MATHEMATICAL MODELING OF FILTRATION COMBUSTION WITH TEMPERATURE DEPENDENT THERMAL CONDUCTIVITY AND DIFFUSION COEFFICIENT IN A WET POROUS MEDIUM

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ABSTRACT

This thesis investigated the effect of temperature dependent thermal conductivity and diffusion coefficient on the filtration combustion in a wet porous medium. The model which relies on several assumptions and based on the conservation of total mass, chemical species and energy written in transient state mode of operation which governed the phenomenon is presented. The existence of unique solution of the problem was examined by actual solution method. The properties of solution were investigated. The coupled nonlinear governing equations were solved simultaneously for the temperature and concentration field analytically via parameter expanding method, direct integration and eigenfunction expansion technique. The influence of dimensionless parameter such as scaled thermal conductivity λ_1, species diffusion coefficient D_1, Frank kamenetskii parameter δ peclet mass number p_{em} on the filtration combustion was investigated. He thesis established that the maximum temperature is attained when δ =0.5 for fixed time t. Simulation results also revealed that high temperature front created by combustion; the oxygen molar fraction, vapor molar fraction, passive gas molar fraction, molar concentration of the solid fuel and molar concentration of liquid depend appreciably on the values of the parameters involved.

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CHAPTER ONE

1.0 INTRODUCTION

1.1 Background to the Study

Air injection leading to in situ combustion is generally considered applicable to recovery of heavy oils because it causes a significant reduction in oil viscosity. However, it can also be used to recover light oils by mechanisms such as combustion gas drive recovery, distillation and thermal expansion. The air injection process usually refers to high pressure air injection (HPAI), whereas the term in situ combustion traditionally has been used for heavy oil reservoirs (Negar *et al.,* 2014). The method of air injection has also been reported to increase recovery rates of light oils (Negar *et al.*, 2015) in this case; thermal expansion and gas drive promoted by the oxidation reaction are responsible for enhancing the recovery of oil. The reaction that takes place between light oil and injected oxygen occurs at lower temperatures, bounded by the boiling point; it is termed low temperature oxidation (LTO). Aldushin *et al.* (1997) described Filtration combustion as the propagation of exothermic reaction waves in a porous medium through which there is gas filtration. The porous solid is composed of both reactive and inert components. Filtration combustion covers a wide range of natural and technological combustion processes in porous media having a common mechanism of reaction front propagation. The principal feature of this mechanism is the delivery of gaseous reactants to the reaction front by filtration from the surrounding environment, where it reacts with the solid reactants. Filtration can be caused by two different mechanisms, referred to as forced and natural. In the former case an external force pushes the gas into the porous matrix, and is often used in technological processes while in the natural filtration combustion, the gas flow is induced by combustion process itself, which is due to consumption of gas in the reaction.

Filtration combustion (FC) waves involve a heterogeneous exothermic reaction front propagating through a porous solid that reacts with a gas carrying oxidizer flowing through its pores (Aldushin, 2003). Filtration combustion involves exothermic reactions within the matrix of a porous media (Micheal and Janet, 1999) the solid may be a condensed fuel with an oxidizer filtrating through the matrix, or the solid may be inert with the filtrating gas consisting of both fuel and oxidizer. In either case, the characteristics of the reaction front differ substantially from homogeneous combustion. The propagation of combustion fronts in porous media is a subject of interest to a variety of applications, ranging from in situ combustion for the recovery of oil to catalyst regeneration, coal gasification, waste incineration, calcinations and agglomeration of ores, smoldering, and high-temperature synthesis of solid materials. The percolation of the oxidizing fluid plays a crucial role; therefore, such processes are often referred to generically as Filtration Combustion (FC). While these problems may differ in application and context, they share a common characteristic that the reaction involves a stationary fuel reactant. The fuel may pre-exist as part of a solid matrix or, as in the case of in situ combustion, may be created in an inert porous medium by processes preceding the combustion region, such as vaporization and low temperature oxidation (Yucel and Yannis, 2003). Filtration combustion FC is a process of importance to a variety of applications, from the recovery of oil from oil reservoirs to the processing of materials (Chuan and Yannis, 2005). The process involves the combustion of a stationary fuel in a porous medium through the injection of an oxidizing agent. It can also serve as an example of a strong exothermic chemical reaction taking place in a confined geometry. When ignition occurs at the gas inlet, reaction and thermal fronts propagate in the direction of the injected gas, and the process is referred to as forward FC. When it is on the opposite side, the fronts propagate in the direction opposite to the gas flow, and the process is reverse FC. The combustion process is a subject of interest to a variety of applications, ranging from in-situ combustion for the recovery of oil to catalyst regeneration, coal gasification, waste incineration, calcinations and agglomeration of ores, smouldering, and high-temperature synthesis of solid materials (Oliveira and Kaviany, 2001).The use of air injection as a method of enhanced oil recovery has been explored for a long time. In this method, part of the oil burns with the injected air, increasing the well temperature and lowering the oil viscosity, thus enhancing its mobility. Traditionally, air injection has been used to recover heavy oils, oils with a very high viscosity. In this case, chemical reactions crack the oil into a non-volatile part (coke) and volatile components, which are expelled from the high temperature region (Endo and Mailybaev, 2017).

1.1.1 Eigenfunction expansion method

The method of eigenfunctions is closely related to the Fourier method, or the method of separation of variables, which is intended for finding a particular solution of a differential equation. When using these methods, we are often concerned with special functions being solution of an eigenfunction problem. The method of separation of variables was proposed by d'Alembert(1749). In the 18th century it was used by Euler, Bernoulli, an lagrange for solving the problem of oscillation of a string. Early in the 19th century, Fourier developed this method in considerable detail and applied it to the heat conductivity problem.

1.1.2 Existence and uniqueness of solution

When a problem is formulated, we need to examine the solution(s) so as to predict the behavior of such solution(s). Moreover, for a problem that has two solutions, any design from such a problem could behave either way. Thus the necessity for uniqueness of solution is as important as the existence of solution.

Generally, there are some rules that must be satisfied before concluding that an equation has a unique solution. The rules make use of first order differential equation. Thus for an ordinary differential equation of order greater than one, the equation will be re-written as a system of first order equations.

1.2 Statement of the Problem

The applications of filtration combustion includes, but are not limited to, such important processes as smouldering and self- propagating high-temperature synthesis (SHS). Smouldering and SHS are both complicated processes involving chemistry; diffusive and convective transport of reactants, products, and heat through a porous medium; heat losses to the environment by radiation and convection (wahle *et al.*, 2013). For this reason, it is necessary to increase our knowledge about this phenomenon. Hence the need for this research work.

1.3 Aim and Objectives of the Study

1.3.1 Aim

The aim of this research work is to provide an analytical solution to a mathematical model describing Filtration Combustion in a wet porous medium taking into consideration the temperature dependent thermal conductivity and diffusion coefficient.

1.3.2 Objectives

The objectives are to:

i. Formulate a mathematical model governing the phenomena;
ii. Establish the criteria for the existence and uniqueness of solution of the model;
iii. Obtain the analytical solution using parameter expanding method and eigenfunctions expanding technique; and

iv. Provide the graphical representation of the results obtained.

1.4 Significance of the Study

Filtration combustion, where air is injected into a porous medium containing fuel, is a method of enhancing oil recovery and has numerous applications in technology and nature. The essence of the research work is to study the effect of temperature dependent thermal conductivity and diffusion coefficient on the process.

1.5 Scope and Limitation of the Study

The essence of the research work is to study the process of filtration combustion in a porous taking into consideration the temperature dependent thermal conductivity and diffusion coefficient. The work is limited to the mathematical modeling of the phenomenon.

1.6 Definition of Terms

Combustion: is the exothermic oxidation of fuel. In the case of a carbon-base compound, the products are primarily carbon dioxide, water and energy (Olayiwola, 2015).

Convection: is the transfer of heat by mass motion of a fluid such as air or water when heated fluid is caused to move away from the sources of heat, convening the energy.

Differential Equations: An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a differential equation. In physics, engineering, economics and other sciences mathematical models are built that involve rates at which things happen. These models are equations and the rates are derivatives. Equation containing derivatives are called differential equations.

Diffusion: is the movement of atoms or molecules from an area of higher concentration to an area of lower concentration.

Diffusion coefficient: is a measure of rate of material transport as a result of the random thermal movement of particles.

Filtration: is any mechanical, physical or biological operations that separate solids from fluids (liquid or gases) by adding a medium through which only the fluid can pass.

Heat: Is the transfer of the kinetic energy from one medium or object to another. Such energy transfer can occur in three ways: radiation, convection and conduction. The standard unit of heat is calorie (cal).

Heat Capacity: The heat capacity of a defined system is the amount heat (usually express in calories, kilocalories, or joules) needed to raise the system's temperature by one degree (usually express in Kelvin or celcius).

In-situ combustion: is basically injection of an oxidizing gas (air or oxygen- enriched air) to generate heat by burning a portion of resident oil.

Ordinary Differential Equation: is a differential equation involving ordinary derivatives of one or more dependent variables.

Order of Differential Equation: the order of differential equation is the order of the highest derivative appearing in the equation.

Degree of a Differential Equation: is given by the exponent that is raises the highest derivative that occurs in the equation.

Partial Differential Equation: is an equation involving partial derivatives of one or more dependent variables with respect to more than one independent variable.

Specific Heat capacity: Is the amount of heat required to change a unit mass (or unit quantity, such as mole) of a substance by one degree in temperature.

Temperature: Is defined as the degree of hotness or coolness of a human subject or an object over a period of time. It is measured in Celsius, Fahrenheit and Kelvin.

Thermal conductivity: Thermal conductivity is a material property describing the ability to conduct heat. Thermal conductivity can be defined as "the quantity of heat transmitted through a unit thickness of a material – in a direction normal to a surface of unit area due to a unit temperature gradient under steady state conditions".

Mathematical modeling: is the process of using various mathematical structures-graphs, equations and diagrams to represent real world situations. The process of developing a mathematical model is termed mathematical modeling. A mathematical model may help to study the effects of different components, and to make a prediction about a behavior (Bellomo *et al.*, 1995).

CHAPTER TWO

2.0 LITERATURE REVIEW

2.1 Related Literature

Since last few decades, Filtration Combustion has been studied extensively; these include the work of Olayiwola (2015) who formulated a model for forward propagation of a combustion front through a porous medium with reaction involving oxygen and a solid fuel. Dependence of thermal conductivity and diffusion coefficient on temperature and gas composition was neglected. Existence and uniqueness of solution of the model was proved by actual solution method and the show that temperature is a non-decreasing function of time. The system of partial differential equations, describing the problem under consideration was transform into a boundary value problem of coupled ordinary differential equation and the numerical technique was used to solve the reduced system. The heat transfer and species consumption are significantly influence by the Frank-kamenetskii number was observed by the researcher. Grigori *et al*. (2012) studied the asymptotic approximation of long time solution for low temperature filtration combustion by considering a combustion process when air is injected into a porous medium containing immobile fuel and inert gas. They focus on the case when the reaction is active for all temperatures, but heat losses were neglected and developed a method for computing the traveling wave profile in the form of an asymptotic expansion and derived its zero-order approximation. Numerical simulations were performed in order to validate the asymptotic formulae. Chapiro and Marchesin (2015) studied the effect of thermal losses on traveling waves for in-situ combustion in porous medium. The purpose of research is to identify waves that arise in one-dimensional models of combustion in porous media, and to understand how the waves fit together in solutions of Riemann problems. Diffusion effects and the dependence of gas density on temperature was disregard. They simplify the proof of uniqueness and existence of the travelling wave solution. Michael and Janet (1999) developed a model of filtration combustion in a packed bed by investigating the low velocity filtration combustion reaction of lean methane/air mixtures flowing through a packed bed and compare to experimental results. The reaction is represented with a complete methane/air kinetic mechanism. Their results for solid temperature agree with the experiments for a mixture with an equivalence ratio 0.15 which is consistent with the existing theory on filtration combustion and discovered that gas-phase transport is not important to wave propagation at this condition. They discovered that gas-phase dispersion is important only at higher equivalence ratios. Olayiwola *et al.* (2014) presented a mathematical model for forward propagation of combustion front with Arrhenius kinetics through a porous medium with the reaction involving oxygen and solid fuel. They assume that the solid fuel depends on the space variable and that the amount of gas produced by the reaction is equal to the amount consumed by it. Existence and uniqueness of solution of the model was proved by actual solution and provided the analytical solution of the model through Homotopy perturbation method and represented the results graphically. They discovered that the Frank-kamenestsskii number on the heat transfer and species consumption is of great importance. Mailybaev *et al.* (2013) formulated a model for recovery of light oil by medium temperature oxidation. They considered two phase flow possessing a combustion front when a gaseous oxidizer (air) is injected into porous rock filled with light oil. The temperature of the medium is bounded by the boiling point of the liquid and, thus, relatively low. They disregarded the gas phase reactions. They observed that the initial period, the recovery curve is typical of gas displacement but after a critical amount of air has been injected the cumulative oil recovery increases linearly until all oil has been recovered, they conclude that oil recovery is independent of reaction rate parameters but recovery is much faster than for gas displacement and among their findings is that oil recovery is faster when the injected pressure is higher.

Bruining *et al.* (2009) developed a model of filtration combustion in wet porous medium. By considering a porous rock cylinder thermally insulated on the side filled with inert gas, liquid and solid fuel. An oxidizer was injected. They assumed that the amount of liquid is small, so its mobility is negligible, and that only a small part of the available space is occupied by solid fuel and liquid, so that changes of rock porosity in the reaction, evaporation, and condensation processes can be neglected. They neglected the dependence of thermal conductivity and diffusion coefficients on the temperature and gas compositions. They discovered that when the diffusion is dominant at the reaction layer, it lead the oxygen to extinction and also discovered two possible sequences of waves, and the internal structure of all waves was characterized. They compared the analytical results with direct numerical simulations. Their model Equation is as shown in equation (2.1) to equation (2.4)

$$\rho c_g \frac{\partial}{\partial t}\left(T - T_{res}\right) + \rho c_g u \frac{\partial}{\partial x}\left(T - T_{res}\right) = \lambda \frac{\partial^2 T}{\partial x^2} + Q_r W_r - Q_e W \qquad (2.1)$$

The mass balance equations for the components X , Y , Z are:

$$\varphi\rho \frac{\partial X}{\partial t} + \varphi\rho u \frac{\partial X}{\partial x} = D_X \varphi \frac{\partial}{\partial x}\left(\rho \frac{\partial X}{\partial x}\right) + W_e \qquad (2.2)$$

$$\varphi\rho \frac{\partial Y}{\partial t} + \varphi\rho u \frac{\partial Y}{\partial x} = D_Y \varphi \frac{\partial}{\partial x}\left(\rho \frac{\partial Y}{\partial x}\right) - \mu_o W_r \qquad (2.3)$$

$$\varphi\rho \frac{\partial Z}{\partial t} + \varphi\rho u \frac{\partial Z}{\partial x} = D_Z \varphi \frac{\partial}{\partial x}\left(\rho \frac{\partial Z}{\partial x}\right) + \mu_g W_r \qquad (2.4)$$

As the solid fuel and the liquid do not move, their concentrations satisfy the equations for reaction and evaporation respectively as shown in equation (2.5) to equation (2.6)

$$\frac{\partial n_f}{\partial t} = -\mu_f W_r \tag{2.5}$$

$$\frac{\partial n_l}{\partial t} = - W_e \tag{2.6}$$

Where ρ [mole/m^3] is the molar density of gas, T [k] is the temperature, c_g is the heat capacity of rock, u [m/s] is the Darcy velocity of gas, T_{res} is the initial reservoir temperature, λ [w/mk] is thermal conductivity of the porous medium, (Q_r and Q_e) [J/mole] are the heats enthalpies of combustion and evaporation of the solid and the liquid at reservoir temperature, Y is the molar fraction of oxygen, X is the vapor molar fraction in the gas phase (mole of vapo/mole of gas), Z is the molar fraction of passive gas in the gas-phase, φ is the porosity, n_f Is the molar concentration of solid fuel, n_l Is the molar concentration of liquid, D_x [m^2/s] is the diffusion coefficients for vapor of porous medium, D_y [m^2/s] is the diffusion coefficients for oxygen of porous medium, D_z [m^2/s] is the diffusion coefficients for passive gas in the gas-phase of porous medium, μ_f is the moles of solid fuel, μ_o is the moles of oxygen and μ_g is the moles of gaseous product.

This research work extended the work of Bruining *et al.* (2009) by incorporating temperature dependent thermal conductivity and diffusion coefficient. We shall provide the criteria for the existence and uniqueness of solution of the equations, examine the properties of solution and provide the analytical solution of the model by parameter expanding and eignfunction expansion methods.

CHAPTER THREE

3.0 MATERIALS AND METHODS

3.1 Mathematical Formulation

Following Bruining *et al. (*2009), we consider a porous rock cylinder thermally insulated on the side and filled with vaporizable liquid, inert gas, and combustible solid fuel. An oxidizer (air) is injected. The liquid can be water or light oil, and the combustible solid can be coke. We assume that the amount of liquid is small, so its mobility is negligible. We assume that only a small part of the available space is occupied by solid fuel and liquid, so that we can neglect changes of rock porosity in the reaction, evaporation, and condensation processes. We assume that the solid, gas, and liquid are in local thermal equilibrium, so they have the same temperature. Based on the above assumptions, a one-dimensional model with time t and space coordinate x is considered the energy equation governing the system is giving by equation (3.1):

$$\left.\begin{array}{l}\rho c_g \frac{\partial}{\partial t}(T - T_{res}) + \rho c_g u \frac{\partial}{\partial x}(T - T_{res}) = \frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right) + Q_r K_r Y n_f\, e^{-\frac{E_r}{RT}} - \\ Q_e k n_l \left(\frac{P_{atm}}{\rho RT} e^{-\frac{Q_e}{R}\left(\frac{1}{T} - \frac{1}{Tb}\right)} - X\right)\end{array}\right\} \quad (3.1)$$

We consider a single component liquid (water), and denote by X its vapor molar fraction in the gas phase (mole of vapor/mole of gas). The gas has several components: oxygen, vapor, and passive (inert and combusted) gas. We denote the molar fractions of oxygen and passive gas in the gas-phase by Y and Z, respectively. Then, we write the mass balance equations for the components X, Y, Z as equation (3.2) to equation (3.4):

$$\phi\rho\left(\frac{\partial X}{\partial t}+u\frac{\partial X}{\partial x}\right)=\phi\frac{\partial}{\partial x}\left(\rho D_X\frac{\partial X}{\partial x}\right)+kn_l\left(\frac{p_{atm}}{\rho RT}e^{-\frac{Q_e}{R}(\frac{1}{T}-\frac{1}{Tb})}-X\right) \quad (3.2)$$

$$\phi\rho\left(\frac{\partial Y}{\partial t}+u\frac{\partial Y}{\partial x}\right)=\phi\frac{\partial}{\partial x}\left(\rho D_Y\frac{\partial Y}{\partial x}\right)-\mu_o K_r Y n_f\, e^{-\frac{E_r}{RT}} \quad (3.3)$$

$$\phi\rho\left(\frac{\partial Z}{\partial t}+u\frac{\partial Z}{\partial x}\right)=\phi\frac{\partial}{\partial x}\left(\rho D_Z\frac{\partial Z}{\partial x}\right)+\mu_g K_r Y n_f\, e^{-\frac{E_r}{RT}} \quad (3.4)$$

As the solid fuel and the liquid do not move, their concentrations satisfy the equations for reaction and evaporation respectively giving by equation (3.5) to equation (3.6):

$$\frac{\partial n_f}{\partial t}=\mu_f K_r Y n_f\, e^{-\frac{E_r}{RT}} \quad (3.5)$$

$$\frac{\partial n_l}{\partial t}= -\ kn_l\left(\frac{p_{atm}}{\rho RT}e^{-\frac{Q_e}{R}\left(\frac{1}{T}-\frac{1}{Tb}\right)}-X\right) \quad (3.6)$$

Where;

ρ [mole/m^3] is the molar density of gas

T [k] is the temperature

T_{res} is the initial reservoir temperature

c_g is the heat capacity of rock

u [m/s] is the Darcy velocity of gas

λ [w/mk] is thermal conductivity of the porous medium

(Q_r and Q_e)[J/mole] are the heats enthalpies of combustion and evaporation of the solid and the liquid at reservoir temperature

K_r [1/s] is the pre exponential parameter.

Y is the molar fraction of oxygen

X is the vapor molar fraction in the gas phase (mole of vapo/mole of gas)

Z is the molar fraction of passive gas in the gas-phase,

n_f Is the molar concentration of solid fuel

n_l Is the molar concentration of liquid

E_r [J/mole] is activation energe

R = 8.314[J/mole k] is the ideal gas constant

T_b is the boiling temperature of the liquid at atmospheric pressure p_{atm}

ϕ is the porosity

D_x [m^2/s] is the diffusion coefficients for vapor of porous medium

D_y [m^2/s] is the diffusion coefficients for oxygen of porous medium

D_z [m^2/s] is the diffusion coefficients for passive gas in the gas-phase of porous medium

μ_f is the moles of solid fuel

μ_o is the moles of oxygen

μ_g is the moles of gaseous product

3.2 Coordinate Transformation

The balance of mass can be eliminated by the means of streamline function (Olayiwola, 2015) giving by equation (3.7)

$$\eta(x.t) = \left(\rho^2\right)^{-\frac{1}{2}} \int_0^x \rho(x,t)ds \qquad (3.7)$$

Then coordinate transformation is giving by equation (3.8) to (3.9)

$$\frac{\partial}{\partial x} \rightarrow \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial}{\partial \eta} \qquad (3.8)$$

$$\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial t} = -u\frac{\partial}{\partial \eta} + \frac{\partial}{\partial t} \qquad (3.9)$$

We make the additional assumptions that ρc_g, ρD, and λ are constant. Although these assumptions could be relaxed in the future, they considerably simplify the equations. The equations (3.1) to equation (3.4) can be simplified as equation (3.10) to equation (3.13):

$$\rho c_g \frac{\partial}{\partial t}(T - T_{res}) = \frac{\partial}{\partial \eta}\left(\lambda \frac{\partial T}{\partial \eta}\right) + Q_r K_r Y n_f\, e^{-\frac{E_r}{RT}} - Q_e k n_l \left(\frac{P_{atm}}{\rho RT} e^{-\frac{Q_e}{R}\left(\frac{1}{T} - \frac{1}{Tb}\right)} - X\right) \qquad (3.10)$$

$$\phi\rho\frac{\partial X}{\partial t} = \phi\frac{\partial}{\partial \eta}\left(\rho D_X \frac{\partial X}{\partial \eta}\right) + k n_l \left(\frac{p_{atm}}{\rho RT} e^{-\frac{Q_e}{R}\left(\frac{1}{T} - \frac{1}{Tb}\right)} - X\right) \qquad (3.11)$$

$$\phi\rho\frac{\partial Y}{\partial t} = \phi\frac{\partial}{\partial \eta}\left(\rho D_Y \frac{\partial Y}{\partial \eta}\right) - \mu_o K_r Y n_f\, e^{-\frac{E_r}{RT}} \qquad (3.12)$$

$$\phi\rho\frac{\partial Z}{\partial t} = \phi\frac{\partial}{\partial \eta}\left(\rho D_Z \frac{\partial X}{\partial \eta}\right) + \mu_g K_r Y n_f\, e^{-\frac{E_r}{RT}} \qquad (3.13)$$

The initial and boundary conditions were formulated as follows:

Initial condition is giving by equation (3.14):

At $t = 0$ and $\forall \eta$

$$\left.\begin{aligned} &T = \frac{RT_0^{\ 2}}{E}\left(1-\frac{\eta}{L}\right)+T_0, \\ &X = X_0\left(1-\frac{\eta}{L}\right), \qquad Y = Y_0\left(1-\frac{\eta}{L}\right), \\ &Z = Z_0\left(1-\frac{\eta}{L}\right), \qquad n_f = n_{fres,} \quad n_l = n_{lres} \end{aligned}\right\} \tag{3.14}$$

Boundary Condition is giving by equation (3.15):

$$\left.\begin{aligned} &T\Big|_{\eta=0} = T_1, \qquad T\Big|_{\eta=l} = T_0 \\ &Y\Big|_{\eta=0} = Y_{inj}, \qquad Y\Big|_{\eta=l} = 0 \\ &X\Big|_{\eta=0} = 0, \qquad X\Big|_{\eta=l} = 0 \\ &Z\Big|_{\eta=0} = 0, \qquad Z\Big|_{\eta=0} = 0 \end{aligned}\right\} \tag{3.15}$$

3.3 Method of Solution

Here, we shall establish the criteria for the existence and uniqueness of solution of the equations and solve the equations analytically.

We let λ and D to be constants, then equation (3.10) to equation (3.13) reduces as shown in equation (3.16) to equation (3.19).

$$\frac{\partial}{\partial t}\left(T - T_{res}\right) = \frac{\lambda}{\rho c_g}\frac{\partial^2 T}{\partial \eta^2} + \frac{Q_r K_r}{\rho c_g} Y n_f\, e^{-\frac{E_r}{RT}} - \frac{Q_e k n_l}{\rho c_g}\left(\frac{P_{atm}}{\rho RT} e^{-\frac{Q_e}{R}\left(\frac{1}{T}-\frac{1}{T_b}\right)} - X\right) \tag{3.16}$$

$$\phi\rho\frac{\partial X}{\partial t}=\phi\rho D\frac{\partial^2 X}{\partial^2\eta^2}+kn_l\left(\frac{P_{atm}}{\rho RT}e^{-\frac{Q_e}{R}\left(\frac{1}{T}-\frac{1}{T_b}\right)}-X\right) \tag{3.17}$$

$$\phi\rho\frac{\partial Y}{\partial t}=\phi\rho D\frac{\partial^2 Y}{\partial^2\eta^2}-\mu_0K_rYn_f\,e^{-\frac{E_r}{RT}} \tag{3.18}$$

$$\phi\rho\frac{\partial Z}{\partial t}=\phi\rho D\frac{\partial^2 Z}{\partial^2\eta^2}+\mu_gK_rYn_f\,e^{-\frac{E_r}{RT}} \tag{3.19}$$

Multiplying equation (3.17) by $\frac{Q_e\mu_0}{\rho c_g}$, we obtain equation (3.20)

$$\frac{\partial}{\partial t}\frac{\phi\rho Q_e\mu_0X}{\rho c_g}=\rho D\frac{\partial^2}{\partial^2\eta^2}\frac{\phi\rho Q_e\mu_0X}{\rho c_g}+\frac{Q_e\mu_0}{\rho c_g}kn_l\left(\frac{P_{atm}}{\rho RT}e^{-\frac{Q_e}{R}\left(\frac{1}{T}-\frac{1}{T_b}\right)}-X\right) \tag{3.20}$$

Multiplying equation (3.16) by μ_0, we obtain equation (3.21)

$$\left.\begin{array}{l}\frac{\partial}{\partial t}\left(\mu_0\left(T-T_{res}\right)\right)=\frac{\lambda}{\rho c_g}\frac{\partial^2\left(\mu_0T\right)}{\partial\eta^2}+\frac{Q_rK_r}{\rho c_g}\mu_0Yn_f\,e^{-\frac{E_r}{RT}}-\\ \frac{Q_ekn_l\mu_0}{\rho c_g}\left(\frac{P_{atm}}{\rho RT}e^{-\frac{Q_e}{R}\left(\frac{1}{T}-\frac{1}{T_b}\right)}-X\right)\end{array}\right\} \tag{3.21}$$

Multiplying equation (3.18) by $\frac{Q_r+\mu_g}{\rho c_g}$, we obtain equation (3.22)

$$\frac{\partial Y}{\partial t}\left(\frac{\phi\rho\left(Q_r+\mu_g\right)Y}{\rho c_g}\right)=D\frac{\partial^2}{\partial^2\eta^2}\frac{\phi\rho\left(Q_r+\mu_g\right)Y}{\rho c_g}-\frac{\left(Q_r+\mu_g\right)}{\rho c_g}\mu_0K_rYn_f\,e^{-\frac{E_r}{RT}} \tag{3.22}$$

Multiplying equation (3.19) by $\frac{\mu_0}{\rho c_g}$, we obtain equation (3.23)

$$\frac{\partial}{\partial t}\left(\frac{\phi\rho\mu_0Z}{\rho c_g}\right)=D\frac{\partial^2}{\partial^2\eta^2}\left(\frac{\phi\rho\mu_0Z}{\rho c_g}\right)+\left(\frac{\mu_0}{\rho c_g}\right)\mu_gK_rYn_f\,e^{-\frac{E_r}{RT}} \tag{3.23}$$

Adding equation (3.20) to equation (3.23), we have equation (3.24)

$$\left.\begin{aligned}&\frac{\partial}{\partial t}\left(\frac{\phi Q_e\mu_0}{c_g}X+\mu_0 T+\frac{\phi\left(Q_r+\mu_g\right)Y}{c_g}+\frac{\phi\mu_0}{c_g}Z\right)=\\&\frac{\partial^2}{\partial\eta^2}\left(D\frac{\phi Q_e\mu_0}{c_g}X+\frac{\lambda\mu_0}{\rho c_g}T+D\frac{\phi\left(Q_r+\mu_g\right)}{c_g}Y+D\frac{\phi\mu_0}{c_g}Z\right)\end{aligned}\right\} \tag{3.24}$$

Let D = $\frac{\lambda\mu_0}{\rho c_g}$ and $\psi=\left(\frac{\phi Q_e\mu_0}{c_g}X+\mu_0 T+\frac{\phi\left(Q_r+\mu_g\right)Y}{c_g}+\frac{\phi\mu_0}{c_g}Z\right)$

Then equation (3.24) yield equation (3.25)

$$\frac{\partial\psi}{\partial t}=D\frac{\partial^2\psi}{\partial\eta^2} \tag{3.25}$$

With the initial and boundary conditions as equation (3.26)

$$\psi(\eta,0)=A\left(1-\frac{\eta}{L}\right)+B,\qquad \psi(0,t)=A_1,\qquad \psi(L,t)=B \tag{3.26}$$

Where;

A = $\left(\frac{\phi Q_e\mu_0}{c_g}X_0+\frac{\mu_0 R T_0^2}{}+\frac{\phi\left(Q_r+\mu_g\right)Y_0}{c_g}+\frac{\phi\mu_0}{c_g}Z_0\right)$

$B=\mu_0 T_0$

$A_1=\left(\mu_0 T_1+\frac{\phi\left(Q_r+\mu_g\right)}{c_g}Y_{inj}\right)$

From equation (3.25) and equation (3.26) , we obtain equation (3.27) to equation (3.39):

Let $\mu(\eta,t) = A_1 t^0 + \frac{\eta}{L}(B - A_1)t^0$ (3.27)

Then

$$\left.\begin{aligned} \mu(\eta,0) &= 0 \\ \mu(0,t) &= A_1 \\ \mu(L,t) &= B \end{aligned}\right\} \qquad (3.28)$$

Also

Let $\psi(\eta,t) = u(\eta,t) + \mu(\eta,t)$ (3.29)

Therefore equation (3.27) becomes

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial \eta^2} \qquad (3.30)$$

$$\left.\begin{aligned} u(\eta,0) &= A\left(1 - \frac{\eta}{L}\right) + B \\ u(0,t) &= 0 \\ u(L,t) &= 0 \end{aligned}\right\} \qquad (3.31)$$

we seek a solution of the form

$$u(\eta,t) = \sum_{n=1}^{\infty} u_n(t)\sin\frac{n\pi}{L}\eta, \qquad (3.32)$$

Where

$$u_n(t) = \int_0^t e^{\left(\alpha - k\left(\frac{n\pi}{L}\right)^2 (t-\tau)\right)} F_n(\tau)d\tau + bn\, e^{\alpha - k\left(\frac{n\pi}{L}\right)^2 t} \qquad (3.33)$$

$$F_n(t) = \frac{2}{L}\int_0^L f(x,t)\sin\frac{n\pi}{L}\eta d\eta \qquad (3.34)$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}\eta d\eta$$

Here, $\alpha = 0,\ f(\eta,t) = 0,\ f(\eta) = A\left(1-\frac{\eta}{L}\right)+B, k = D$

Then

$$b_n = \frac{2}{L}\int_0^L \left(A\left(1-\frac{\eta}{L}\right)+B\right)\sin\frac{n\pi}{L}\eta d\eta \tag{3.35}$$

Integrating equation (3.34), we obtain equation (3.36)

$$b_n = \frac{2}{n\pi}\left((-1)^n - (A+B)\left((-1)^n - 1\right)\right) \tag{3.36}$$

But $F(\eta,t) = 0 \Rightarrow F_n(t) = 0$

Then

$$u_n(t) = \frac{2}{n\pi}\left((-1)^n - (A+B)\left((-1)^n - 1\right)\right)e^{-D\left(\frac{n\pi}{L}\right)^2 t} \tag{3.37}$$

Therefore

$$u(\eta,t) = \sum_{n=1}^{\infty}\frac{2}{n\pi}\left((-1)^n - (A+B)\left((-1)^n - 1\right)\right)e^{-D\left(\frac{n\pi}{L}\right)^2 t}\sin\frac{n\pi}{L}\eta \tag{3.38}$$

Thus

$$\psi(\eta,t) = A_1 + \frac{\eta}{L}(B - A_1) + \sum_{n=1}^{\infty}\frac{2}{n\pi}\left((-1)^n - (A+B)\left((-1)^n - 1\right)\right)e^{-D\left(\frac{n\pi}{L}\right)^2 t}\sin\frac{n\pi}{L}\eta \tag{3.39}$$

This led us to the theorem 3.1 and its proof.

3.3.1 Existence and uniqueness of Solution

Theorem 3.1: let $D=\dfrac{\lambda}{\rho c_g}$ and T_{res} = constant. Then there exists a unique solution of equation (3.10) to equation (3.13) satisfy Equation (3.14) and equation (3.15).

Proof:

Let $D=\dfrac{\lambda}{\rho c_g}$ and T_{res} = constant and $\psi=\left(\dfrac{\phi Q_e\mu_0}{c_g}X+\mu_0 T+\dfrac{\phi(Q_r+\mu_g)Y}{c_g}+\dfrac{\phi\mu_0}{c_g}Z\right)$

Then, equation (3.10) to equation (3.13) reduces to equation (3.40) to (3.41)

$$\frac{\partial\psi}{\partial t}=D\frac{\partial^2\psi}{\partial\eta^2} \tag{3.40}$$

$$\psi(\eta,0)=A\left(1-\frac{\eta}{L}\right)+B,\ \psi(0,t)=A_1,\ \ \psi(L,t)=B \tag{3.41}$$

Using eigenfunction expansion technique, we obtain the solution of equation (3.40) and equation (3.41) as equation (3.42) to (3.46).

$$\psi(\eta,t)=A_1+\frac{\eta}{L}(B-A_1)+\sum_{n=1}^{\infty}\frac{2}{n\pi}\left((-1)^n-(A+B)\left((-1)^n-1\right)\right)e^{-D\left(\frac{n\pi}{L}\right)^2 t}\sin\frac{n\pi}{L}\eta \tag{3.42}$$

Then, we have

$$\left.\begin{aligned}T(\eta,t)=\frac{1}{\mu_0}&\left(A_1+\frac{\eta}{L}(B-A_1)+\sum_{n=1}^{\infty}\frac{2}{n\pi}\left((-1)^n-(A+B)\left((-1)^n-1\right)\right)e^{-D\left(\frac{n\pi}{L}\right)^2 t}\sin\frac{n\pi}{L}\eta\right.\\&\left.-\left(\frac{\phi Q_e\mu_0}{c_g}X+\frac{\phi(Q_r+\mu_g)}{c_g}Y+\frac{\phi\mu_0}{c_g}Z\right)\right)\end{aligned}\right\} \tag{3.43}$$

$$
\left.\begin{array}{l}
X(\eta,t)= \\
\dfrac{c_g}{\phi Q_e\mu_0}\left(A_1+\dfrac{\eta}{L}(B-A_1)+\displaystyle\sum_{n=1}^{\infty}\dfrac{2}{n\pi}\left((-1)^n-(A+B)\left((-1)^n-1\right)\right)e^{-D\left(\frac{n\pi}{L}\right)^2 t}\sin\dfrac{n\pi}{L}\eta\right. \\
\left.-\left(\mu_0 T+\dfrac{\phi\left(Q_r+\mu_g\right)}{c_g}Y+\dfrac{\phi\mu_0}{c_g}Z\right)\right)
\end{array}\right\} \qquad (3.44)
$$

$$
\left.\begin{array}{l}
Y(\eta,t)= \\
\dfrac{c_g}{\phi\left(Q_r+\mu_g\right)}\left(A_1+\dfrac{\eta}{L}(B-A_1)+\displaystyle\sum_{n=1}^{\infty}\dfrac{2}{n\pi}\left((-1)^n-(A+B)\left((-1)^n-1\right)\right)e^{-D\left(\frac{n\pi}{L}\right)^2 t}\sin\dfrac{n\pi}{L}\eta\right. \\
\left.-\left(\mu_0 T+\dfrac{\phi Q_e\mu_0}{c_g}Y+\dfrac{\phi\mu_0}{c_g}Z\right)\right)
\end{array}\right\} \qquad (3.45)
$$

$$
\left.\begin{array}{l}
Z(\eta,t)= \\
\dfrac{c_g}{\phi\mu_0}\left(A_1+\dfrac{\eta}{L}(B-A_1)+\displaystyle\sum_{n=1}^{\infty}\dfrac{2}{n\pi}\left((-1)^n-(A+B)\left((-1)^n-1\right)\right)e^{-D\left(\frac{n\pi}{L}\right)^2 t}\sin\dfrac{n\pi}{L}\eta\right. \\
\left.-\left(\mu_0 T+\dfrac{\phi Q_e\mu_0}{}X+\dfrac{\phi\left(Q_r+\mu_0\right)}{c_g}Y\right)\right)
\end{array}\right\} \qquad (3.46)
$$

Hence, there exist unique solutions of equation (3.10) to equation (3.13). This completes the proof.

We shall return to our original equations, that's equation (3.1) to equation (3.6) satisfying equation (3.14) and equation (3.15) and consider an alternative method for the existence of unique solution of the problem.

Here, the dependence of thermal conductivity and diffusion coefficient on the temperature is taken into account by the mathematical expression giving by equation (3.47) and equation (3.48):

$$\lambda = \lambda_0 \left(\frac{T}{T_0} \right) \tag{3.47}$$

$$D = D_0 \left(\frac{T}{T_0} \right) \tag{3.48}$$

Where λ_0 is the initial thermal conductivity, D_0 is the initial diffusion coefficient, and T_0 is the initial temperature of the medium.

Substituting equation (3.44), equation (3.45), equation (3.46) and equation (3.47) into equation (3.5) and equation (3.6), equation (3.10) to equation (3.13), we have equation (3.49) to equation (3.53)

$$\left.\begin{aligned}
&\rho c_g \frac{\partial}{\partial t}\left(T - T_{res}\right) = \lambda_0 \frac{\partial}{\partial \eta}\left(\frac{T}{T_0}\frac{\partial T}{\partial \eta}\right) + Q_r K_r n_f \left(\frac{c_g}{\phi\left(Q_r + \mu_g\right)}\left(A_1 + \frac{\eta}{L}\left(B - A_1\right) + \right.\right.\\
&\sum_{n=1}^{\infty}\frac{2}{n\pi}\left(\left(-1\right)^n - \left(A + B\right)\left(\left(-1\right)^n - 1\right)\right)e^{-D\left(\frac{n\pi}{L}\right)^2 t}\sin\frac{n\pi}{L}\eta - \\
&\left.\left.\left(\mu_0 T + \frac{\phi Q_e \mu_0}{c_g}X + \frac{\phi\mu_0}{c_g}Z\right)\right)\right)e^{-\frac{E_r}{RT}} - \\
&k n_l \left(\frac{p_{atm}}{\rho R T}e^{-\frac{Q_e}{R}\left(\frac{1}{T} - \frac{1}{Tb}\right)} - X\right)
\end{aligned}\right\} \tag{3.49}$$

$$\phi\rho\frac{\partial X}{\partial t} = \phi\rho D_0 \frac{\partial}{\partial \eta}\left(\frac{T}{T_0}\frac{\partial X}{\partial \eta}\right) + k n_l \left(\frac{p_{atm}}{\rho R T}e^{-\frac{Q_e}{R}\left(\frac{1}{T} - \frac{1}{Tb}\right)} - X\right) \tag{3.50}$$

$$\left.\begin{aligned}
&\phi\rho\frac{\partial Y}{\partial t} = \phi\rho D_0 \frac{\partial}{\partial \eta}\left(\frac{T}{T_0}\frac{\partial Y}{\partial \eta}\right) - \mu_0 K_r n_f \left(\frac{c_g}{\phi\left(Q_r + \mu_g\right)}\left(A_1 + \frac{\eta}{L}\left(B - A_1\right) + \right.\right.\\
&\sum_{n=1}^{\infty}\frac{2}{n\pi}\left(\left(-1\right)^n - \left(A + B\right)\left(\left(-1\right)^n - 1\right)\right)e^{-D\left(\frac{n\pi}{L}\right)^2 t}\sin\frac{n\pi}{L}\eta - \\
&\left.\left.\left(\mu_0 T + \frac{\phi Q_e \mu_0}{c_g}X + \frac{\phi\mu_0}{c_g}Z\right)\right)\right)e^{-\frac{E_r}{RT}}
\end{aligned}\right\} \tag{3.51}$$

$$\left.\begin{array}{l}\phi\rho\frac{\partial Z}{\partial t}=\phi\rho D_0\frac{\partial}{\partial\eta}\left(\frac{T}{T_0}\frac{\partial Z}{\partial\eta}\right)+\mu_g K_r n_f\left(\frac{c_g}{\phi\left(Q_r+\mu_g\right)}\left(A_1+\frac{\eta}{L}\left(B-A_1\right)+\right.\right.\\ \sum_{n=1}^{\infty}\frac{2}{n\pi}\left(\left(-1\right)^n-\left(A+B\right)\left(\left(-1\right)^n-1\right)\right)e^{-D\left(\frac{n\pi}{L}\right)^2 t}\sin\frac{n\pi}{L}\eta-\\ \left.\left.\left(\mu_0 T+\frac{\phi Q_e\mu_0}{c_g}X+\frac{\phi\mu_0}{c_g}Z\right)\right)\right)e^{-\frac{E_r}{RT}}\end{array}\right\} \quad (3.52)$$

$$\left.\begin{array}{l}\phi\rho\frac{\partial n_f}{\partial t}=\mu_f K_r n_f\left(\frac{c_g}{\phi\left(Q_r+\mu_g\right)}\left(A_1+\frac{\eta}{L}\left(B-A_1\right)+\right.\right.\\ \sum_{n=1}^{\infty}\frac{2}{n\pi}\left(\left(-1\right)^n-\left(A+B\right)\left(\left(-1\right)^n-1\right)\right)\times\\ \left.\left.e^{-D\left(\frac{n\pi}{L}\right)^2 t}\sin\frac{n\pi}{L}\eta-\left(\mu_0 T+\frac{\phi Q_e\mu_0}{c_g}X+\frac{\phi\mu_0}{c_g}Z\right)\right)\right)e^{-\frac{E_r}{RT}}\end{array}\right\} \quad (3.53)$$

3.3.2 Non – dimensionalization

Here we shall non-dimensionalized equation (3.49) to equation (3.53) using the following dimensionless variables as shown in equation (3.54)

$$\left.\begin{array}{l}X^1=\frac{X}{X_0},\quad Y^1=\frac{Y}{Y_0},\quad Z^1=\frac{Z}{Z_0},\quad \theta=\frac{E}{RT_0}\left(T-T_0\right),\\ t^1=\frac{t}{t_0},\quad \eta^1=\frac{\eta}{L},\quad n_f{}^1=\frac{n_f}{n_{fres}},\quad n_l{}^1=\frac{n_l}{n_{lres}},\quad \varepsilon=\frac{RT_0}{E}\end{array}\right\} \quad (3.54)$$

Then, we have equation (3.55)

$$\left.\begin{array}{ll}X=X_0X^1 & \Rightarrow\partial X=X_0\partial X^1\\ Y=Y_0Y^1 & \Rightarrow\partial Y=Y_0\partial Y^1\\ Z=Z_0Z^1 & \Rightarrow\partial Z=Z_0\partial Z^1\\ t=t_0t^1 & \Rightarrow\partial t=t_0\partial t^1\\ \eta=L\eta^1 & \Rightarrow\partial\eta=L\partial\eta^1\\ T=\varepsilon T_0\vartheta+T_0 & \Rightarrow\partial T=\varepsilon T_0\partial\vartheta\\ n_f=n_{fres}n_f^1 & \Rightarrow\partial n_f=n_{fres}\partial n_f^1\\ n_l=n_{lres}n_l^1 & \Rightarrow\partial n_l=n_{lres}\partial n_l^1\end{array}\right\} \quad (3.55)$$

Now,

$$e^{-\frac{E_r}{RT}} = e^{-\frac{E_r}{RT_0}} e^{-\frac{E_r}{RT}} e^{-\frac{E_r}{RT_0}}$$

$$= e^{-\frac{E_r}{RT_0}} e^{\frac{E_r}{R}\left(\frac{1}{T_0}-\frac{1}{T}\right)}$$

$$= e^{-\frac{E_r}{RT_0}} e^{\frac{E_r}{R}\left(\frac{T-T_0}{TT_0}\right)}$$

$$= e^{-\frac{E_r}{RT_0}} e^{\frac{E_r}{RT_0}\left(\frac{\varepsilon T_0\theta}{T_0(1+\varepsilon\theta)}\right)}$$

$$e^{-\frac{E_r}{RT}} = e^{-\frac{E_r}{RT_0}} e^{\frac{\theta}{1+\varepsilon\theta}} \tag{3.56}$$

Also

$$e^{-\frac{Q_e}{R}\left(\frac{1}{T}-\frac{1}{T_b}\right)} = e^{-\frac{Q_e}{R}\left(\frac{T_b-T}{T_bT}\right)} = e^{-\frac{Q_e}{R}\left(\frac{T_b-T_0-\varepsilon T_0\theta}{T_b(\varepsilon T_0\theta+T_0)}\right)} = e^{-\frac{Q_e}{R}\left(\frac{T_b-T_0-\varepsilon T_0\theta}{T_bT_0(1+\varepsilon\theta)}\right)} = e^{-\frac{Q_e}{R}\left(\frac{\varepsilon T_0}{T_0T_b}\left(\frac{\frac{T_b-T_0}{\varepsilon T_0}-\theta}{1+\varepsilon\theta}\right)\right)}$$

$$= e^{-\frac{Q_e\varepsilon}{RT_0}\left(\frac{b-\theta}{1+\varepsilon\theta}\right)} = e^{-a\left(\frac{b-\theta}{1+\varepsilon\theta}\right)}$$

That is

$$e^{-\frac{Q_e}{R}\left(\frac{1}{T}-\frac{1}{T_b}\right)} = e^{-a\left(\frac{b-\theta}{1+\varepsilon\theta}\right)} \tag{3.57}$$

Where;

a = $\frac{Q_e\varepsilon}{RT_0}$, b = $\frac{T_b-T_0}{\varepsilon T_0}$

Substituting equation (3.54), equation (3.55), equation (3.56) and equation (3.57) into equation (3.49), that is

$$\frac{\rho c_g \varepsilon T_0}{t_0}\frac{\partial\theta}{\partial t^1} = \frac{\lambda_0}{L}\frac{\partial}{\partial\eta^1}\left(\frac{\varepsilon T_0\theta + T_0}{T_0}\frac{\varepsilon T_0}{L}\frac{\partial\theta}{\partial\eta^1}\right) + Q_r k_r n_{fres} n_f^1\left(\frac{c_g}{\phi(Q_r+\mu_g)}\left(A_1 + \frac{L\eta^1}{L}(B-A_1)+\right.\right.$$

$$\left.\left.\sum_{n=1}^{\infty} B_1 e^{-D\left(\frac{n\pi}{L}\right)^2 t_0 t^1}\sin\frac{n\pi}{L}L\eta^1 - \left(\mu_0 T_0(1+\varepsilon\theta) + \frac{\phi Q_e\mu_0 X_0}{c_g}X^1 + \frac{\phi\mu_0 Z_0}{c_g}Z^1\right)\right)\right) e^{-\frac{Er}{RT_0}} e^{\frac{\theta}{1+\varepsilon\theta}} -$$

$$Q_e k n_{lres} n_l^1\left(\frac{P_{atm}}{\rho R(\varepsilon T_0 + T_0)} e^{-a\left(\frac{b-\theta}{1+\varepsilon\theta}\right)} - X_0 X^1\right)$$

That is

$$\frac{\partial\theta}{\partial t^1} = \frac{\lambda_0 t_0}{\rho c_g L^2}\frac{\partial}{\partial\eta^1}\left((1+\varepsilon\theta)\frac{\partial\theta}{\partial\eta^1}\right) + \frac{t_0 Q_r k_r n_{fres}}{\rho c_g \varepsilon T_0} e^{-\frac{Er}{RT_0}}\left(\frac{c_g}{\phi(Q_r+\mu_0)}(A_1 + (B-A_1)\eta^1 + \right.$$

$$\left.\sum_{n=1}^{\infty} B_1 e^{-\frac{Dt_0}{L^2}n^2\pi^2 t^1}\sin n\pi\eta^1 - \left(\mu_0 T_0(1+\varepsilon\theta) + \frac{\phi Q_e\mu_0 X_0}{c_g}X^1 + \frac{\phi\mu_0 Z_0}{c_g}Z^1\right)\right)\Bigg) n_f^1 e^{\frac{\theta}{1+\varepsilon\theta}} -$$

$$\frac{t_0 Q_e k n_{lres} X_0}{\rho c_g \varepsilon T_0}\left(\frac{P_{atm}}{\rho R T_0 X_0}\frac{e^{-a\left(\frac{b-\theta}{1+\varepsilon\theta}\right)}}{1+\varepsilon\theta} - X^1\right)$$

Dropping prime, we have equation (3.58)

$$\left.\begin{array}{l}\frac{\partial\theta}{\partial t} = \lambda_1\frac{\partial}{\partial\eta}\left((1+\varepsilon\theta)\frac{\partial\theta}{\partial\eta}\right) + \delta\Big(a_1\Big(A_1 + (B-A_1)\eta + \sum_{n=1}^{\infty} B_1 e^{-P_{em}n^2\pi^2 t}\sin n\pi\eta - \\ \big(b_1(1+\varepsilon\theta) + a_2 X + b_2 Z\big)\Big)\Big) n_f e^{\frac{\theta}{1+\varepsilon\theta}} - \alpha\left(a_3\frac{e^{-a\left(\frac{b-\theta}{1+\varepsilon\theta}\right)}}{1+\varepsilon\theta} - X\right)\end{array}\right\} \qquad (3.58)$$

Where;

$$\lambda_1 = \frac{\lambda_0 t_0}{\rho c_g L^2}, \quad \delta = \frac{t_0 Q_r k_r n_{fres}}{\rho c_g \varepsilon T_0} e^{-\frac{Er}{RT_0}}, \quad P_{em} = \frac{Dt_0}{L^2}, \quad a_1 = \frac{c_g}{\phi(Q_r+\mu_0)},$$

$$b_1 = \mu_0 T_0, \quad a_2 = \frac{\phi Q_e \mu_0 X_0}{c_g}, \quad b_2 = \frac{\phi \mu_0 Z_0}{c_g}, \quad \alpha = \frac{t_0 Q_e k n_{lres} X_0}{\rho c_g \varepsilon T_0}, \quad a_3 = \frac{P_{atm}}{\rho R T_0 X_0}$$

Substituting equation (3.54), equation (3.55), equation (3.56) and equation (3.57) into equation (3.50), that is

$$\frac{\phi \rho X_0}{t_0}\frac{\partial X^1}{\partial t^1} = \frac{\phi \rho D_0}{L}\frac{\partial}{\partial \eta^1}\left(\left(\varepsilon T_0 \theta + T_0\right) X_0 \frac{\partial X^1}{\partial \eta^1}\right) + k n_{lres} n_l^1 \left(\frac{P_{atm}}{\rho R\left(\varepsilon T_0 + T_0\right)} e^{-a\left(\frac{b-\theta}{1+\varepsilon\theta}\right)} - X_0 X^1\right)$$

That is

$$\frac{\partial X^1}{\partial t^1} = \frac{t_0 D_0}{L^2}\frac{\partial}{\partial \eta^1}\left(\left(1+\varepsilon\theta\right)\frac{\partial X^1}{\partial \eta^1}\right) + \frac{t_0 k n_{lres}}{\phi \rho} n_l^1 \left(\frac{P_{atm}}{\rho R T_0 X_0}\frac{e^{-a\left(\frac{b-\theta}{1+\varepsilon\theta}\right)}}{1+\varepsilon\theta} - X^1\right)$$

Dropping prime, we have equation (3.59)

$$\frac{\partial X}{\partial t} = D_1 \frac{\partial}{\partial \eta}\left(\left(1+\varepsilon\theta\right)\frac{\partial X}{\partial \eta}\right) + \alpha_1 n_l \left(a_3 \frac{e^{-a\left(\frac{b-\theta}{1+\varepsilon\theta}\right)}}{1+\varepsilon\theta} - X\right) \qquad (3.59)$$

Where;

$$D_1 = \frac{t_0 D_0}{L^2}, \quad \alpha_1 = \frac{t_0 k n_{lres}}{\phi \rho}, \quad a_3 = \frac{P_{atm}}{\rho R T_0 X_0}$$

Substituting equation (3.55), equation (3.56), equation (3.57) and equation (3.58) into equation (3.52), that is

$$\frac{\phi \rho Y_0}{t_0}\frac{\partial Y^1}{\partial t^1} = \frac{\phi \rho D_0}{L}\frac{\partial}{\partial \eta^1}\left(\left(\varepsilon T_0 \theta + T_0\right) Y_0 \frac{\partial Y^1}{\partial \eta^1}\right) - \mu_0 k_r n_{fres} n_f^1 \left(\frac{c_g}{\phi\left(Q_r + \mu_g\right)}\left(A_1 + \frac{L\eta^1}{L}\left(B - A_1\right) + \right.\right.$$

$$\sum_{n=1}^{\infty} B_1 e^{-D\left(\frac{n\pi}{L}\right)^2 t_0 t^1} \sin\frac{n\pi}{L} L\eta^1 - \left(\mu_0 T_0 (1+\varepsilon\theta) + \frac{\phi Q_e \mu_0 X_0}{c_g} X^1 + \frac{\phi \mu_0 Z_0}{c_g} Z^1\right)\Bigg)\Bigg) e^{-\frac{Er}{RT_0}} e^{\frac{\theta}{1+\varepsilon\theta}}$$

That is

$$\frac{\partial Y^1}{\partial t^1} = \frac{t_0 D_0}{L^2}\frac{\partial}{\partial \eta^1}\left((1+\varepsilon\theta)\frac{\partial Y^1}{\partial \eta^1}\right) - \frac{t_0 \mu_0 k_r n_{fres} n_f^1}{\phi \rho Y_0} e^{-\frac{Er}{RT_0}} \left(\frac{c_g}{\phi(Q_r + \mu_0)}(A_1 + (B - A_1)\eta^1 + \right.$$

$$\left.\sum_{n=1}^{\infty} B_1 e^{-\frac{Dt_0}{L^2} n^2 \pi^2 t^1} \sin n\pi\eta^1 - \left(\mu_0 T_0 (1+\varepsilon\theta) + \frac{\phi Q_e \mu_0 X_0}{c_g} X^1 + \frac{\phi \mu_0 Z_0}{c_g} Z^1\right)\Bigg)\right) n_f^1 e^{\frac{\theta}{1+\varepsilon\theta}}$$

Dropping prime, we have equation (3.60)

$$\left.\begin{aligned} &\frac{\partial Y}{\partial t} = D_1 \frac{\partial}{\partial \eta}\left((1+\varepsilon\theta)\frac{\partial Y}{\partial \eta}\right) - \gamma\Big(a_1\Big(A_1 + (B - A_1)\eta + \sum_{n=1}^{\infty} B_1 e^{-P_{em} n^2 \pi^2 t} \sin n\pi\eta - \\ &\big(b_1(1+\varepsilon\theta) + a_2 X + b_2 Z\big)\Big)\Big) n_f\, e^{\frac{\theta}{1+\varepsilon\theta}} \end{aligned}\right\} \qquad (3.60)$$

Where;

$D_1 = \frac{t_0 D_0}{L^2}$, $\gamma = \frac{t_0 \mu_0 k_r n_{fres} n_f^1}{\phi \rho Y_0} e^{-\frac{Er}{RT_0}}$, $a_1 = \frac{c_g}{\phi(Q_r + \mu_0)}$, $b_1 = \mu_0 T_0$, $a_2 = \frac{\phi Q_e \mu_0 X_0}{c_g}$, $b_2 = \frac{\phi \mu_0 Z_0}{c_g}$.

Substituting equation (3.54), equation (3.55), equation (3.56) and equation (3.57) into equation (3.52), that is

$$\frac{\phi \rho Z_0}{t_0}\frac{\partial Z^1}{\partial t^1} = \frac{\phi \rho D_0}{L}\frac{\partial}{\partial \eta^1}\left((\varepsilon T_0 \theta + T_0) Z_0 \frac{\partial Z^1}{\partial \eta^1}\right) + \mu_g k_r n_{fres} n_f^1 \left(\frac{c_g}{\phi(Q_r + \mu_g)}\left(A_1 + \frac{L\eta^1}{L}(B - A_1) + \right.\right.$$

$$\sum_{n=1}^{\infty} B_1 e^{-D\left(\frac{n\pi}{L}\right)^2 t_0 t^1} \sin\frac{n\pi}{L} L\eta^1 - \left(\mu_0 T_0 (1+\varepsilon\theta) + \frac{\phi Q_e \mu_0 X_0}{c_g} X^1 + \frac{\phi \mu_0 Z_0}{c_g} Z^1\right)\Bigg)\Bigg) e^{-\frac{Er}{RT_0}} e^{\frac{\theta}{1+\varepsilon\theta}}$$

That is

$$\frac{\partial Z^1}{\partial t^1} = \frac{t_0 D_0}{L^2}\frac{\partial}{\partial \eta^1}\left(\left(1+\varepsilon\theta\right)\frac{\partial Z^1}{\partial \eta^1}\right) + \frac{t_0 \mu_0 k_r n_{fres} n_f^1}{\phi \rho Z_0} e^{-\frac{Er}{RT_0}}\left(\frac{c_g}{\phi\left(Q_r+\mu_0\right)}\left(A_1 + (B-A_1)\eta^1 + \right.\right.$$

$$\left.\left.\sum_{n=1}^{\infty} B_1 e^{-\frac{Dt_0}{L^2}n^2\pi^2 t^1} \sin n\pi\eta^1 - \left(\mu_0 T_0\left(1+\varepsilon\theta\right) + \frac{\phi Q_e \mu_0 X_0}{c_g}X^1 + \frac{\phi\mu_0 Z_0}{c_g}Z^1\right)\right)\right) n_f^1 e^{\frac{\theta}{1+\varepsilon\theta}}$$

Dropping prime, we have equation (3.61)

$$\left.\begin{aligned}&\frac{\partial Z}{\partial t} = D_1 \frac{\partial}{\partial \eta}\left(\left(1+\varepsilon\theta\right)\frac{\partial Z}{\partial \eta}\right) + \gamma_1\left(a_1\left(A_1 + (B-A_1)\eta + \sum_{n=1}^{\infty} B_1 e^{-P_{em}n^2\pi^2 t}\sin n\pi\eta - \right.\right.\\&\left.\left.\left(b_1\left(1+\varepsilon\theta\right) + a_2 X + b_2 Z\right)\right)\right) n_f\, e^{\frac{\theta}{1+\varepsilon\theta}}\end{aligned}\right\} \qquad (3.61)$$

Where;

$$D_1 = \frac{t_0 D_0}{L^2}, \quad \gamma_1 = \frac{t_0 \mu_g k_r n_{fres}}{\phi \rho Y_0} e^{-\frac{Er}{RT_0}}, \quad a_1 = \frac{c_g}{\phi\left(Q_r + \mu_0\right)}, \quad b_1 = \mu_0 T_0, \quad a_2 = \frac{\phi Q_e \mu_0 X_0}{c_g},$$

$$b_2 = \frac{\phi \mu_0 Z_0}{c_g}.$$

Substituting equation (3.54), equation (3.55), equation (3.56) and equation (3.57) into equation (3.53), that is

$$\frac{n_{fres}}{t_0}\frac{\partial n_f^1}{\partial t^1} = \mu_f k_r n_{fres} n_f^1 \left(\frac{c_g}{\phi\left(Q_r+\mu_0\right)}\left(A_1 + (B-A_1)\eta^1 + \sum_{n=1}^{\infty} B_1 e^{-D\left(\frac{n\pi}{L}\right)^2 t_0 t^1}\sin\frac{n\pi}{L}L\eta^1 - \right.\right.$$

$$\left.\left.\left(\mu_0 T_0\left(1+\varepsilon\theta\right) + \frac{\phi Q_e \mu_0 X_0}{c_g}X^1 + \frac{\phi\mu_0 Z_0}{c_g}Z^1\right)\right)\right) e^{-\frac{Er}{RT_0}} e^{\frac{\theta}{1+\varepsilon\theta}}$$

That is

$$\frac{\partial n_f^1}{\partial t^1} = t_0 \mu_f k_r e^{-\frac{Er}{RT_0}} n_f^1 \left(\frac{c_g}{\phi(Q_r + \mu_0)} (A_1 + (B - A_1)\eta^1 + \sum_{n=1}^{\infty} B_1 e^{-\frac{Dt_0}{L^2} n^2 \pi^2 t^1} \sin n\pi\eta^1 - \right.$$

$$\left. \left(\mu_0 T_0 (1 + \varepsilon\theta) + \frac{\phi Q_e \mu_0 X_0}{c_g} X^1 + \frac{\phi \mu_0 Z_0}{c_g} Z^1 \right)\right)\Bigg) e^{\frac{\theta}{1+\varepsilon\theta}}$$

Dropping prime, we have equation (3.62)

$$\left. \begin{array}{l} \frac{\partial n_f}{\partial t} = \gamma_2 n_f \left(a_1 (A_1 + (B - A_1)\eta + \right. \\ \sum_{n=1}^{\infty} B_1 e^{-P_{em} n^2 \pi^2 t} \sin n\pi\eta - \left(b_1 (1 + \varepsilon\theta) + a_2 X + b_2 Z \right)) \Big) e^{\frac{\theta}{1+\varepsilon\theta}} \end{array} \right\} \quad (3.62)$$

Where;

$$D_1 = \frac{t_0 D_0}{L^2}, \quad \gamma_2 = t_0 \mu_f k_r e^{-\frac{Er}{RT_0}}, \quad a_1 = \frac{c_g}{\phi(Q_r + \mu_0)}, \quad b_1 = \mu_0 T_0, \quad a_2 = \frac{\phi Q_e \mu_0 X_0}{c_g},$$

$$b_2 = \frac{\phi \mu_0 Z_0}{c_g}.$$

Substituting equation (3.54), equation (3.55), equation (3.56) and equation (3.57) into equation (3.6), that is

$$\frac{n_{lres}}{t_0} \frac{\partial n_l^1}{\partial t^1} = -k n_{lres} n_l^1 \left(\frac{P_{atm}}{\rho R (\varepsilon T_0 + T_0)} e^{-a\left(\frac{b-\theta}{1+\varepsilon\theta}\right)} - X_0 X^1 \right)$$

That is

$$\frac{\partial n_l^1}{\partial t^1} = -t_0 k n_l^1 X_0 \left(\frac{P_{atm}}{\rho R T_0 X_0} \frac{e^{-a\left(\frac{b-\theta}{1+\varepsilon\theta}\right)}}{1 + \varepsilon\theta} - X^1 \right)$$

Dropping prime, we have equation (3.63)

$$\frac{\partial n_l}{\partial t} = -\gamma_3 n_l \left(\frac{P_{atm}}{\rho R T_0 X_0} \frac{e^{-a\left(\frac{b-\theta}{1+\varepsilon\theta}\right)}}{1+\varepsilon\theta} - X \right) \tag{3.63}$$

Where; $\gamma_3 = t_0 k X_0$, $a_3 = \dfrac{P_{atm}}{\rho R T_0 X_0}$

Substituting equation (3.55), equation (3.56), equation (3.57) and equation (3.58) into equation (3.15), that is

$$T = \frac{RT_0^2}{E}\left(1 - \frac{\eta}{L}\right) + T_0$$

That is

$$\varepsilon T_0 \theta + T_0 = \varepsilon T_0 \left(1 - \frac{L\eta^1}{L}\right) + T_0 \Rightarrow \theta = \left(1 - \eta^1\right)$$

Dropping prime, we have equation (3.64)

$$\theta(\eta, 0) = 1 - \eta \tag{3.64}$$

Also

$$X = X_0 \left(1 - \frac{\eta}{L}\right)$$

That is

$$X_0 X^1 = X_0 \left(1 - \frac{L\eta^1}{L}\right) \qquad \Rightarrow \quad X^1 = \left(1 - \eta^1\right)$$

Dropping prime, we have equation (3.65)

$$X(\eta,0)=1-\eta \tag{3.65}$$

Also

$$Y=Y_0\left(1-\frac{\eta}{L}\right)$$

That is

$$Y_0Y^1=Y_0\left(1-\frac{L\eta^1}{L}\right) \qquad \Rightarrow \quad Y^1=\left(1-\eta^1\right)$$

Dropping prime, we have equation (3.66)

$$Y(\eta,0)=1-\eta \tag{3.66}$$

Also

$$Z=Z_0\left(1-\frac{\eta}{L}\right)$$

That is

$$Z_0Z^1=Z_0\left(1-\frac{L\eta^1}{L}\right) \qquad \Rightarrow \quad Z^1=\left(1-\eta^1\right)$$

Dropping prime, we have equation (3.67)

$$Z(\eta,0)=1-\eta \tag{3.67}$$

Also

$$n_f=n_{fres}$$

That is

$$n_{fres} n_f^1 = n_{fres} \qquad \Rightarrow \quad n_f^1 = 1$$

Dropping prime, we have equation (3.68)

$$n_f = 1 \qquad (3.68)$$

And

$$n_l = n_{lres}$$

That is

$$n_{lres} n_l^1 = n_{lres} \qquad \Rightarrow \quad n_l^1 = 1$$

Dropping prime, we have equation (3.69)

$$n_l = 1 \qquad (3.69)$$

Substituting Equation (3.55), equation (3.56), equation (3.57) and equation (3.58) into equation (3.15), that is

$$T\big|_{\eta=0} = T_1$$

That is

$$\varepsilon T_0 \theta + T_0\big|_{L\eta^1=0} = T_1 \qquad \Rightarrow \quad \theta\big|_{\eta^1=0} = \frac{T_1 - T_0}{\varepsilon T_0}$$

Dropping prime, we have equation (3.70)

$$\theta(0,t) = b_3 \qquad (3.70)$$

Where; $b_3 = \dfrac{T_1 - T_0}{\varepsilon T_0}$

And

$$T\big|_{\eta=L} = T_0$$

That is

$$\varepsilon T_0 \theta + T_0\big|_{L\eta^1=L} = T_0 \qquad \Rightarrow \quad \theta\big|_{\eta^1=1} = 0$$

Dropping prime, we have equation (3.71)

$$\theta(1,t) = 0 \qquad (3.71)$$

Also

$$Y\big|_{\eta=0} = Y_{inj}$$

That is

$$Y_{inj}Y^1\big|_{L\eta^1=0} = Y_{inj} \qquad \Rightarrow \quad Y^1\big|_{\eta^1=0} = 1$$

Dropping prime, we have equation (3.72)

$$Y(0,t) = 1 \qquad (3.72)$$

And

$$Y\big|_{\eta=L} = 0$$

That is

$$Y_0 Y^1\Big|_{L\eta^1=L} = 0 \qquad \Rightarrow \; Y^1\Big|_{\eta^1=1} = 0$$

Dropping prime, we have equation (3.73)

$$Y(1,t) = 0 \tag{3.73}$$

Also

$$X_{\eta=0} = 0$$

That is

$$X_0 X^1\Big|_{L\eta^1=0} = 0 \qquad \Rightarrow \; X^1\Big|_{\eta^1=0} = 0$$

Dropping prime, we have equation (3.74)

$$X(0,t) = 0 \tag{3.74}$$

And

$$X\Big|_{\eta=L} = 0$$

That is

$$X_0 X^1\Big|_{L\eta^1=L} = 0 \qquad \Rightarrow \; X^1\Big|_{\eta^1=1} = 0$$

Dropping prime, we have equation (3.75)

$$X(1,t) = 0 \tag{3.75}$$

also

$$Z|_{\eta=0} = 0$$

That is

$$Z_0 Z^1\big|_{L\eta^1=0} = 0 \qquad \Rightarrow \; Z^1\big|_{\eta^1=0} = 0$$

Dropping prime, we have equation (3.76)

$$Z(0,t) = 0 \tag{3.76}$$

And

$$Z|_{\eta=L} = 0$$

That is

$$Z_0 Z^1\big|_{L\eta^1=L} = 0 \qquad \Rightarrow \; Z^1\big|_{\eta^1=1} = 0$$

Dropping prime, we have equation (3.77)

$$Z(1,t) = 0 \tag{3.77}$$

Therefore, the dimensionless equations together with initial and boundary conditions are giving as equation (3.78) to equation (3.84):

$$\left.\begin{array}{l}\dfrac{\partial\theta}{\partial t} = \lambda_1 \dfrac{\partial}{\partial\eta}\left((1+\varepsilon\theta)\dfrac{\partial\theta}{\partial\eta}\right) + \delta\Big(a_1\Big(A_1 + (B - A_1)\eta + \displaystyle\sum_{n=1}^{\infty} B_1 e^{-P_{em}n^2\pi^2 t}\sin n\pi\eta - \\ \big(b_1(1+\varepsilon\theta) + a_2X + b_2Z\big)\Big)\Big)n_f\, e^{\frac{\theta}{1+\varepsilon\theta}} - \alpha\left(a_3\dfrac{e^{-a\left(\frac{b-\theta}{1+\varepsilon\theta}\right)}}{1+\varepsilon\theta} - X\right)\end{array}\right\} \tag{3.78}$$

$$\frac{\partial X}{\partial t} = D_1 \frac{\partial}{\partial \eta}\left((1+\varepsilon\theta)\frac{\partial X}{\partial \eta} \right) + \alpha_1 n_l \left(a_3 \frac{e^{-a\left(\frac{b-\theta}{1+\varepsilon\theta}\right)}}{1+\varepsilon\theta} - X \right) \tag{3.79}$$

$$\left.\begin{aligned} &\frac{\partial Y}{\partial t} = D_1 \frac{\partial}{\partial \eta}\left((1+\varepsilon\theta)\frac{\partial Y}{\partial \eta} \right) - \gamma\Big(a_1\big(A_1 + (B-A_1)\eta + \sum_{n=1}^{\infty} B_1 e^{-P_{em} n^2 \pi^2 t} \sin n\pi\eta - \\ &\big(b_1(1+\varepsilon\theta) + a_2 X + b_2 Z\big)\big)\Big) n_f\, e^{\frac{\theta}{1+\varepsilon\theta}} \end{aligned}\right\} \tag{3.80}$$

$$\left.\begin{aligned} &\frac{\partial Z}{\partial t} = D_1 \frac{\partial}{\partial \eta}\left((1+\varepsilon\theta)\frac{\partial Z}{\partial \eta} \right) + \gamma_1\Big(a_1\big(A_1 + (B-A_1)\eta + \sum_{n=1}^{\infty} B_1 e^{-P_{em} n^2 \pi^2 t} \sin n\pi\eta - \\ &\big(b_1(1+\varepsilon\theta) + a_2 X + b_2 Z\big)\big)\Big) n_f\, e^{\frac{\theta}{1+\varepsilon\theta}} \end{aligned}\right\} \tag{3.81}$$

$$\left.\begin{aligned} &\frac{\partial n_f}{\partial t} = \gamma_2 n_f \Big(a_1\big(A_1 + (B-A_1)\eta + \\ &\sum_{n=1}^{\infty} B_1 e^{-P_{em} n^2 \pi^2 t} \sin n\pi\eta - \big(b_1(1+\varepsilon\theta) + a_2 X + b_2 Z\big)\big)\Big) e^{\frac{\theta}{1+\varepsilon\theta}} \end{aligned}\right\} \tag{3.82}$$

$$\frac{\partial n_l}{\partial t} = -\gamma_3 n_l \left(a_3 \frac{e^{-a\left(\frac{b-\theta}{1+\varepsilon\theta}\right)}}{1+\varepsilon\theta} - X \right) \tag{3.83}$$

$$\left.\begin{aligned} &\theta(\eta,0) = 1-\eta, \quad \theta(0,t) = b_3, \quad \theta(1,t) = 0 \\ &X(\eta,0) = 1-\eta, \quad X(0,t) = 0, \quad X(1,t) = 0 \\ &Y(\eta,0) = 1-\eta, \quad Y(0,t) = 1, \quad Y(1,t) = 0 \\ &Z(\eta,0) = 1-\eta, \quad Z(0,t) = 0, \quad Z(1,t) = 0 \\ &n_f = 1 \\ &n_l = 1 \end{aligned}\right\} \tag{3.84}$$

3.3.3 Properties of Solution

To examine the properties of solution, we consider the following asymptotic expansion of temperature of θ and concentrations X, Y, Z, n_f *and* n_l *in* ε, as shown in equation (3.85) .

Let

$$\left.\begin{array}{l}\theta = \theta_0 + \varepsilon\theta_1 + \dots \\ X = X_0 + \varepsilon X_1 + \dots \\ Y = Y_0 + \varepsilon Y_1 + \dots \\ Z = Z_0 + \varepsilon Z_1 + \dots \\ n_f = n_{f0} + \varepsilon n_{f1} + \dots \\ n_l = n_{l0} + \varepsilon n_{l1} + ..\end{array}\right\}. \tag{3.85}$$

Then, we have equation (3.86)

$$\frac{e^{-a\left(\frac{b-\theta}{1+\varepsilon\theta}\right)}}{1+\varepsilon\theta} = e^{-a\left(\frac{b-\theta}{1+\varepsilon\theta}\right)} \times \frac{1}{1+\varepsilon\theta} = e^{-\frac{ab}{1+\varepsilon\theta}} \times e^{\frac{a\theta}{1+\varepsilon\theta}} \times \frac{1}{1+\varepsilon\theta}$$

$$\frac{1}{1+\varepsilon\theta} = (1+\varepsilon\theta)^{-1} \approx 1 - \varepsilon\theta + \dots$$

$$= 1 - \varepsilon(\theta_0 + \varepsilon\theta_1) \approx 1 - \varepsilon\theta_0$$

$$e^{\frac{a\theta}{1+\varepsilon\theta}} = \left(e^{\frac{\theta}{1+\varepsilon\theta}}\right)^a = \left(e^{\theta_0} + \varepsilon\theta_1 e^{\theta_0} + \dots\right)^a = \left(e^{a\theta_0} + \varepsilon a e^{2\theta_0}\theta_1 + \dots\right)$$

$$e^{-\frac{ab}{1+\varepsilon\theta}} = -ab(1+\varepsilon\theta)^{-1} \approx -ab(1-\varepsilon\theta_0)$$

Now,

$$\frac{e^{-a\left(\frac{b-\theta}{1+\varepsilon\theta}\right)}}{1+\varepsilon\theta} = -ab(1-\varepsilon\theta_0)\left(e^{a\theta_0} + \varepsilon a e^{2\theta_0}\theta_1 + \dots\right)(1-\varepsilon\theta_0)$$

$$\frac{e^{-a\left(\frac{b-\theta}{1+\varepsilon\theta}\right)}}{1+\varepsilon\theta} = (1-\varepsilon\theta_0)\left(-ab e^{a\theta_0} - a^2 b\varepsilon\theta_1 e^{2\theta_0} + ab\varepsilon\theta_0 e^{a\theta_0} + a^2 b\varepsilon^2\theta_0\theta_1 e^{2\theta_0} + \dots\right) \tag{3.86}$$

Substituting equation (3.85) and equation (3.86) into equation (3.78) to (3.81), we have equation (3.87) to equation (3.90)

$$\left.\begin{aligned}&\frac{\partial}{\partial t}(\theta_0+\varepsilon\theta_1)=\lambda_1(1+\varepsilon(\theta_0+\varepsilon\theta_1))\frac{\partial^2}{\partial\eta^2}(\theta_0+\varepsilon\theta_1)+\lambda_1\varepsilon\left(\frac{\partial}{\partial\eta}(\theta_0+\varepsilon\theta_1)\right)^2+\\&\delta\Big(a_1(A_1+(B-A_1)\eta+\sum_{n=1}^{\infty}B_1 e^{-P_{em}n^2\pi^2 t}\sin n\pi\eta\\&-(b_1(1+\varepsilon(\theta_0+\varepsilon\theta_1))+a_2(X_0+\varepsilon X_1)+b_2(Z_0+\varepsilon Z_1))))(n_{f0}+\varepsilon n_{f1})(e^{\theta_0}+\varepsilon\theta_1 e^{\theta_0}\ldots)\\&-\alpha(n_{l0}+\varepsilon n_{l1})(a_3(-ab e^{a\theta_0}-a^2b\varepsilon\theta_1 e^{2\theta_0}+ab\varepsilon\theta_0 e^{a\theta_0}+\ldots)(1-\varepsilon\theta_0)-(X_0+\varepsilon X_1))\Big)\end{aligned}\right\}\quad(3.87)$$

$$\left.\begin{aligned}&\frac{\partial}{\partial t}(X_0+\varepsilon X_1)=\\&D_1(1+\varepsilon(\theta_0+\varepsilon\theta_1))\frac{\partial^2}{\partial\eta^2}(X_0+\varepsilon X_1)+D_1\varepsilon\left(\frac{\partial}{\partial\eta}(\theta_0+\varepsilon\theta_1)\frac{\partial}{\partial\eta}(X_0+\varepsilon X_1)\right)+\\&\alpha_1(n_{l0}+\varepsilon n_{l1})(a_3(-ab e^{a\theta_0}-a^2b\varepsilon\theta_1 e^{2\theta_0}+ab\varepsilon\theta_0 e^{a\theta_0}+\ldots)(1-\varepsilon\theta_0)-(X_0+\varepsilon X_1))\end{aligned}\right\}\quad(3.88)$$

$$\left.\begin{aligned}&\frac{\partial}{\partial t}(Y_0+\varepsilon Y_1)=\\&D_1(1+\varepsilon(\theta_0+\varepsilon\theta_1))\frac{\partial^2}{\partial\eta^2}(Y_0+\varepsilon Y_1)+D_1\varepsilon\left(\frac{\partial}{\partial\eta}(\theta_0+\varepsilon\theta_1)\frac{\partial}{\partial\eta}(Y_0+\varepsilon Y_1)\right)-\\&\gamma(a_1(A_1+(B-A_1)\eta+\sum_{n=1}^{\infty}B_1 e^{-P_{em}n^2\pi^2 t}\sin n\pi\eta-\\&(b_1(1+\varepsilon(\theta_0+\varepsilon\theta_1))+a_2(X_0+\varepsilon X_1)+b_2(Z_0+\varepsilon Z_1))))(n_{f0}+\varepsilon n_{f1})(e^{\theta_0}+\varepsilon\theta_1 e^{\theta_0}\ldots)\end{aligned}\right\}\quad(3.89)$$

$$\left.\begin{aligned}&\frac{\partial}{\partial t}(Z_0+\varepsilon Z_1)=\\&D_1(1+\varepsilon(\theta_0+\varepsilon\theta_1))\frac{\partial^2}{\partial\eta^2}(Z_0+\varepsilon Z_1)+D_1\varepsilon\left(\frac{\partial}{\partial\eta}(\theta_0+\varepsilon\theta_1)\frac{\partial}{\partial\eta}(Z_0+\varepsilon Z_1)\right)+\\&\gamma_1(a_1(A_1+(B-A_1)\eta+\sum_{n=1}^{\infty}B_1 e^{-P_{em}n^2\pi^2 t}\sin n\pi\eta-\\&(b_1(1+\varepsilon(\theta_0+\varepsilon\theta_1))+a_2(X_0+\varepsilon X_1)+b_2(Z_0+\varepsilon Z_1))))(n_{f0}+\varepsilon n_{f1})(e^{\theta_0}+\varepsilon\theta_1 e^{\theta_0}\ldots)\end{aligned}\right\}\quad(3.90)$$

Collecting like power of ε^0 and ε^1 in equation (3.87) to equation (3.90), we have equation (3.91) to equation (3.100):

ε^0 :

$$\begin{cases} \dfrac{\partial \theta_0}{\partial t} = \lambda_1 \dfrac{\partial^2 \theta_0}{\partial \eta^2} + \delta\Big(a_1\Big(A_1 + (B - A_1)\eta + \sum_{n=1}^{\infty} B_1 e^{-P_{em} n^2 \pi^2 t} \sin n\pi\eta \\ -(b_1 + a_2 X_0 + b_2 Z_0)\Big)\Big) n_{f0} e^{\theta_0} - \alpha n_{l0}\left(-aba_3 e^{a\theta_0} - X_0\right) \end{cases} \tag{3.91}$$

$$\frac{\partial X_0}{\partial t} = D_1 \frac{\partial^2 X_0}{\partial \eta^2} + \alpha_1 n_{l0}\left(-aba_3 e^{a\theta_0} - X_0\right) \tag{3.92}$$

$$\left. \begin{array}{l} \dfrac{\partial Y_0}{\partial t} = D_1 \dfrac{\partial^2 Y_0}{\partial \eta^2} - \\ \gamma\left(a_1\left(A_1 + (B - A_1)\eta + \sum_{n=1}^{\infty} B_1 e^{-P_{em} n^2 \pi^2 t} \sin n\pi\eta - (b_1 + a_2 X_0 + b_2 Z_0)\right)\right) n_{f0} e^{\theta_0} \end{array} \right\} \tag{3.93}$$

$$\left. \begin{array}{l} \dfrac{\partial Z_0}{\partial t} = D_1 \dfrac{\partial^2 Z_0}{\partial \eta^2} + \\ \gamma_1\left(a_1\left(A_1 + (B - A_1)\eta + \sum_{n=1}^{\infty} B_1 e^{-P_{em} n^2 \pi^2 t} \sin n\pi\eta - (b_1 + a_2 X_0 + b_2 Z_0)\right)\right) n_{f0} e^{\theta_0} \end{array} \right\} \tag{3.94}$$

$$\left. \begin{array}{lll} \theta_0(\eta,0) = 1 - \eta, & \theta_0(0,t) = b_3, & \theta_0(1,t) = 0 \\ X_0(\eta,0) = 1 - \eta, & X_0(0,t) = 0, & X_0(1,t) = 0 \\ Y_0(\eta,0) = 1 - \eta, & Y_0(0,t) = 1, & Y_0(1,t) = 0 \\ Z_0(\eta,0) = 1 - \eta, & Z_0(0,t) = 0, & Z_0(1,t) = 0 \\ n_{f0} = 1 & & \\ n_{l0} = 1 & & \end{array} \right\} \tag{3.95}$$

ε^1 :

$$\left. \begin{array}{l} \dfrac{\partial \theta_1}{\partial t} = \lambda_1 \theta_0 \dfrac{\partial^2 \theta_0}{\partial \eta^2} + \lambda_1 \left(\dfrac{\partial \theta_0}{\partial \eta}\right)^2 - (b_1\theta_0 + a_2 X_1 + b_2 Z_1)))(n_{f1})(\theta_1 e^{\theta_0}) - \\ \alpha(n_{l1})\left(2aba_3\theta_0 e^{a\theta_0} - a^2 ba_1\theta_0 e^{2\theta_0} - X_1\right) \end{array} \right\} \tag{3.96}$$

$$\frac{\partial X_1}{\partial t} = D_1 \frac{\partial^2 \theta_0}{\partial \eta^2} + D_1\left(\frac{\partial \theta_0}{\partial \eta}\frac{\partial X_0}{\partial \eta}\right) + \alpha_1 n_{l1}\left(2aba_3\theta_0 e^{a\theta_0} - a^2 ba_1\theta_0 e^{2\theta_0} - X_1\right)\Bigg\} \qquad (3.97)$$

$$\frac{\partial Y_1}{\partial t} = D_1\theta_0 \frac{\partial^2 Y_0}{\partial \eta^2} + D_1\left(\frac{\partial \theta_0}{\partial \eta}\frac{\partial Y_0}{\partial \eta}\right) - \left(b_1\theta_0 + a_2 X_1 + b_2 Z_1)\right)\left(n_{f1}\right)\left(\theta_1 e^{\theta_0}\right)\Bigg\} \qquad (3.98)$$

$$\frac{\partial Z_1}{\partial t} = D_1\theta_0 \frac{\partial^2 Z_0}{\partial \eta^2} + D_1\left(\frac{\partial \theta_0}{\partial \eta}\frac{\partial Z_0}{\partial \eta}\right) - \left(b_1\theta_0 + a_2 X_1 + b_2 Z_1)\right)\left(n_{f1}\right)\left(\theta_1 e^{\theta_0}\right)\Bigg\} \qquad (3.99)$$

$$\left.\begin{array}{lll} \theta_1(\eta,0) = 1-\eta, & \theta_1(0,t) = b_3, & \theta_1(1,t) = 0 \\ X_1(\eta,0) = 1-\eta, & X_1(0,t) = 0, & X_1(1,t) = 0 \\ Y_1(\eta,0) = 1-\eta, & Y_1(0,t) = 1, & Y_1(1,t) = 0 \\ Z_1(\eta,0) = 1-\eta, & Z_1(0,t) = 0, & Z_1(1,t) = 0 \\ n_{f_1} = 1 & & \\ n_{l_1} = 1 & & \end{array}\right\} \qquad (3.100)$$

This question of existence and uniqueness of solutions to these equations has been addressed by Ayeni (1978) who consider a similar set of equations and showed among other results that existence and uniqueness are somewhat well known. In his work, he studied the following system of parabolic equation (3.101)

$$\left.\begin{array}{ll} \dfrac{\partial \phi}{\partial t} = \Delta\phi + f\left(x,t,\phi,u,v\right) & x \in R^n,\ t > 0 \\ \dfrac{\partial u}{\partial t} = \Delta u + g\left(x,t,\phi,u,v\right) & x \in R^n,\ t > 0 \\ \dfrac{\partial v}{\partial t} = \Delta v + f\left(x,t,\phi,u,v\right) & x \in R^n,\ t > 0 \end{array}\right\} \qquad (3.101)$$

$\phi(x,0) = f_0(x)$

$u(x,0) = g_0(x)$

$v(x,0) = h_0(x)$

$x = (x_1, x_2, x_3, \ldots, x_n)$

(S.1): $f_0(x), g(x)$ and $h_0(x)$ are bounded for $x \in R^n$. Each has at most countable number of discontinuities.

(S.2): f, g, h satisfies the uniform Lipschitz condition

$$|\varphi(x,t,\phi_1,u_1,v_1) - \varphi(x,t,\phi_2,u_2,v_2)| \le M(|\phi_1 - \phi_2| + |u_1 - u_2| + |v_1 - v_2|),\ (x,t) \in G$$

Where;

$$G = \{(x,t) : x \in R^n, 0 < t < \tau\}.$$

Our proof of existence of unique solution of the system of parabolic equation (3.91) to equation (3.95) will be analogous to his proof.

Theorem 3.2: There exists a unique solution $\theta_0(\eta,t),\ X_0(\eta,t), Y_0(\eta,t), and\ Z_0(\eta,t)$ of equation (3.91), equation (3.92), equation (3.93) and equation (3.94) which satisfy equation (3.95).

In the proof we shall need the following Lemma:

Lemma 3.3 (Ayeni (1978))

Let $(f_{0,}\ g_{0,}\ h_{0,}\ j_0)$ and $(f_{,}\ g_{,}\ h_{,}\ j\ )$ satisfy $(S.1)$ *and* $(S.2)$ respectively. Then there exists a solution of equation (3.91), equation (3.92), equation (3.93) and equation (3.94).

Proof of Lemma: see Ayeni (1978)

Proof of theorem 3.2

We rewrite equation (3.91), equation (3.92), equation (3.93) and equation (3.94) as equation (3.102) to equation (3.105)

$$\frac{\partial \theta_0}{\partial t} = \lambda_1 \frac{\partial^2 \theta_0}{\partial \eta^2} + f(\eta,\, t,\, \theta_0,\, X_0, Y_0,\, Z_0) \qquad \eta \in R^n,\ \ t > 0 \tag{3.102}$$

$$\frac{\partial X_0}{\partial t} = D_1 \frac{\partial^2 X_0}{\partial \eta^2} + g(\eta,\, t,\, \theta_0,\, X_0, Y_0,\, Z_0) \qquad \eta \in R^n,\ \ t > 0 \tag{3.103}$$

$$\frac{\partial Y_0}{\partial t} = D_1 \frac{\partial^2 Y_0}{\partial \eta^2} + h(\eta,\, t,\, \theta_0,\, X_0, Y_0,\, Z_0) \qquad \eta \in R^n,\ \ t > 0 \tag{3.104}$$

$$\frac{\partial Z_0}{\partial t} = D_1 \frac{\partial^2 Z_0}{\partial \eta^2} + j(\eta,\, t,\, \theta_0,\, X_0, Y_0,\, Z_0) \qquad \eta \in R^n,\ \ t > 0 \tag{3.105}$$

Where;

$$f(\eta,\, t,\, \theta_0,\, X_0, Y_0,\, Z_0) = \delta\left(a_1\left(A_1 + (B - A_1)\eta + \sum_{n=1}^{\infty} B_1\, e^{-P_{em} n^2 \pi^2 t} \sin n\pi\eta - (b_1 + a_2 X_0 + b_2 Z_0)\right)\right) n_{f0}\, e^{\theta_0} - \alpha n_{l0}\left(-aba_3\, e^{a\theta_0} - X_0\right).$$

$$g(\eta,\, t,\, \theta_0,\, X_0, Y_0,\, Z_0) = \alpha_2 n_{l0}\left(-aba_3\, e^{a\theta_0} - X_0\right).$$

$$h(\eta,\, t,\, \theta_0,\, X_0, Y_0,\, Z_0) = -\gamma\left(a_1\left(A_1 + (B - A_1)\eta + \sum_{n=1}^{\infty} B_1\, e^{-P_{em} n^2 \pi^2 t} \sin n\pi\eta - (b_1 + a_2 X_0 + b_2 Z_0)\right)\right) n_{f0}\, e^{\theta_0}$$

$$j(\eta,\, t,\, \theta_0,\, X_0,\, Y_0,\, Z_0) = \gamma_2\left(a_1\left(A_1 + (B - A_1)\eta + \sum_{n=1}^{\infty} B_1\, e^{-P_{em} n^2 \pi^2 t} \sin n\pi\eta - (b_1 + a_2 X_0 + b_2 Z_0)\right)\right) n_{f0}\, e^{\theta_0}$$

Ignoring the second term at the right hand side, the fundamental solution of equation equation (3.91), equation (3.92), equation (3.93) and equation (3.94) are (see Toki and Tokis (2007)).

$$F(\eta, t) = \frac{\eta}{2\lambda_1 \pi^{1/2} t^{3/2}}\, e^{-\frac{\eta^2}{4\lambda_1 t}}$$

$$G(\eta, t) = \frac{\eta}{2D_1 \pi^{1/2} t^{3/2}}\, e^{-\frac{\eta^2}{4D_1 t}}$$

$$H(\eta, t) = \frac{\eta}{2D_1 \pi^{1/2} t^{3/2}}\, e^{-\frac{\eta^2}{4D_1 t}}$$

$$J(\eta, t) = \frac{\eta}{2D_1 \pi^{1/2} t^{3/2}}\, e^{-\frac{\eta^2}{4D_1 t}}$$

Clearly,

$$f(\eta,\, t,\, \theta_0,\, X_0,\, Y_0,\, Z_0) = \delta\left(a_1\left(A_1 + (B - A_1)\eta + \sum_{n=1}^{\infty} B_1\, e^{-P_{em} n^2 \pi^2 t} \sin n\pi\eta - (b_1 + a_2 X_0 + b_2 Z_0)\right)\right) n_{f0}\, e^{\theta_0} -$$

$$\alpha n_{l0}\left(-aba_3\, e^{a\theta_0} - X_0\right),\ g(\eta,\, t,\, \theta_0,\, X_0,\, Y_0,\, Z_0) = \alpha_2 n_{l0}\left(-aba_3\, e^{a\theta_0} - X_0\right),$$

$$h(\eta,\, t,\, \theta_0,\, X_0,\, Y_0,\, Z_0) = -\gamma\left(a_1\left(A_1 + (B - A_1)\eta + \sum_{n=1}^{\infty} B_1\, e^{-P_{em} n^2 \pi^2 t} \sin n\pi\eta - (b_1 + a_2 X_0 + b_2 Z_0)\right)\right) n_{f0}\, e^{\theta_0}$$

and

$$j(\eta,\, t,\, \theta_0,\, X_0,\, Y_0,\, Z_0) = \gamma_2\left(a_1\left(A_1 + (B - A_1)\eta + \sum_{n=1}^{\infty} B_1\, e^{-P_{em} n^2 \pi^2 t} \sin n\pi\eta - (b_1 + a_2 X_0 + b_2 Z_0)\right)\right) n_{f0}\, e^{\theta_0}$$

are lipschitz continues. Hence by Lemma 3.1, the result follows. This completes the proof.

3.3.4 Analytical Solution

Ayeni (1982) has shown that $e^{\frac{\theta}{1+\varepsilon\theta}}$ can be approximated as $1+(e-2)\theta+\theta^2$. For convenience, we assume an approximation as giving in equation (3.106)

$$e^{\frac{\theta}{1+\varepsilon\theta}} \approx 1+(e-2)\theta \tag{3.106}$$

Substituting equation (3.106) into equation (3.78) to equation (3.83), we have equation (3.107) to equation (3.114)

$$\left.\begin{aligned}&\frac{\partial\theta}{\partial t}=\lambda_1\frac{\partial}{\partial\eta}\left((1+\varepsilon\theta)\frac{\partial\theta}{\partial\eta}\right)+\delta\Big(a_1\Big(A_1+(B-A_1)\eta+\sum_{n=1}^{\infty}B_1 e^{-P_{em}n^2\pi^2t}\sin n\pi\eta-\\&(b_1(1+\varepsilon\theta)+a_2X+b_2Z)\Big)\Big)n_f\left(1+(e-2)\theta\right)-\\&\alpha n_l\left(a_3(1-\varepsilon\theta)a(1+(e-2)\theta)(-ab(1-\varepsilon\theta))-X\right)\end{aligned}\right\} \tag{3.107}$$

$$\frac{\partial X}{\partial t}=D_1\frac{\partial}{\partial\eta}\left((1+\varepsilon\theta)\frac{\partial X}{\partial\eta}\right)+\alpha_1 n_l\left(a_3(1-\varepsilon\theta)a(1+(e-2)\theta)(-ab(1-\varepsilon\theta))-X\right) \tag{3.108}$$

$$\left.\begin{aligned}&\frac{\partial Y}{\partial t}=D_1\frac{\partial}{\partial\eta}\left((1+\varepsilon\theta)\frac{\partial Y}{\partial\eta}\right)-\gamma\Big(a_1\Big(A_1+(B-A_1)\eta+\sum_{n=1}^{\infty}B_1 e^{-P_{em}n^2\pi^2t}\sin n\pi\eta-\\&(b_1(1+\varepsilon\theta)+a_2X+b_2Z)\Big)\Big)n_f\left(1+(e-2)\theta\right)\end{aligned}\right\} \tag{3.109}$$

$$\left.\begin{aligned}&\frac{\partial Z}{\partial t}=D_1\frac{\partial}{\partial\eta}\left((1+\varepsilon\theta)\frac{\partial Z}{\partial\eta}\right)+\gamma_1\Big(a_1\Big(A_1+(B-A_1)\eta+\sum_{n=1}^{\infty}B_1 e^{-P_{em}n^2\pi^2t}\sin n\pi\eta-\\&(b_1(1+\varepsilon\theta)+a_2X+b_2Z)\Big)\Big)n_f\left(1+(e-2)\theta\right)\end{aligned}\right\} \tag{3.110}$$

$$\left.\begin{aligned}&\frac{\partial n_f}{\partial t}=\gamma_2 n_f\Big(a_1\Big(A_1+(B-A_1)\eta+\sum_{n=1}^{\infty}B_1 e^{-P_{em}n^2\pi^2t}\sin n\pi\eta-\\&(b_1(1+\varepsilon\theta)+a_2X+b_2Z)\Big)\Big)\left(1+(e-2)\theta\right)\end{aligned}\right\} \tag{3.111}$$

$$\frac{\partial n_l}{\partial t}=-\gamma_3 n_l\left(a_3(1-\varepsilon\theta)a(1+(e-2)\theta)(-ab(1-\varepsilon\theta))-X\right) \tag{3.112}$$

Let $0 < \varepsilon << 1$ and $\gamma = m\varepsilon,\quad \gamma_1 = m_1\varepsilon,\ \delta = m_2\varepsilon$,

$$\gamma_2 = m_3\varepsilon\,, \gamma_3 = m_4\varepsilon\,, \alpha = m_5\varepsilon \tag{3.113}$$

Such that

$$\left.\begin{aligned}
&\theta = \theta_0 + \varepsilon\theta_1 + h.o.t\\
&X = X_0 + \varepsilon X_1 + h.o.t\\
&Y = Y_0 + \varepsilon Y_1 + h.o.t\\
&Z = Z_0 + \varepsilon Z_1 + h.o.t\\
&n_f = n_{f0} + \varepsilon n_{f1} + h.o.t\\
&n_l = n_{l0} + \varepsilon n_{l1} + h.o.t
\end{aligned}\right\} \tag{3.114}$$

Where h.o.t read Higher Order Terms

Substituting equation (3.113) and equation (3.114) into equation (3.107) to equation (3.112), we have equation (3.115) to (3.120)

$$\left.\begin{aligned}
&\frac{\partial}{\partial t}(\theta_0 + \varepsilon\theta_1) = \lambda_1(1+\varepsilon(\theta_0 + \varepsilon\theta_1))\frac{\partial^2}{\partial\eta^2}(\theta_0 + \varepsilon\theta_1) + \lambda_1\varepsilon\left(\frac{\partial}{\partial\eta}(\theta_0 + \varepsilon\theta_1)\right)^2 +\\
&m_2\varepsilon\Big(a_1\Big(A_1 + (B - A_1)\eta + \sum_{n=1}^{\infty} B_1 e^{-P_{em}n^2\pi^2 t}\sin n\pi\eta -\\
&(b_1(1+\varepsilon(\theta_0 + \varepsilon\theta_1)) + a_2(X_0 + \varepsilon X_1) + b_2(Z_0 + \varepsilon Z_1))\Big)\Big)\times\\
&(n_{f0} + \varepsilon n_{f1})(1+(e-2)(\theta_0 + \varepsilon\theta_1))\\
&- m_5\varepsilon(n_{l0} + \varepsilon n_{l1})\times\\
&(a_3(1-\varepsilon(\theta_0 + \varepsilon\theta_1))a(1+(e-2)(\theta_0 + \varepsilon\theta_1))(-ab(1-\varepsilon(\theta_0 + \varepsilon\theta_1))) - (X_0 + \varepsilon X_1))
\end{aligned}\right\} \tag{3.115}$$

$$\left.\begin{aligned}
&\frac{\partial}{\partial t}(X_0 + \varepsilon X_1) = D_1(1+\varepsilon(\theta_0 + \varepsilon\theta_1))\frac{\partial^2}{\partial\eta^2}(X_0 + \varepsilon X_1) +\\
&D_1\varepsilon\left(\frac{\partial}{\partial\eta}(\theta_0 + \varepsilon\theta_1)\frac{\partial}{\partial\eta}(X_0 + \varepsilon X_1)\right) + \alpha_1(n_{l0} + \varepsilon n_{l1})\\
&(a_3(1-\varepsilon(\theta_0 + \varepsilon\theta_1))a(1+(e-2)(\theta_0 + \varepsilon\theta_1))(-ab(1-\varepsilon(\theta_0 + \varepsilon\theta_1)) - (X_0 + \varepsilon X_1))
\end{aligned}\right\} \tag{3.116}$$

$$\left.\begin{aligned}&\frac{\partial}{\partial t}(Y_0+\varepsilon Y_1)=D_1(1+\varepsilon(\theta_0+\varepsilon\theta_1))\frac{\partial^2}{\partial\eta^2}(Y_0+\varepsilon Y_1)+\\&D_1\varepsilon\left(\frac{\partial}{\partial\eta}(\theta_0+\varepsilon\theta_1)\frac{\partial}{\partial\eta}(Y_0+\varepsilon Y_1)\right)-m\varepsilon(a_1(A_1+(B-A_1)\eta+\\&\sum_{n=1}^{\infty}B_1e^{-P_{em}n^2\pi^2t}\sin n\pi\eta-(b_1(1+\varepsilon(\theta_0+\varepsilon\theta_1))+a_2(X_0+\varepsilon X_1)+b_2(Z_0+\varepsilon Z_1))))\times\\&(n_{f0}+\varepsilon n_{f1})(1+(e-2)(\theta_0+\varepsilon\theta_1))\end{aligned}\right\}\quad(3.117)$$

$$\left.\begin{aligned}&\frac{\partial}{\partial t}(Z_0+\varepsilon Z_1)=D_1(1+\varepsilon(\theta_0+\varepsilon\theta_1))\frac{\partial^2}{\partial\eta^2}(Z_0+\varepsilon Z_1)+\\&D_1\varepsilon\left(\frac{\partial}{\partial\eta}(\theta_0+\varepsilon\theta_1)\frac{\partial}{\partial\eta}(Z_0+\varepsilon Z_1)\right)+m_1\varepsilon(a_1(A_1+(B-A_1)\eta+\\&\sum_{n=1}^{\infty}B_1e^{-P_{em}n^2\pi^2t}\sin n\pi\eta-(b_1(1+\varepsilon(\theta_0+\varepsilon\theta_1))+a_2(X_0+\varepsilon X_1)+b_2(Z_0+\varepsilon Z_1))))\times\\&(n_{f0}+\varepsilon n_{f1})(1+(e-2)(\theta_0+\varepsilon\theta_1))\end{aligned}\right\}\quad(3.118)$$

$$\left.\begin{aligned}&\frac{\partial}{\partial t}(n_{f0}+\varepsilon n_{f_1})=m_3\varepsilon n_f(a_1(A_1+(B-A_1)\eta+\sum_{n=1}^{\infty}B_1e^{-P_{em}n^2\pi^2t}\sin n\pi\eta-\\&(b_1(1+\varepsilon(\theta_0+\varepsilon\theta_1))+a_2(X_0+\varepsilon X_1)+b_2(Z_0+\varepsilon Z_1))))(1+(e-2)(\theta_0+\varepsilon\theta_1))\end{aligned}\right\}\quad(3.119)$$

$$\left.\begin{aligned}&\frac{\partial}{\partial t}(n_{l0}+\varepsilon n_{l1})=-m_4\varepsilon(n_{l0}+\varepsilon n_{l1})\times\\&(a_3(1-\varepsilon(\theta_0+\varepsilon\theta_1))a(1+(e-2)(\theta_0+\varepsilon\theta_1))(-ab(1-\varepsilon(\theta_0+\varepsilon\theta_1)))-(X_0+\varepsilon X_1))\end{aligned}\right\}\quad(3.120)$$

Collecting like power of ε in equations (3.115) to equation (3.120), we have equation (3.121) to equation (3.32)

ε^0 :

$$\left.\begin{aligned}&\frac{\partial\theta_0}{\partial t}=\lambda_1\frac{\partial^2\theta_0}{\partial\eta^2}\\&\theta_0(\eta,0)=1-\eta,\quad\theta_0(0,t)=b_3,\quad\theta_0(1,t)=0\end{aligned}\right\}\quad(3.121)$$

$$\left.\begin{aligned}&\frac{\partial X_0}{\partial t}=D_1\frac{\partial^2 X_0}{\partial \eta^2}-\alpha_1 n_{l0}\left(X_0+a_3a^2b\left(1+\left(e-2\right)\theta_0\right)\right)\\&X_0(\eta,0)=1-\eta,\quad X_0(0,t)=0,\quad X_0(1,t)=0\end{aligned}\right\} \tag{3.122}$$

$$\left.\begin{aligned}&\frac{\partial Y_0}{\partial t}=D_1\frac{\partial^2 Y_0}{\partial \eta^2}\\&Y_0(\eta,0)=1-\eta,\quad Y_0(0,t)=1,\quad Y_0(1,t)=0\end{aligned}\right\} \tag{3.123}$$

$$\left.\begin{aligned}&\frac{\partial Z_0}{\partial t}=D_1\frac{\partial^2 Z_0}{\partial \eta^2}\\&Z_0(\eta,0)=1-\eta,\quad Z_0(0,t)=0,\quad Z_0(1,t)=0\end{aligned}\right\} \tag{3.124}$$

$$\left.\begin{aligned}&\frac{\partial n_{f0}}{\partial t}=0\\&n_{f0}(\eta,0)=1\end{aligned}\right\} \tag{3.125}$$

$$\left.\begin{aligned}&\frac{\partial n_{l0}}{\partial t}=0\\&n_{l0}(\eta,0)=1\end{aligned}\right\} \tag{3.126}$$

ε^1 :

$$\left.\begin{aligned}&\frac{\partial \theta_1}{\partial t}=\lambda_1\theta_0\frac{\partial^2\theta_0}{\partial \eta^2}+\lambda_1\frac{\partial^2\theta_1}{\partial \eta^2}+\lambda_1\left(\frac{\partial\theta_0}{\partial\eta}\right)^2+\\&m_2\left(a_1\left(A_1+(B-A_1)\eta+\sum_{n=1}^{\infty}B_1 e^{-P_{em}n^2\pi^2 t}\sin n\pi\eta-\right.\right.\\&\left.\left.\left(b_1+a_2X_0+b_2Z_0\right)\right)\right)\left(n_{f0}\right)\left(1+\left(e-2\right)\theta_0\right)+m_4n_{l0}\left(a_3a^2b\left(1+\left(e-2\right)\theta_0\right)+X_0\right)\\&\theta_1(\eta,0)=1-\eta,\quad \theta_1(0,t)=0\quad \theta_1(1,t)=0\end{aligned}\right\} \tag{3.127}$$

$$\left.\begin{aligned}
&\frac{\partial X_1}{\partial t}=D_1\theta_0\frac{\partial^2 X_0}{\partial\eta^2}+D_1\frac{\partial^2 X_1}{\partial\eta^2}+D_1\frac{\partial\theta_0}{\partial\eta}\frac{\partial X_0}{\partial\eta}+\\
&\alpha_1 n_{l1}\left(X_0+a_3a^2b\left(1+(e-2)\theta_0\right)\right)+\alpha_1 n_{l0}\left(a_3a^2b(-\theta_0)\left(1+(e-2)\theta_0\right)-X_1\right)\\
&X_1(\eta,0)=0\quad X_1(0,t)=0,\quad X_1(1,t)=0
\end{aligned}\right\}\qquad(3.128)$$

$$\left.\begin{aligned}
&\frac{\partial Y_1}{\partial t}=D_1\theta_0\frac{\partial^2 Y_0}{\partial\eta^2}+D_1\frac{\partial^2 Y_1}{\partial\eta^2}+D_1\frac{\partial\theta_0}{\partial\eta}\frac{\partial Y_0}{\partial\eta}+\\
&m\left(a_1\left(A_1+(B-A_1)\eta+\sum_{n=1}^{\infty}B_1 e^{-P_{em}n^2\pi^2 t}\sin n\pi\eta-\left(b_1+a_2X_0+b_2Z_0\right)\right)\right)\times\\
&n_{f0}\left(1+(e-2)\theta_0\right)\\
&Y_1(\eta,0)=0,\quad Y_1(0,t)=0,\quad Y_1(1,t)=0
\end{aligned}\right\}\qquad(3.129)$$

$$\left.\begin{aligned}
&\frac{\partial Z_1}{\partial t}=D_1\theta_0\frac{\partial^2 Z_0}{\partial\eta^2}+D_1\frac{\partial^2 Z_1}{\partial\eta^2}+D_1\frac{\partial\theta_0}{\partial\eta}\frac{\partial Z_0}{\partial\eta}+\\
&m_1\left(a_1\left(A_1+(B-A_1)\eta+\sum_{n=1}^{\infty}B_1 e^{-P_{em}n^2\pi^2 t}\sin n\pi\eta-\right.\right.\\
&\left.\left.\left(b_1+a_2X_0+b_2Z_0\right)\right)\right)n_{f0}\left(1+(e-2)\theta_0\right)\\
&Z_1(\eta,0)=0\quad Z_1(0,t)=0,\quad Z_1(1,t)=0
\end{aligned}\right\}\qquad(3.130)$$

$$\left.\begin{aligned}
&\frac{\partial n_{f1}}{\partial t}=m_2 n_{f0}\left(a_1\left(A_1+(B-A_1)\eta+\sum_{n=1}^{\infty}B_1 e^{-P_{em}n^2\pi^2 t}\sin n\pi\eta-\right.\right.\\
&\left.\left.\left(b_1+a_2X_0+b_2Z_0\right)\right)\right)\left(1+(e-2)\theta_0\right)\\
&n_{f1}(\eta,0)=0
\end{aligned}\right\}\qquad(3.131)$$

$$\left.\begin{aligned}
&\frac{\partial n_{l1}}{\partial t}=-m_3 n_{l0}\left(-a_3a^2b\left(1+(e-2)\theta_0\right)-X_0\right)\\
&n_{l1}(\eta,0)=0
\end{aligned}\right\}\qquad(3.132)$$

Considering equation (3.125), that is

$$\frac{\partial n_{f0}}{\partial t} = 0, \quad n_{f0}(\eta,0) = 1$$

Integrating, we have

$$n_{f0}(\eta,t) = c$$

Applying initial condition, we have equation (3.133)

$$n_{f0}(\eta,0) = c = 1 \Rightarrow c = 1$$

$$n_{f0}(\eta,0) = 1 \qquad (3.133)$$

Considering equation (3.126), that is

$$\frac{\partial n_{l0}}{\partial t} = 0, \quad n_{l0}(\eta,0) = 1$$

Integrating, we have

$$n_{l0}(\eta,t) = c$$

Applying initial condition, we have equation (3.134)

$$n_{l0}(\eta,0) = c = 1 \Rightarrow c = 1$$

$$n_{l0}(\eta,0) = 1 \qquad (3.134)$$

Considering equation (3.121), that is

$$\left.\begin{aligned}&\frac{\partial\theta_0}{\partial t}=\lambda_1\frac{\partial^2\theta_0}{\partial\eta^2}\\&\theta_0(\eta,0)=1-\eta,\quad \theta_0(0,t)=b_3,\quad \theta_0(1,t)=0\end{aligned}\right\}$$

Solving equation (3.121), we have equation (3.135) to equation (3.147)

Let

$$\mu(\eta,t)=b_3+\eta(0-b_3)$$

That is,

$$\mu(\eta,t)=b_3(1-\eta)t^0 \qquad (3.135)$$

Then

$$\begin{aligned}&\mu(\eta,0)=0\\&\mu(0,t)=b_3\\&\mu(1,t)=0\end{aligned}$$

Also

Let

$$\theta_0(\eta,t)=s(\eta,t)+\mu(\eta,t) \qquad (3.136)$$

Differentiating equation (3.136) with respect to t, we have

$$\frac{\partial\theta_0}{\partial t}=\frac{\partial s}{\partial t}+\frac{\partial\mu}{\partial t}=\frac{\partial s}{\partial t}+0=\frac{\partial s}{\partial t}$$

Differentiating equation (3.136) with respect to η, we have

$$\frac{\partial \theta_0}{\partial \eta} = \frac{\partial s}{\partial \eta} + \frac{\partial \mu}{\partial \eta} = -b_3$$

Differentiating equation (3.136) twice with respect to t, we have

$$\frac{\partial^2 \theta_0}{\partial \eta^2} = \frac{\partial^2 s}{\partial \eta^2} + \frac{\partial^2 \mu}{\partial \eta^2} = \frac{\partial^2 s}{\partial \eta^2} + 0 = \frac{\partial^2 s}{\partial \eta^2}$$

Therefore equation (3.136) becomes

$$\left.\begin{aligned} &\frac{\partial s}{\partial t} = \lambda_1 \frac{\partial^2 s}{\partial \eta^2} \\ &s(\eta,0) = 1-\eta,\ \ s(0,t) = 0,\ \ s(1,t) = 0 \end{aligned}\right\} \tag{3.137}$$

To solve equation (3.137), we shall consider the following problem:

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} = k \frac{\partial^2 s}{\partial \eta^2} + \alpha u + F(x,t) \\ &u(\eta,0) = 1-\eta,\ \ u(0,t) = 0,\ \ u(L,t) = 0 \end{aligned}\right\} \tag{3.138}$$

We assume the solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin \frac{n\pi}{L} x \tag{3.139}$$

Where,

$$u_n(t) = \int_0^t e^{\left(\alpha - k\left(\frac{n\pi}{L}\right)^2\right)(t-\tau)} F_n(\tau) d\tau + bn e^{\alpha - k\left(\frac{n\pi}{L}\right)^2 t} \tag{3.140}$$

$$F_n(t) = \frac{2}{L}\int_0^L f(x,t) \sin \frac{n\pi}{L} \eta d\eta \tag{3.141}$$

$$b_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi}{L} x dx \tag{3.142}$$

Comparing equation (3.137) and equation (3.138)

$u = s\,,\; x = \eta\,,\; k = \lambda_1\,,\; \alpha = 0\,,\; f(x,t) = 0\,,\; f(x) = 1 - \eta\,, L = 1$

Then

$$b_n = 2\int_0^1 (1-\eta)\sin n\pi\eta d\eta \qquad (3.143)$$

Integrating Equation (3.143), we obtain (3.144)

That is,

$$= 2\int_0^1 \sin n\pi\eta d\eta - 2\int_0^1 \eta \sin n\pi\eta d\eta$$

$$= -\frac{2}{n\pi}\cos n\pi\eta\Big|_0^1 - 2\left(-\frac{\eta}{n\pi}\cos n\pi\eta\Big|_0^1 + \frac{1}{n\pi}\int_0^1 \cos n\pi\eta d\eta \right)$$

$$= -\frac{2}{n\pi}\left((-1)^n - 1\right) + \frac{2}{n\pi}\left((-1)^n - 0\right) - \frac{2}{n^2\pi^2}\sin n\pi\eta\Big|_0^1$$

$$= \frac{2}{n\pi} - \frac{2}{n\pi}(-1)^n + \frac{2}{n\pi}(-1)^n - 0$$

$$b_n = \frac{2}{n\pi} \qquad (3.144)$$

But $f(\eta,t) = o \;\Rightarrow F_n = 0$

Then

$$s_n = \frac{2}{n\pi} e^{-\lambda_1 \left(\frac{n\pi}{L}\right)^2 t} \qquad (3.145)$$

Therefore

$$s(\eta,t)=\sum_{n=1}^{\infty}\frac{2}{n\pi}e^{-\lambda_1\left(\frac{n\pi}{L}\right)^2 t}\sin n\pi\eta \tag{3.146}$$

Thus

$$\theta_0(\eta,t)=b_3(1-\eta)+\sum_{n=1}^{\infty}\frac{2}{n\pi}e^{-\lambda_1(n\pi)^2 t}\sin n\pi\eta \tag{3.147}$$

Considering equation (3.122), that is

$$\left.\begin{aligned}&\frac{\partial X_0}{\partial t}=D_1\frac{\partial^2 X_0}{\partial\eta^2}-\alpha_1 n_{l0}\left(X_0+a_3a^2b\left(1+(e-2)\theta_0\right)\right)\\&X_0(\eta,0)=1-\eta,\quad X_0(0,t)=0,\quad X_0(1,t)=0\end{aligned}\right\}$$

Substituting equation (3.147) into (3.121), we have

$$\left.\begin{aligned}&\frac{\partial X_0}{\partial t}=\\&D_1\frac{\partial^2 X_0}{\partial\eta^2}-\alpha_1 n_{l0}\left(X_0+a_3a^2b\left(1+(e-2)b_3(1-\eta)+\sum_{n=1}^{\infty}\frac{2}{n\pi}e^{-\lambda_1\left(\frac{n\pi}{L}\right)^2 t}\sin n\pi\eta\right)\right)\\&X_0(\eta,0)=1-\eta,\quad X_0(0,t)=0,\quad X_0(1,t)=0\end{aligned}\right\} \tag{3.148}$$

Comparing equation (3.138) and equation (3.148)

That is,

$$f(\eta,t)=-p\left(1+p(1-\eta)+\sum_{n=1}^{\infty}p_2 e^{-qt}\sin n\pi\eta\right),$$

Where;

$$p=\alpha_1a_3a^2b,\qquad p_1=b_3(e-2),\qquad p_2=\frac{2}{n\pi}(e-2),\qquad q=\lambda_1n^2\pi^2$$

$$b_n = 2\int_0^1 (1-\eta)\sin n\pi\eta d\eta = \frac{2}{n\pi} \tag{3.149}$$

$$F_n(t) = -2p\int_0^1 \left(1 + p(1-\eta) + \sum_{n=1}^{\infty} p_2 e^{-qt} \sin n\pi\eta\right) \sin n\pi\eta$$

$$\left.\begin{aligned} &= -2p\int_0^1 (1-p_1)\sin n\pi\eta d\eta + 2pp_1\int_0^1 \eta \sin n\pi\eta d\eta - \\ &2p\sum_{n=1}^{\infty} p_2 e^{-qt}\int_0^1 \sin n\pi\eta \sin n\pi\eta d\eta \end{aligned}\right\} \tag{3.150}$$

Integrating equation (3.150), we have equation (3.151)

$$= \frac{2p(1+p_1)}{n\pi}\cos n\pi\eta\Big|_0^1 + 2pp_1\left(-\frac{\eta}{n\pi}\cos n\pi\eta\Big|_0^1 + \frac{1}{n\pi}\int_0^1 \cos n\pi\eta d\eta\right) + 2p\sum_{n=1}^{\infty} p_2 e^{-qt} \times \frac{1}{2}$$

$$F_n(t) = \frac{2p(1+p_1)}{n\pi}\left((-1)^n - 1\right) - 2pp_1\left(\frac{(-1)^n}{n\pi}\right) + p\sum_{n=1}^{\infty} p_2 e^{-qt}$$

$$= \frac{2p}{n\pi}\left((-1)^n - 1\right) - \frac{2pp_1}{n\pi} + p\sum_{n=1}^{\infty} p_2 e^{-qt}$$

$$F_n(t) = p_3 + \sum_{n=1}^{\infty} p_4 e^{-qt} \tag{3.151}$$

Where;

$$p_3 = \frac{2p}{n\pi}\left((-1)^n - 1\right) - \frac{2pp_1}{n\pi}, \qquad p_4 = pp_2$$

Then

$$X_{0n}(t) = \int_0^t e^{-\left(\alpha_1 + D_1(n\pi)^2\right)(t-\tau)}\left(p_3 + \sum_{n=1}^{\infty} p_4 e^{-qt}\right)d\tau + \frac{2}{n\pi}e^{-\left(\alpha_1 + D_1(n\pi)^2\right)t} \tag{3.152}$$

$$= e^{-q_1 t}\left(p_3 \int_0^t e^{q_1 \tau} d\tau + \sum_{n=1}^{\infty} p_4 \int_0^t e^{(q_1 - q)t} d\tau \right) + \frac{2}{n\pi} e^{-q_1 t}$$

Integrate equation (3.152) with respect to τ . We have equation (3.153)

$$= \frac{p_3}{q_1}\left(1 - e^{-q_1 t}\right) + \sum_{n=1}^{\infty} \frac{p_4}{(q_1 - q)}\left(e^{-qt} - e^{-q_1 t}\right) + \frac{2}{n\pi} e^{-q_1 t}$$

$$= \frac{p_3}{q_1} + \left(\frac{2}{n\pi} - \frac{p_3}{q_1}\right) e^{-q_1 t} + \sum_{n=1}^{\infty} \frac{p_4}{(q_1 - q)}\left(e^{-qt} - e^{-q_1 t}\right)$$

$$X_{0n}(t) = p_5 + p_6 e^{-q_1 t} + \sum_{n=1}^{\infty} p_7 \left(e^{-qt} - e^{-q_1 t}\right) \tag{3.153}$$

Where;

$$p_5 = \frac{p_3}{q_1}, \quad p_6 = \left(\frac{2}{n\pi} - \frac{p_3}{q_1}\right), \quad p_7 = \frac{p_4}{(q_1 - q)} e^{-q_1 t}, \quad q_1 = \alpha_1 + D_1 n^2 \pi^2$$

Therefore, we have equation (3.154)

$$X_0(\eta, t) = \sum_{n=1}^{\infty}\left(p_5 + p_6 e^{-q_1 t} + \sum_{n=1}^{\infty} p_7 \left(e^{-qt} - e^{-q_1 t}\right)\right) \sin n\pi\eta \tag{3.154}$$

considering equation (3.122), that is

$$\left.\begin{array}{l} \dfrac{\partial Y_0}{\partial t} = D_1 \dfrac{\partial^2 Y_0}{\partial \eta^2} \\ Y_0(\eta, 0) = 1 - \eta, \quad Y_0(0, t) = 1, \quad Y_0(1, t) = 0 \end{array}\right\}$$

Solving equation (3.122), we have equation (3.155) to equation (3.147)

Let

$$\mu(\eta,t)=1+\frac{\eta}{1}(0-1)$$

That is,

$$\mu(\eta,t)=(1-\eta)t^0 \tag{3.155}$$

Then

$$\mu(\eta,0)=0$$
$$\mu(0,t)=1$$
$$\mu(1,t)=0$$

Also

Let

$$Y_0(\eta,t)=\mu(\eta,t)+s(\eta,t) \tag{3.156}$$

Differentiating equation (3.156) with respect to t, we have

$$\frac{\partial Y_0}{\partial t}=\frac{\partial \mu}{\partial t}+\frac{\partial s}{\partial t}=0+\frac{\partial s}{\partial t}=\frac{\partial s}{\partial t}$$

Differentiating equation (3.156) with respect to η, we have

$$\frac{\partial Y_0}{\partial \eta}=\frac{\partial \mu}{\partial \eta}+\frac{\partial s}{\partial \eta}=-1$$

Differentiating equation (3.156) twice with respect to t, we have

$$\frac{\partial^2 Y_0}{\partial \eta^2}=\frac{\partial^2 \mu}{\partial \eta^2}+\frac{\partial^2 s}{\partial \eta^2}=0+\frac{\partial^2 s}{\partial \eta^2}=\frac{\partial^2 s}{\partial \eta^2}$$

Therefore equation (3.156) becomes

$$\left.\begin{array}{l}\frac{\partial s}{\partial t}=D_1\frac{\partial^2 s}{\partial \eta^2}\\ s(\eta,0)=1-\eta,\ \ s(0,t)=0,\ \ \ s(1,t)=0\end{array}\right\} \tag{3.157}$$

Comparing equation (1.157) and (1.138)

$$u=s\,,\ x=\eta\,,\ k=D_1\,,\ \alpha=0\,,\ f(x,t)=0\,,\ f(x)=1-\eta\,, L=1$$

Then

$$b_n=2\int_0^1(1-\eta)\sin n\pi\eta d\eta \tag{3.158}$$

Integrating, we obtain equation (3.156)

$$b_n=\frac{2}{n\pi} \tag{3.159}$$

But $f(\eta,t)=o\ \ \Rightarrow F_n=0$

Then

$$s_n=\frac{2}{n\pi}e^{-D_1\left(\frac{n\pi}{L}\right)^2 t} \tag{3.160}$$

Therefore

$$s(\eta,t)=\sum_{n=1}^{\infty}\frac{2}{n\pi}e^{-D_1\left(\frac{n\pi}{L}\right)^2 t}\sin n\pi\eta \tag{3.161}$$

Thus

$$Y_0(\eta,t)=(1-\eta)+\sum_{n=1}^{\infty}\frac{2}{n\pi}e^{-D_1(n\pi)^2 t}\sin n\pi\eta \tag{3.163}$$

Considering equation (3.123), that is

$$\left.\begin{aligned}&\frac{\partial Z_0}{\partial t}=D_1\frac{\partial^2 Z_0}{\partial \eta^2}\\&Z_0(\eta,0)=1-\eta,\quad Z_0(0,t)=0,\quad Z_0(1,t)=0\end{aligned}\right\}$$

Comparing equation (1.123) and (1.138)

$$u=Z\,,\;x=\eta\,,\;k=D_1\,,\;\alpha=0\,,\;f(x,t)=0\,,\;f(x)=1-\eta\,,L=1$$

Then

$$b_n=2\int_0^1(1-\eta)\sin n\pi\eta d\eta \tag{3.163}$$

Integrating, we obtain equation (3.164)

$$b_n=\frac{2}{n\pi} \tag{3.164}$$

But $f(\eta,t)=o\;\Rightarrow F_n=0$

Then

$$Z_{0n}=\frac{2}{n\pi}e^{-D_1\left(\frac{n\pi}{L}\right)^2 t} \tag{3.165}$$

Therefore

$$Z_0(\eta,t)=\sum_{n=1}^{\infty}\frac{2}{n\pi}e^{-D_1(n\pi)^2 t}\sin n\pi\eta \tag{3.166}$$

Considering equation (3.132), that is

$$\left.\begin{aligned}&\frac{\partial n_{l1}}{\partial t}=-m_3 n_{l0}\left(-a_3a^2b\left(1+(e-2)\theta_0\right)-X_0\right)\\&n_{l1}(\eta,0)=0\end{aligned}\right\}$$

Substituting equations (3.134), (3.147) and equation (3.154) into (3.132), we have

$$\left.\begin{array}{l}\frac{\partial n_{l1}}{\partial t}=m_3\sum_{n=1}^{\infty}\left(p_5+p_6e^{-q_1t}+\sum_{n=1}^{\infty}p_7\left(e^{-qt}-e^{-q_1t}\right)\right)\sin n\pi\eta+\\ p_8\left(1+p_1(1-\eta)+\sum_{1}^{\infty}p_2e^{-qt}\sin n\pi\eta\right)\end{array}\right\} \quad (3.167)$$

Integrating equation (3.167) with respect to t, we have

$$\left.\begin{array}{l}n_{l1}(\eta,0)=m_3\sum_{n=1}^{\infty}\left(p_5t-\frac{p_6e^{-q_1t}}{q_1}-\sum_{n=1}^{\infty}p_7\left(-\frac{e^{-qt}}{q}+\frac{e^{-q_1t}}{q_1}\right)\right)\sin n\pi\eta+\\ p_8\left(t+p_1(1-\eta)t-\sum_{1}^{\infty}\frac{p_2}{q}e^{-qt}\sin n\pi\eta\right)+c\end{array}\right\} \quad (3.168)$$

Apply $n_{l1}(\eta,0)=0 \qquad \Rightarrow c=0$

Therefore , we have equation (3.169)

$$\left.\begin{array}{l}n_{l1}(\eta,0)=m_3\sum_{n=1}^{\infty}\left(p_5t-\frac{p_6}{q_1}e^{-q_1t}-\sum_{n=1}^{\infty}p_7\left(\frac{e^{-q_1t}}{q_1}-\frac{e^{-qt}}{q}\right)\right)\sin n\pi\eta+\\ p_8\left((1+p_1(1-\eta))t-\sum_{1}^{\infty}\frac{p_2}{q}e^{-qt}\sin n\pi\eta\right)+c\end{array}\right\} \quad (3.169)$$

Where;

$$p_8=m_3a_3a^2b$$

Considering equation (3.131), that is

$$\left.\begin{array}{l}\frac{\partial n_{f1}}{\partial t}=m_2n_{f0}\left(a_1\left(A_1+(B-A_1)\eta+\sum_{n=1}^{\infty}B_1e^{-P_{em}n^2\pi^2t}\sin n\pi\eta-\right.\right.\\ \left.\left.(b_1+a_2X_0+b_2Z_0)\right)\right)\left(1+(e-2)\theta_0\right)\\ \\ n_{f1}(\eta,0)=0\end{array}\right\}$$

Substituting equation (3.133), equation (3.147), equation (3.154), equation (3.166) into equation (3.131)

$$\frac{\partial n_{f1}}{\partial t} = p_9\left(1+p_1(1-\eta)+\sum_1^{\infty} p_2\, e^{-qt}\sin n\pi\eta\right)+p_{10}\left(1+p_1(1-\eta)+\sum_1^{\infty} p_2\, e^{-qt}\sin n\pi\eta\right)\eta+$$
$$p_{11}\left(\sum_1^{\infty} p_2\, e^{-q_2 t}\sin n\pi\eta+p_1(1-\eta)\sum_1^{\infty} p_2\, e^{-q_2 t}\sin n\pi\eta+\sum_1^{\infty}\sum_1^{\infty} p_2\, e^{-(q+q_2)t}\sin^2 n\pi\eta\right)-$$
$$p_{12}\left(\sum_1^{\infty} e^{-q_3 t}\sin n\pi\eta+p_1(1-\eta)\sum_1^{\infty} e^{-q_2 t}\sin n\pi\eta+\sum_1^{\infty}\sum_1^{\infty} p_2\, e^{-(q+q_3)t}\sin^2 n\pi\eta\right)-$$
$$p_{13}\left(\begin{array}{l}\sum_{n=1}^{\infty}\left(p_5+p_6\, e^{-q_1 t}+\sum_{n=1}^{\infty} p_7\left(e^{-qt}-e^{-q_1 t}\right)\right)\sin n\pi\eta+p_1(1-\eta)\left(p_5+p_6\, e^{-q_1 t}+\sum_{n=1}^{\infty} p_7\left(e^{-qt}-e^{-q_1 t}\right)\right)\times\\ \sin n\pi\eta+\sum_1^{\infty}\sum_1^{\infty} p_2\left(p_5\, e^{-qt}\sin n\pi\eta+p_6\, e^{-(q+q_1)t}\sin n\pi\eta+\sum_{n=1}^{\infty} p_7\left(e^{-2qt}-e^{-(q+q_1)t}\right)\right)\end{array}\right) \quad (3.170)$$

Integrating equation (3.170) with respect to t yields equation (3.171)

$$n_{f1}(\eta,t)=p_9\left(t+p_1(1-\eta)t-\sum_{n=1}^{\infty}\frac{p_2}{q}e^{-qt}\sin n\pi\eta\right)+p_{10}\left(\left(1+p_1(1-\eta)\right)t-\sum_{n=1}^{\infty}\frac{p_2}{q}e^{-qt}\sin n\pi\eta\right)\eta+$$

$$p_{11}\left(-\sum_{n=1}^{\infty}\frac{1}{q_2}e^{-q_2 t}\sin n\pi\eta-p_1(1-\eta)\sum_{n=1}^{\infty}\frac{1}{q_2}e^{-q_2 t}\sin n\pi\eta-\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{p_2}{q+q_1}e^{-(q+q_2)t}\sin^2 n\pi\eta\right)-$$
$$p_{12}\left(-\sum_{n=1}^{\infty}\frac{1}{q_3}e^{-q_3 t}\sin n\pi\eta-p_1(1-\eta)\sum_{n=1}^{\infty}\frac{1}{q_1}e^{-q_3 t}\sin n\pi\eta+\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{p_2}{q+q_1}e^{-(q+q_3)t}\sin^2 n\pi\eta\right)-$$
$$p_{13}\left(\begin{array}{l}\sum_{n=1}^{\infty}\left(p_5 t+\frac{p_6}{q_1}e^{-q_1 t}+\sum_{n=1}^{\infty} p_7\left(-\frac{1}{q}e^{-qt}+\frac{1}{q}e^{-q_1 t}\right)\right)\sin n\pi\eta+\\ p_1(1-\eta)\left(p_5 t-\frac{p_6}{q_1}e^{-q_1 t}+\sum_{n=1}^{\infty} p_7\left(-\frac{1}{q}e^{-qt}+\frac{1}{q}e^{-q_1 t}\right)\right)\sin n\pi\eta+\\ \sum_{n=1}^{\infty}\sum_{n=1}^{\infty} p_2\left(\begin{array}{l}-\frac{p_5}{q}e^{-qt}\sin n\pi\eta-\frac{p_6}{(q+q_1)}e^{-(q+q_1)t}\sin n\pi\eta+\\ \sum_{n=1}^{\infty} p_7\left(\frac{1}{(q+q_1)}e^{-(q+q_1)t}-\frac{1}{2q}e^{-2qt}\right)\sin^2 n\pi\eta\end{array}\right)\end{array}\right) \quad (3.171)$$

Where; $p_9 = m_2 n_{f0} a_1 (A-b_1)$, $\quad p_{10} = m_2 n_{f0} a_1 (B-A_1)$, $\quad p_{11} = m_2 n_{f0} a_1 B_1$

$$p_{12} = \frac{2m_2 n_{f0} a_1 b_2}{n\pi}, \quad p_{13} = m_2 n_{f0} a_1 a_2, \quad q_2 = p_{em} n^2\pi^2, \quad q_3 = D_1 n^2 \pi^2$$

Now, differentiating equation (3.147), we have equation (3.172)

$$\left.\begin{aligned}
&\frac{\partial\theta_0}{\partial\eta} = -b_3 + \sum_{n=1}^{\infty} 2e^{-qt}\cos n\pi\eta, \qquad \frac{\partial^2\theta}{\partial\eta^2} = -\sum_{n=1}^{\infty} 2n\pi e^{-qt}\sin n\pi\eta \\
&\left(\frac{\partial\theta}{\partial\eta}\right)^2 = b_3^2 - \sum_{n=1}^{\infty} 2b_3 e^{-qt}\cos n\pi\eta + \sum_{n=1}^{\infty}\sum_{n=1}^{\infty} 4e^{-2qt}\cos^2 n\pi\eta \\
&\theta_0\frac{\partial^2\theta_0}{\partial\eta^2} = \left(b_3(1-\eta) + \sum_{n=1}^{\infty}\frac{2}{n\pi}e^{-qt}\sin n\pi\eta\right)\left(-\sum_{n=1}^{\infty} 2n\pi e^{-qt}\sin n\pi\eta\right) \\
&\qquad = -\sum_{n=1}^{\infty} 2b_3 e^{-qt}\sin n\pi\eta + \sum_{n=1}^{\infty} 2b_3 e^{-qt}\eta\sin n\pi\eta - \sum_{n=1}^{\infty}\sum_{n=1}^{\infty} 4e^{-2qt}\sin^2 n\pi\eta
\end{aligned}\right\} \quad (3.172)$$

Then equation (3.172) becomes

$$\begin{aligned}
&\frac{\partial\theta_1}{\partial t} = \lambda_1\frac{\partial^2\theta_1}{\partial\eta^2} + \lambda_1\sum_{n=1}^{\infty} 2n\pi b_3 e^{-qt}\eta\sin n\pi\eta - \lambda_1\sum_{n=1}^{\infty} 2n\pi b_3 e^{-qt}\sin n\pi\eta \, - \\
&\lambda_1\sum_{n=1}^{\infty}\sum_{n=1}^{\infty} 4e^{-2qt}\sin^2 n\pi\eta + \lambda_1\left(b_3^2 - \sum_{n=1}^{\infty} 2b_3 e^{-qt}\cos n\pi\eta + \sum_{n=1}^{\infty}\sum_{n=1}^{\infty} 4e^{-2qt}\cos^2 n\pi\eta\right) \\
&+ p_9\left(1 + p_1(1-\eta) + \sum_{1}^{\infty} p_2 e^{-qt}\sin n\pi\eta\right) + p_{10}\left(\eta + p_1\eta(1-\eta) + \sum_{1}^{\infty} p_2 e^{-qt}\eta n n\pi\eta\right) \\
&+ p_{11}\left(\sum_{n=1}^{\infty} e^{-q_2 t}\sin n\pi\eta + p_1(1-\eta)\sum_{n=1}^{\infty} e^{-q_2 t}\sin n\pi\eta + \sum_{n=1}^{\infty}\sum_{n=1}^{\infty} p_2 e^{-(q+q_2)t}\sin^2 n\pi\eta\right) \\
&- p_{12}\left(\sum_{n=1}^{\infty} e^{-q_3 t}\sin n\pi\eta + p_1(1-\eta)\sum_{n=1}^{\infty} e^{-q_3 t}\sin n\pi\eta + \sum_{n=1}^{\infty}\sum_{n=1}^{\infty} p_2 e^{-(q+q_3)t}\sin^2 n\pi\eta\right)
\end{aligned}$$

$$\begin{aligned}
&- p_{13}\left(\begin{aligned}
&\sum_{n=1}^{\infty}\left(p_5 + p_6 e^{-q_1 t} + \sum_{n=1}^{\infty} p_7\left(e^{-qt} - e^{-q_1 t}\right)\right)\sin n\pi\eta + \\
&p_1(1-\eta)\left(p_5 + p_6 e^{-q_1 t} + \sum_{n=1}^{\infty} p_7\left(e^{-qt} - e^{-q_1 t}\right)\right)\sin n\pi\eta + \\
&\sum_{n=1}^{\infty}\sum_{n=1}^{\infty} p_2\left(p_5 e^{-qt}\sin n\pi\eta + p_6 e^{-(q+q_1)t}\sin n\pi\eta + \sum_{n=1}^{\infty} p_7\left(e^{-2qt} - e^{-(q+q_1)t}\right)\right)
\end{aligned}\right) \quad (3.173) \\
&+ m_4\sum_{n=1}^{\infty}\left(p_5 + p_6 e^{-q_1 t} + \sum_{n=1}^{\infty} p_7\left(e^{-qt} - e^{-q_1 t}\right)\right)\sin n\pi\eta - p_8\left(1 + p_1(1-\eta) + \sum_{n=1}^{\infty} p_2 e^{-qt}\sin n\pi\eta\right)
\end{aligned}$$

$$\theta_1(\eta, t) = 0, \quad \theta_1(0, t) = 0, \quad \theta_1(1, t) = 0$$

Comparing equation (3.136) and equation (3.172), we have equation (3.174)

$f(\eta)=0 \Rightarrow b_n=0$ and

$$f(\eta,t)=\sum_{n=1}^{\infty}\left(\begin{array}{l}(2n\pi\lambda_1 b_3+p_{10}p_2)e^{-qt}-p_{11}p_1 e^{-q_2 t}+p_{12}p_1 e^{-q_3 t}+p_1 p_5 p_{13}+p_1 p_6 p_{13}e^{-q_1 t}\\ +\sum_{n=1}^{\infty}p_1 p_7 p_{13}\left(e^{-qt}-e^{-q_1 t}\right)\end{array}\right)\eta\sin n\pi\eta-$$

$$\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\left(\begin{array}{l}4\lambda_1 e^{-2qt}-\left(p_{11}p_2 e^{-(q+q_2)t}-p_{11}p_2 e^{-(q+q_3)t}\right)+\sum_{n=1}^{\infty}\ p_2 p_7 p_{13}\times\\ \left(e^{-2qt}-e^{-(q+q_1)t}\right)\end{array}\right)\sin^2 n\pi\eta-$$

$$\sum_{n=1}^{\infty}\left(\begin{array}{l}(2n\pi\lambda_1 b_3-p_2 p_9-p_8 p_2)e^{-qt}-p_{11}(1+p_1)e^{-q_2 t}+p_{12}(1-p_1)e^{-q_3 t}+p_5 p_{13}\\ +p_6 p_{13}e^{-q_1 t}+\sum_{n=1}^{\infty}p_7 p_{13}\left(e^{-qt}-e^{-q_1 t}\right)+p_1 p_5 p_{13}+p_1 p_6 p_{13}e^{-q_1 t}+\sum_{n=1}^{\infty}p_2 p_5 p_{13}e^{-qt}\\ +\sum_{n=1}^{\infty}p_1 p_7 p_{13}\left(e^{-qt}-e^{-q_1 t}\right)+\sum_{n=1}^{\infty}p_2 p_6 p_{13}e^{-(q+q_1)t}-m_4 p_5-m_4 p_6 e^{-q_1 t}-\\ \sum_{n=1}^{\infty}m_4 p_7\left(e^{-qt}-e^{-q_1 t}\right)\end{array}\right)\sin n\pi\eta+$$

$$\begin{array}{l}\left(\lambda_1 b_3^2+p_9(1+p_1)+p_8(1+p_1)\right)-\sum_{n=1}^{\infty}\ 2b_3\lambda_1 e^{-qt}\cos n\pi\eta+\left(p_{10}(1+p_1)-p_1 p_9-p_1 p_8\right)\eta\\ -p_1 p_{10}\eta+\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}4\lambda_1 e^{-q_2 t}\cos^2 n\pi\eta\end{array} \qquad (3.174)$$

Then

$$
\left.
\begin{aligned}
f_n(t) = {} & 2\sum_{n=1}^{\infty}\begin{pmatrix} \left(2n\pi\lambda_1 b_3 + p_{10}p_2\right)e^{-qt} - p_{11}p_1 e^{-q_2 t} + p_{12}p_1 e^{-q_3 t} + \\ p_1 p_5 p_{13} + p_1 p_6 p_{13} e^{-q_1 t} + \sum_{n=1}^{\infty} p_1 p_7 p_{13}\left(e^{-qt} - e^{-q_1 t}\right) \end{pmatrix}\int_0^1 \eta \sin n\pi\eta \sin n\pi\eta d\eta - \\
& 2\sum_{n=1}^{\infty}\begin{pmatrix} \left(2n\pi\lambda_1 b_3 - p_2 p_9 - p_8 p_2\right)e^{-qt} - p_{11}\left(1+p_1\right)e^{-q_2 t} + p_{12}\left(1-p_1\right)e^{-q_3 t} \\ + p_5 p_{13} + p_6 p_{13} e^{-q_1 t} + \sum_{n=1}^{\infty} p_7 p_{13}\left(e^{-qt} - e^{-q_1 t}\right) + p_1 p_5 p_{13} + \\ p_1 p_6 p_{13} e^{-q_1 t} + \sum_{n=1}^{\infty} p_1 p_7 p_{13}\left(e^{-qt} - e^{-q_1 t}\right) + \sum_{n=1}^{\infty} p_2 p_5 p_{13} e^{-qt} + \\ \sum_{n=1}^{\infty} p_2 p_6 p_{13} e^{-(q+q_1)t} - m_4 p_5 - m_4 p_6 e^{-q_1 t} - \sum_{n=1}^{\infty} m_4 p_7\left(e^{-qt} - e^{-q_1 t}\right) \end{pmatrix}\int_0^1 \sin^2 n\pi\eta d\eta \\
& - 2\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\begin{pmatrix} 4\lambda_1 e^{-2qt} - \left(p_{11}p_2 e^{-(q+q_2)t} - p_{11}p_2 e^{-(q+q_3)t}\right) + \sum_{n=1}^{\infty} p_2 p_7 p_{13} \times \\ \left(e^{-2qt} - e^{-(q+q_1)t}\right) \end{pmatrix}\int_0^1 \sin^3 n\pi\eta d\eta \\
& + 2\left(\lambda_1 b_3^2 + p_9\left(1+p_1\right) + p_8\left(1+p_1\right)\right)\int_0^1 \sin n\pi\eta d\eta - 2\sum_{n=1}^{\infty} 2b_3\lambda_1 e^{-qt}\int_0^1 \sin n\pi\eta \cos n\pi\eta d\eta + \\
& 2\sum_{n=1}^{\infty}\sum_{n=1}^{\infty} 4\lambda_1 e^{-q_2 t}\int_0^1 \sin n\pi\eta \cos^2 n\pi\eta d\eta + 2\left(p_{10}\left(1+p_1\right) - p_1 p_9 - p_1 p_8\right)\int_0^1 \eta \sin n\pi\eta d\eta - \\
& 2p_1 p_{10}\int_0^1 \eta^2 \sin n\pi\eta d\eta
\end{aligned}
\right\} \quad (3.175)
$$

Integrating equation (3.175) with respect to η, we have equation (3.176)

$$
\left.
\begin{aligned}
&f_n(t)=\sum_{n=1}^{\infty}\begin{pmatrix}(2n\pi\lambda_1 b_3+p_{10}p_2)e^{-qt}-p_{11}p_1e^{-q_2t}+p_{12}p_1e^{-q_3t}+\\ p_1p_5p_{13}+p_1p_6p_{13}e^{-q_1t}+\sum_{n=1}^{\infty}p_1p_7p_{13}\left(e^{-qt}-e^{-q_1t}\right)\end{pmatrix}\times\frac{\left(1-n^2\pi^2+(-1)^{2n}\right)}{2n^2\pi^2}-\\
&\sum_{n=1}^{\infty}\begin{pmatrix}(2n\pi\lambda_1 b_3-p_2p_9-p_8p_2)e^{-qt}-p_{11}(1+p_1)e^{-q_2t}+p_{12}(1-p_1)e^{-q_3t}\\ +p_5p_{13}+p_6p_{13}e^{-q_1t}+\sum_{n=1}^{\infty}p_7p_{13}\left(e^{-qt}-e^{-q_1t}\right)+p_1p_5p_{13}+\\ p_1p_6p_{13}e^{-q_1t}+\sum_{n=1}^{\infty}p_1p_7p_{13}\left(e^{-qt}-e^{-q_1t}\right)+\sum_{n=1}^{\infty}p_2p_5p_{13}e^{-qt}+\\ \sum_{n=1}^{\infty}p_2p_6p_{13}e^{-(q+q_1)t}-m_4p_5-m_4p_6e^{-q_1t}-\sum_{n=1}^{\infty}m_4p_7\left(e^{-qt}-e^{-q_1t}\right)\end{pmatrix}\\
&-\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\begin{pmatrix}4\lambda_1e^{-2qt}-\left(p_{11}p_2e^{-(q+q_2)t}-p_{11}p_2e^{-(q+q_3)t}\right)+\\ \sum_{n=1}^{\infty}p_2p_7p_{13}\left(e^{-2qt}-e^{-(q+q_1)t}\right)\end{pmatrix}\frac{2}{3}\frac{\left(2-3(-1)^n+(-1)^{3n}\right)}{n\pi}\\
&+2\left(\lambda_1b_3^2+p_9(1+p_1)+p_8(1+p_1)\right)\times\frac{\left(1-(-1)^n\right)}{n\pi}-\sum_{n=1}^{\infty}2b_3\lambda_1e^{-qt}\frac{\left(1-(-1)^{2n}\right)}{n\pi}+\\
&\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}4\lambda_1e^{-q_2t}\frac{2}{3}\frac{\left(1-(-1)^{3n}\right)}{n\pi}+2\left(p_{10}(1+p_1)-p_1p_9-p_1p_8\right)\frac{2(-1)^n}{n\pi}-\\
&2p_1p_{10}\frac{2\left(-2+2(-1)^n-n^2\pi^2(-1)^n\right)}{n^3\pi^3}
\end{aligned}
\right\}\quad(3.176)
$$

Then

$$
\begin{aligned}
&\theta_{1n}(t)=\sum_{n=1}^{\infty}\left(\begin{matrix}(2n\pi\lambda_1b_3+p_{10}p_2)e^{-qt}\int_0^t d\tau-p_{11}p_1e^{-qt}\int_0^t e^{(q-q_2)\tau}d\tau+p_{12}p_1e^{-qt}\int_0^t e^{(q-q_3)\tau}d\tau+\\ p_1p_5p_{13}e^{-qt}\int_0^t e^{q\tau}d\tau+p_1p_6p_{13}e^{-qt}\int_0^t e^{(q-q_1)\tau}d\tau+\sum_{n=1}^{\infty}p_1p_7p_{13}e^{-qt}\left(\int_0^t d\tau-\int_0^t e^{(q-q_1)\tau}d\tau\right)\end{matrix}\right)\times\\
&\frac{\left(1-n^2\pi^2+(-1)^{2n}\right)}{2n^2\pi^2}-\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\begin{pmatrix}4\lambda_1e^{-qt}\int_0^t e^{-q\tau}d\tau-\left(p_{11}p_2e^{-qt}\int_0^t e^{-q_2\tau}d\tau-p_{11}p_2e^{-qt}\int_0^t e^{-q_3\tau}d\tau\right)+\\ \sum_{n=1}^{\infty}p_2p_7p_{13}e^{-qt}\left(\int_1^t e^{-qt}d\tau-\int_0^t e^{-q_1\tau}d\tau\right)\end{pmatrix}\times\\
&\frac{2}{3}\frac{\left(2-3(-1)^n+(-1)^{3n}\right)}{n\pi}+2\left(\lambda_1b_3^2+p_9(1+p_1)+p_8(1+p_1)\right)e^{-qt}\int_0^t e^{q\tau}d\tau\times\frac{\left(1-(-1)^n\right)}{n\pi}-
\end{aligned}
$$

$$\sum_{n=1}^{\infty}\left(\begin{array}{l}(2n\pi\lambda_1 b_3 - p_2p_9 - p_8p_2)e^{-qt}\int_0^t d\tau - p_{11}(1+p_1)e^{-qt}\int_0^t e^{(q-q_2)\tau}d\tau + p_{12}(1-p_1)e^{-qt}\int_0^t e^{(q-q_3)\tau}d\tau + \\ p_5p_{13}e^{-qt}\int_0^t e^{q\tau}d\tau + p_6p_{13}e^{-qt}\int_0^t e^{(q-q_1)\tau}d\tau + \sum_{n=1}^{\infty}p_7p_{13}e^{-qt}\left(\int_0^t d\tau - \int_0^t e^{(q-q_1)\tau}d\tau\right) + \\ p_1p_5p_{13}e^{-qt}\int_0^t e^{q\tau}d\tau + p_1p_6p_{13}e^{-qt}\int_0^t e^{(q-q_1)\tau}d\tau + \sum_{n=1}^{\infty}p_1p_7p_{13}e^{-qt}\left(\int_0^t d\tau - \int_0^t e^{(q-q_1)\tau}d\tau\right) + \\ \sum_{n=1}^{\infty}p_2p_5p_{13}e^{-qt}\int_0^t d\tau - m_4p_5e^{-qt}\int_0^t e^{q\tau}d\tau + \sum_{n=1}^{\infty}p_2p_6p_{13}e^{-qt}\int_0^t e^{-q_1\tau}d\tau - m_4p_6e^{-qt}\int_0^t e^{(q-q_1)\tau}d\tau - \\ \sum_{n=1}^{\infty}m_4p_7e^{-qt}\left(\int_0^t d\tau - \int_0^t e^{(q-q_1)\tau}d\tau\right)\end{array}\right) -$$

$$\sum_{n=1}^{\infty}2b_3\lambda_1 e^{-qt}\int_0^t d\tau \times \frac{\left(1-(-1)^{2n}\right)}{n\pi} + \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}4\lambda_1 e^{-qt}\int_0^t d\tau \frac{2}{3}\times\frac{\left(1-(-1)^{3n}\right)}{n\pi} + 2\left(p_{10}(1+p_1) - p_1p_9 - p_1p_8\right)\times$$

$$e^{-qt}\int_0^t e^{q\tau}d\tau \times \frac{2(-1)^n}{n\pi} - 4p_1p_{10}e^{-qt}\int_0^t e^{q\tau}d\tau\frac{\left(-2+2(-1)^n - n^2\pi^2(-1)^n\right)}{n^3\pi^3} \qquad (3.177)$$

Integrating equation (3.177) with respect to τ, we have equation (3.178)

$$\theta_{1n}(t) = \sum_{n=1}^{\infty}\left(\begin{array}{l}(2n\pi\lambda_1 b_3 + p_{10}p_2)te^{-qt} - \frac{p_{11}p_1}{(q-q_2)}\left(e^{-q_2t} - e^{-qt}\right) + \frac{p_1p_{12}}{(q-q_3)}\left(e^{-q_3t} - e^{-qt}\right) + \\ \frac{p_1p_5p_{13}}{q}\left(1-e^{-qt}\right) + \frac{p_1p_6p_{13}}{(q-q_1)}\left(e^{-q_1t} - e^{-qt}\right) + \sum_{n=1}^{\infty}p_1p_7p_{13}\left(te^{-qt} - \frac{1}{(q-q_1)}\left(e^{-q_1t} - e^{-qt}\right)\right)\end{array}\right)\times$$

$$\frac{\left(1-n^2\pi^2 + (-1)^{2n}\right)}{2n^2\pi^2} -$$

$$\sum_{n=1}^{\infty}\left(\begin{array}{l}(2n\pi\lambda_1 b_3 - p_2p_9 - p_8p_2)te^{-qt} - \frac{p_{11}(1+p_1)}{(q-q_2)}\left(e^{-q_2t} - e^{-qt}\right) + \frac{p_{12}(1-p_1)}{(q-q_3)}\left(e^{-q_3t} - e^{-qt}\right) + \\ \frac{p_5p_{13}}{q}\left(1-e^{-qt}\right) + \frac{p_6p_{13}}{(q-q_1)}\left(e^{-q_1t} - e^{-qt}\right) + \sum_{n=1}^{\infty}p_7p_{13}\left(te^{-qt} - \frac{1}{(q-q_1)}\left(e^{-q_1t} - e^{-qt}\right)\right) + \\ p_1p_5p_{13}\left(1-e^{-qt}\right) + \frac{p_1p_6p_{13}}{(q-q_1)}\left(e^{-q_1t} - e^{-qt}\right) + \sum_{n=1}^{\infty}p_1p_7p_{13}\left(te^{-qt} - \frac{1}{(q-q_1)}\left(e^{-q_1t} - e^{-qt}\right)\right) \\ + \sum_{n=1}^{\infty}p_2p_5p_{13}te^{-qt} - m_4p_5\left(1-e^{-qt}\right) + \sum_{n=1}^{\infty}\frac{p_2p_6p_{13}}{q_1}\left(e^{-(q-q_1)t} - e^{-qt}\right) - \\ \frac{m_4p_6}{(q-q_1)}\left(e^{-q_1t} - e^{-qt}\right) - \sum_{n=1}^{\infty}m_4p_7\left(te^{-qt} - \frac{1}{(q-q_1)}\left(e^{-q_1t} - e^{-qt}\right)\right)\end{array}\right) -$$

$$
\left.\begin{aligned}
&-\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\left(\begin{aligned}&\frac{4\lambda_1}{q}\left(e^{-2qt}-e^{-qt}\right)-\left(\frac{p_{11}p_2}{q_2}\left(e^{-(q+q_2)t}-e^{-qt}\right)-\frac{p_{12}p_2}{q_3}\left(e^{-(q+q_3)t}-e^{-qt}\right)\right)+\\&\sum_{n=1}^{\infty}p_2p_7p_{13}\left(\frac{1}{-q}\left(e^{-2qt}-e^{-qt}\right)-\frac{1}{q_1}\left(e^{-(q+q_1)t}-e^{-qt}\right)\right)\end{aligned}\right)\times\\
&\frac{2}{3}\frac{\left(2-3(-1)^n+(-1)^{3n}\right)}{n\pi}+2\left(\lambda_1b_3^2+p_9(1+p_1)+p_8(1+p_1)\right)\left(1-e^{-qt}\right)\times\frac{\left(1-(-1)^n\right)}{n\pi}-\\
&\sum_{n=1}^{\infty}2b_3\lambda_1t\,e^{-qt}\times\frac{\left(1-(-1)^{2n}\right)}{n\pi}+\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{8\lambda_1}{3}t\,e^{-qt}\times\frac{\left(1-(-1)^{3n}\right)}{n\pi}+\\
&2\left(p_{10}(1+p_1)-p_1p_9-p_1p_8\right)\frac{2(-1)^n}{qn\pi}\left(1-e^{-qt}\right)-4p_1p_{10}\frac{\left(-2+2(-1)^n-n^2\pi^2(-1)^n\right)}{qn^3\pi^3}\left(1-e^{-qt}\right)
\end{aligned}\right\}\quad(3.178)
$$

Therefore, we have equation (3.179)

$$
\theta_1(\eta,t)=\sum_{n=1}^{\infty}\theta_{1n}(t)\sin n\pi\eta \qquad (3.179)
$$

Differentiating equation (3.15), we have Equation (3.180)

$$\left.\begin{aligned}
&\frac{\partial X_0}{\partial \eta}=\sum_{n=1}^{\infty} n\pi\left(p_5+p_6 e^{-q_1 t}+\sum_{n=1}^{\infty} p_7\left(e^{-qt}-e^{-q_1 t}\right)\right)\cos n\pi\eta \\
&\frac{\partial^2 X_0}{\partial \eta^2}=-\sum_{n=1}^{\infty} n^2\pi^2\left(p_5+p_6 e^{-q_1 t}+\sum_{n=1}^{\infty} p_7\left(e^{-qt}-e^{-q_1 t}\right)\right)\sin n\pi\eta \\
&\theta_0 \frac{\partial^2 X_0}{\partial \eta^2}=\left(b_3(1-\eta)+\sum_{n=1}^{\infty}\frac{2}{n\pi} e^{-qt}\sin n\pi\eta\right)\times \\
&\left(-\sum_{n=1}^{\infty} n^2\pi^2\left(p_5+p_6 e^{-q_1 t}+\sum_{n=1}^{\infty} p_7\left(e^{-qt}-e^{-q_1 t}\right)\right)\sin n\pi\eta\right) \\
&=-\sum_{n=1}^{\infty} n^2\pi^2 b_3\left(p_5+p_6 e^{-q_1 t}+\sum_{n=1}^{\infty} p_7\left(e^{-qt}-e^{-q_1 t}\right)\right)\sin n\pi\eta+ \\
&\sum_{n=1}^{\infty} n^2\pi^2 b_3\left(p_5+p_6 e^{-q_1 t}+\sum_{n=1}^{\infty} p_7\left(e^{-qt}-e^{-q_1 t}\right)\right)\eta\sin n\pi\eta- \\
&\sum_{n=1}^{\infty}\sum_{n=1}^{\infty} n^2\pi^2\left(p_5 e^{-qt}+p_6\left(e^{-(q_1+q)t}-e^{-q_1 t}\right)+\sum_{n=1}^{\infty} p_7\left(e^{-2qt}-e^{-(q_1+q)t}\right)\right)\sin^2 n\pi\eta \\
&\frac{\partial \theta_0}{\partial \eta}\frac{\partial X_0}{\partial \eta}=\left(-b_3+\sum_{n=1}^{\infty} 2e^{-qt}\cos n\pi\eta\right)\left(\sum_{n=1}^{\infty} n\pi\left(p_5+p_6 e^{-q_1 t}+\sum_{n=1}^{\infty} p_7\left(e^{-qt}-e^{-q_1 t}\right)\right)\cos n\pi\eta\right) \\
&=-\sum_{n=1}^{\infty} n\pi b_3\left(p_5+p_6 e^{-q_1 t}+\sum_{n=1}^{\infty} p_7\left(e^{-qt}-e^{-q_1 t}\right)\right)\cos n\pi\eta+ \\
&\sum_{n=1}^{\infty}\sum_{n=1}^{\infty} n^2\pi^2\left(p_5 e^{-qt}+p_6\left(e^{-(q_1+q)t}-e^{-q_1 t}\right)+\sum_{n=1}^{\infty} p_7\left(e^{-2qt}-e^{-(q_1+q)t}\right)\right)\cos^2 n\pi\eta
\end{aligned}\right\}(3.180)$$

Then equation (3.128) becomes

$$
\left.
\begin{aligned}
&\frac{\partial X_1}{\partial t} = D_1\frac{\partial^2 X_1}{\partial \eta^2} + D_1\left(-\sum_{n=1}^{\infty} n^2\pi^2 b_3\left(p_5 + p_6 e^{-q_1 t} + \sum_{n=1}^{\infty} p_7\left(e^{-qt} - e^{-q_1 t}\right)\right)\sin n\pi\eta\right) - \\
&\sum_{n=1}^{\infty} n^2\pi^2 b_3\left(p_5 + p_6 e^{-q_1 t} + \sum_{n=1}^{\infty} p_7\left(e^{-qt} - e^{-q_1 t}\right)\right)\eta\sin n\pi\eta - \\
&\sum_{n=1}^{\infty}\sum_{n=1}^{\infty} 2n\pi\left(p_5 e^{-qt} + p_6\left(e^{-(q_1+q)t} - e^{-q_1 t}\right) + \sum_{n=1}^{\infty} p_7\left(e^{-2qt} - e^{-(q_1+q)t}\right)\right)\sin n\pi\eta - \\
&D_1\left(-\sum_{n=1}^{\infty} n\pi b_3\left(p_5 + p_6 e^{-q_1 t} + \sum_{n=1}^{\infty} p_7\left(e^{-qt} - e^{-q_1 t}\right)\right)\cos n\pi\eta\right) + \\
&\sum_{n=1}^{\infty}\sum_{n=1}^{\infty} 2n\pi\left(p_5 e^{-qt} + p_6\left(e^{-(q_1+q)t} - e^{-q_1 t}\right) + \sum_{n=1}^{\infty} p_7\left(e^{-2qt} - e^{-(q_1+q)t}\right)\right)\cos^2 n\pi\eta - \\
&\alpha_1\left(m_3\sum_{n=1}^{\infty}\begin{pmatrix}\left(tp_5 - \frac{p_6}{q_1}e^{-q_1 t} + \sum_{n=1}^{\infty} p_7\left(\frac{e^{-q_1 t}}{q_1} - \frac{e^{-qt}}{q}\right)\right)\sin n\pi\eta + p_8\left(1 + p_1(1-\eta)\right)t - \\ \sum_{n=1}^{\infty} p_8\frac{p_2}{q}e^{-qt}\sin n\pi\eta\end{pmatrix}\right)\times \\
&\left(\begin{aligned}&\sum_{n=1}^{\infty}\left(p_5 + p_6 e^{-q_1 t} + \sum_{n=1}^{\infty} p_7\left(e^{-qt} - e^{-q_1 t}\right)\right)\sin n\pi\eta + \\ &a_3 a^2 b\left(1 + (e-2)(1-\eta)b_3 + \sum_{n=1}^{\infty}\frac{2}{n\pi}e^{-qt}\sin n\pi\eta\right)\end{aligned}\right) + \\
&\alpha_1\left(-a_3 ab\left(-b_3(1-\eta) - \sum_{n=1}^{\infty}\frac{2}{n\pi}e^{-qt}\sin n\pi\eta\right)a\left(1 + (e-2)b_3(1-\eta) + \sum_{n=1}^{\infty}\frac{2}{n\pi}e^{-qt}\sin n\pi\eta\right)\right) \\
&-\alpha_1 X_1 \\
&X_1(\eta,t) = 0, \quad X_1(0,t) = 0, \quad X_1(1,t) = 0
\end{aligned}
\right\} \quad (3.181)
$$

Comparing equation (3.138) and equation (3.181), we have equation (3.182) to equation (3.186):

$f(\eta) = 0 \qquad \Rightarrow b_n = 0$ and

$$f(\eta,t)=\sum_{n=1}^{\infty}\left(\begin{array}{l}-D_1n^2\pi^2b_3p_5-D_1n^2\pi^2b_3p_6e^{-qt}-\sum_{n=1}^{\infty}D_1n^2\pi^2b_3\left(e^{-qt}-e^{-q_1t}\right)-\\ p_{26}p_5t-p_{26}p_1p_5-p_{26}\frac{p_6}{q_1}e^{-q_1t}-p_1p_{26}\frac{p_6}{q_1}e^{-q_1t}-\sum_{n=1}^{\infty}p_7p_{26}\times\\ \left(\frac{e^{-q_1t}}{q_1}-\frac{e^{-qt}}{q}\right)-\sum_{n=1}^{\infty}p_1p_7p_{26}-\alpha_1p_5\left(p_8(1+p)\right)t-\alpha_1p_6p_8(1+p)t\,e^{-q_1t}\\ -\alpha_1\sum_{n=1}^{\infty}p_7p_8(1+p_1)t\left(e^{-qt}-e^{-q_1t}\right)-\sum_{n=1}^{\infty}p_{27}p_8p_2(1+p_1)t+a_3a^2\alpha_1\times\\ \frac{p_2}{q}e^{-qt}+p_1p_8p_{27}\frac{p_2}{q}e^{-qt}+p_{27}\frac{2}{n\pi}e^{-qt}+p_2p_{27}b_3e^{-qt}+p_{27}b_3\frac{2}{n\pi}e^{-qt}\end{array}\right)\sin n\pi\eta$$

$$-\sum_{n=1}^{\infty}\left(\begin{array}{l}D_1n^2\pi^2b_3p_5+n^2\pi^2b_3e^{-q_1t}+\sum_{n=1}^{\infty}D_1n^2\pi^2b_3p_7\left(e^{-qt}-e^{-q_1t}\right)+p_1p_5p_{26}+\\ p_1p_{26}\frac{p_2}{q_1}e^{-q_1t}+p_1p_{26}\left(\frac{e^{-q_1t}}{q_1}-\frac{e^{-qt}}{q}\right)+\alpha_1p_1p_5p_8t+\sum_{n=1}^{\infty}\alpha_1p_1p_5p_7\left(e^{-qt}-e^{-q_1t}\right)\\ +\alpha_1p_1p_6p_8te^{-q_1t}+p_1p_2p_8p_{27}e^{-qt}-p_1p_8p_{27}e^{-qt}-2p_{27}b_3\frac{2}{n\pi}e^{-qt}\end{array}\right)\eta\sin n\pi\eta$$

$$+\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\left(\begin{array}{l}D_12n\pi p_5e^{-qt}-D_12n\pi p_6e^{-(q+q_1)t}-\sum_{n=1}^{\infty}\alpha_1D_12n\pi p_7\left(e^{-2qt}-e^{-(q+q_1)t}\right)-\alpha_1m_3p_5^2\\ -\alpha_1m_3p_5p_6e^{-q_1t}-\sum_{n=1}^{\infty}\alpha_1m_3p_5p_7t\left(e^{-qt}-e^{-q_1t}\right)-p_2p_5p_{26}e^{-qt}-\alpha_1m_3p_5\times\\ \frac{p_6}{q_1}e^{-q_1t}-\alpha_1m_3\frac{p_6^2}{q_1}e^{-2q_1t}-\sum_{n=1}^{\infty}\alpha_1m_3\frac{p_6}{q_1}p_7\left(e^{-(q+q_1)t}-e^{-2qt}\right)-p_2p_{26}\frac{p_2}{q_1}e^{-(q+q_1)t}\\ -\sum_{n=1}^{\infty}\alpha_1m_3p_5p_7\left(\frac{e^{-q_1t}}{q_1}-\frac{e^{-qt}}{q}\right)-\alpha_1m_3p_5p_6\left(\frac{e^{-2q_1t}}{q_1}-\frac{e^{-(q+q_1)t}}{q}\right)-\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\alpha_1m_3p_7^2\times\\ \left(\frac{e^{-(q+q_1)t}}{q_1}-\frac{e^{-2qt}}{q}-\frac{e^{-2q_1t}}{q_1}+\frac{e^{-(q+q_1)t}}{q}\right)+\alpha_1p_5p_8\frac{p_2}{q}e^{-qt}+\alpha_1p_6p_8\frac{p_2}{q}e^{-(q+q_1)t}-\\ \sum_{n=1}^{\infty}p_2p_7p_{26}\left(\frac{e^{-(q+q_1)t}}{q_1}-\frac{e^{-2qt}}{q}\right)+\sum_{n=1}^{\infty}\alpha_1p_8p_7\frac{p_2}{q}\left(e^{-2qt}-e^{-(q+q_1)t}\right)+p_2p_{27}p_8\times\\ e^{-2qt}+p_{27}\frac{4}{n^2\pi^2}e^{-2qt}\end{array}\right)\sin^2 n\pi\eta$$

$$+\sum_{n=1}^{\infty}\left(D_1n\pi b_3p_5-D_1n\pi b_3p_6e^{-q_1t}-\sum_{n=1}^{\infty}D_1n\pi b_3p_7\left(e^{-2qt}-e^{-(q+q_1)t}\right)\right)\cos n\pi\eta+$$

$$\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\left(2n\pi p_5-2n\pi p_6e^{-(q+q_1)t}-\sum_{n=1}^{\infty}2n\pi p_7\left(e^{-2qt}-e^{-(q+q_1)t}\right)\right)\cos^2 n\pi\eta+$$

$$+\left(p_1 p_8 p_{27}(1+p_1)t + p_1 p_8 p_{27} t + p_1^2 p_8 p_{27} t - p_{27} b_3 - 2 p_1 p_{27} b_3\right)\eta + \left(p_1^2 p_8 p_{27} t + p_1 p_{27} b_3\right)\eta^2$$
$$+\left(p_8 p_{27}(1+p_1)t -_1 p_1 p_8 p_{27}(1+p_1) + p_{27} b_3 + p_1 p_8 p_{27} b_3 (1+p_1)\right) \qquad (3.182)$$

where;

$$p_{26} = \alpha_1 m_3 a_3 a^2 b, \quad p_{27} = \alpha_1 a_3 a^2 b, \quad p_1 = b_3(e-2), \quad p_2 = \frac{2}{n\pi}(e-2)$$

Then

$$f(\eta,t) = 2\sum_{n=1}^{\infty}\left(\begin{array}{l} -D_1 n^2\pi^2 b_3 p_5 - D_1 n^2 \pi^2 b_3 p_6 e^{-qt} - \sum_{n=1}^{\infty} D_1 n^2 \pi^2 b_3 \left(e^{-qt} - e^{-q_1 t}\right) - \\ p_{26} p_5 t - p_{26} p_1 p_5 - p_{26}\frac{p_6}{q_1} e^{-q_1 t} - p_1 p_{26}\frac{p_6}{q_1} e^{-q_1 t} - \sum_{n=1}^{\infty} p_7 p_{26} \times \\ \left(\frac{e^{-q_1 t}}{q_1} - \frac{e^{-qt}}{q}\right) - \sum_{n=1}^{\infty} p_1 p_7 p_{26} - \alpha_1 p_5 (p_8(1+p))t - \alpha_1 p_6 p_8 (1+p) t e^{-q_1 t} \\ -\alpha_1 \sum_{n=1}^{\infty} p_7 p_8 (1+p_1) t \left(e^{-qt} - e^{-q_1 t}\right) - \sum_{n=1}^{\infty} p_{27} p_8 p_2 (1+p_1) t + a_3 a^2 \alpha_1 \times \\ \frac{p_2}{q} e^{-qt} + p_1 p_8 p_{27}\frac{p_2}{q} e^{-qt} + p_{27}\frac{2}{n\pi} e^{-qt} + p_2 p_{27} b_3 e^{-qt} + p_{27} b_3 \frac{2}{n\pi} e^{-qt} \end{array}\right)\int_0^1 \sin^2 n\pi\eta \, d\eta$$

$$-2\sum_{n=1}^{\infty}\left(\begin{array}{l} D_1 n^2 \pi^2 b_3 p_5 + n^2 \pi^2 b_3 e^{-q_1 t} + \sum_{n=1}^{\infty} D_1 n^2 \pi^2 b_3 p_7 \left(e^{-qt} - e^{-q_1 t}\right) + p_1 p_5 p_{26} + \\ p_1 p_{26}\frac{p_2}{q_1} e^{-q_1 t} + p_1 p_{26}\left(\frac{e^{-q_1 t}}{q_1} - \frac{e^{-qt}}{q}\right) + \alpha_1 p_1 p_5 p_8 t + \sum_{n=1}^{\infty} \alpha_1 p_1 p_5 p_7 \left(e^{-qt} - e^{-q_1 t}\right) \\ + \alpha_1 p_1 p_6 p_8 t e^{-q_1 t} + p_1 p_2 p_8 p_{27} e^{-qt} - p_1 p_8 p_{27} e^{-qt} - 2 p_{27} b_3 \frac{2}{n\pi} e^{-qt} \end{array}\right)\int_0^1 \eta \sin^2 n\pi\eta \, d\eta$$

$$+2\sum_{n=1}^{\infty}\left(D_1 n\pi b_3 p_5 - D_1 n\pi b_3 p_6 e^{-q_1 t} - \sum_{n=1}^{\infty} D_1 n\pi b_3 p_7 \left(e^{-qt} - e^{-q_1 t}\right)\right)\int_0^1 \sin n\pi\eta \cos n\pi\eta \, d\eta$$
$$+2\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\left(2n\pi p_5 - 2n\pi p_6 e^{-(q+q_1)t} - \sum_{n=1}^{\infty} 2n\pi p_7 \left(e^{-2qt} - e^{-(q+q_1)t}\right)\right)\int_0^1 \sin n\pi\eta \cos^2 n\pi\eta +$$

$$
+2\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\left(\begin{array}{l}
D_1 2n\pi p_5 e^{-qt} - D_1 2n\pi p_6 e^{-(q+q_1)t} - \sum_{n=1}^{\infty}\alpha_1 D_1 2n\pi p_7\left(e^{-2qt} - e^{-(q+q_1)t}\right) - \alpha_1 m_3 p_5^2 \\
-\alpha_1 m_3 p_5 p_6 e^{-q_1 t} - \sum_{n=1}^{\infty}\alpha_1 m_3 p_5 p_7 t\left(e^{-qt} - e^{-q_1 t}\right) - p_2 p_5 p_{26} e^{-qt} - \alpha_1 m_3 p_5 \times \\
\frac{p_6}{q_1} e^{-q_1 t} - \alpha_1 m_3 \frac{p_6^2}{q_1} e^{-2q_1 t} - \sum_{n=1}^{\infty}\alpha_1 m_3 \frac{p_6}{q_1} p_7\left(e^{-(q+q_1)t} - e^{-2qt}\right) - p_2 p_{26}\frac{p_2}{q_1} e^{-(q+q_1)t} \\
-\sum_{n=1}^{\infty}\alpha_1 m_3 p_5 p_7\left(\frac{e^{-q_1 t}}{q_1} - \frac{e^{-qt}}{q}\right) - \alpha_1 m_3 p_5 p_6\left(\frac{e^{-2q_1 t}}{q_1} - \frac{e^{-(q+q_1)t}}{q}\right) - \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\alpha_1 m_3 p_7^2 \times \\
\left(\frac{e^{-(q+q_1)t}}{q_1} - \frac{e^{-2qt}}{q} - \frac{e^{-2q_1 t}}{q_1} + \frac{e^{-(q+q_1)t}}{q}\right) + \alpha_1 p_5 p_8 \frac{p_2}{q} e^{-qt} + \alpha_1 p_6 p_8 \frac{p_2}{q} e^{-(q+q_1)t} - \\
\sum_{n=1}^{\infty} p_2 p_7 p_{26}\left(\frac{e^{-(q+q_1)t}}{q_1} - \frac{e^{-2qt}}{q}\right) + \sum_{n=1}^{\infty}\alpha_1 p_8 p_7 \frac{p_2}{q}\left(e^{-2qt} - e^{-(q+q_1)t}\right) + p_2 p_{27} p_8 \times \\
e^{-2qt} + p_{27}\frac{4}{n^2\pi^2} e^{-2qt}
\end{array}\right)\int_0^1 \sin^3 n\pi\eta\, d\eta
$$

$$
+2\left(p_1 p_8 p_{27}(1+p_1)t + p_1 p_8 p_{27} t + p_1^2 p_8 p_{27} t - p_{27} b_3 - 2 p_1 p_{27} b_3\right)\int_0^1 \eta \sin n\pi\eta
$$

$$
+2\left(p_1^2 p_8 p_{27} t + p_1 p_{27} b_3\right)\times\int_0^1 \eta^2 \sin n\pi\eta +
$$

$$
2\left(p_8 p_{27}(1+p_1)t -_1 p_1 p_8 p_{27}(1+p_1) + p_{27} b_3 + p_1 p_8 p_{27} b_3 (1+p_1)\right)\int_0^1 \eta \sin n\pi\eta
$$

Integrating with respect to η, we have equation (3.183)

$$
f(\eta,t) = -\sum_{n=1}^{\infty}\left(\begin{array}{l}
D_1 n^2\pi^2 b_3 p_5 + n^2\pi^2 b_3 e^{-q_1 t} + \sum_{n=1}^{\infty} D_1 n^2\pi^2 b_3 p_7\left(e^{-qt} - e^{-q_1 t}\right) + p_1 p_5 p_{26} + \\
p_1 p_{26}\frac{p_2}{q_1} e^{-q_1 t} + p_1 p_{26}\left(\frac{e^{-q_1 t}}{q_1} - \frac{e^{-qt}}{q}\right) + \alpha_1 p_1 p_5 p_8 t + \sum_{n=1}^{\infty}\alpha_1 p_1 p_5 p_7\left(e^{-qt} - e^{-q_1 t}\right) \\
+\alpha_1 p_1 p_6 p_8 t e^{-q_1 t} + p_1 p_2 p_8 p_{27} e^{-qt} - p_1 p_8 p_{27} e^{-qt} - 2 p_{27} b_3 \frac{2}{n\pi} e^{-qt}
\end{array}\right)\frac{\left(1 - n^2\pi^2(-1)^{2n}\right)}{2n^2\pi^2}
$$

$$+\sum_{n=1}^{\infty}\begin{pmatrix} -D_1n^2\pi^2b_3p_5-D_1n^2\pi^2b_3p_6\,e^{-qt}-\sum_{n=1}^{\infty}D_1n^2\pi^2b_3\left(e^{-qt}-e^{-q_1t}\right)- \\ p_{26}p_5t-p_{26}p_1p_5-p_{26}\frac{p_6}{q_1}e^{-q_1t}-p_1p_{26}\frac{p_6}{q_1}e^{-q_1t}-\sum_{n=1}^{\infty}p_7p_{26}\times \\ \left(\frac{e^{-q_1t}}{q_1}-\frac{e^{-qt}}{q}\right)-\sum_{n=1}^{\infty}p_1p_7p_{26}-\alpha_1p_5\left(p_8(1+p)\right)t-\alpha_1p_6p_8(1+p)t\,e^{-q_1t} \\ -\alpha_1\sum_{n=1}^{\infty}p_7p_8(1+p_1)t\left(e^{-qt}-e^{-q_1t}\right)-\sum_{n=1}^{\infty}p_{27}p_8p_2(1+p_1)t+a_3a^2\alpha_1\times \\ \frac{p_2}{q}e^{-qt}+p_1p_8p_{27}\frac{p_2}{q}e^{-qt}+p_{27}\frac{2}{n\pi}e^{-qt}+p_2p_{27}b_3\,e^{-qt}+p_{27}b_3\frac{2}{n\pi}e^{-qt} \end{pmatrix}$$

$$+\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\begin{pmatrix} D_12n\pi p_5\,e^{-qt}-D_12n\pi p_6\,e^{-(q+q_1)t}-\sum_{n=1}^{\infty}\alpha_1D_12n\pi p_7\left(e^{-2qt}-e^{-(q+q_1)t}\right)- \\ \alpha_1m_3p_5^2-\alpha_1m_3p_5p_6\,e^{-q_1t}-\sum_{n=1}^{\infty}\alpha_1m_3p_5p_7t\left(e^{-qt}-e^{-q_1t}\right)-p_2p_5p_{26}\,e^{-qt}- \\ \alpha_1m_3p_5\frac{p_6}{q_1}e^{-q_1t}-\alpha_1m_3\frac{p_6^2}{q_1}e^{-2q_1t}-\sum_{n=1}^{\infty}\alpha_1m_3\frac{p_6}{q_1}p_7\left(e^{-(q+q_1)t}-e^{-2qt}\right)- \\ p_2p_{26}\frac{p_2}{q_1}e^{-(q+q_1)t}-\sum_{n=1}^{\infty}\alpha_1m_3p_5p_7\left(\frac{e^{-q_1t}}{q_1}-\frac{e^{-qt}}{q}\right)-\alpha_1m_3p_5p_6\left(\frac{e^{-2q_1t}}{q_1}-\frac{e^{-(q+q_1)t}}{q}\right) \\ -\alpha_1p_6p_8\frac{p_2}{q}e^{-(q+q_1)t}-\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\alpha_1m_3p_7^2\left(\frac{e^{-(q+q_1)t}}{q_1}-\frac{e^{-2qt}}{q}-\frac{e^{-2q_1t}}{q_1}+\frac{e^{-(q+q_1)t}}{q}\right)+ \\ \alpha_1p_5p_8\frac{p_2}{q}e^{-qt}+\sum_{n=1}^{\infty}p_2p_7p_{26}\left(\frac{e^{-(q+q_1)t}}{q_1}-\frac{e^{-2qt}}{q}\right)+ \\ \sum_{n=1}^{\infty}\alpha_1p_8p_7\frac{p_2}{q}\left(e^{-2qt}-e^{-(q+q_1)t}\right)+p_2p_{27}p_8\,e^{-2qt}+p_{27}\frac{4}{n^2\pi^2}e^{-2qt} \end{pmatrix}\frac{2\left(2-3(-1)^n+(-1)^{3n}\right)}{3n\pi}$$

$$+\sum_{n=1}^{\infty}\left(D_1n\pi b_3p_5-D_1n\pi b_3p_6\,e^{-q_1t}-\sum_{n=1}^{\infty}D_1n\pi b_3p_7\left(e^{-qt}-e^{-q_1t}\right)\right)\frac{\left(1-(-1)^{2n}\right)}{n\pi}$$

$$+\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\left(2n\pi p_5-2n\pi p_6\,e^{-(q+q_1)t}-\sum_{n=1}^{\infty}2n\pi p_7\left(e^{-2qt}-e^{-(q+q_1)t}\right)\right)\frac{2\left(1-(-1)^{3n}\right)}{3}$$

$$+2\left(p_1p_8p_{27}(1+p_1)t+p_1p_8p_{27}t+p_1^2p_8p_{27}t-p_{27}b_3-2p_1p_{27}b_3\right)\frac{2(-1)^n}{n\pi}$$

$$+2\left(p_1^2p_8p_{27}t+p_1p_{27}b_3\right)\times\frac{2\left(-2+2(-1)^n-n\pi(-1)^n\right)}{n^3\pi^3}$$

$$+2\left(p_8p_{27}(1+p_1)t-_1\,p_1p_8p_{27}(1+p_1)+p_{27}b_3+p_1p_8p_{27}b_3(1+p_1)\right)\times\frac{\left(1-(-1)^n\right)}{n\pi}\quad(3.183)$$

Then

$$X_{1n}(t)=\sum_{n=1}^{\infty}\left(\begin{array}{l}\left(-\sum_{n=1}^{\infty}p_1p_7p_{26}-D_1n^2\pi^2b_3p_5-p_{26}p_1p_5\right)e^{-q_1t}\int_0^t e^{q_1\tau}d\tau-\left(D_1n^2\pi^2b_3p_6-p_{27}\frac{2}{n\pi}\right)\times\\ e^{-q_1t}\int_0^t e^{(q_1-q)\tau}d\tau+\left(-\sum_{n=1}^{\infty}p_{27}p_8p_2(1+p_1)t-\alpha_1p_5(p_8(1+p))t-p_{26}p_5t\right)e^{-q_1t}\int_0^t e^{q_1\tau}d\tau-\\ \sum_{n=1}^{\infty}p_7p_{26}e^{-q_1t}\left(\frac{1}{q_1}\int_0^t d\tau-\frac{1}{q}\int_0^t e^{(q_1-q)\tau}d\tau\right)-\left(\alpha_1\sum_{n=1}^{\infty}p_7p_8(1+p_1)t+\sum_{n=1}^{\infty}D_1n^2\pi^2b_3\right)e^{-q_1t}\times\\ \left(\int_0^t e^{(q_1-q)\tau}d\tau-\int_0^t d\tau\right)-\left(p_{26}\frac{p_6}{q_1}+p_1p_{26}\frac{p_6}{q_1}+\alpha_1p_6p_8(1+p)t\right)e^{-q_1t}\int_0^t d\tau+\\ \left(a_3a^2\alpha_1\frac{p_2}{q}+p_1p_8p_{27}\frac{p_2}{q}+p_2p_{27}b_3+p_{27}b_3\frac{2}{n\pi}\right)e^{-q_1t}\int_0^t e^{(q_1-q)\tau}d\tau\end{array}\right)$$

$$-\sum_{n=1}^{\infty}\left(\begin{array}{l}\left(D_1n^2\pi^2b_3p_5+p_1p_5p_{26}+\alpha_1p_1p_5p_8t\right)e^{-q_1t}\int_0^t e^{q_1\tau}d\tau+\left(n^2\pi^2b_3+p_1p_{26}\frac{p_2}{q_1}\right)\times\\ e^{-q_1t}\int_0^t d\tau+\alpha_1p_1p_6p_8t\,e^{-q_1t}\int_0^t d\tau+p_1p_{26}e^{-q_1t}\left(\frac{1}{q_1}\int_0^t d\tau-\frac{1}{q}\int_0^t e^{(q_1-q)\tau}d\tau\right)+\\ \left(\sum_{n=1}^{\infty}\alpha_1p_1p_5p_7+\sum_{n=1}^{\infty}D_1n^2\pi^2b_3p_7\right)e^{-q_1t}\left(\int_0^t e^{(q_1-q)\tau}d\tau-\int_0^t d\tau\right)+\\ \left(p_1p_2p_8p_{27}-p_1p_8p_{27}-2p_{27}b_3\frac{2}{n\pi}\right)e^{-q_1t}\int_0^t e^{(q_1-q)\tau}d\tau\end{array}\right)\frac{\left(1-n^2\pi^2(-1)^{2n}\right)}{2n^2\pi^2}$$

$$+\sum_{n=1}^{\infty}\left(D_1n\pi b_3p_5e^{-q_1t}\int_0^t e^{q\tau}d\tau-D_1n\pi b_3p_6e^{-q_1t}\int_0^t d\tau-\sum_{n=1}^{\infty}D_1n\pi b_3p_7e^{-q_1t}\left(\int_0^t e^{(q_1-q)t}d\tau-\int_0^t d\tau\right)\right)\times$$

$$\frac{\left(1-(-1)^{2n}\right)}{n\pi}+\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\left(\begin{array}{l}2n\pi p_5e^{-q_1t}\int_0^t e^{q\tau}d\tau-2n\pi p_6e^{-q_1t}\int_0^t e^{-q\tau}d\tau-\\ \sum_{n=1}^{\infty}2n\pi p_7e^{-q_1t}\left(\int_0^t e^{(q_1-2q)\tau}d\tau-\int_0^t e^{-q\tau}d\tau\right)\end{array}\right)\frac{2\left(1-(-1)^{3n}\right)}{3}$$

$$+2\left(p_1p_8p_{27}(1+p_1)t+p_1p_8p_{27}t+p_1^2p_8p_{27}t-p_{27}b_3-2p_1p_{27}b_3\right)e^{-q_1t}\int_0^t e^{q_1\tau}d\tau\frac{2(-1)^n}{n\pi}$$

$$+2\left(p_1^2p_8p_{27}t+p_1p_{27}b_3\right)e^{-q_1t}\int_0^t e^{q_1\tau}d\tau\frac{2\left(-2+2(-1)^n-n\pi(-1)^n\right)}{n^3\pi^3}$$

$$+2\left(p_8p_{27}(1+p_1)t-_1p_1p_8p_{27}(1+p_1)+p_{27}b_3+p_1p_8p_{27}b_3(1+p_1)\right)e^{-q_1t}\int_0^t e^{q_1\tau}d\tau\frac{\left(1-(-1)^n\right)}{n\pi}$$

$$
+\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\left(\begin{array}{l}
\left(D_1 2n\pi p_5+\alpha_1 p_5 p_8\frac{p_2}{q}-p_2 p_5 p_{26}\right)e^{-q_1 t}\int_0^t e^{(q_1-q)\tau}d\tau-\alpha_1 m_3 p_5^2 e^{-q_1 t}\int_0^t e^{q_1\tau}d\tau\\
-\left(p_2 p_{26}\frac{p_2}{q_1}-D_1 2n\pi p_6+\alpha_1 p_6 p_8\frac{p_2}{q}\right)e^{-q_1 t}\int_0^t e^{-q\tau}d\tau-\alpha_1 m_3\frac{p_6^2}{q_1}e^{-q_1 t}\int_0^t e^{q_1\tau}d\tau\\
-\left(\alpha_1 m_3 p_5 p_6+\alpha_1 m_3 p_5\frac{p_6}{q_1}\right)e^{-q_1 t}\int_0^t d\tau-\sum_{n=1}^{\infty}\alpha_1 m_3 p_5 p_7 t e^{-q_1 t}\times\\
\left(\int_0^t e^{(q_1-q)\tau}d\tau-\int_0^t d\tau\right)-\sum_{n=1}^{\infty}\alpha_1 m_3\frac{p_6}{q_1}p_7 e^{-q_1 t}\left(\int_0^t e^{-q\tau}d\tau-\int_0^t e^{(q_1-2q)\tau}d\tau\right)-\\
\sum_{n=1}^{\infty}\alpha_1 D_1 2n\pi p_7 e^{-q_1 t}\left(\int_0^t e^{(q_1-2q)\tau}d\tau-\int_0^t e^{-q\tau}d\tau\right)-\sum_{n=1}^{\infty}\alpha_1 m_3 p_5 p_7 e^{-q_1 t}\times\\
\left(\frac{1}{q_1}\int_0^t d\tau-\frac{1}{q}\int_0^t e^{(q_1-q)\tau}d\tau\right)-\alpha_1 m_3 p_5 p_6 e^{-q_1 t}\left(\frac{1}{q_1}\int_0^t e^{-q_1\tau}d\tau-\frac{1}{q}\int_0^t e^{-q\tau}d\tau\right)+\\
\left(p_2 p_{27} p_8+p_{27}\frac{4}{n^2\pi^2}\right)e^{-q_1 t}\int_0^t e^{(q_1-2q)\tau}d\tau+\sum_{n=1}^{\infty}p_2 p_7 p_{26}e^{-q_1 t}\\
\left(\frac{1}{q_1}\int_0^t e^{-q\tau}-\frac{1}{q}\int_0^t e^{(q_1-2q)\tau}d\tau\right)-\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\alpha_1 m_3 p_7^2\times\\
e^{-q_1 t}\left(\frac{1}{q_1}\int_0^t e^{-q\tau}d\tau-\frac{1}{q}\int_0^t e^{(q_1-2q)\tau}d\tau-\frac{1}{q_1}\int_0^t e^{-q_1\tau}d\tau+\frac{1}{q}\int_0^t e^{-q\tau}d\tau\right)+\\
\sum_{n=1}^{\infty}\alpha_1 p_8 p_7\frac{p_2}{q}e^{-q_1 t}\left(\int_0^t e^{(q_1-2q)\tau}d\tau-\int_0^t e^{-q\tau}d\tau\right)
\end{array}\right)\frac{2\left(2-3(-1)^n+(-1)^{3n}\right)}{3n\pi} \quad (3.184)
$$

Integrating equation (3.184) with respect to τ, we have equation (3.185)

$$
X_{1n}(t)=\sum_{n=1}^{\infty}\left(\begin{array}{l}
\left(-\sum_{n=1}^{\infty}p_1 p_7 p_{26}-D_1 n^2\pi^2 b_3 p_5-p_{26}p_1 p_5\right)\left(\frac{1}{q_1}\left(1-e^{-q_1 t}\right)\right)-\left(D_1 n^2\pi^2 b_3 p_6-p_{27}\frac{2}{n\pi}\right)\times\\
\frac{1}{(q_1-q)}\left(e^{-qt}-e^{-q_1 t}\right)+\left(-\sum_{n=1}^{\infty}p_{27}p_8 p_2(1+p_1)t-\alpha_1 p_5\left(p_8(1+p)\right)t-p_{26}p_5 t\right)\frac{1}{q_1}\left(1-e^{-q_1 t}\right)-\\
\sum_{n=1}^{\infty}p_7 p_{26}\left(\frac{te^{-q_1 t}}{q_1}-\frac{1}{q(q_1-q)}\left(e^{-qt}-e^{-q_1 t}\right)\right)-\left(\alpha_1\sum_{n=1}^{\infty}p_7 p_8(1+p_1)t+\sum_{n=1}^{\infty}D_1 n^2\pi^2 b_3\right)e^{-q_1 t}\times\\
\left(\frac{1}{(q_1-q)}\left(e^{-qt}-e^{-q_1 t}\right)-te^{-q_1 t}\right)-\left(p_{26}\frac{p_6}{q_1}+p_1 p_{26}\frac{p_6}{q_1}+\alpha_1 p_6 p_8(1+p)t\right)te^{-q_1 t}+\\
\left(a_3 a^2\alpha_1\frac{p_2}{q}+p_1 p_8 p_{27}\frac{p_2}{q}+p_2 p_{27}b_3+p_{27}b_3\frac{2}{n\pi}\right)\frac{1}{(q_1-q)}\left(e^{-qt}-e^{-q_1 t}\right)
\end{array}\right)
$$

$$-\sum_{n=1}^{\infty}\left(\begin{array}{l}\left(D_1 n^2\pi^2 b_3 p_5 + p_1 p_5 p_{26} + \alpha_1 p_1 p_5 p_8 t\right)\left(1-e^{-q_1 t}\right)+\left(n^2\pi^2 b_3 + p_1 p_{26}\frac{p_2}{q_1}\right)t\times \\ e^{-q_1 t} + \alpha_1 p_1 p_6 p_8 t^2 e^{-q_1 t} + p_1 p_{26}\left(\frac{t e^{-q_1 t}}{q_1} - \frac{1}{q(q_1-q)}\left(e^{-qt}-e^{-q_1 t}\right)\right)+ \\ \left(\sum_{n=1}^{\infty}\alpha_1 p_1 p_5 p_7 + \sum_{n=1}^{\infty} D_1 n^2\pi^2 b_3 p_7\right)\left(\frac{1}{(q_1-q)}\left(e^{-qt}-e^{-q_1 t}\right)-t e^{-q_1 t}\right)+ \\ \left(p_1 p_2 p_8 p_{27} - p_1 p_8 p_{27} - 2 p_{27} b_3 \frac{2}{n\pi}\right)\frac{1}{(q_1-q)}\left(e^{-qt}-e^{-q_1 t}\right)\end{array}\right)\frac{\left(1-n^2\pi^2(-1)^{2n}\right)}{2n^2\pi^2}$$

$$+\sum_{n=1}^{\infty}\left(D_1 n\pi b_3 p_5 \frac{1}{q}\left(e^{(q-q_1)t}-e^{-q_1 t}\right)-D_1 n\pi b_3 p_6 t e^{-q_1 t} - \sum_{n=1}^{\infty} D_1 n\pi b_3 p_7\left(\frac{1}{(q_1-q)}\left(e^{-qt}-e^{-q_1 t}\right)-t e^{-q_1 t}\right)\right)\times$$

$$\frac{\left(1-(-1)^{2n}\right)}{n\pi}+\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\left(\begin{array}{l}2n\pi p_5 \frac{1}{q}\left(e^{-(q+q_1)t}-e^{-q_1 t}\right)-2n\pi p_6\left(e^{-(q_1-q)t}-e^{-q_1 t}\right)- \\ \sum_{n=1}^{\infty} 2n\pi p_7 e^{-q_1 t}\left(\frac{1}{(q_1-2q)}e^{-2qt}-\frac{1}{-q}e^{-(q+q_1)t}\right)\end{array}\right)\frac{2\left(1-(-1)^{3n}\right)}{3}$$

$$+2\left(p_1 p_8 p_{27}(1+p_1)t + p_1 p_8 p_{27} t + p_1^2 p_8 p_{27} t - p_{27} b_3 - 2 p_1 p_{27} b_3\right)\frac{1}{q_1}\left(1-e^{-q_1 t}\right)\frac{2(-1)^n}{n\pi}$$

$$+2\left(p_1^2 p_8 p_{27} t + p_1 p_{27} b_3\right)\frac{1}{q_1}\left(1-e^{-q_1 t}\right)\frac{2\left(-2+2(-1)^n - n\pi(-1)^n\right)}{n^3\pi^3}$$

$$+2\left(p_8 p_{27}(1+p_1)t -_1 p_1 p_8 p_{27}(1+p_1) + p_{27} b_3 + p_1 p_8 p_{27} b_3 (1+p_1)\right)\frac{1}{q_1}\left(1-e^{-q_1 t}\right)\frac{\left(1-(-1)^n\right)}{n\pi}$$

$$
+\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\left(\begin{array}{l}
\left(D_1 2n\pi p_5+\alpha_1 p_5 p_8\frac{p_2}{q}-p_2 p_5 p_{26}\right)\frac{1}{(q_1-q)}\left(e^{-qt}-e^{-q_1 t}\right)-\alpha_1 m_3 p_5^2\frac{1}{q_1}\left(1-e^{-q_1 t}\right)\\
-\left(p_2 p_{26}\frac{p_2}{q_1}-D_1 2n\pi p_6+\alpha_1 p_6 p_8\frac{p_2}{q}\right)\frac{1}{-q}\left(e^{-(q+q_1)t}-e^{-q_1 t}\right)-\alpha_1 m_3\frac{p_6^2}{q_1^2}\left(1-e^{-q_1 t}\right)\\
-\left(\alpha_1 m_3 p_5 p_6+\alpha_1 m_3 p_5\frac{p_6}{q_1}\right)t e^{-q_1 t}-\sum_{n=1}^{\infty}\alpha_1 m_3 p_5 p_7 t\left(\frac{1}{(q_1-q)}\left(e^{-qt}-e^{-q_1 t}\right)-t e^{-q_1 t}\right)\\
-\sum_{n=1}^{\infty}\alpha_1 m_3\frac{p_6}{q_1}p_7\left(\frac{1}{-q}\left(e^{-(q_1+q)t}-e^{-q_1 t}\right)-\frac{1}{(q_1-2q)}\left(e^{e^{-2qt}t}-e^{-q_1 t}\right)\right)-\sum_{n=1}^{\infty}\alpha_1 D_1 2\times\\
n\pi p_7\left(\frac{1}{(q_1-2q)}\left(e^{e^{-2qt}t}-e^{-q_1 t}\right)-\frac{1}{-q}\left(e^{-(q_1+q)t}-e^{-q_1 t}\right)\right)-\sum_{n=1}^{\infty}\alpha_1 m_3 p_5 p_7\times\\
\left(\frac{t}{q_1}-\frac{1}{q}\frac{1}{(q_1-q)}\left(e^{-qt}-e^{-q_1 t}\right)\right)-\alpha_1 m_3 p_5 p_6 e^{-q_1 t}\left(\frac{1}{-q_1^2}\left(e^{-2q_1\tau}-e^{-q_1\tau}\right)+\frac{1}{q^2}\left(e^{-(q_1+q)t}-e^{-q_1 t}\right)\right)+\\
\left(p_2 p_{27} p_8+p_{27}\frac{4}{n^2\pi^2}\right)\frac{1}{(q_1-2q)}\left(e^{e^{-2qt}t}-e^{-q_1 t}\right)+\sum_{n=1}^{\infty}p_2 p_7 p_{26}\times\\
\left(\frac{1}{-qq_1}\left(e^{-(q_1+q)t}-e^{-q_1 t}\right)-\frac{1}{q(q_1-2q)}\left(e^{e^{-2qt}t}-e^{-q_1 t}\right)\right)-\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\alpha_1 m_3 p_7^2\times\\
\left(\frac{1}{-q_1^2}\left(e^{-(q_1+q)t}-e^{-q_1 t}\right)-\frac{1}{q}\frac{1}{(q_1-2q)}\left(e^{e^{-2qt}t}-e^{-q_1 t}\right)-\frac{1}{q_1}\frac{1}{(q_1-2q)}\left(e^{e^{-2qt}t}-e^{-q_1 t}\right)-\frac{1}{q^2}\left(e^{-(q_1+q)t}-e^{-q_1 t}\right)\right)+\\
\sum_{n=1}^{\infty}\alpha_1 p_8 p_7\frac{p_2}{q}\left(\frac{1}{(q_1-2q)}\left(e^{e^{-2qt}t}-e^{-q_1 t}\right)+\frac{1}{q}\left(e^{-(q_1+q)t}-e^{-q_1 t}\right)\right)
\end{array}\right)\frac{2\left(2-3(-1)^n+(-1)^{3n}\right)}{3n\pi} \quad (3.185)
$$

Therefore,

$$
X_1(\eta,t)=\sum_{n=1}^{\infty}X_{1n}(t)\sin n\pi\eta \qquad (3.186)
$$

Differentiating equation (3.162) with respect to η, we have equations (3.18) and (3.188)

$$
\left.
\begin{aligned}
&\frac{\partial Y_0}{\partial \eta} = -1 + \sum_{n=1}^{\infty} 2e^{-q_3 t}\cos n\pi\eta \\
&\frac{\partial^2 Y_0}{\partial \eta^2} = -\sum_{n=1}^{\infty} 2n\pi\, e^{-q_3 t}\sin n\pi\eta \\
&\theta_0 \frac{\partial^2 Y_0}{\partial \eta^2} = \left(b_3(1-\eta) + \sum_{n=1}^{\infty} \frac{2}{n\pi} e^{-qt}\sin n\pi\eta \right)\left(-\sum_{n=1}^{\infty} 2n\pi\, e^{-q_3 t}\sin n\pi\eta \right) \\
&= -\sum_{n=1}^{\infty} b_3 2n\pi\, e^{-q_3 t}\sin n\pi\eta + \sum_{n=1}^{\infty} b_3 2n\pi\, e^{-q_3 t}\eta\sin n\pi\eta - \sum_{n=1}^{\infty}\sum_{n=1}^{\infty} 4e^{-(q+q_3)t}\sin^2 n\pi\eta \\
&\frac{\partial \theta}{\partial \eta}\frac{\partial Y_0}{\partial \eta} = \left(-b_3 + \sum_{n=1}^{\infty} 2e^{-qt}\cos n\pi\eta \right)\left(-1 + \sum_{n=1}^{\infty} 2e^{-q_3 t}\cos n\pi\eta \right) \\
&= b_3 - \sum_{n=1}^{\infty} 2b_3\, e^{-q_3 t}\cos n\pi\eta - \sum_{n=1}^{\infty} 2e^{-qt}\cos n\pi\eta + \sum_{n=1}^{\infty}\sum_{n=1}^{\infty} 4b_3\, e^{-(q+q_3)t}\cos^2 n\pi\eta
\end{aligned}
\right\} \quad (3.187)
$$

$$
\left.
\begin{aligned}
&\frac{\partial Y_1}{\partial t} = D_1 \frac{\partial^2 Y_1}{\partial \eta^2} + D_1 \left(\begin{aligned} &-\sum_{n=1}^{\infty} b_3 2n\pi\, e^{-q_3 t}\sin n\pi\eta + \sum_{n=1}^{\infty} b_3 2n\pi\, e^{-q_3 t}\eta\sin n\pi\eta - \\ &\sum_{n=1}^{\infty}\sum_{n=1}^{\infty} 4e^{-(q+q_3)t}\sin^2 n\pi\eta \end{aligned} \right) + \\
&D_1\left(b_3 - \sum_{n=1}^{\infty} 2b_3\, e^{-q_3 t}\cos n\pi\eta - \sum_{n=1}^{\infty} 2e^{-qt}\cos n\pi\eta + \sum_{n=1}^{\infty}\sum_{n=1}^{\infty} 4b_3\, e^{-(q+q_3)t}\cos^2 n\pi\eta \right) - \\
&p_{14}(1+p_1) + p_{14}(1+p_1)\eta - \sum_{n=1}^{\infty} p_{14}p_2\, e^{-qt}\sin n\pi\eta - p_{15}(1+p_1)\eta + p_{15}(1+p_1)\eta^2 + \\
&\sum_{n=1}^{\infty} p_{15}p_2\, e^{-qt}\sin n\pi\eta - \sum_{n=1}^{\infty} p_{16}(1+p_1)e^{-q_2 t}\sin n\pi\eta + \sum_{n=1}^{\infty} p_{16}(1+p_1)e^{-q_2 t}\eta\sin n\pi\eta - \\
&\sum_{n=1}^{\infty}\sum_{n=1}^{\infty} p_2 p_{16}\, e^{-(q+q_2)t}\sin^2 n\pi\eta - p_{17}(1+p_1) + p_{17}(1+p_1)\eta - \sum_{n=1}^{\infty} p_{17}p_2\, e^{-qt}\sin n\pi\eta - \\
&\sum_{n=1}^{\infty} p_{18}(1+p_1)\left(p_5 + p_6\, e^{-q_1 t} + \sum_{n=1}^{\infty} p_7\left(e^{-qt} - e^{-q_1 t}\right) \right)\sin n\pi\eta + \sum_{n=1}^{\infty} p_{18}b_3(1+p_1)\times \\
&\left(p_5 + p_6\, e^{-q_1 t} + \sum_{n=1}^{\infty} p_7\left(e^{-qt} - e^{-q_1 t}\right) \right)\eta\sin n\pi\eta - \sum_{n=1}^{\infty} p_{19}\frac{2}{n\pi}(1+p_1)e^{-q_3 t}\sin n\pi\eta - \\
&\sum_{n=1}^{\infty}\sum_{n=1}^{\infty} p_2 p_{18}\, e^{-qt}\left(p_5 + p_6\, e^{-q_1 t} + \sum_{n=1}^{\infty} p_7\left(e^{-qt} - e^{-q_1 t}\right) \right)\sin^2 n\pi\eta + \sum_{n=1}^{\infty} p_1 p_{19}\frac{2}{n\pi}e^{-q_3 t}\times \\
&\eta\sin n\pi\eta - \sum_{n=1}^{\infty}\sum_{n=1}^{\infty} p_2 p_{19}\frac{4}{n\pi}e^{-(q+q_2)t}\sin^2 n\pi\eta \\
&Y_1(\eta,0) = 0, \quad Y_1(0,t) = 0, \quad Y_1(1,t) = 0
\end{aligned}
\right\} \quad (3.188)
$$

Where, $p_{14} = ma_1 A_1,\ \ p_{15} = ma_1(B - A_1),\ \ p_{16} = ma_1 B_1,\ \ p_{17} = ma_1 b_1,\ \ p_{18} = ma_1 a^2$

$p_{14} = ma_1 b_2,\ \ p_1 = b_3(e-2),\ \ p_2 = \frac{2}{n\pi}(e-2)$

Comparing equation (3.138) and (3.189)

$f(\eta)=0 \qquad \Rightarrow b_n=0$ and

$$f(\eta,t)=\sum_{n=1}^{\infty}\left(\begin{array}{l}\left(2D_1b_3n\pi+p_2p_{19}\right)e^{-q_3t}+p_2p_{15}e^{-qt}+p_1p_{16}e^{-q_2t}\,p_1p_{18}\times\\ \left(p_5+p_6e^{-q_1t}+\sum_{n=1}^{\infty}p_7p_1p_{18}\left(e^{-qt}-e^{-q_1t}\right)\right)\end{array}\right)\eta\sin n\pi\eta+$$
$$\sum_{n=1}^{\infty}\left(\begin{array}{l}-2D_1b_3n\pi e^{-q_3t}-p_2p_{14}e^{-qt}-p_{16}(1+p_1)e^{-q_2t}-p_2p_{17}e^{-qt}-p_{19}(1+p_1)\times\\ \frac{2}{n\pi}e^{-q_3t}-p_{18}(1+p_1)\left(p_5+p_6e^{-q_1t}+\sum_{n=1}^{\infty}p_7\left(e^{-qt}-e^{-q_1t}\right)\right)\end{array}\right)\sin n\pi\eta$$
$$+\sum_{n=1}^{\infty}\left(\begin{array}{l}-\sum_{n=1}^{\infty}4D_1e^{-(q+q_3)t}-\sum_{n=1}^{\infty}p_2p_{16}e^{-(q+q_2)t}-\sum_{n=1}^{\infty}p_2p_{19}\frac{4}{n\pi}e^{-(q+q_3)t}-\sum_{n=1}^{\infty}p_{18}\times\\ \frac{2}{n\pi}e^{-qt}\left(p_5+p_6e^{-q_1t}+\sum_{n=1}^{\infty}p_7\left(e^{-qt}-e^{-q_1t}\right)\right)\end{array}\right)\sin^2 n\pi\eta$$
$$-\sum_{n=1}^{\infty}\left(2D_1b_3e^{-q_3t}+2D_1e^{-qt}\right)\cos n\pi\eta+\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}D_14b_3e^{-(q+q_3)t}\cos^2 n\pi\eta+p_1p_{15}\eta^2+$$
$$\left(p_1p_{14}-p_{15}(1+p_1)+p_1p_{17}\right)\eta+\left(p_{14}(1+p_1)-p_1p_{17}\right) \qquad (3.189)$$

Then

$$\left.\begin{array}{l}f_n(t)=2\sum_{n=1}^{\infty}\left(\begin{array}{l}\left(2D_1b_3n\pi+p_2p_{19}\right)e^{-q_3t}+p_2p_{15}e^{-qt}+p_1p_{16}e^{-q_2t}+p_1p_5p_{18}+\\ p_1p_6p_{18}e^{-q_1t}+\sum_{n=1}^{\infty}p_7p_1p_{18}\left(e^{-qt}-e^{-q_1t}\right)\end{array}\right)\int_0^1\eta\sin^2 n\pi\eta+\\
2\sum_{n=1}^{\infty}\left(\begin{array}{l}-\left(2D_1b_3n\pi+p_{19}(1+p_1)\right)\frac{2}{n\pi}e^{-q_3t}-\left(p_2p_{14}+p_2p_{17}\right)e^{-qt}-p_{16}(1+p_1)e^{-q_2t}\\ -p_5p_{18}(1+p_1)-p_6p_{18}(1+p_1)e^{-q_1t}-\sum_{n=1}^{\infty}p_7p_{18}(1+p_1)\left(e^{-qt}-e^{-q_1t}\right)\end{array}\right)\int_0^1\sin^2 n\pi\eta\\
+2\sum_{n=1}^{\infty}\left(\begin{array}{l}-\sum_{n=1}^{\infty}4D_1e^{-(q+q_3)t}-\sum_{n=1}^{\infty}p_2p_{16}e^{-(q+q_2)t}-\sum_{n=1}^{\infty}p_2p_{19}\frac{4}{n\pi}e^{-(q+q_3)t}-\sum_{n=1}^{\infty}p_5p_{18}\times\\ \frac{2}{n\pi}e^{-qt}-\sum_{n=1}^{\infty}p_6p_{18}\frac{2}{n\pi}e^{-(q+q_1)t}-\sum\sum_{n=1}^{\infty}p_7p_{18}\frac{2}{n\pi}\left(e^{-2qt}-e^{-(q+q_1)t}\right)\end{array}\right)\int_0^1\sin^3 n\pi\eta\\
-2\sum_{n=1}^{\infty}\left(2D_1b_3e^{-q_3t}+2D_1e^{-qt}\right)\int_0^1\sin n\pi\eta\cos n\pi\eta+2\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}D_14b_3e^{-(q+q_3)t}\int_0^1\sin n\pi\eta\cos^2 n\pi\eta\\
+2p_1p_{15}\int_0^1\eta^2\sin n\pi\eta+2\left(p_1p_{14}-p_{15}(1+p_1)+p_1p_{17}\right)\int_0^1\eta\sin n\pi\eta+2\left(p_{14}(1+p_1)-p_1p_{17}\right)\int_0^1\sin n\pi\eta\end{array}\right\} \qquad (3.190)$$

Integrating with respect to η, we have

$$
\left.
\begin{aligned}
f_n(t) = {} & \sum_{n=1}^{\infty}\left(\begin{aligned} & \left(2D_1b_3n\pi + p_2p_{19}\right)e^{-q_3t} + p_2p_{15}e^{-qt} + p_1p_{16}e^{-q_2t} + p_1p_5p_{18} + \\ & p_1p_6p_{18}e^{-q_1t} + \sum_{n=1}^{\infty} p_7p_1p_{18}\left(e^{-qt} - e^{-q_1t}\right) \end{aligned}\right)\frac{1 - n^2\pi^2 + (-1)^{2n}}{2n^2\pi^2} + \\
& \sum_{n=1}^{\infty}\left(\begin{aligned} & -\left(2D_1b_3n\pi + p_{19}(1+p_1)\right)\frac{2}{n\pi}e^{-q_3t} - \left(p_2p_{14} + p_2p_{17}\right)e^{-qt} - p_{16}(1+p_1)e^{-q_2t} \\ & - p_5p_{18}(1+p_1) - p_6p_{18}(1+p_1)e^{-q_1t} - \sum_{n=1}^{\infty} p_7p_{18}(1+p_1)\left(e^{-qt} - e^{-q_1t}\right) \end{aligned}\right) \\
& + \sum_{n=1}^{\infty}\left(\begin{aligned} & -\sum_{n=1}^{\infty} 4D_1e^{-(q+q_3)t} - \sum_{n=1}^{\infty} p_2p_{16}e^{-(q+q_2)t} - \sum_{n=1}^{\infty} p_2p_{19}\frac{4}{n\pi}e^{-(q+q_3)t} - \sum_{n=1}^{\infty} p_5p_{18} \times \\ & \frac{2}{n\pi}e^{-qt} - \sum_{n=1}^{\infty} p_6p_{18}\frac{2}{n\pi}e^{-(q+q_1)t} - \sum\sum_{n=1}^{\infty} p_7p_{18}\frac{2}{n\pi}\left(e^{-2qt} - e^{-(q+q_1)t}\right) \end{aligned}\right)\frac{2}{3}\frac{\left(2 - 3(-1)^n + (-1)^{3n}\right)}{n\pi} \\
& - \sum_{n=1}^{\infty}\left(2D_1b_3e^{-q_3t} + 2D_1e^{-qt}\right)\frac{\left(1 - (-1)^{2n}\right)}{n\pi} + \sum_{n=1}^{\infty}\sum_{n=1}^{\infty} D_14b_3e^{-(q+q_3)t}\frac{2}{3}\frac{\left(1 - (-1)^{3n}\right)}{n\pi} + 4p_1p_{15}\frac{\left(-2 + 2(-1)^n - n^2\pi^2(-1)^n\right)}{n^3\pi^3} \\
& + 2\left(p_1p_{14} - p_{15}(1+p_1) + p_1p_{17}\right)\frac{2(-1)^n}{n\pi} + 2\left(p_{14}(1+p_1) - p_{17}(1+p_1)\right)\frac{\left(1 - (-1)^n\right)}{n\pi}
\end{aligned}
\right\} \quad (3.191)
$$

Then

$$
\left.
\begin{aligned}
Y_n(t) = {} & \sum_{n=1}^{\infty}\left(\begin{aligned} & \left(2D_1b_3n\pi + p_2p_{19}\right)e^{-q_3t}\int_0^t d\tau + p_2p_{15}e^{-q_3t}\int_0^t e^{(q_3-q)\tau}d\tau + p_1p_{16}e^{-q_3t}\int_0^t e^{(q_3-q_2)\tau}d\tau + \\ & p_1p_5p_{18}e^{-q_3t}\int_0^t e^{q_3\tau}d\tau + p_1p_6p_{18}e^{-q_3t}\int_0^t e^{(q_3-q_1)\tau}d\tau + \sum_{n=1}^{\infty} p_7p_1p_{18}e^{-q_3t}\left(\int_0^t e^{(q_3-q)\tau}d\tau - \int_0^t e^{(q_3-q_1)\tau}d\tau\right) \end{aligned}\right)\frac{1 - n^2\pi^2 + (-1)^{2n}}{2n^2\pi^2} \\
& + \sum_{n=1}^{\infty}\left(\begin{aligned} & -\left(2D_1b_3n\pi + p_{19}(1+p_1)\right)\frac{2}{n\pi}e^{-q_3t}\int_0^t d\tau - \left(p_2p_{14} + p_2p_{17}\right)e^{-q_3t}\int_0^t e^{(q_3-q)\tau}d\tau - p_{16}(1+p_1)e^{-q_3t}\int_0^t e^{(q_3-q_2)\tau}d\tau \\ & - p_5p_{18}(1+p_1)e^{-q_3t}\int_0^t e^{q_3\tau}d\tau - p_6p_{18}(1+p_1)e^{-q_3t}\int_0^t e^{(q_3-q_1)\tau}d\tau - \sum_{n=1}^{\infty} p_7p_{18}(1+p_1)e^{-q_3t}\left(\int_0^t e^{(q_3-q)\tau}d\tau - \int_0^t e^{(q_3-q_1)\tau}d\tau\right) \end{aligned}\right) \\
& + \sum_{n=1}^{\infty}\left(\begin{aligned} & -\sum_{n=1}^{\infty} 4D_1e^{-q_3t}\int_0^t e^{-q\tau}d\tau - \sum_{n=1}^{\infty} p_2p_{16}e^{-q_3t}\int_0^t e^{(q_3-(q+q_2))\tau}d\tau - \sum_{n=1}^{\infty} p_2p_{19}\frac{4}{n\pi}e^{-q_3t}\int_0^t e^{-q\tau}d\tau - \\ & \sum_{n=1}^{\infty} p_5p_{18}\frac{2}{n\pi}e^{-q_3t}\int_0^t e^{(q_3-q)\tau}d\tau - \sum_{n=1}^{\infty} p_6p_{18}\frac{2}{n\pi}e^{-q_3t}\int_0^t e^{(q_3-(q+q_1))\tau}d\tau - \sum\sum_{n=1}^{\infty} p_7p_{18}\frac{2}{n\pi}e^{-q_3t} \times \\ & \left(\int_0^t e^{(q_3-q_2)\tau}d\tau - \int_0^t e^{(q_3-(q+q_1))\tau}d\tau\right) \end{aligned}\right)\frac{2}{3}\frac{\left(2 - 3(-1)^n + (-1)^{3n}\right)}{n\pi} \\
& - \sum_{n=1}^{\infty}\left(2D_1b_3e^{-q_3t}\int_0^t d\tau + 2D_1e^{-q_3t}\int_0^t e^{(q_3-q_1)\tau}d\tau\right)\frac{\left(1 - (-1)^{2n}\right)}{n\pi} + \sum_{n=1}^{\infty}\sum_{n=1}^{\infty} D_14b_3e^{-q_3t}\int_0^t e^{-q\tau}d\tau\frac{2}{3}\frac{\left(1 - (-1)^{3n}\right)}{n\pi} + \\
& 4p_1p_{15}e^{-q_3t}\int_0^t e^{q_3\tau}d\tau\frac{\left(-2 + 2(-1)^n - n^2\pi^2(-1)^n\right)}{n^3\pi^3} + 2\left(p_1p_{14} - p_{15}(1+p_1) + p_1p_{17}\right)e^{-q_3t}\int_0^t e^{q_3\tau}d\tau\frac{2(-1)^n}{n\pi} + \\
& 2\left(p_{14}(1+p_1) - p_{17}(1+p_1)\right)e^{-q_3t}\int_0^t e^{q_3\tau}d\tau\frac{\left(1 - (-1)^n\right)}{n\pi}
\end{aligned}
\right\} \quad (3.192)
$$

Integrating equation (3.192) with respect to τ , we have equation (3.193)

$$
\left.\begin{aligned}
Y_n(t) = {} & \sum_{n=1}^{\infty}\left(\begin{aligned}&(2D_1b_3n\pi + p_2p_{19})te^{-q_3t} + \left(\frac{p_2p_{15}}{(q_3-q)}\left(e^{-qt}-e^{-q_3t}\right)\right) + \left(\frac{p_1p_{16}}{(q_3-q_2)}\left(e^{-q_2t}-e^{-q_3t}\right)\right) + p_1p_5p_{18}\frac{1}{q_3}\left(1-e^{-q_3t}\right)\\ &+\frac{p_1p_6p_{18}}{(q_3-q_1)}\left(e^{-q_1t}-e^{-q_3t}\right)+\sum_{n=1}^{\infty}p_7p_1p_{18}e^{-q_3t}\left(\frac{1}{(q_3-q)}\left(e^{-qt}-e^{-q_3t}\right)-\frac{1}{(q_3-q_1)}\left(e^{-q_1t}-e^{-q_3t}\right)\right)\end{aligned}\right)\frac{1-n^2\pi^2+(-1)^{2n}}{2n^2\pi^2}\\
& +\sum_{n=1}^{\infty}\left(\begin{aligned}&-\left(2D_1b_3n\pi + p_{19}(1+p_1)\right)\frac{2}{n\pi}te^{-q_3t}-\left(p_2p_{14}+p_2p_{17}\right)\frac{1}{(q_3-q)}\left(e^{-qt}-e^{-q_3t}\right)-p_{16}(1+p_1)\frac{1}{(q_3-q_2)}\left(e^{-q_2t}-e^{-q_3t}\right)\\ &-p_5p_{18}(1+p_1)\frac{1}{q_3}\left(1-e^{-q_3t}\right)-p_6p_{18}(1+p_1)\frac{1}{(q_3-q_1)}\left(e^{-q_1t}-e^{-q_3t}\right)-\sum_{n=1}^{\infty}p_7p_{18}(1+p_1)e^{-q_3t}\times\\ &\left(\frac{1}{(q_3-q)}\left(e^{-qt}-e^{-q_3t}\right)-\frac{1}{(q_3-q_1)}\left(e^{-q_1t}-e^{-q_3t}\right)\right)\end{aligned}\right)\\
& +\sum_{n=1}^{\infty}\left(\begin{aligned}&\sum_{n=1}^{\infty}4D_1\frac{1}{q}\left(e^{-(q_3+q)t}-e^{-q_3t}\right)-\sum_{n=1}^{\infty}p_2p_{16}\frac{1}{(q_3-(q+q_2))}\left(e^{-(q+q_2)t}-e^{-q_3t}\right)+\sum_{n=1}^{\infty}p_2p_{19}\frac{4}{n\pi}\frac{1}{q}\left(e^{-(q_3+q)t}-e^{-q_3t}\right)-\\ &\sum_{n=1}^{\infty}p_5p_{18}\frac{2}{n\pi}\frac{1}{(q_3-q)}\left(e^{-qt}-e^{-q_3t}\right)-\sum_{n=1}^{\infty}p_6p_{18}\frac{2}{n\pi}\frac{1}{(q_3-(q+q_1))}\left(e^{-(q+q_1)t}-e^{-q_3t}\right)-\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}p_7p_{18}\frac{2}{n\pi}e^{-q_3t}\times\\ &\left(\frac{1}{(q_3-q_2)}\left(e^{-q_2t}-e^{-q_3t}\right)-\frac{1}{(q_3-(q+q_1))}\left(e^{-(q+q_1)t}-e^{-q_3t}\right)\right)\end{aligned}\right)\frac{2}{3}\frac{\left(2-3(-1)^n+(-1)^{3n}\right)}{n\pi}\\
& -\sum_{n=1}^{\infty}\left(2D_1b_3te^{-q_3t}+2D_1\frac{1}{(q_3-q_1)}\left(e^{-q_1t}-e^{-q_3t}\right)\right)\frac{\left(1-(-1)^{2n}\right)}{n\pi}-\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}D_14b_3\frac{1}{q}\left(e^{-(q_3+q)t}-e^{-q_3t}\right)\frac{2}{3}\frac{\left(1-(-1)^{3n}\right)}{n\pi}+\\
& 4p_1p_{15}\left(1-e^{-q_3t}\right)\frac{\left(-2+2(-1)^n-n^2\pi^2(-1)^n\right)}{n^3\pi^3}+2\left(p_1p_{14}-p_{15}(1+p_1)+p_1p_{17}\right)\left(1-e^{-q_3t}\right)\frac{2(-1)^n}{n\pi}+\\
& 2\left(p_{14}(1+p_1)-p_{17}(1+p_1)\right)\left(1-e^{-q_3t}\right)\frac{\left(1-(-1)^n\right)}{n\pi}
\end{aligned}\right\} \quad (3.193)
$$

Therefore, we obtain equation (3.194)

$$
Y_1(\eta, t) = \sum_{n=1}^{\infty} Y_{1n}(t)\sin n\pi\eta \qquad (3.194)
$$

Differentiating equation (3.162) with respect to η, we have equation (3.195)

$$\left.\begin{aligned}
&Z_0(\eta,t)=\sum_{n=1}^{\infty}\frac{2}{n\pi}e^{-q_3 t}\sin n\pi\eta\\
&\frac{\partial Z_0}{\partial\eta}=\sum_{n=1}^{\infty}2e^{-q_3 t}\cos n\pi\eta\\
&\frac{\partial^2 Z_0}{\partial\eta^2}=-\sum_{n=1}^{\infty}2n\pi e^{-q_3 t}\sin n\pi\eta\\
&\theta_0\frac{\partial^2 Z_0}{\partial\eta^2}=\left(b_3(1-\eta)+\sum_{n=1}^{\infty}\frac{2}{n\pi}e^{-qt}\sin n\pi\eta\right)\left(-\sum_{n=1}^{\infty}2n\pi e^{-q_3 t}\sin n\pi\eta\right)\\
&=-\sum_{n=1}^{\infty}b_3 2n\pi e^{-q_3 t}\sin n\pi\eta+\sum_{n=1}^{\infty}b_3 2n\pi e^{-q_3 t}\sin n\pi\eta-\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}2n\pi e^{-(q+q_3)t}\sin^2 n\pi\eta\\
&\frac{\partial\theta_0}{\partial\eta}\frac{\partial Z_0}{\partial\eta}=\left(-b_3+\sum_{n=1}^{\infty}2e^{-qt}\cos n\pi\eta\right)\left(\sum_{n=1}^{\infty}2e^{-q_3 t}\cos n\pi\eta\right)\\
&=-\sum_{n=1}^{\infty}2b_3 e^{-q_3 t}\cos n\pi\eta+\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}4e^{-(q+q_3)t}\cos^2 n\pi\eta
\end{aligned}\right\}\quad(3.195)$$

Then equation (3.130) yields equation (3.196)

$$\left.\begin{aligned}
&\frac{\partial Z_1}{\partial t}=D_1\frac{\partial^2 Z_1}{\partial\eta^2}+D_1\left(\begin{aligned}&-\sum_{n=1}^{\infty}b_3 2n\pi e^{-q_3 t}\sin n\pi\eta+\sum_{n=1}^{\infty}b_3 2n\pi e^{-q_3 t}\sin n\pi\eta-\\&\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}2n\pi e^{-(q+q_3)t}\sin^2 n\pi\eta\end{aligned}\right)+\\
&D_1\left(-\sum_{n=1}^{\infty}2b_3 e^{-q_3 t}\cos n\pi\eta+\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}4e^{-(q+q_3)t}\cos^2 n\pi\eta\right)+p_{20}(1+p_1)-p_1p_{20}\eta+\\
&\sum_{n=1}^{\infty}p_2p_{20}e^{-qt}\sin n\pi\eta+p_{21}(1+p_1)\eta-p_1p_{21}\eta^2+\sum_{n=1}^{\infty}p_2p_{21}e^{-qt}\eta\sin n\pi\eta+\sum_{n=1}^{\infty}p_{22}(1+p_1)\times\\
&e^{-q_2 t}\sin n\pi\eta-\sum_{n=1}^{\infty}p_1p_{22}e^{-q_2 t}\eta\sin n\pi\eta+\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}p_1p_{22}\,e^{-(q+q_3)t}\sin^2 n\pi\eta+p_{23}(1+p_1)-\\
&p_1p_{23}\eta+\sum_{n=1}^{\infty}p_2p_{23}e^{-qt}\sin n\pi\eta+\sum_{n=1}^{\infty}p_{24}(1+p_1)\left(p_5+p_6e^{-q_1 t}+\sum_{n=1}^{\infty}p_7\left(e^{-qt}-e^{-q_1 t}\right)\right)\sin n\pi\eta\\
&-\sum_{n=1}^{\infty}p_1p_{24}\left(p_5+p_6e^{-q_1 t}+\sum_{n=1}^{\infty}p_7\left(e^{-qt}-e^{-q_1 t}\right)\right)\eta\sin n\pi\eta+\sum_{n=1}^{\infty}p_{25}(1+p_1)\frac{2}{n\pi}e^{-q_3 t}\sin n\pi\eta\\
&+\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}p_2p_{24}e^{-qt}\left(p_5+p_6e^{-q_1 t}+\sum_{n=1}^{\infty}p_7\left(e^{-qt}-e^{-q_1 t}\right)\right)\sin^2 n\pi\eta-\sum_{n=1}^{\infty}\frac{2}{n\pi}p_1p_{25}e^{-q_3 t}\eta\sin n\pi\eta\\
&+\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{p_2p_{25}}{n\pi}e^{-(q+q_3)t}\cos^2 n\pi\eta\qquad(3.196)\\
&z_1(\eta,0)=0,\quad z_1(0,t)=0,\quad z_1(1,t)=0
\end{aligned}\right\}$$

Where

$$p_{20} = m_1 a_1 A_1, \quad p_{21} = m_1 a_1 (B - A_1)$$
$$p_{22} = m_1 a_1 B_1, \quad p_{23} = m_1 a_1 b_1$$
$$p_{24} = m_1 a_1 a^2, \quad p_{25} = m_1 a_1 b_2$$
$$p_1 = b_3 (e-2), \quad p_2 = \frac{2}{n\pi}(e-2)$$

Comparing equation (3.138) and equation (3.196)

$f(\eta) = 0 \qquad \Rightarrow b_n = 0$ and

$$f(\eta,t) = \sum_{n=1}^{\infty} \begin{pmatrix} D_1 2b_3 n\pi e^{-q_3 t} + p_2 p_{21} e^{-qt} - p_1 p_{22} - p_1 p_{24}\left(p_5 + p_6 e^{-q_1 t} + \sum_{n=1}^{\infty} p_7 \left(e^{-qt} - e^{-q_1 t}\right)\right) \\ - p_1 p_{25} e^{-q_3 t} \end{pmatrix} \eta \sin n\pi\eta$$
$$+ \sum_{n=1}^{\infty} \begin{pmatrix} - D_1 2b_3 n\pi e^{-q_3 t} + p_2 p_{20} e^{-qt} + p_{22}(1+p_1) e^{-q_2 t} + p_2 p_{23} e^{-qt} + p_{25}(1+p_1)\frac{2}{n\pi} e^{-q_2 t} + \\ p_{24}(1+p_1)\left(p_5 + p_6 e^{-q_1 t} + \sum_{n=1}^{\infty} p_7 \left(e^{-qt} - e^{-q_1 t}\right)\right) \end{pmatrix} \sin n\pi\eta$$
$$+ \sum_{n=1}^{\infty} \begin{pmatrix} -\sum_{n=1}^{\infty} D_1 4 e^{-(q+q_3)t} + \sum_{n=1}^{\infty} p_1 p_{22} e^{-(q+q_2)t} + \sum_{n=1}^{\infty} p_2 p_{24}\left(p_5 + p_6 e^{-q_1 t} + \sum_{n=1}^{\infty} p_7 \left(e^{-qt} - e^{-q_1 t}\right)\right) \\ + \sum_{n=1}^{\infty} \frac{p_2 p_{25}}{n\pi} e^{-(q+q_3)t} \end{pmatrix} \sin^2 n\pi\eta$$
$$- \sum_{n=1}^{\infty} D_1 2b_3 e^{-q_3 t} \cos n\pi\eta + \sum_{n=1}^{\infty}\sum_{n=1}^{\infty} 4 e^{-(q+q_3)t} \cos^2 n\pi\eta + \left(- p_1 p_{20} + p_{21}(1+p_1) - p_1 p_{23}\right)\eta - p_1 p_{21}\eta^2$$
$$\left(p_{20}(1+p_1) + p_{23}(1+p_1)\right) \tag{3.197}$$

Then

$$\begin{aligned}
f_n(t) = {} & 2\sum_{n=1}^{\infty}\left(\begin{array}{l}(D_1 2b_3 n\pi - p_1 p_{25})e^{-q_3 t} + p_2 p_{21} e^{-qt} - p_1 p_{24} p_6 e^{-q_1 t} - (p_1 p_{22} + p_1 p_{24} p_5) \\ -\sum_{n=1}^{\infty} p_1 p_{24} p_7 \left(e^{-qt} - e^{-q_1 t}\right)\end{array}\right)\int_0^1 \eta \sin^2 n\pi\eta \, d\eta \\
& + 2\sum_{n=1}^{\infty}\left(\begin{array}{l}\left(p_{25}(1+p_1)\dfrac{2}{n\pi} - D_1 2b_3 n\pi\right)e^{-q_3 t} + (p_2 p_{20} + p_2 p_{23})e^{-qt} + p_{22}(1+p_1)e^{-q_2 t} \\ + p_5 p_{24}(1+p_1) + p_6 p_{24}(1+p_1)e^{-q_1 t} + p_{24}(1+p_1)\sum_{n=1}^{\infty} p_7\left(e^{-qt} - e^{-q_1 t}\right)\end{array}\right)\int_0^1 \sin^2 n\pi\eta \, d\eta \\
& + 2\sum_{n=1}^{\infty}\left(\begin{array}{l}\sum_{n=1}^{\infty} p_1 p_{22} e^{-(q+q_2)t} + \sum_{n=1}^{\infty} p_2 p_{24} p_5 + \sum_{n=1}^{\infty} p_2 p_{24} p_6 e^{-q_1 t} + \\ \sum_{n=1}^{\infty}\sum_{n=1}^{\infty} p_2 p_{24} p_7\left(e^{-qt} - e^{-q_1 t}\right) + \sum_{n=1}^{\infty}\left(\dfrac{p_2 p_{25}}{n\pi} - D_1 4\right)e^{-(q+q_3)t}\end{array}\right)\int_0^1 \sin^3 n\pi\eta \, d\eta \\
& - 2\sum_{n=1}^{\infty} D_1 2b_3 e^{-q_3 t}\int_0^1 \sin n\pi\eta \cos n\pi\eta \, d\eta + 2\sum_{n=1}^{\infty}\sum_{n=1}^{\infty} 4e^{-(q+q_3)t}\int_0^1 \sin n\pi\eta \cos^2 n\pi\eta \, d\eta - 2p_1 p_{21}\int_0^1 \eta^2 \sin n\pi\eta \\
& + 2(-p_1 p_{20} + p_{21}(1+p_1) - p_1 p_{23})\int_0^1 \eta \sin n\pi\eta \, d\eta + 2(p_{20}(1+p_1) + p_{23}(1+p_1))\int_0^1 \sin n\pi\eta \, d\eta \qquad (3.198)
\end{aligned}$$

Integrating equation (3.198) with respect to η, to obtain equation (3.199)

$$\begin{aligned}
f_n(t) = {} & \sum_{n=1}^{\infty}\left(\begin{array}{l}(D_1 2b_3 n\pi - p_1 p_{25})e^{-q_3 t} + p_2 p_{21} e^{-qt} - p_1 p_{24} p_6 e^{-q_1 t} - (p_1 p_{22} + p_1 p_{24} p_5) \\ -\sum_{n=1}^{\infty} p_1 p_{24} p_7\left(e^{-qt} - e^{-q_1 t}\right)\end{array}\right)\frac{1 - n^2\pi^2 + (-1)^{2n}}{2n^2\pi^2} \\
& + 2\sum_{n=1}^{\infty}\left(\begin{array}{l}\left(p_{25}(1+p_1)\dfrac{2}{n\pi} - D_1 2b_3 n\pi\right)e^{-q_3 t} + (p_2 p_{20} + p_2 p_{23})e^{-qt} + p_{22}(1+p_1)e^{-q_2 t} \\ + p_5 p_{24}(1+p_1) + p_6 p_{24}(1+p_1)e^{-q_1 t} + p_{24}(1+p_1)\sum_{n=1}^{\infty} p_7\left(e^{-qt} - e^{-q_1 t}\right)\end{array}\right) \\
& + 2\sum_{n=1}^{\infty}\left(\begin{array}{l}\sum_{n=1}^{\infty} p_1 p_{22} e^{-(q+q_2)t} + \sum_{n=1}^{\infty} p_2 p_{24} p_5 + \sum_{n=1}^{\infty} p_2 p_{24} p_6 e^{-q_1 t} + \\ \sum_{n=1}^{\infty}\sum_{n=1}^{\infty} p_2 p_{24} p_7\left(e^{-qt} - e^{-q_1 t}\right) + \sum_{n=1}^{\infty}\left(\dfrac{p_2 p_{25}}{n\pi} - D_1 4\right)e^{-(q+q_3)t}\end{array}\right)\frac{2}{3}\frac{\left(2 - 3(-1)^n + (-1)^{3n}\right)}{n\pi} \\
& - \sum_{n=1}^{\infty} D_1 2b_3 e^{-q_3 t}\frac{\left(1 - (-1)^{2n}\right)}{n\pi} + \sum_{n=1}^{\infty}\sum_{n=1}^{\infty} 4e^{-(q+q_3)t}\frac{2}{3}\frac{\left(1 - (-1)^{3n}\right)}{n\pi} - 2p_1 p_{21}\frac{\left(-2 + 2(-1)^n - n^2\pi^2(-1)^n\right)}{n^3\pi^3} \\
& + 2(-p_1 p_{20} + p_{21}(1+p_1) - p_1 p_{23})\frac{2(-1)^n}{n\pi} + 2(p_{20}(1+p_1) + p_{23}(1+p_1))\frac{\left(1 - (-1)^n\right)}{n\pi} \qquad (3.199)
\end{aligned}$$

Then

$$Z_{1n}(t)=\sum_{n=1}^{\infty}\left(\begin{array}{l}(D_1 2b_3 n\pi - p_1 p_{25})e^{-q_3 t}\int_0^t d\tau + p_2 p_{21} e^{-q_3 t}\int_0^t e^{(q_3-q)\tau}d\tau - p_1 p_{24} p_6 e^{-q_3 t}\int_0^t e^{(q_3-q_1)\tau}d\tau \\ -(p_1 p_{22}+p_1 p_{24} p_5)e^{-q_3 t}\int_0^t e^{q_3\tau}d\tau - \sum_{n=1}^{\infty} p_1 p_{24} p_7 e^{-q_3 t}\left(\int_0^t e^{(q_3-q)\tau}d\tau - \int_0^t e^{(q_3-q_1)\tau}d\tau\right)\end{array}\right)\frac{1-n^2\pi^2+(-1)^{2n}}{2n^2\pi^2}$$

$$+2\sum_{n=1}^{\infty}\left(\begin{array}{l}\left(p_{25}(1+p_1)\frac{2}{n\pi}-D_1 2b_3 n\pi\right)e^{-q_3 t}\int_0^t d\tau + (p_2 p_{20}+p_2 p_{23})e^{-q_3 t}\int_0^t e^{(q_3-q)\tau}d\tau + p_{22}(1+p_1)\times \\ e^{-q_3 t}\int_0^t e^{(q_3-q_1)\tau}d\tau + p_5 p_{24}(1+p_1)e^{-q_3 t}\int_0^t e^{q_3\tau}d\tau + p_6 p_{24}(1+p_1)e^{-q_3 t}\int_0^t e^{(q_3-q_1)\tau}d\tau + p_{24}(1+p_1)\times \\ \sum_{n=1}^{\infty} p_7 e^{-q_3 t}\left(\int_0^t e^{(q_3-q)\tau}d\tau - \int_0^t e^{(q_3-q_1)\tau}d\tau\right)\end{array}\right)$$

$$+2\sum_{n=1}^{\infty}\left(\begin{array}{l}\sum_{n=1}^{\infty} p_1 p_{22} e^{-q_3 t}\int_0^t e^{(q_3-(q+q_2))\tau}d\tau + \sum_{n=1}^{\infty} p_2 p_{24} p_5 e^{-q_3 t}\int_0^t e^{q_3\tau}d\tau + \sum_{n=1}^{\infty} p_2 p_{24} p_6 e^{-q_3 t}\times \\ \int_0^t e^{(q_3-q_1)\tau}d\tau + \sum_{n=1}^{\infty}\sum_{n=1}^{\infty} p_2 p_{24} p_7 e^{-q_3 t}\left(\int_0^t e^{(q_3-q)\tau}d\tau - \int_0^t e^{(q_3-q_1)\tau}d\tau\right)+ \\ \sum_{n=1}^{\infty}\left(\frac{p_2 p_{25}}{n\pi}-D_1 4\right)e^{-q_3 t}\int_0^t e^{-q\tau}d\tau\end{array}\right)\frac{2}{3}\frac{\left(2-3(-1)^n+(-1)^{3n}\right)}{n\pi}$$

$$-\sum_{n=1}^{\infty} D_1 2b_3 e^{-q_3\tau}\int_0^t d\tau \frac{\left(1-(-1)^{2n}\right)}{n\pi} + \sum_{n=1}^{\infty}\sum_{n=1}^{\infty} 4 e^{-q_3 t}\int_0^t e^{-q\tau}d\tau \frac{2}{3}\frac{\left(1-(-1)^{3n}\right)}{n\pi} - 2p_1 p_{21} e^{-q_3 t}\int_0^t e^{q_3\tau}d\tau \times$$

$$\frac{\left(-2+2(-1)^n-n^2\pi^2(-1)^n\right)}{n^3\pi^3} + 2\left(-p_1 p_{20}+p_{21}(1+p_1)-p_1 p_{23}\right)e^{-q_3 t}\int_0^t e^{q_3\tau}d\tau \frac{2(-1)^n}{n\pi} +$$

$$2\left(p_{20}(1+p_1)+p_{23}(1+p_1)\right)e^{-q_3 t}\int_0^t e^{q_3\tau}d\tau \frac{\left(1-(-1)^n\right)}{n\pi} \tag{3.200}$$

Integrating equation (3.200) with respect to τ, we have equations (3.201) and (3.202):

$$Z_{1n}(t)=\sum_{n=1}^{\infty}\left(\begin{array}{l}(D_1 2b_3 n\pi - p_1 p_{25})t e^{-q_3 t} + \frac{p_2 p_{21}}{(q_3-q)}\left(e^{-qt}-e^{-q_3 t}\right) - \frac{p_1 p_{24} p_6}{(q_3-q_1)}\left(e^{-q_1 t}-e^{-q_3 t}\right) \\ -(p_1 p_{22}+p_1 p_{24} p_5)\frac{1}{q_3}\left(1-e^{-q_3 t}\right) - \sum_{n=1}^{\infty} p_1 p_{24} p_7 e^{-q_3 t}\times \\ \left(\frac{1}{(q_3-q)}\left(e^{-qt}-e^{-q_3 t}\right) - \frac{1}{(q_3-q_1)}\left(e^{-q_1 t}-e^{-q_3 t}\right)\right)\end{array}\right)\frac{1-n^2\pi^2+(-1)^{2n}}{2n^2\pi^2}$$

$$+2\sum_{n=1}^{\infty}\left(\begin{array}{l}\left(p_{25}(1+p_1)\frac{2}{n\pi}-D_1 2b_3 n\pi\right)t e^{-q_3 t} + (p_2 p_{20}+p_2 p_{23})\frac{1}{(q_3-q)}\left(e^{-qt}-e^{-q_3 t}\right)+ \\ p_{22}(1+p_1)\frac{1}{(q_3-q_2)}\left(e^{-q_2 t}-e^{-q_3 t}\right) + p_5 p_{24}(1+p_1)\frac{1}{q_3}\left(1-e^{-q_3 t}\right)+ \\ p_6 p_{24}(1+p_1)\frac{1}{(q_3-q_1)}\left(e^{-q_1 t}-e^{-q_3 t}\right) + \sum_{n=1}^{\infty} p_7 p_{24}(1+p_1)\times \\ e^{-q_3 t}\left(\frac{1}{(q_3-q)}\left(e^{-qt}-e^{-q_3 t}\right) - \frac{1}{(q_3-q_1)}\left(e^{-q_1 t}-e^{-q_3 t}\right)\right)\end{array}\right)$$

$$+2\sum_{n=1}^{\infty}\left(\begin{array}{l}\left(p_{25}(1+p_1)\frac{2}{n\pi}-D_1 2b_3 n\pi\right)t\,e^{-q_3 t}+(p_2 p_{20}+p_2 p_{23})\frac{1}{(q_3-q)}\left(e^{-qt}-e^{-q_3 t}\right)+\\ p_{22}(1+p_1)\frac{1}{(q_3-q_2)}\left(e^{-q_2 t}-e^{-q_3 t}\right)+p_5 p_{24}(1+p_1)\frac{1}{q_3}\left(1-e^{-q_3 t}\right)+\\ p_6 p_{24}(1+p_1)\frac{1}{(q_3-q_1)}\left(e^{-q_1 t}-e^{-q_3 t}\right)+\sum_{n=1}^{\infty}p_7 p_{24}(1+p_1)\times\\ e^{-q_3 t}\left(\frac{1}{(q_3-q)}\left(e^{-qt}-e^{-q_3 t}\right)-\frac{1}{(q_3-q_1)}\left(e^{-q_1 t}-e^{-q_3 t}\right)\right)\end{array}\right)$$

$$+2\sum_{n=1}^{\infty}\left(\begin{array}{l}\sum_{n=1}^{\infty}p_1 p_{22}\frac{1}{(q_3-(q+q_2))}\left(e^{-(q+q_2)t}-e^{-q_3 t}\right)+\sum_{n=1}^{\infty}p_2 p_{24} p_5\frac{1}{q_3}\left(1-e^{-q_3 t}\right)+\sum_{n=1}^{\infty}p_2\times\\ p_{24} p_6\frac{1}{(q_3-q_1)}\left(e^{-q_1 t}-e^{-q_3 t}\right)-\sum_{n=1}^{\infty}\left(\frac{p_2 p_{25}}{n\pi}-D_1 4\right)\frac{1}{q}\left(e^{-(q_3+q)t}-e^{-q_3 t}\right)+\\ \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}p_2 p_{24} p_7\,e^{-q_3 t}\left(\frac{1}{(q_3-q)}\left(e^{-qt}-e^{-q_3 t}\right)-\frac{1}{(q_3-q_1)}\left(e^{-q_1 t}-e^{-q_3 t}\right)\right)\end{array}\right)\frac{2}{3}\frac{\left(2-3(-1)^n+(-1)^{3n}\right)}{n\pi}$$

$$-\sum_{n=1}^{\infty}D_1 2b_3 t\,e^{-q_3 t}\frac{\left(1-(-1)^{2n}\right)}{n\pi}-\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}4\frac{1}{q}\left(e^{-(q_3+q)t}-e^{-q_3 t}\right)\frac{2}{3}\frac{\left(1-(-1)^{3n}\right)}{n\pi}-2p_1 p_{21}\frac{1}{q_3}\left(1-e^{-q_3 t}\right)\times$$

$$\frac{\left(-2+2(-1)^n-n^2\pi^2(-1)^n\right)}{n^3\pi^3}+2\left(-p_1 p_{20}+p_{21}(1+p_1)-p_1 p_{23}\right)\frac{1}{q_3}\left(1-e^{-q_3 t}\right)\frac{2(-1)^n}{n\pi}+$$

$$2\left(p_{20}(1+p_1)+p_{23}(1+p_1)\right)\frac{1}{q_3}\left(1-e^{-q_3 t}\right)\frac{\left(1-(-1)^n\right)}{n\pi} \tag{3.201}$$

Therefore, we obtain equation (3.202)

$$Z_1(\eta,t)=\sum_{n=1}^{\infty}Z_{1n}(t)\sin n\pi\eta \tag{3.202}$$

CHAPTER FOUR

4.0 RESULTS AND DISCUSSIONt

The systems of equations describing filtration combustion with temperature dependent thermal conductivity and diffusion coefficients in wet porous medium is solved analytically using parameter expanding method and eigenfunctions expansion technique. Analytical solution given by equations (3.122)-(3.133) are computed for the following parameters values of λ_1=0.4, D_1=0.3, δ=0.4, p_{em}=1 using computer symbolic algebraic package MAPLE 17.

Where,

λ_1 = scaled thermal conductivity

D_1 = species diffusion coefficient

δ = Frank-kamenesskii parameter

p_{em} = peclet mass number

The results obtained from the method are shown in Figure 4.1 to 4.32.

Figure 4.1: shows the effect of scaled thermal conductivity λ_1 on the temperature. It is observed that the temperature increases and later decreases along distance η, but decreases with increase in scaled thermal conductivity.

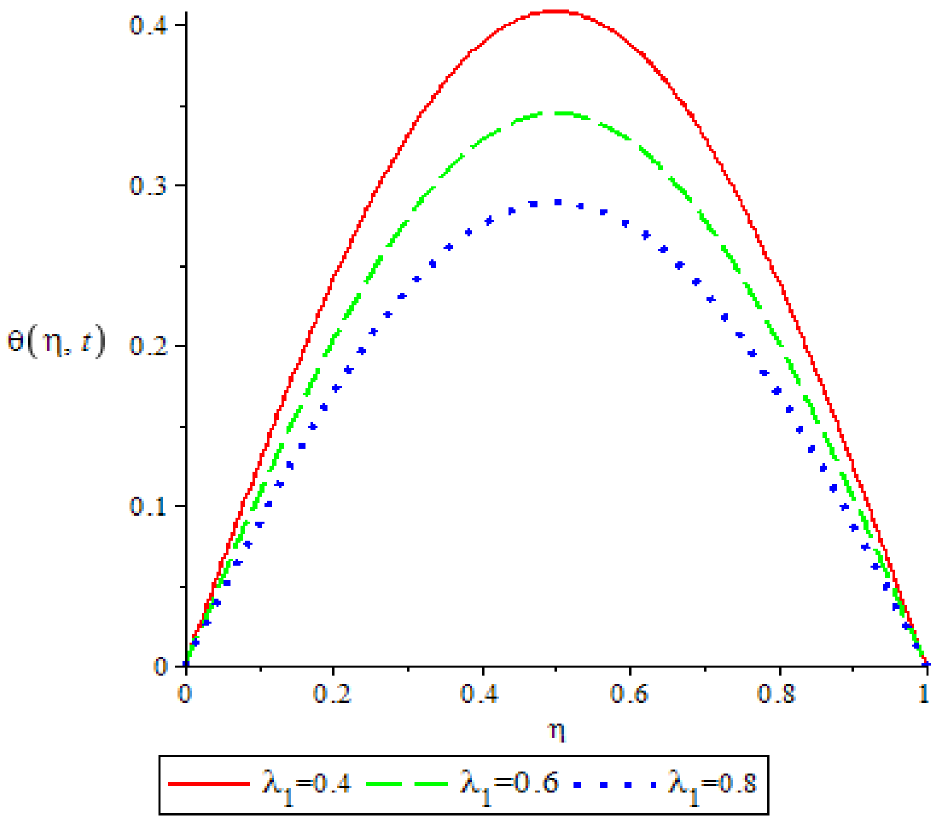

Figure 4.1: Relation between temperature $\theta(\eta,t)$ and distance η at various values of scaled thermal conductivity λ_1.

Figure 4.2: shows the effect of scaled thermal conductivity λ_1 on the temperature. It is observed that the temperature decreases with time t, and decreases with increase in scaled thermal conductivity.

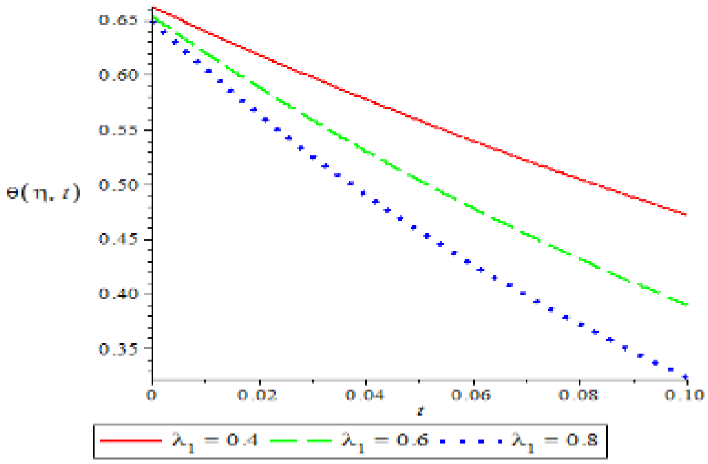

Figure 4.2: Temperature $\theta(\eta,t)$ –time t relationships at various values of scaled thermal conductivity λ_1.

Figure 4.3: shows the graph of temperature $\theta(\eta,t)$ against distance η and time t for different values of scaled thermal conduct λ_1. It is observed that the temperature increases and later decreases along distance with increase in time, but decreases with increase in scaled thermal conduct λ_1.

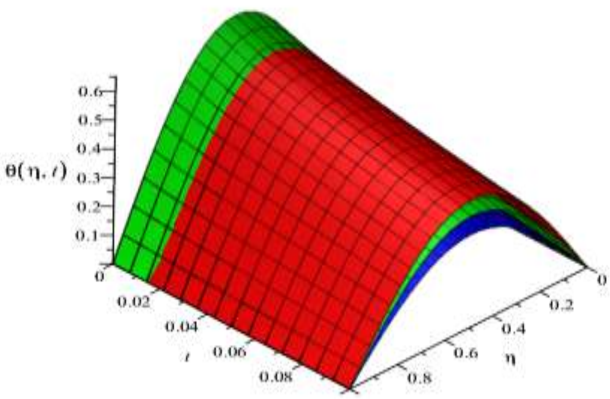

Figure 4.3: Relation among temperature $\theta(\eta,t)$, time t and distance η at various values of scaled thermal conduct λ_1.

Figure 4.4: shows the effect of scaled thermal conductivity λ_1 on the vapour molar fraction in the gas phase. It is observed that the vapour molar fraction in the gas phase increases and later decreases along distance η, but increases with increase in scaled thermal conductivity.

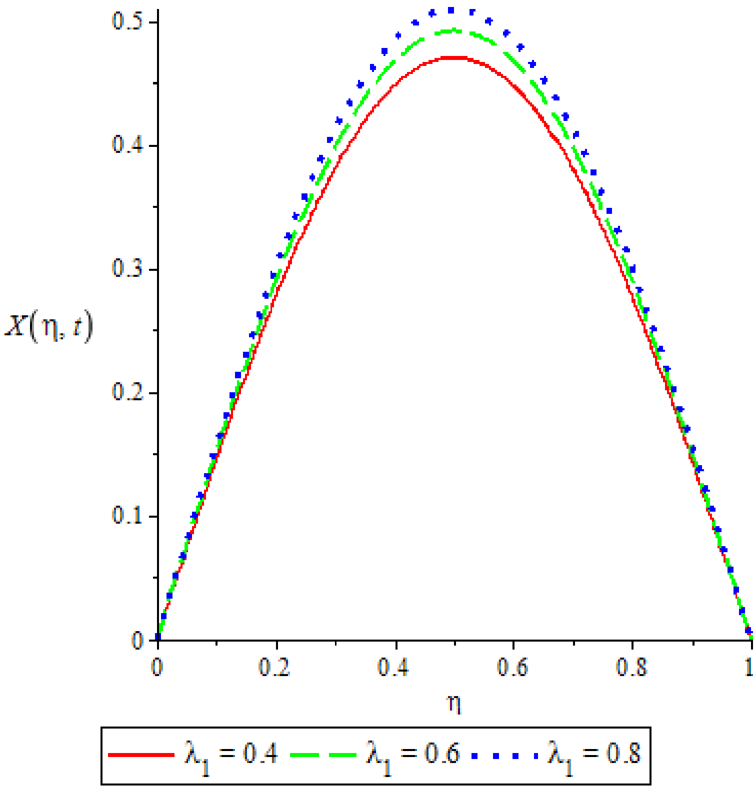

Figure 4.4: Relation between vapour molar fraction in the gas phase $X(\eta,t)$ and distance η at various values of scaled thermal conductivity λ_1.

Figure 4.5: shows the effect of scaled thermal conductivity λ_1 on the vapour molar fraction in the gas phase. It is observed that the vapour molar fraction in the gas phase decreases with time t, but increases with increase in scaled thermal conductivity.

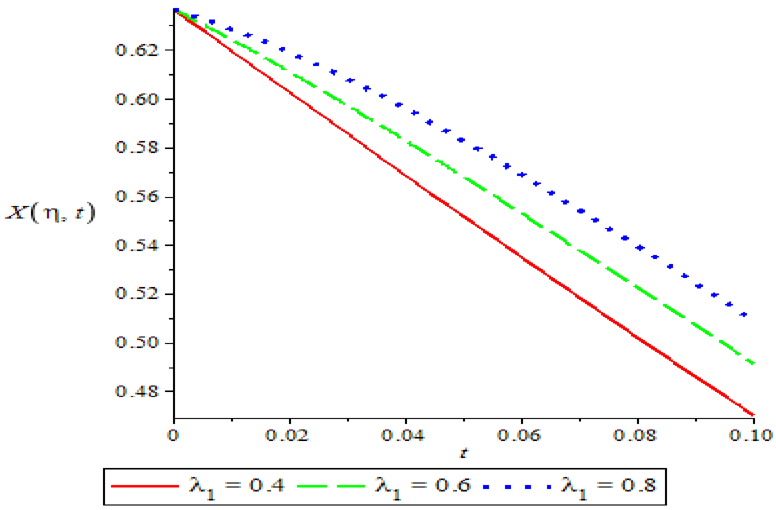

Figure 4.5: vapour molar fraction in the gas phase $X(\eta,t)$ – time t relationships at various values of scaled thermal conductivity λ_1.

Figure 4.6: shows the graph of vapour molar fraction in the gas phase $X(\eta,t)$ against distance η and time t for different values of scaled thermal conduct λ_1. It is observed that the vapour molar fraction in the gas phase increases and later decreases along the distance with increase in time, but increases with increase in scaled thermal conduct λ_1.

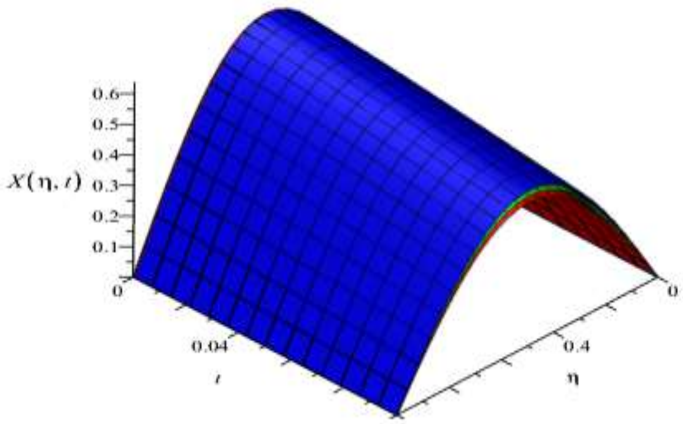

Figure 4.6: Relation among vapour molar fraction in the gas phase $X(\eta,t)$, time t and distance η at various values of scaled thermal conduct λ_1.

Figure 4.7: shows the effect of scaled thermal conductivity λ_1 on the molar concentration of solid fuel. It is observed that the molar concentration of solid fuel decreases and later increases along distance η, but increases with increase in scaled thermal conductivity λ_1.

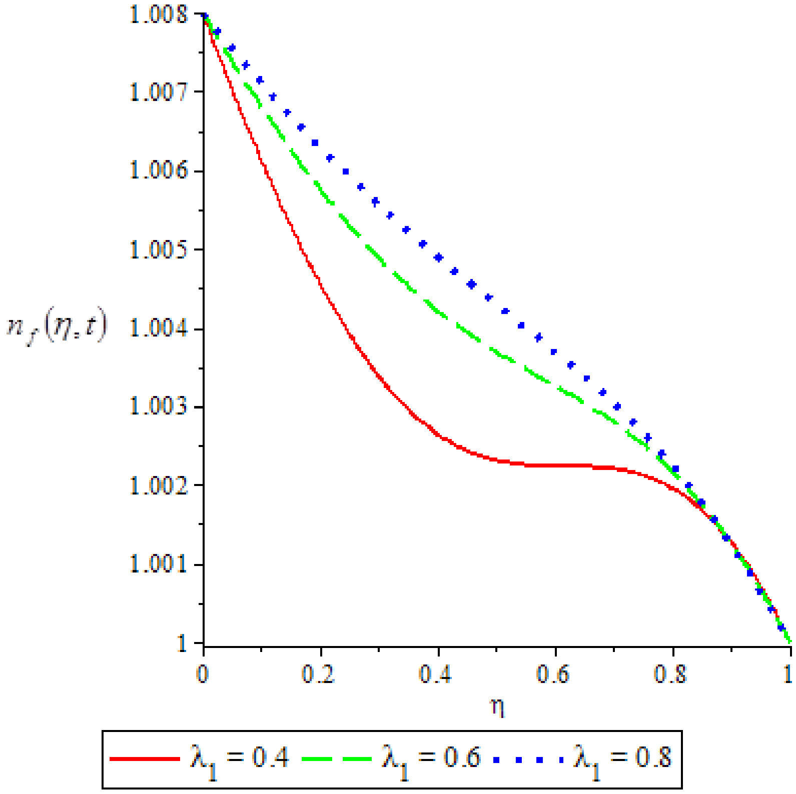

Figure 4.7: Relation between molar concentration of solid fuel $n_f(\eta,t)$ and distance η at various values of scaled thermal conductivity λ_1.

Figure 4.8: shows the effect of scaled thermal conductivity λ_1 on the molar concentration of solid fuel. It is observed that the molar concentration of solid fuel increases with time t, but increases with increase in scaled thermal conductivity λ_1.

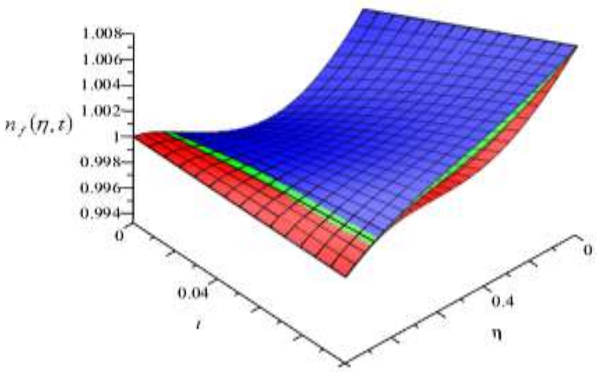

Figure 4.8: molar concentration of solid fuel $n_f(\eta,t)$ – time t relationships at various values of scaled thermal conductivity λ_1.

Figure 4.9: shows the graph of molar concentration of solid fuel $n_f(\eta,t)$ against distance η and time t for different values of scaled thermal conduct λ_1. It is observed that the molar concentration of solid fuel increases along the distance with increase in time, but increases with increase in scaled thermal conduct λ_1.

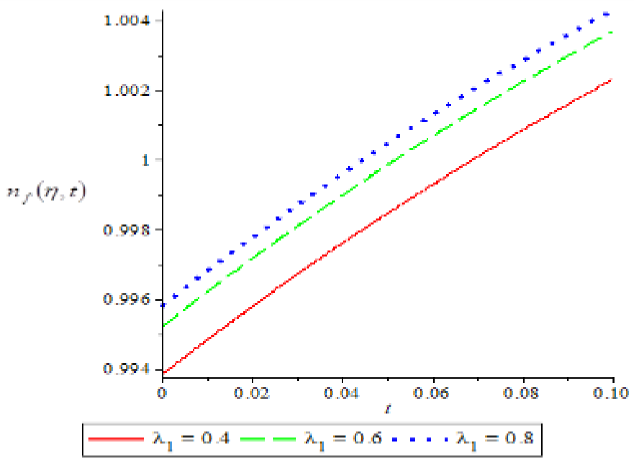

Figure 4.9: Relation among molar concentration of solid fuel $n_f(\eta,t)$, time t and distance η at various values of scaled thermal conductivity λ_1.

Figure 4.10: shows the effect of species diffusion coefficient D_1 on the temperature. It is observed that the temperature increases and later decreases along distance η, but decreases with increase in species diffusion coefficient D_1.

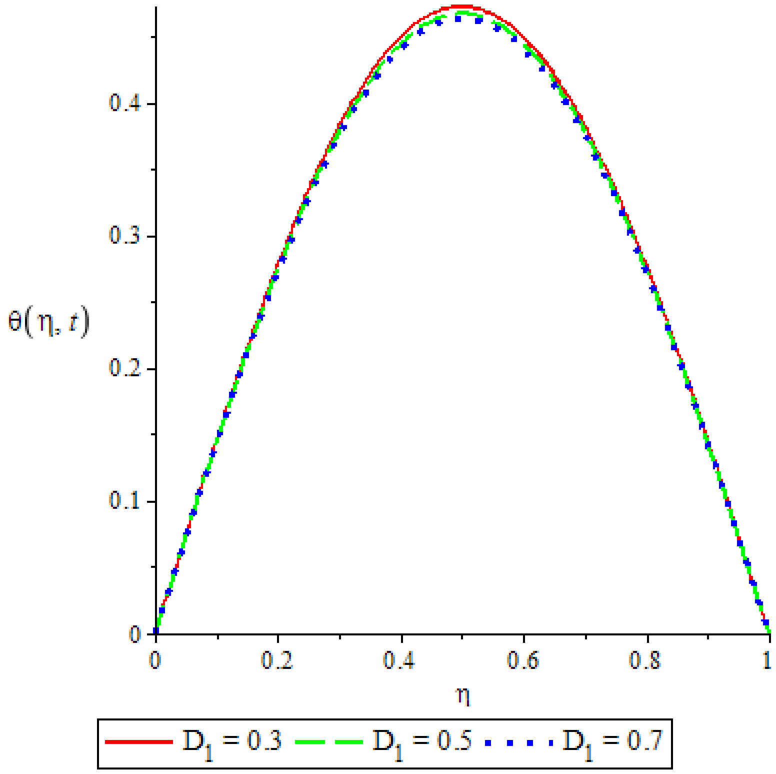

Figure 4.10: Relation between temperature $\theta(\eta,t)$ and distance η at various values of species diffusion coefficient D_1.

Figure 4.11: shows the effect of species diffusion coefficient D_1 on the temperature. It is observed that the temperature decreases with time t, but decreases with increase in species diffusion coefficient D_1.

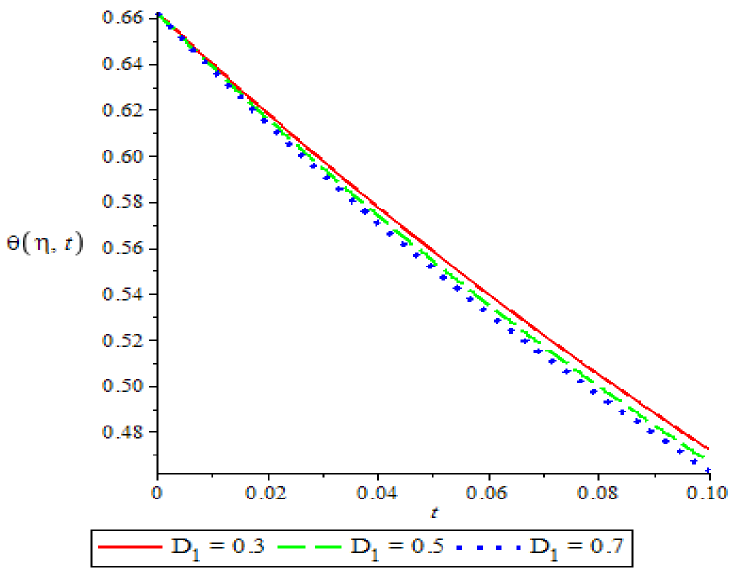

Figure 4.11: Temperature $\theta(\eta,t)$ –time t relationships at various values of species diffusion coefficient D_1.

Figure 4.12: shows the effect of species diffusion coefficient D_1 on the vapour molar fraction in the gas phase. It is observed that the vapour molar fraction in the gas phase increases and later decreases along distance η, but decreases with increase in species diffusion coefficient D_1.

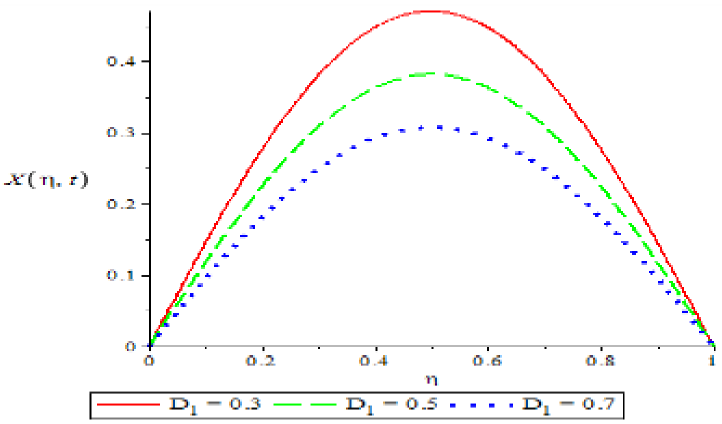

Figure 4.12: Relation between vapour molar fraction in the gas phase $X(\eta,t)$ and distance η at various values of species diffusion coefficient D_1.

Figure 4.13: shows the effect of species diffusion coefficient D_1 on the vapour molar fraction in the gas phase. It is observed that the vapour molar fraction in the gas phase decreases with time t, but decreases with increase in species diffusion coefficient D_1.

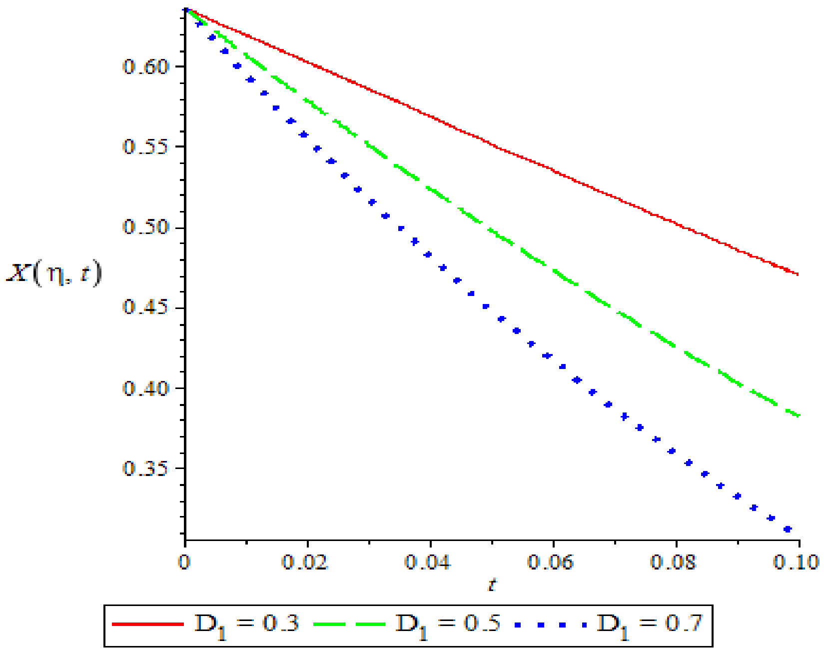

Figure 4.13: vapour molar fraction in the gas phase $X(\eta,t)$ – time t relationships at various values of species diffusion coefficient D_1.

Figure 4.14: shows the graph of vapour molar fraction in the gas phase $X(\eta,t)$ against distance η and time t for different values of species diffusion coefficient D_1. It is observed that the vapour molar fraction in the gas phase increases and later decreases along the distance with increase in time, but decreases with increase in species diffusion coefficient D_1.

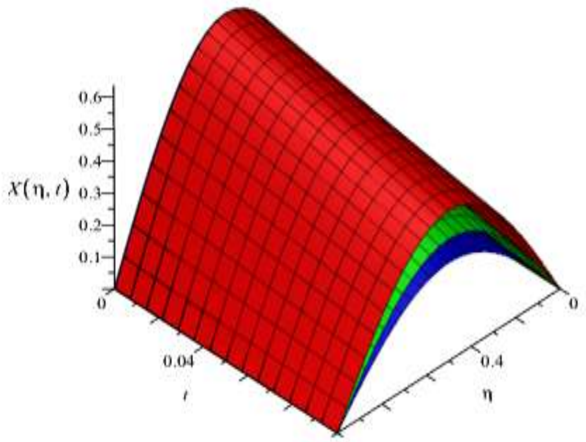

Figure 4.14: Relation among vapour molar fraction in the gas phase $X(\eta,t)$, time and distance η at various values of species diffusion coefficient D_1.

Figure 4.15: shows the effect of species diffusion coefficient D_1 on the molar fraction of oxygen. It is observed that the molar fraction of oxygen increases and later decreases along distance η, but decreases with increase in species diffusion coefficient D_1.

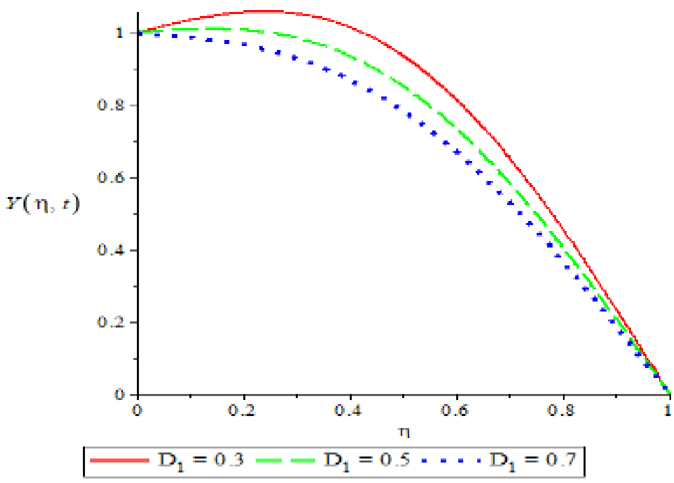

Figure 4.15: Relation between molar fraction of oxygen $Y(\eta,t)$ and distance η at various values of species diffusion coefficient D_1.

Figure 4.16: shows the effect of species diffusion coefficient D_1 on the molar fraction of oxygen. It is observed that the molar fraction of oxygen decreases with time t, but decreases with increase in species diffusion coefficient D_1.

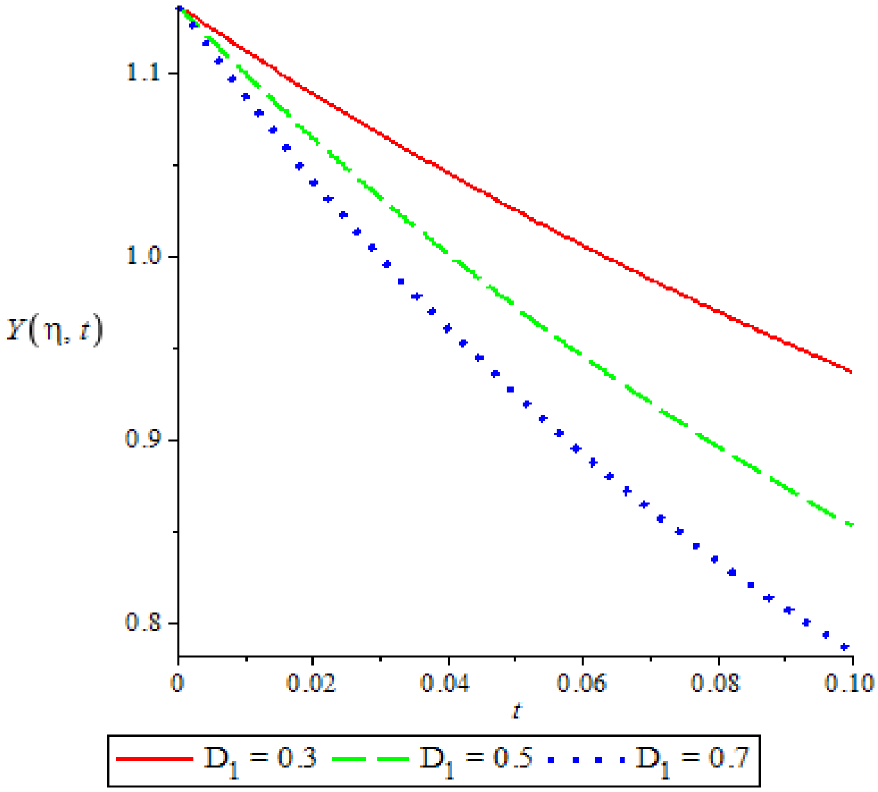

Figure 4.16: molar fraction of oxygen $Y(\eta,t)$ – time t relationships at various values of species diffusion coefficient D_1.

Figure 4.17: shows the graph of molar fraction of oxygen $Y(\eta,t)$ against distance η and time t for different values of species diffusion coefficient D_1. It is observed that the molar fraction of oxygen increases and later decreases along the distance with increase in time, but decreases with increase in species diffusion coefficient D_1.

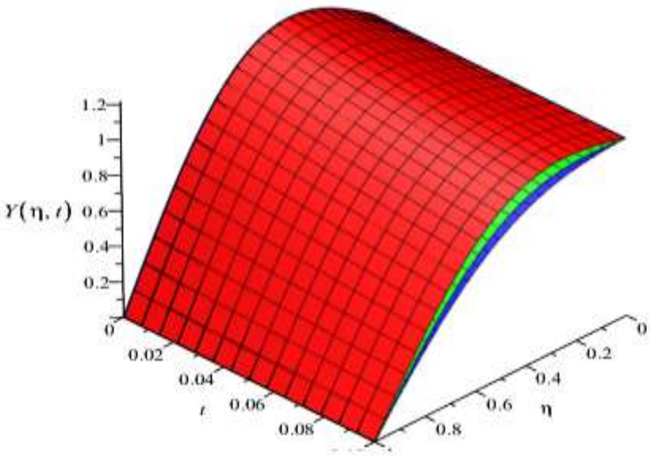

Figure 4.17: Relation among molar fraction of oxygen $Y(\eta,t)$, time t and distance η at various values of species diffusion coefficient D_1.

Figure 4.18: shows the effect of species diffusion coefficient D_1 on the molar fraction of passive gas in the gas phase. It is observed that the molar fraction of passive gas in the gas phase increases and later decreases along distance η, but decreases with increase in species diffusion coefficient D_1.

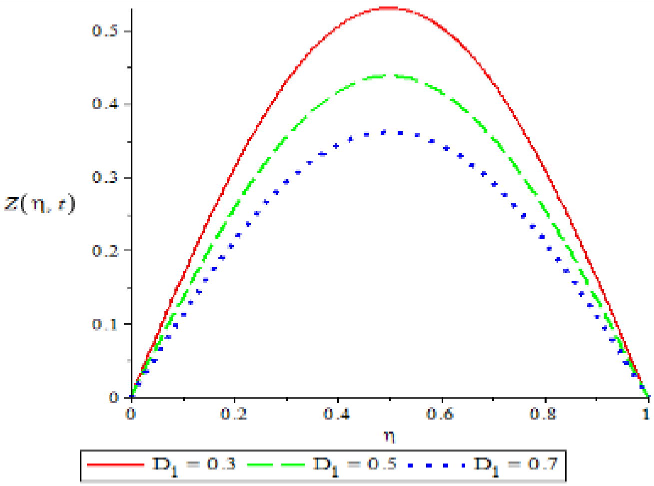

Figure 4.18: Relation between molar fraction of passive gas in the gas phase $Z(\eta,t)$ and distance η at various values of species diffusion coefficient D_1.

Figure 4.19: shows the effect of species diffusion coefficient D_1 on the molar fraction of passive gas in the gas phase. It is observed that the molar fraction of passive gas in the gas phase decreases with time t, but decreases with increase in species diffusion coefficient D_1.

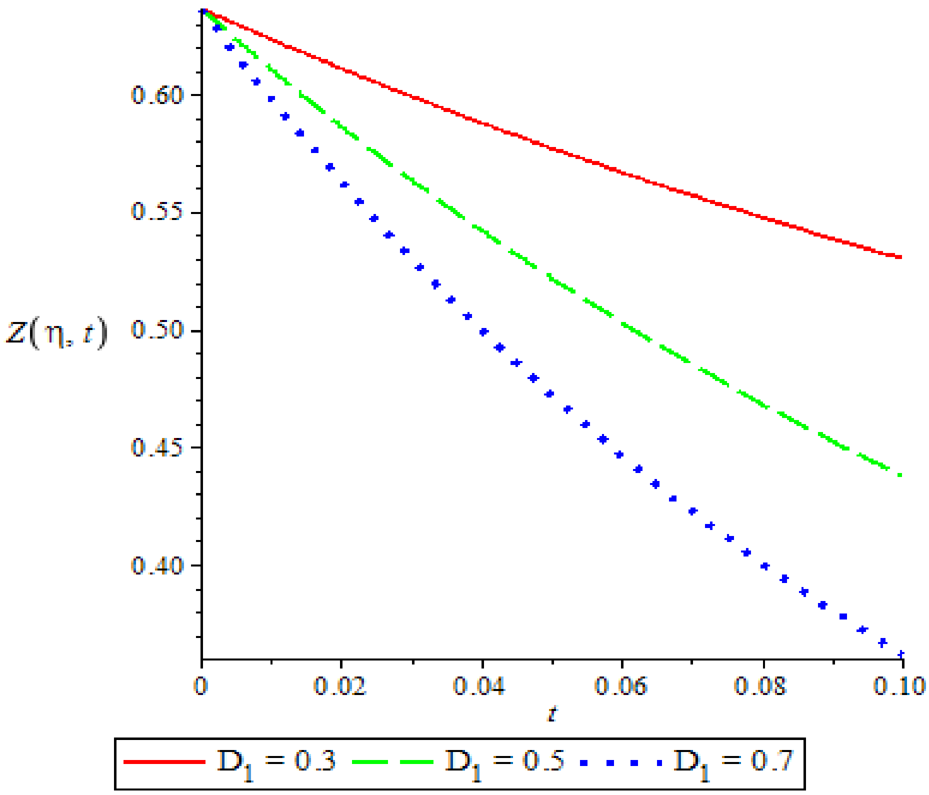

Figure 4.19: molar fraction of passive gas in gas phase $Z(\eta, t)$ – time t relationships at various values of species diffusion coefficient D_1.

Figure 4.20: shows the graph of molar fraction of passive gas in the gas phase $Z(\eta, t)$ against distance η and time t for different values of species diffusion coefficient D_1. It is observed that the molar fraction of passive gas in the gas phase increases and later decreases along the distance with increase in time, but decreases with increase in species diffusion coefficient D_1.

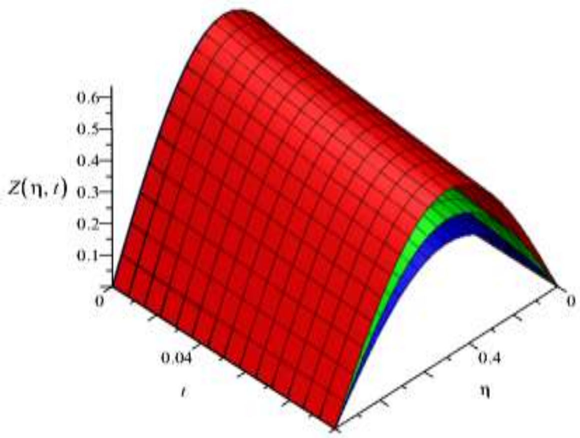

Figure 4.20: Relation among molar fraction of passive gas in the gas phase $Z(\eta,t)$, time t and distance η at various values of species diffusion coefficient D_1.

Figure 4.21: shows the effect of species diffusion coefficient D_1 on the molar concentration of solid fuel. It is observed that the molar concentration of solid fuel decreases and later increases along distance η, but decreases with increase in species diffusion coefficient D_1.

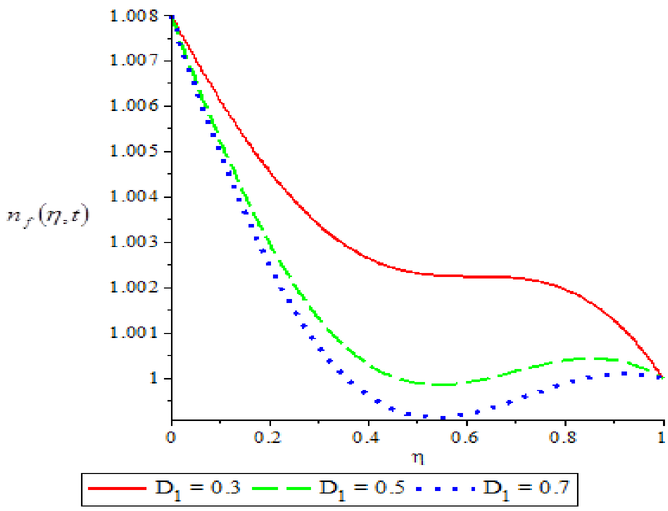

Figure 4.21: Relation between molar concentration of solid fuel $n_f(\eta,t)$ and distance η at various values of species diffusion coefficient D_1.

Figure 4.22: shows the effect of species diffusion coefficient D_1 on the molar concentration of solid fuel. It is observed that the molar concentration of solid fuel increases with time t, but decreases with increase in species diffusion coefficient D_1.

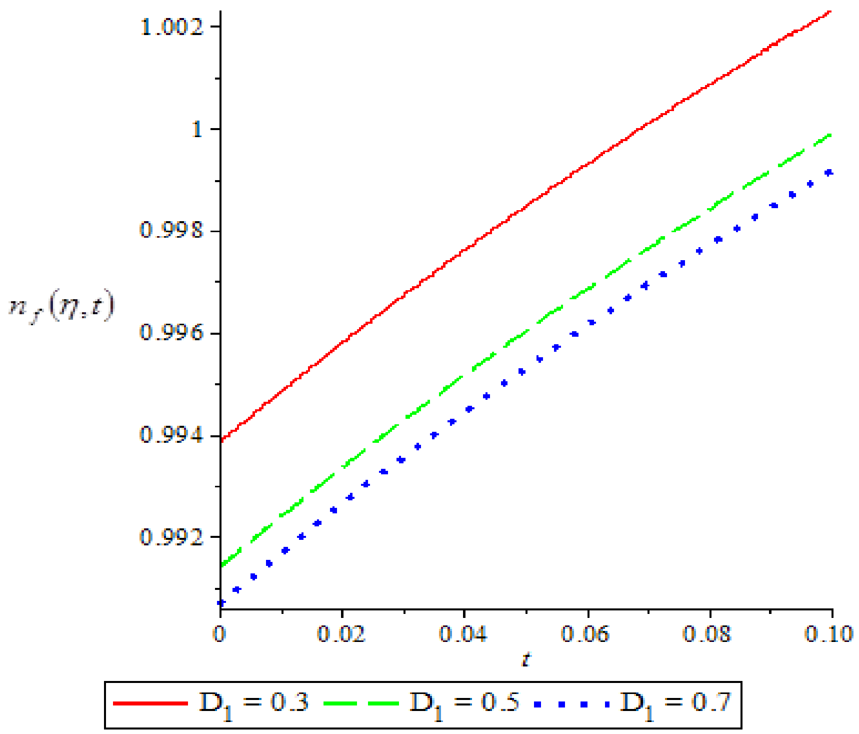

Figure 4.22: molar concentration of solid fuel $n_f(\eta, t)$ – time t relationships at various values of species diffusion coefficient D_1.

Figure 4.23: shows the graph of molar concentration of solid fuel $n_f(\eta, t)$ against distance η and time t for different values of species diffusion coefficient D_1. It is observed that the molar concentration of solid fuel decreases and later increases along the distance with increase in time, but decreases with increase in species diffusion coefficient D_1.

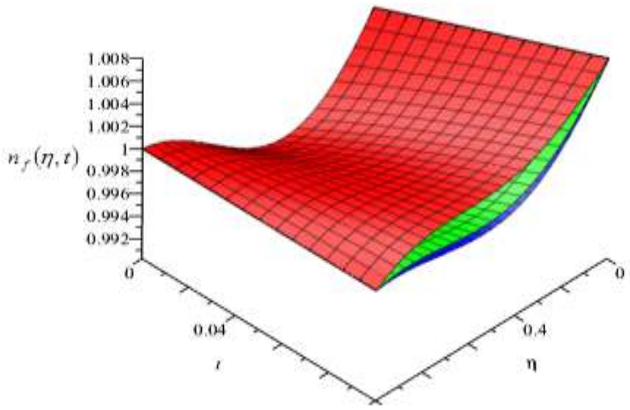

Figure 4.23: Relation among molar concentration of solid fuel $n_f(\eta,t)$, time t and distance η at various values of species diffusion coefficient D_1.

Figure 4.24: shows the effect of species diffusion coefficient D_1 on the molar concentration of liquid. It is observed that the molar concentration of liquid decreases and later increases along distance η, but increases with increase in species diffusion coefficient D_1.

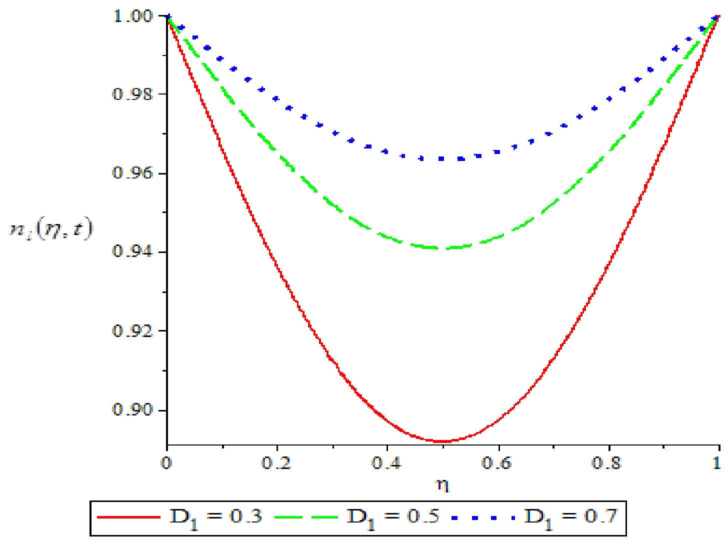

Figure 4.24: Relation between molar concentration of liquid $n_l(\eta,t)$ and distance η at various values of species diffusion coefficient D_1.

Figure 4.25: shows the effect of species diffusion coefficient D_1 on the molar concentration of liquid. It is observed that the molar concentration of liquid increases with time t, but increases with increase in species diffusion coefficient D_1.

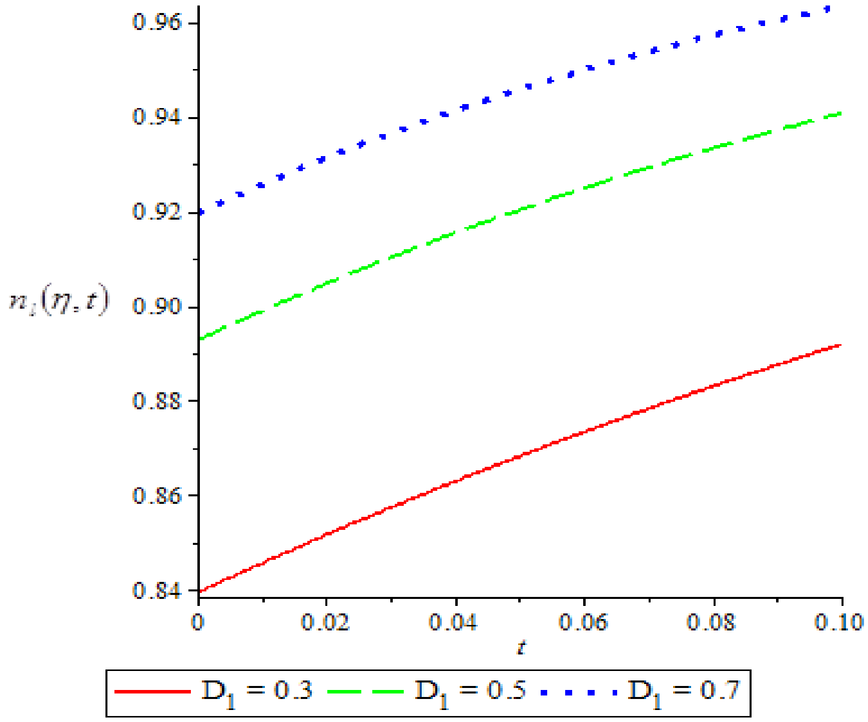

Figure 4.25: molar concentration of liquid $n_l(\eta,t)$ – time t relationships at various values of species diffusion coefficient D_1.

Figure 4.26: shows the graph of molar concentration of liquid $n_l(\eta,t)$ against distance η and time t for different values of species diffusion coefficient D_1. It is observed that the molar concentration of liquid decreases and later increases along the distance with increase in time, but increases with increase in species diffusion coefficient D_1.

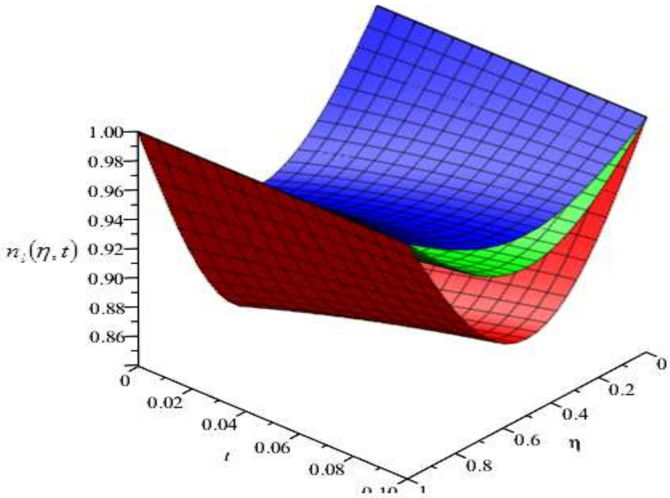

Figure 4.26: Relation among molar concentration of liquid $n_l(\eta, t)$, time t and distance η at various values of species diffusion coefficient D_1.

Figure 4.27: shows the effect of Frank-kamenesskii parameter δ on the temperature. It is observed that the temperature increases and later decreases along distance η, but increases with increase in Frank-kamenesskii parameter δ.

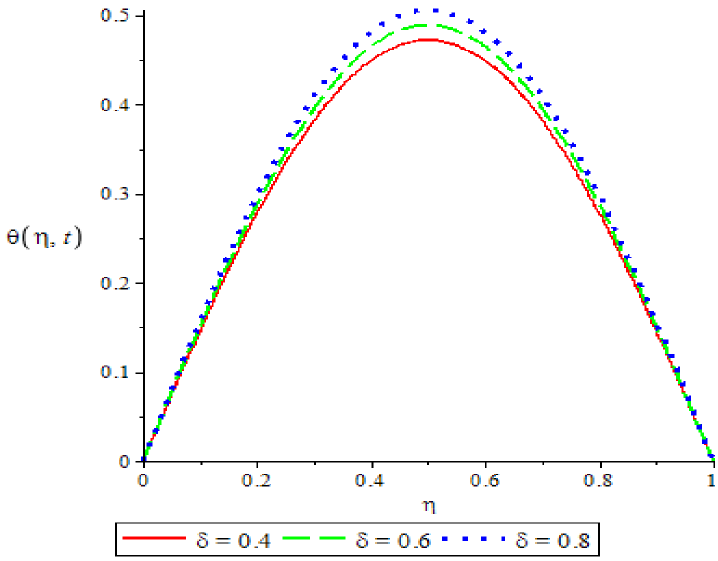

Figure 4.27: Relation between temperature $\theta(\eta, t)$ and distance η at various values of Frank-kamenesskii parameter δ.

Figure 4.28: shows the effect of Frank-kamenesskii parameter δ on the temperature. It is observed that the temperature decreases with time t, but inecreases with increase in Frank-kamenesskii parameter δ.

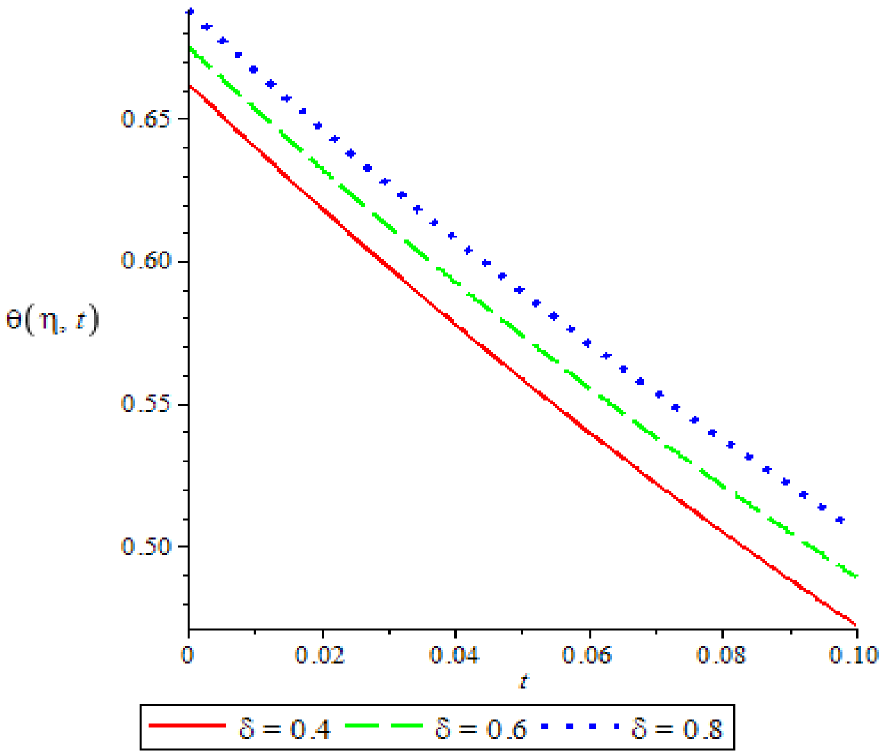

Figure 4.28: Temperature $\theta(\eta,t)$ –time t relationships at various values of Frank-kamenesskii parameter δ.

Figure 4.29: shows the graph of temperature $\theta(\eta,t)$ against distance η and time t for different values of Frank-kamenesskii parameter δ. It is observed that the temperature increases and later decreases along distance with increase in time, but increases with increase in Frank-kamenesskii parameter δ.

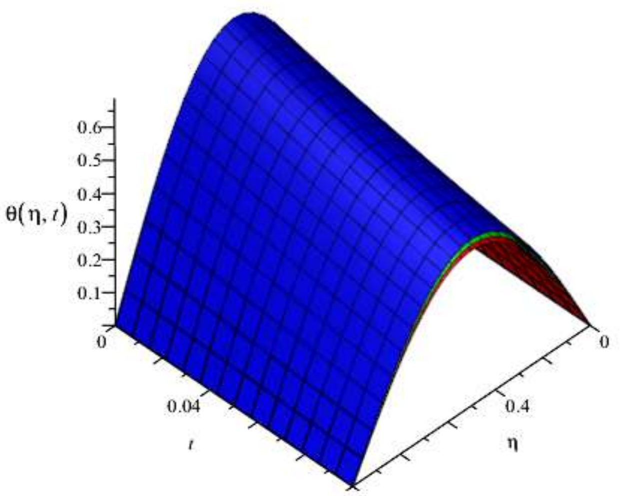

Figure 4.29: Relation among temperature $\theta(\eta, t)$, time t and distance η at various values of Frank-kamenesskii parameter δ.

Figure 4.30: shows the effect of peclet mass p_{em} on the molar concentration of solid fuel. It is observed that the molar concentration of solid fuel decreases along distance η, but increases with increase in peclet mass p_{em}.

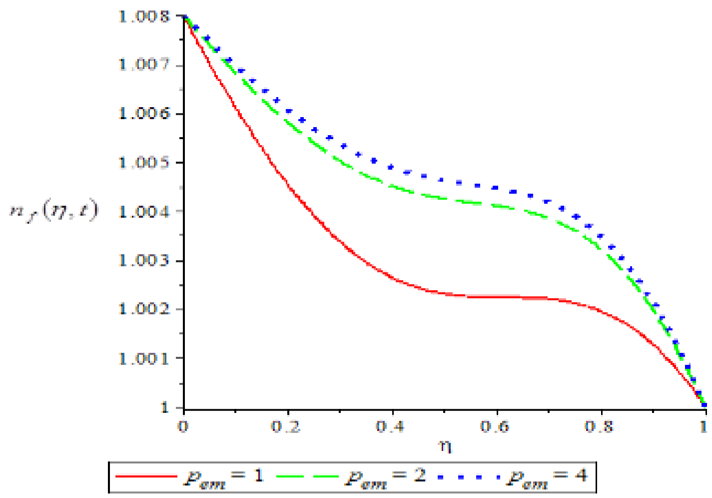

Figure 4.30: Relation between molar concentration of solid fuel $n_f(\eta, t)$ and distance η at various values of peclet mass p_{em}.

Figure 4.31: shows the effect of peclet mass p_{em} on the molar concentration of solid fuel. It is observed that the molar concentration of solid fuel increases with time t, but increases with increase in peclet mass p_{em}.

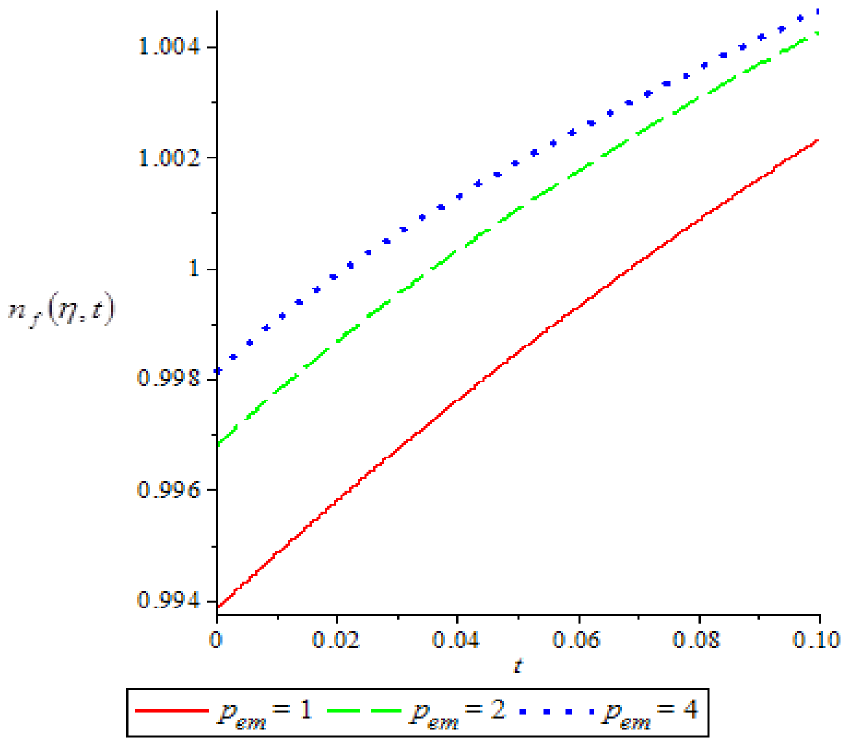

Figure 4.31: molar concentration of solid fuel $n_f(\eta,t)$ – time t relationships at various values of peclet mass p_{em}.

Figure 4.32: shows the graph of molar concentration of solid fuel $n_f(\eta,t)$ against distance η and time t for different values of peclet mass p_{em}.It is observed that the molar concentration of solid fuel increases oscillate along the distance with increase in time, but increases with increase in peclet mass p_{em}.

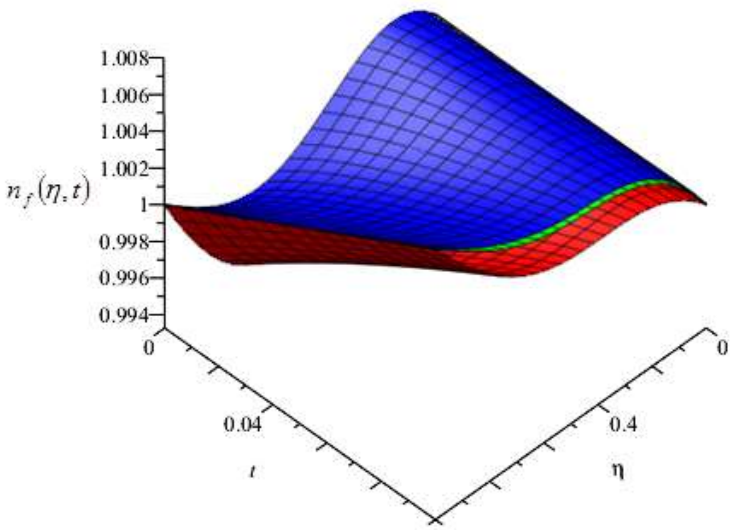

Figure 4.32: Relation among molar concentration of solid fuel $n_f(\eta, t)$, time t and distance η at various values of peclet mass p_{em}.

4.3 Comparison of Results

From the literature review, Bruining et al. (2009) in their studies, they discovered that when the diffusion is dominant at the reaction layer, it lead the oxygen to extinction. These agreed with Figure 4.15: shows the effect of species diffusion coefficient D_1 on the molar fraction of oxygen. It is observed that the molar fraction of oxygen increases and later decreases along distance η, but decreases with increase in species diffusion coefficient D_1 and Figure 4.16: shows the effect of species diffusion coefficient D_1 on the molar fraction of oxygen. It is observed that the molar fraction of oxygen decreases with time t, but decreases with increase in species diffusion coefficient D_1.

Figure 4.27: shows the effect of Frank-kamenesskii parameter δ on the temperature. It is observed that the temperature increases and later decreases along distance η, but

increases with increase in Frank-kamenesskii parameter δ, Figure 4.28: shows the effect of Frank-kamenesskii parameter δ on the temperature. It is observed that the temperature decreases with time t, but inecreases with increase in Frank-kamenesskii parameter δ and Figure 4.29: shows the graph of temperature $\theta(\eta, t)$ against distance η and time t for different values of Frank-kamenesskii parameter δ. It is observed that the temperature increases and later decreases along distance with increase in time, but increases with increase in Frank-kamenesskii parameter δ. These agreed with Olayiwola (2015) who formulated a model for forward propagation of a combustion front through a porous medium with reaction involving oxygen and a solid fuel and Olayiwola *et al.* (2014) presented a mathematical model for forward propagation of combustion front with Arrhenius kinetics through a porous medium with the reaction involving oxygen and solid fuel. Both researchers observed that with the increase in Frank-kamenesskii parameter δ, solid phase temperature decreases as time increases and decreases along the distance but increases with increase in Frank-kamenesskii parameter δ.

CHAPTER FIVE

5.0 CONLUSION AND RECOMMENDATIONS

5.1 Conclusion

We have formulated and solved analytically a mathematical model of filtration combustion with temperature dependent thermal conductivity and diffusion coefficient in a wet porous medium. The existences of unique solution of the problem were examined by actual solution method. The properties of solution were investigated. We solved the model equations analytically using parameter expanding method, direct integration and eigenfunction expansion technique. Finally, the graphical summaries of solutions were provided.

5.2 Contribution to Knowledge

From the studies made on this research work, we achieve the following:

i. Formulation of model of filtration combustion with temperature dependence thermal conductivity and diffusion coefficient in a wet porous medium.
ii. Existence and uniqueness of solution by actual solution approach.
iii. Analytical solution by parameter expanding method and eigenfunctions expansion method.
iv. We provide the Graphical summaries of system responses

5.3 Recommendation

We study one-dimensional problem in the present research, interested researchers may wish to study two-dimensional problems. Therefore, it is recommended for further research.

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