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# THE LINEAR TRANSFORMATION OF A BLOCK HYBRID RUNGE-KUTTA TYPE METHOD FOR DIRECT INTEGRATION OF FIRST AND SECOND ORDER INITIAL VALUE PROBLEM

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**ABSTRACT**
The Algebraic structure of a block hybrid Runge-Kutta Type Method (BHRKTM) for the solution of initial value problems was analysed. The coefficients of the first order Runge-Kutta Type Method (RKTM) of the Butcher table was applied to prove to the second order RKTM. The method of linear transformation and monomorphism was employed to substantiate the uniform order and error constant for the first order BHRKTM and the corresponding extended second order BHRKTM. Two equations evolved that satisfied the Runge-Kutta consistency conditions of second and first order respectively. The Algebraic structure was carefully retained during the transformation.

**Keywords:** Implicit, Initial Value Problems, Linear Transformation, Monomorphism., Runge-Kutta Type

## 1 INTRODUCTION

A linear transformation (Homomorphism) can be defined as when a function $T$ between two vector spaces $T: V \to W$ preserves the operations of addition if $v_1$ and $v_2 \epsilon V$ then

$$T(v_1 + v_2) = T(v_1) + T(v_2) \quad (1)$$

And scalar multiplication if $v \epsilon V$ and $r \epsilon R$, then

$$T(r.v) = rT(v) \quad (2)$$

Agam (2013).

$A$ homomorphism that is one to one or a mono is called a monomorphism.

The monomorphism Transformation preserves its algebraic structure and the order of the Domain into its Range.

Butcher and Hojjati (2005) laid another strong foundation by extending the general linear method (GLM) to the case in which second derivative as well as first derivative can be calculated. They constructed methods of third and fourth order which are A-stable, possess the Runge-Kutta stability property and have a diagonally implicit structure for efficient implementation to solve any initial value problem of ordinary differential equation. Okunuga, Sofoluwe , Ehigie and Akanbi (2012)presented a direct integration of second order ordinary differential equations using only Explicit Runge-Kutta Nystrom (RKN) method with higher derivative. They derived and tested various numerical schemes on standard problems. Due to the limitations of Explicit Runge-Kutta (ERK) in handling stiff problems, the extension to higher order Explicit Runge-Kutta Nystrom (RKN) was considered and results obtained showed an improvement over conventional Explicit Runge-Kutta schemes. The Implicit Runge-Kutta scheme was however not considered. Yahaya and Adegboye (2013) derived an implicit 6-stage block Runge-Kutta Type Method for direct integration of second order (special or general), third order (special or general) as well as first order initial value and boundary value problems. The theory of Nystrom was adopted in the reformulation of the methods. The convergence and stability analysis of the method were conducted and the region of absolute stability plotted. The method was A-stable, possessed the Runge-Kutta stability property, had an implicit structure for efficient implementation and produced at the same time approximation to the solution of both linear and non linear initial value problems.

## 2 METHODOLOGY

Let $T$ be a linear transformation which is continuously differentiable on a set of ordered three- tuple vector $\in \mathbb{R}^3$ as follows

$$V_i = (x + c_i h, y + \sum_{j=1}^{s} a_{ij} T(v_j), y' + \sum_{j=1}^{s} a_{ij} T'(v_j)) \in \mathbb{R}^3 \quad (3)$$

$$T(V_i) = h(y' + \sum_{j=1}^{s} a_{ij} T'(v_j) \quad (4)$$

and

$T^{'}(v_i) = hf\left(x + c_i h, y + \sum_{j=1}^{s} a_{ij} T\left(v_j\right), y^{'} + \sum_{j=1}^{s} a_{ij} T^{'}\left(v_j\right)\right) = hm_i$ (5)

That is,
$m_i = f\left(x + c_i h, y + \sum_{j=1}^{s} a_{ij} T\left(v_j\right), y^{'} + \sum_{j=1}^{s} a_{ij} T^{'}\left(v_j\right)\right)$ (6)

Then the Transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}$ is a well defined monomorphism:

Proof

Let $u, v \epsilon \mathbb{R}$ defined by

$U = (x + c_i h, y_1 + \sum_{j=1}^{s} a_{ij} T\left(u_j\right), y_1^{'} + \sum_{j=1}^{s} a_{ij} T^{'}\left(u_j\right)$ (7)

$V = (x + c_i h, y_2 + \sum_{j=1}^{s} a_{ij} T\left(v_j\right), y_2^{'} + \sum_{j=1}^{s} a_{ij} T^{'}\left(v_j\right))$ (8)

$T(U + V) = h(y_1^{'} + y_2^{'} + \sum_{j=1}^{s} a_{ij} (T^{'}(u_j) + T^{'}(v_j)$ (9)

By the definition of T on $\mathbb{R}^3$

$h\left(y_1^{'} + \sum_{j=1}^{s} a_{ij} T^{'}\left(u_j\right)\right) + h\left(y_2^{'} + \sum_{j=1}^{s} a_{ij} T^{'}\left(v_j\right)\right) = T(U + V)$ (10)

$T(U + V) = T(U) + T(V)$ (11)

$T(k.U) = k.T(U)$ (12)

Hence $T$ is a homomorphism

Now we show that $T$ is $1 - 1$

Let $u, v \epsilon \mathbb{R}^3$ with

$T(u) = T(v)$ (13)

By definition of T, we have

$h(y_1^{'} + \sum_{j=1}^{s} a_{ij} T^{'}\left(u_j\right)) = h(y_2^{'} + \sum_{j=1}^{s} a_{ij} T^{'}\left(v_j\right))$ (14)

Since

$T(u) = T(v)$ then $T\left(u_j\right) = T(v_j)$ and

$T^{'}\left(u_j\right) = T^{'}\left(v_j\right)$ (15)

$y_1 = y_2$ and $x_1 + c_i h = x_2 + c_i h$

that is $x_1 = x_2$ (16)

Hence $U = V$ (17)

Thus T is $1 - 1 \Leftrightarrow$ a monomorphism from $\mathbb{R}^3 \rightarrow \mathbb{R}$

Remark: The necessity for the above proposition is to ensure that the algebraic structure and the order do not change during the transformation.

## 2.1 Figures and Tables

Consider the Butcher Table 1 and Table 2

**Table I: Butcher Table for K=2**

| | | | | |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $\frac{1}{2}$ | $0$ | $\frac{8}{9}$ | $\frac{-11}{24}$ | $\frac{5}{72}$ |
| $2$ | $0$ | $\frac{8}{9}$ | $\frac{2}{3}$ | $\frac{4}{9}$ |
| $1$ | $0$ | $\frac{10}{9}$ | $\frac{-1}{6}$ | $\frac{1}{18}$ |
| | $0$ | $\frac{10}{9}$ | $\frac{-1}{6}$ | $\frac{1}{18}$ |

**Table II: Butcher Table for second order for K=2**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $\frac{1}{2}$ | $0$ | $\frac{8}{9}$ | $\frac{-11}{24}$ | $\frac{5}{72}$ | $0$ | $\frac{37}{108}$ | $\frac{-41}{144}$ | $\frac{29}{432}$ |
| $2$ | $0$ | $\frac{8}{9}$ | $\frac{2}{3}$ | $\frac{4}{9}$ | $0$ | $\frac{52}{57}$ | $\frac{-2}{9}$ | $\frac{8}{27}$ |
| $1$ | $0$ | $\frac{10}{9}$ | $\frac{-1}{6}$ | $\frac{1}{18}$ | $0$ | $\frac{23}{27}$ | $\frac{-2}{9}$ | $\frac{5}{54}$ |
| | $0$ | $\frac{10}{9}$ | $\frac{-1}{6}$ | $\frac{1}{18}$ | $0$ | $\frac{23}{27}$ | $\frac{-2}{9}$ | $\frac{5}{54}$ |

## 3 RESULTS AND DISCUSSION

The table satisfies the Runge-Kutta conditions for solution of first order ode since

$(i) \sum_{j=1}^{s} a_{ij} = c_i$ (18)

$(ii) \sum_{j=1}^{s} b_j = 1$ (19)

We consider the general second order differential equation in the form

$y^{''} = f(x, y, y^{'}), y(x_0) = y_0\, y^{'}(x_0) = y_0^{'}$ (20)

$y^{''} = f(v), \quad v = (x, y, y^{'})$ (21)

$T(V_i) = T\left(x + c_i h, y + \sum_{j=1}^{s} a_{ij} T\left(V_j\right), y^{'} + \sum_{j=1}^{s} a_{ij} T^{'}\left(V_j\right)\right)$ (22)

$= h\ (y^{'} + \sum_{j=1}^{4} a_{ij}\ T^{'}\ (V_j)) = h\ (y^{'} + \sum_{j=1}^{4} a_{ij}\ hm_j)$ (23)

$T^{'}\left(V_j\right) = hm_j$ (24)

$T(V_1) = 0$ (25)

$T(V_2) = h(y^{'} + \frac{8}{9} hm_2 - \frac{11}{24} hm_3 + \frac{5}{72} hm_4)$ (26)

$T(V_3) = h(y^{'} + \frac{10}{9} hm_2 - \frac{1}{6} hm_3 + \frac{1}{18} hm_4)$ (27)

$$T(V_4) = h(y' + \frac{8}{9}hm_2 + \frac{2}{3}hm_3 + \frac{4}{9}hm_4) \quad (28)$$

$$m_j = T'(V_j) = f(x + c_i h, y + \sum_{j=1}^{s} a_{ij}T(V_j), y' + \sum_{j=1}^{s} a_{ij}T'(V_j)) \quad (29)$$

$$m_1 = 0 \quad (30)$$

$$m_2 = f(x + \frac{1}{2}h; y + \frac{1}{2}hy' + \frac{37}{108}h^2m_2 - \frac{41}{144}h^2m_3 + \frac{29}{432}h^2m_4; y' + \frac{8}{9}hm_2 - \frac{11}{24}hm_3 + \frac{5}{72}hm_4) \quad (31)$$

$$m_3 = f(x + h; y + hy' + \frac{23}{27}h^2m_2 - \frac{4}{9}h^2m_3 + \frac{5}{54}h^2m_4; y' + \frac{10}{9}hm_2 - \frac{1}{6}hm_3 + \frac{1}{18}hm_4) \quad (32)$$

$$m_4 = f(x + 2h; y + 2hy' + \frac{52}{27}h^2m_2 - \frac{2}{9}h^2m_3 + \frac{8}{27}h^2m_4; y' + \frac{8}{9}hm_2 + \frac{2}{3}hm_3 + \frac{4}{9}hm_4) \quad (33)$$

The direct method for solving $y'' = f(x, y, y')$ is now

$$y_{n+1} = y_n + b_1T(V_1) + b_2T(V_2) + b_3T(V_3) + b_4T(V_4) \quad (34)$$

$$y_{n+1} = y_n + hy_n' + \frac{h^2}{54}(46m_2 - 24m_3 + 5m_4) \quad (35)$$

$$y_{n+1}' = y_n' + b_1T'(V_1) + b_2T'(V_2) + b_3T'(V_3) + b_4T'(V_4) \quad (36)$$

$$y_{n+1}' = y_n' + 0hm_1 + \frac{10}{9}hm_2 - \frac{1}{6}hm_3 + \frac{1}{18}hm_4 \quad (37)$$

$$y_{n+1}' = y_n' + \frac{h}{18}(20m_2 - 3m_3 + m_4) \quad (38)$$

The coefficients of the first order Runge-Kutta Type Method (RKTM) of the Butcher table was applied to prove to the second order RKTM. Equations (35) and (38) satisfied the Runge-Kutta consistency conditions of second and first order respectively. This further shows that it is a monomorphism.

## 4 CONCLUSION

This research work established the reason for the uniform order and error constant of the first order Runge-Kutta type method and the extended second order method. And also why the Linear transformation and the order of the two methods was preserved and not changed during the transformation.

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