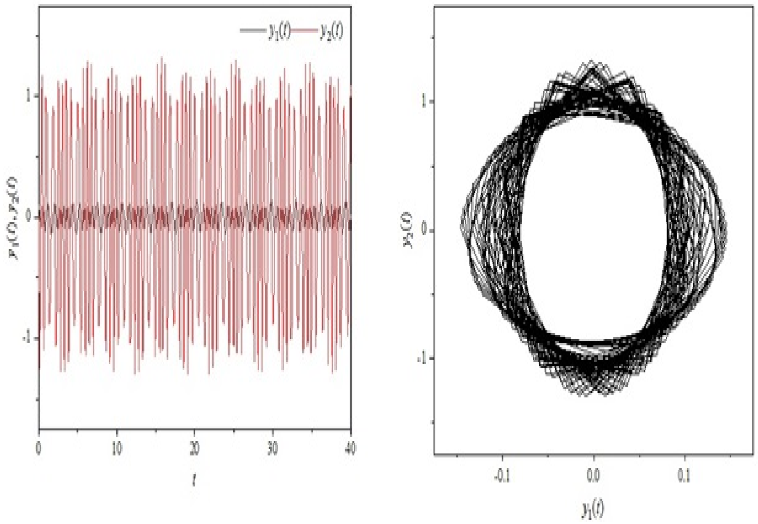

Figure 2: Simulation of the solution of Problem 1 when $\delta = 103.7529$ and $\epsilon = 64.8456$: solution profiles (left) and phase portrait (right)

Problem 2. Consider the following Lotka-Volterra model equation:

$$\begin{cases} y_1' = y_1 - y_1 y_2, \\ y_2' = -\frac{1}{5} y_2 + y_1 y_2. \end{cases}$$

With initial conditions $y_1(0) = 1$, $y_2(0) = 1$. For this problem, the method parameters are

$$\begin{cases} \phi_1(t, y_1, y_2) = 1 - y_2, & \psi_1(t, y_1, y_2) = 0 \\ \phi_2(t, y_1, y_2 = y_1 - 0.2 & \psi_2(t, y_1, y_2) = 0 \end{cases}$$

The solution and phase portrait is given in Figure 3.

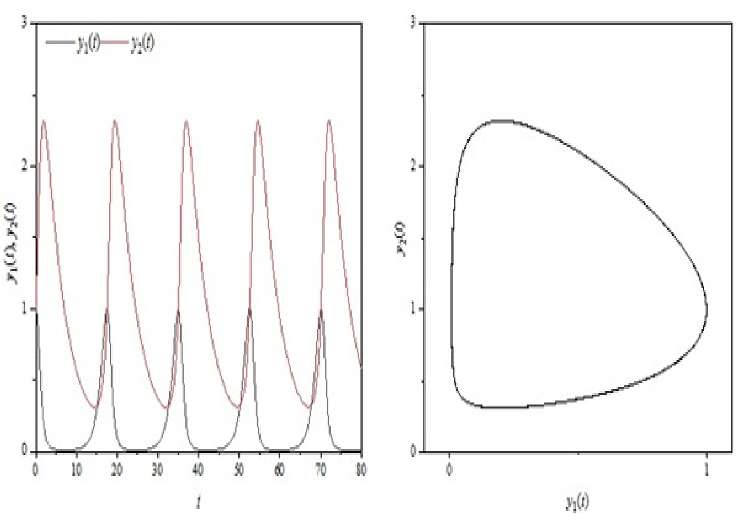

Figure 3: Solution (left) and phase portrait (right) for Problem 2, $h = 0.1$

Problem 3. We also consider the Kermack-McKendrick model. SIR model that tracks the rise and fall in the number of infected patients observed in epidemics. If the population is dived into three classes: y_1 – susceptible, y_2 – infectious, and those removed due to y_3 – immunity, the governing equations are:

$$\begin{cases} y_1' = by_1 - \nu y_1 y_2, \quad y_1(0) = 700, \\ y_2' = \nu y_1 y_3 - cy_2, \quad y_2(0) = 1, \\ y_3' = cy_2, \quad y_3(0) = 0, \end{cases}$$

where ν is the infection rate, b is the birth rate and c is the immunity rate. The simulation of the SIR model with $b = 0.02$, $\nu = 0.0005$, $c = 0.2$ is given in Figure 4.

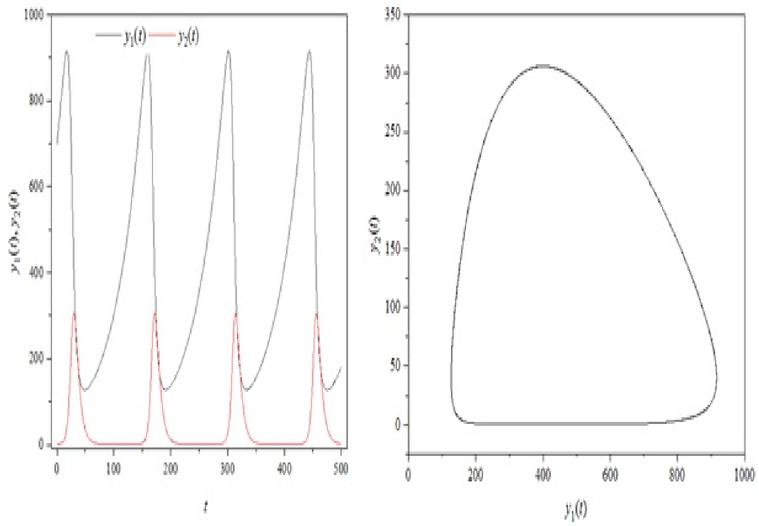

Figure 4: Time series solution (left), phase portraits (right) of problem 3, $h = 0.1$

Problem 4. Consider the predator –prey model with a Beddington-DeAnglis functional response:

$$\begin{cases} y_1' = y_1(1 - y_1) - \frac{\alpha y_1 y_2}{1+\beta_1+\mu y_2}, \quad y_1(0) = 0.15, \\ y_2' = \frac{E y_1 y_2}{1+\beta y_1+\mu y_2} - Dy_2, \quad y_2(0) = 0.5, \end{cases}$$

In this example, the parameter used are $\alpha = 1$, $\beta = 1.3$, $E = 4$, $D = 0.4$.

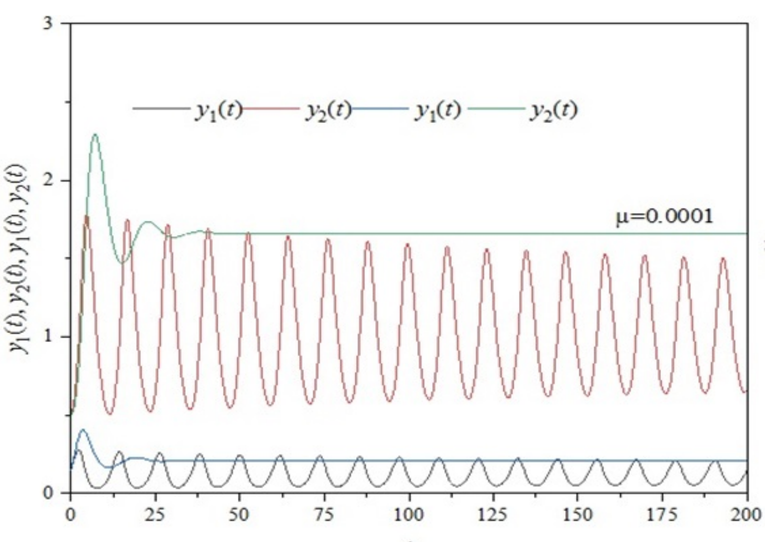

Figure 5: Time series solution (left), phase portraits (right) of problem 4, $h = 0.1$

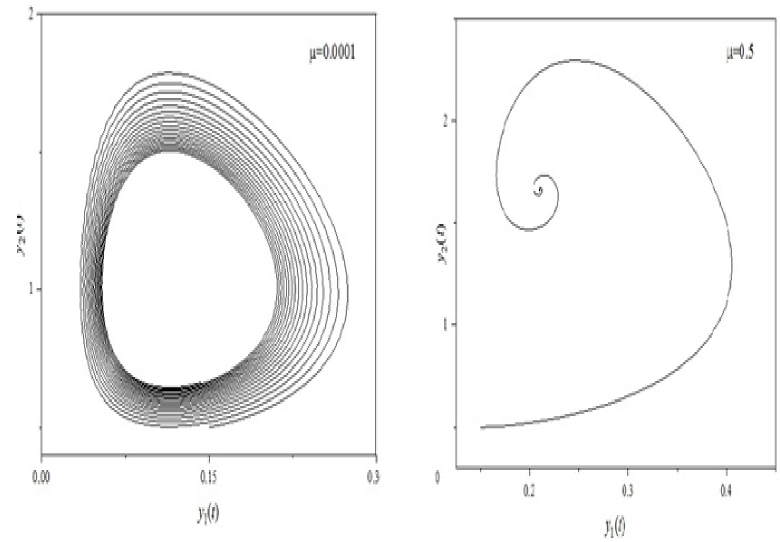

Figure 6: Phase portraits of problem 4, $h = 0.1$ and different μ

Conclusions: In this study, we derived a modified multi-step method to overcome the Dahquist barrier theorem by imposing varieties of countable intra-step points for one-step methods from the Bhaskara cosine approximation formula, and incorporating higher derivatives in the derivation process of our algorithms for solving nonlinear dynamic systems of ordinary differential equations. Analysis of basic properties of numerical methods was carried out and findings show that the method is convergent and is A-stable of higher order. The effectiveness of the derived methods is demonstrated by considering test problem on Non-Linear dynamical system.

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