FEDERAL UNIVERSITY OF TECHNOLOGY

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PROJECT REPORT ON FOURIER SERIES

BY

ADAH ABUBEBELYE JUSTINE

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A project submitted to the Department of Mathematical Sciences, School of Science and Science Education, in partial fulfillment for the award of Bachelor of Technology in Mathematics/Computer Science of Federal University of Technology, Minna.

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D E D I C A T I O N

This project work is dedicated to my father, Mr. ROBERT AKONMAYE ADAH who died on 16th March, 1991.

C E R T I F I C A T I O N

I certify that this work was carried out by Mr. Adah A. J. in the Department of Mathematical Sciences, Federal University of Technology, Minna.

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ABSTRACT

In this report, Fourier Series is extensively discussed to meet expectations. It started with the discovery of trigonometrical series by Daniel Bernoulli and the recognition of the significance of the coefficients by Joseph Fourier, which brought about the series now called Fourier Series to the application of it.

The mathematical part in subsequent chapter gave, explicitly though, the definition of some common terminologies and then discuss the calculation of trigonometric series coefficients which later gives the Fourier Series required. There are also good examples to show how Fourier Series may be generated depending on the function given.

The application of Fourier Series is not left out. This was also discussed exhaustively and buttressed with good examples. For credibility sake, a computer result of one of the examples was obtained and compared later to the manually obtained results. The report shows with the aid of diagrams and examples how Fourier Series is involved in the field of engineering and mathematical physics at large.

In a nutshell, the role and importance of Fourier Series in the field of technology is discussed.

TABLE OF CONTENT

INTRODUCTION

Fourier Series was first introduced in mathematical physics into a pioneering work of a French mathematician called Joseph Fourier. The work was named, "The analytical theory of heat" and was completed in 1822.

Actually the possibility of forming a trigonometrical expansion of a periodic function had been recognised by predecessors of Fourier. For instance, about 1750 Daniel Bernoulli used a series of trigonometric terms in his proposed solution for representing the motion of vibrating string. He advocated this procedure as a general principle, but he did not calculate the coefficients of the terms.

Joseph Fourier was the first one to recognise the significance of the coefficients. He then saw that the trigonometric expansion could be applied independently of whether a function, f(x) is periodic. He again saw that a function need not be smooth to have trigonometric expansion but could have steep sides like those of square wave.

Fourier developed the use of his series into a powerful general method for solving problems such as those involved in heat diffusion and wave propagation. His work stimulated research in mathematical physics , and this is often identified with the solution of boundary-value problems, encompassing many natural occurrences such as sunspot, tides and the weather. The key to his methods is the distinction between the interior of a region and its boundary-value problems in the partial differential equations and which still is a basic tool in physics and engineering.

Fourier also extended this concept into the so-called Fourier series. Doubts of the validity of the Fourier series, which lead later mathematicians to a fundamental renewal of the concept of real function, were resolved by P.G.L. Dirichlet, Bernhard Riemann, Henri Lebesque and others. However, the mathematical~~ly~~ problems presented by the Fourier series have been a great stimulus to the development of mathematics in general.

The application of Fourier Series is wide. More specifically, Fourier Series is used for representing the motion of vibrating string. This idea was introduced by Daniel Bernoulli in 1750 when there was need to obtain~~ed~~ the Fourier analysis of some non-periodic functions. It was then concluded that if a function is defined only over a fixed interval (e.g Violin string) and the physical behaviors of the system is required only in that interval than Fourier Series and analysis may be used.

In wave and electronics Fourier analysis is also used to determine the value and the nature of voltage and current used in a linear circuit with time. The employment of this idea has also been of great importance in the analysis of square wave voltage. Also in transmission line in terms of natural modes, Fourier analysis is also used. This is said to help demonstrate a technique used for solutions of wave problems earlier mentioned.

When solving problem such as those involved in heat diffusion Fourier Series is also used. This is why it is employed in heat transfer analysis to help predict heat flow. This is said to be even easier if the temperature distribution in a material is known Since it would help in the establishing of the heat flow.

Partial differential equation which is a basic tool in physics and engineering make use of Fourier analysis to obtain Series solution to boundary-value problems,

However, the mathematical problem presented by the Fourier Series have been a great stimulus to the development of mathematics .

In this project we have in Chapter 1 an introductory aspect of the Fourier Series. In chapter 2 we have preambles and the definitions of some concept or terms used extensively in the Fourier Series. In chapter 3 we have Fourier Series itself discussed with examples and theorems to buttress facts. In chapter 4, the application of Fourier Series is then giving with good examples. And in chapter 5 we have conclusion which involve a comparison of the computer obtained result to the manually obtained result in chapter 4.

CHAPTER TWO

PREAMBLES AND IMPORTANT TERMINOLOGIES

PREAMBLES

Fourier Series is extensively used in solving some problems in applied mathematics in the development of functions into trigonometric polymonials. Solution of problems in the partial differential equations can be done by using Fourier Series. It is also useful in boundary-value problems.

In the course of studies we shall make use of some well known functional relations and some of them will be listed below for ease of reference.

1 TRIGONOMETRIC FUNCTIONS

We state without proof, the following well known trigonometric formula:

1.1 $\quad \text{Cos}2x = 1 - \text{Sin}^2 x$

1.2 $\quad \text{Cos}^2 x = \frac{1}{2}(1+\text{Cos}2x)$

1.3 $\quad \text{Sin}^2 x = \frac{1}{2}(1-\text{Cos}2x)$

From the above, it is easy to see that $\int \text{Sin}^2 x\, dx$

$= x/2 - \text{Sin}2x/4 + c$

(b) $\int \text{Cos}^2 x\, dx = x/2 + \text{Sin}2x/4 + c$

where c is an arbitrary constant of integration

Also since $\quad 2\text{SinACosB} = \text{Sin}(A+B) + \text{Sin}(A-B)$

1.4 $\quad 2\text{SinACosB} = \frac{1}{2}[\text{Sin}(A+B) + \text{Sin}(A+B)$

Similarly,

a, $\text{CosASinB} = \frac{1}{2}[\text{Sin}(A+B) - \text{Sin}(A+B)]$

b, $\text{CosACosB} = \frac{1}{2}\ \text{Cos(A-B)} - \text{Cos(A-B)}]$

c, $\text{SinASinB} = \frac{1}{2}[\text{Cos(A-B)} - \text{Cos(A-B)}]$

Example 2.1

The following integral can be evaluated thus:

$$I = \int \text{Cos}5x\ \text{Sin}3x dx = \frac{1}{2} \int (\text{Cos}5x\ \text{Sin}3x) dx$$

$$= \frac{1}{2} \int [\text{Sin}(5x+3x)\ \text{Sin}(5x-3x)]\ dx$$

$$= \frac{1}{2} \int (\text{Sin}8x - \text{Sin}2x) dx$$

$$= \frac{1}{2}[-\text{Cos}8x/8 + \text{Cos}2x/2] + c = \text{Cos}2x/4 - \text{Cos}8x/16 + c$$

Example 2.2

$$I = \int \text{Sin}5x\ \text{Sin}3x\ dx = \frac{1}{2}\int (2\text{Sin}5x\ \text{Sin}3x) dx$$

$$= \frac{1}{2} \int [\ \text{Cos}(5-3)x - \text{Cos}(5+3)\ ] dx$$

$$= \frac{1}{2} \int [\ \text{Cos}2x - \text{Cos}8x]\ dx = \frac{1}{2}[\text{Sin}2x/2 - \text{Sin}8x/8] + c$$

$$= \text{Sin}2x/4 - \text{Sin}8x/16 + c$$

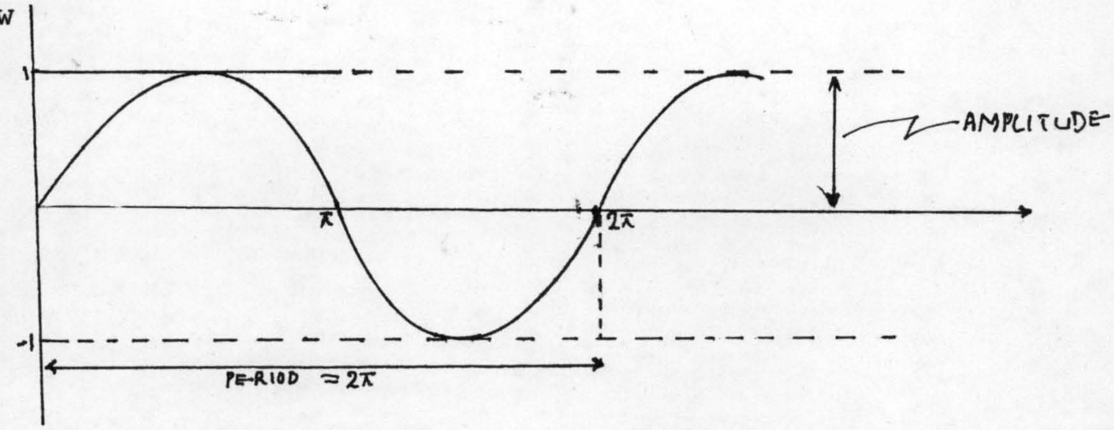
PERIODIC FUNCTION

Definition.

A function F(x) is said to be periodic with period T if for all x, F(x)=F(x+T). The least of T >0 is called the least period or simply the period of F(x). If T is then the least period, 2T, 3T, 4T, ... are also periods of f(x).

Thus, given f(x) as function then f(x) = f(x+t) = f(x+2t)... where T is the least period.

Thus, a periodic function is a function that has values that repeat at regular time interval. This is graphically shown below



Graph of f(x) = Sin x

Fig (2.1)

Remarks

(i) The function Sinx has period $T = 2\pi,\ 4\pi,\ 6\pi$ Since $\sin(x+2\pi)$,

$\sin(x+4\pi)$, $\sin(x+6\pi)$..... all equals Sin x.

(ii) If f(x) = Sinnx, where n is a positive integer then the period is $2\pi/n$. That is for Sin nx to repeat itself after time T, $nT = 2\pi$ and $T = 2\pi/n$.

(iii) A constant has any positive number as period.

Example 2.3

$$F(x) = \begin{cases} \text{Sin}, & 0 \le x \le \pi \\ 0, & \pi < x < 2\pi \end{cases} \qquad \text{PERIOD} = 2\pi$$

Graphically we have :-

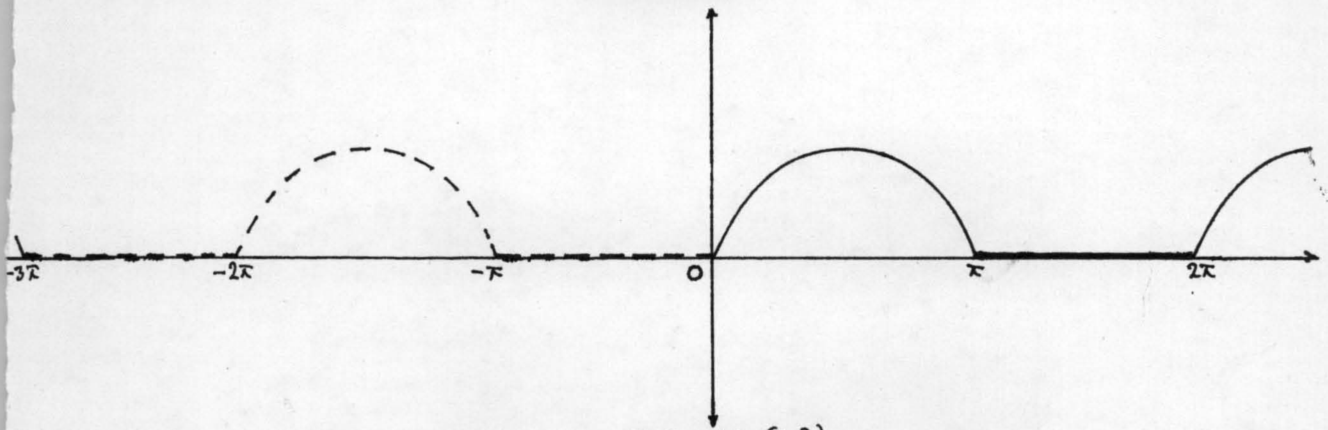


FIGURE (2.2)

From the above graph f(x) repeated itself at 2π, then 2π is the period

$f(x) = \text{Sin}x = \text{Sin}x\ (x+2\pi) = \text{Sin}x\ ,\ 0 \le x \le \pi$

$f(x) = 0,\ \pi < x < 2\pi.$

ORTHOGONAL FUNCTION

Definition:-

Two functions $\phi(x)$, $p(x)$ are said to be orthogonal on an interval [a, b] if $\int^{b}_{a}\phi(x)p(x)\,dx = 0$

for all $x \in [a,b]$

Example 2.4

Given that $\phi(x) = \text{Sinmx}$ and $P(x) = \text{Cosnx}$ where m,n = 1,2,3... and in the interval $[-\pi,\pi]$, show that they are orthogonal.

Solution:

for $m \neq n$

$$\int^{\pi}_{-\pi} \underline{\text{Sinnx}}\ \text{Cosnx}\ dx = \tfrac{1}{2}\int^{\pi}_{-\pi} [\text{Sin}(n+m)x + \text{Sin}(n-m)x]dx$$

$$= \tfrac{1}{2}[-\text{Cos}(n+m)/n+m + (-)\text{Cos}(n-m)x/n-m]^{\pi}_{-\pi}$$

$$= -\tfrac{1}{2}\ [\text{Cos}(n+m)x/n+m + \text{Cos}(n-m)/n-m]^{\pi}_{-\pi}$$

$$= -\tfrac{1}{2}\ [\text{Cos}-(n+m)\pi/(m+n) + \text{Cos}(n-m)\pi/(n-m] - (-\tfrac{1}{2})[\text{Cos}(n+m)\pi/n+m + \text{Cos}\ (n-m)\pi/n-m]\}$$

$$= \{-\tfrac{1}{2}\ [\text{Cos}-(n+m)\pi/n+m + \text{Cos}-(n-m)\pi/n-m] + \tfrac{1}{2}\ [\text{Cos}(n+m)\pi/n+m + \text{Cos}(n-m)\pi/n-m]\}$$

$$= 0 + 0 + = 0$$

Now, for m = n then the integral of the product of the trigo- nometric~~al~~ function in this interval will be :-

$$\int^{\pi}_{-\pi} \underline{\text{Sinnx}}\ \text{Cosnx}\ dx = \tfrac{1}{2}\int^{\pi}_{-\pi} [\text{Sin}\ 2nx\ ]dx$$

$$= \tfrac{1}{2}\ [-\text{Cos}\ 2nx/2n]^{\pi}_{-\pi} = \tfrac{1}{4n}\ [-\text{Cos}\ 2nx]^{\pi}_{-\pi}$$

$= \{-\frac{1}{4}n\ [\text{Cos}2n\ (-^{\pi})] - (-)\ [\text{Cos}\ 2n\pi]\} = \{-\frac{1}{4}n\ [\ \text{Cos}\ 2n\pi]$

$+\frac{1}{4}{}_n[\text{Cos}2n\pi\}$

But $\text{Cos}x = \text{Cos}(-x)$ therefore

$\frac{1}{4}n\{-[\text{Cos}-2n\pi] + [\text{Cos}2n\pi]\} = 0$

Definition:

A system of function $< \phi_n(x) >$ is called an orthogonal system in the interval [a,b] if the integral of the product of any two different functions of the system, taken over the interval [a,b] is equal to zero. That is $\int \phi_i(x)\phi_j(x)dx = 0$, $i \neq j$

and i,j =1,2,3..

Example 2.5

Consider z = [1,Cosx Sinx, Cos2x Sin2x,

prove that z is an orthogonal system in the interval $[-\pi,\pi]$.

<u>Solution</u>

$\int^{\pi}{}_{-\pi}\ 1.\ \text{Cos}nx\ dx\ = \text{Sin}nx/n\ |_{-\pi}{}^{\pi}\ = [\text{Sin}n\pi/n - \text{Sin}-n\pi/n] = 0$

$\int^{\pi}{}_{-\pi} 1.\text{Sin}nxdx = -\ \text{Cos}nx/n\ |^{\pi}{}_{-\pi} = [-\ \text{Cos}\ n\pi/n - (\overset{?}{-})$

$\text{Cos}-(-\pi)/n] = 0$

$\int^{\pi}{}_{-\pi}\ \text{Cos}mx\ \text{Cos}nxdx\ = \frac{1}{2}\int^{\pi}{}_{-\pi}\ [\text{Cos}(m+n)x + \text{Cos}(m-n)x]\ dx$

$= \frac{1}{2}[\text{Sin}(m+n)x + \text{Sin}(m-n)x/m-n]^{\pi}{}_{-\pi}$

$= \frac{1}{2}\{[\ \text{Sin}(m+n)\pi/m+n + \text{Sin}(m-n)\pi/m-n - [\text{Sin}(m+n)-\pi/m+n)$

$+ \text{Sin}(m-n)-\pi/m-n]\}\ \ = 0 + 0 = 0$

Also

$\int^{\pi}{}_{-\pi}\ \text{Sin}mx\ \text{Cos}nxdx = \frac{1}{2}\int^{\pi}{}_{-\pi}\ [\text{Sin}(m+n)x + \text{Sin}(m-n)x]dx$

$= \frac{1}{2}[- \mathrm{Cos}(m+n)x/m+n) + (-)\mathrm{Cos}(m-n)x/m-n\ ]^{\pi}{}_{-\pi}$

$= \frac{1}{2}\{-[\mathrm{Cos}(m+n)\pi/m+n + \mathrm{Cos}(m-n)\pi/m-n] + [\mathrm{Cos}(m+n)-\pi/m+n +$

$\mathrm{Cos}(m-n)-\pi/m-n]\}$

$= 0 + 0 = 0$

Finally,

$\int^{\pi}{}_{-\pi} \mathrm{Sinmx\ Cosnxdx} = \frac{1}{2}\int^{\pi}{}_{-\pi} \mathrm{Sin2nxdx} = [-\mathrm{Cos2nx}/n]^{\pi}{}_{-\pi}$

$= [\mathrm{Cos}2n\pi/2n] + [\mathrm{Cos}2nx(-\pi/2n])]$

$= 0 + 0 =0$

Hence this system is an orthogonal system in the interval $[-\pi,\pi]$

It is not in all cases that we can obtain or thogonality in a system. In some cases orthorgonality may be achieved by the introduction of a function we call weight function.

WEIGHT FUNCTION

This is a function say $q(x)>0$ defined in the interval (a,b) such that for any two elements in a non-orthogonal set say, $[\tau_r(k)]^{\infty}{}_{r=0} =$ P we can get

$\int^{b}{}_{a}\tau_k(x)\tau_r(x)q(x)dx = 0;\ r \neq k$

and $r,k = 0,1,2,3$ ....

The function $q(x)$ is called the weight function.

CHAPTER THREE

FOURIER SERIES

Definition:

A series of function say f(x) given by $f(x) = a_o/2 + a_1 Cosx + b_1 Sinx + a_2 Cos2x + b_2 Sin2x + \ldots + an\ Cosnx + bn\ Sinnx + \ldots$

and defined in any finite interval (-L,L) is Called Fourier Series. It is always expressed in the form

$f(x) = a_o/2 + \Sigma(a_n Cosnx + b_n Sinnx)$

This implies that Fourier Series is used to represent a periodic function as shown above:

It is true that there are many functions in existence, but not all of them can be represented by Fourier Series. Therefore, we have some conditions put forward by Dirichlet that help to determine possible functions to be represented by Fourier Series.

DIRICHLET CONDITIONS

It says that for a function to be represented by Fourier Series such that putting $x = x_1$ and the value converging to f(x) with addition of more terms, the following condition must be satisfied.

(1) The function, f(x) must be defined and single value.

(2) f(x) must be continuous or have a finite number of discontinuities within a periodic interval.

(3) f(x) and f'(x) must be sectionally continuous in an interval say [-L.L]

Example 3.1

(a) $f(x) = x^3$

This function satisfy the Dirichlet condition and it Fourier

Series may be generated.

(b) $f(x) = 2/x$

This function does not satisfy the Dirichlet conditions since it has infinite discontinuity at $x=0$

DETERMINATION OF COEFFICIENTS

Given that

$$f(x) = a_o/2 + a_1 Cosx + b_1 Sinx + a_2 Cos2x + b_2 Sin2x + \ldots$$
$$\ldots + a_n Cosnx + b_n Sinnx + \ldots \qquad (2.1)$$

then the coefficient $a_o, a_1, a_2 \ldots b_1\ b_2\ b_3 \ldots bn$ can be determined.

Therefore, to find a_o we integrate f(x) with respect to x from $-\pi$ to π.

$$\int^{\pi}_{-\pi} f(x)\ dx = \int^{\pi}_{-\pi} a_o/2\ dx + \int^{\pi}_{-\pi} a1\ Cosx\ dx$$
$$+ \int^{\pi}_{-\pi} b1\ Sinx + \ldots$$

$$= a_o/2\ [x]^{\pi}_{-\pi} + a1[Sinx\ ]^{\pi}_{-\pi} + b1[-Cosx\ ]^{\pi}_{-\pi}$$
$$+ \ldots$$

But $[Sinx]^{\pi}_{-\pi} = [Sin\pi - Sin(-\pi)\ ] = 0$

Also $[-Cosx]^{\pi}_{-\pi} = [-Cos\pi + Cos(-\pi)] = 0$

and $[x]^{\pi}_{-\pi} = [\pi+\pi] = 2\pi$

therefore,

$$\int^{\pi}_{-\pi} f(x)\ dx = 2\pi a_o/2 + 0 + 0 + \ldots = a_o\pi$$

and so, $a_o = 1/\pi \int^{\pi}{}_{-\pi} f(x)\, dx$

Example 3.2

Given that $f(x) = x/2$, find a_o in the interval $[-\pi,\pi]$

<u>Solution</u>

We know that,

$$a_o = 1/\pi \int^{\pi}{}_{-\pi} f(x)dx = 1/\pi\int^{\pi}{}_{-\pi} x/2)dx$$

$$= \tfrac{1}{2}\pi \int^{\pi}{}_{-\pi} xdx = 1/2\pi[x^2]^{\pi}{}_{-\pi} = 1/2\pi \ \{ \ [\pi^2+\pi^2]\}$$

$$= 1/2\pi \ [2\pi^2] = 2\pi^2/2\pi = \pi^2/\pi = \pi$$

Therefore,

$$a_o = \pi$$

To find a_n, we then consider the equation (2.1)

Multiplying both side by cos mx we have

$f(x)$ Cosnx $= a_o/2$Cosmx $+ a_1$CosxCosmx $+ b_1$SinxCosmx

$+ \ldots +$ a_nCosnxCosmx $+ b_n$SinnxCosmx

Integrating we then have

$$\int^{\pi}{}_{-\pi} f(x) \text{Cosmx} = \int^{\pi}{}_{-\pi} a_o/2\text{Cosmx} + \int^{\pi}{}_{-\pi} a_n\text{CosnxCosmxdx}$$

$$+ \int^{\pi}{}_{-\pi} b_n\text{SinnxCosmxdx}$$

But $a_o/2 \int^{\pi}{}_{-\pi}$Cosmxdx $= a_o/2[$ Sinmx/m $]^{\pi}{}_{-\pi}$

$= a_o/2[$Sinmπ/m - Sin m/m $(-\pi)] = 0$

Also $b_n\int^{\pi}_{-\pi}$CosmxSinnxdx = $b_n/2\int^{\pi}_{-\pi}$[Sin(m+n)x

- Sin(m-n)x]dx

= $b_n/2$[[Cos(m+n)x/m+n + Cos -(m-n)x/m-n]$_{-\pi}{}^{\pi}$ = 0+0 = 0

Therefore,

$a_n\int^{\pi}_{-\pi}$CosnxCosmxdx = $a_n\int^{\pi}_{-\pi}$ Cos²mxdx

= $a_n/2\int^{\pi}_{-\pi}$[1+Cos2x]dx

Assuming m = n

= $a_n/2$[[1+Cos2x]$^{\pi}{}_{-\pi}$ = $a_n/2$ * 2π = $a_n\pi$

Therefore,

$\int^{\pi}_{-\pi}$f(x)Cosmxdx = $a_o/2$ [0] + $a_n\pi$ +b_n[0] +... = $a_n\pi$

then a_n = $1/\pi\int^{\pi}_{-\pi}$f(x)Cosmx dx

Example 3.3

Given that f(x) = -x, find a_n in the interval $[-\pi,\pi]$

<u>Solution</u>

We know that

a_n = $1/\pi\int^{\pi}_{-\pi}$f(x)Cosmxdx

Then a_n = $1/\pi\int^{0}_{-\pi}$ x Cos mx dx =

Let u = x, du = dx

Let du = Cosmx

V = Sin mx/m

= $1/\pi\int^{0}_{-\pi}$xCosmx dx = $-x/\pi$Sinmx/m $|^{0}_{-\pi}$ - $1/\pi$ $\int^{0}_{-\pi}$ Sinmxdu

$= -[\cos mx/\pi m^2]^{\pi}{}_{-\pi} = -\cos(m\pi) + \cos(-m\pi)/\pi m^2$

Since $\cos m\pi = 1$ we then have

$= =1/\pi[1+ \cos m\pi/m^2] = -1/\pi m^2 \ [1 - \cos m\pi]$

Considering that $\cos m\pi = 1$, for n even and $\cos m\pi = -1$ for n odd therefore, $an = -2/\pi n^2$ for n odd or 0 for n even then a_1 $a_2 \ldots a_n$ can now be determined.

Now, to find bn, let us Consider the equation (2.1) and then multiply both side by Sinmx. And we then have

$f(x) \sin mx = a_o/2 \sin mx + a_1 \cos x \sin mx + a_n \cos nx \sin mx$

$+ b_n \sin nx \cos mx$

Integrating we again have

$\int^{\pi}{}_{-\pi} f(x) \sin mx = \int^{\pi}{}_{-\pi} a_o /2 \sin mx dx$

$+ \int^{\pi}{}_{-\pi} a_n \cos nx \sin mx dx + \int^{\pi}{}_{-\pi} b_n \sin nx \sin mx dx$

But

$a_o/2 \int^{\pi}{}_{-\pi} \sin mx dx = a_o/2[-\cos mx/m \ ]^{\pi}{}_{-\pi}$

$= a_o/2[-\cos m\pi/m + \cos -\pi/m)] = 0$

And

$a_n \int^{\pi}{}_{-\pi} \cos nx \sin mx dx = a_n/2 \int^{\pi}{}_{-\pi} [\sin(m+n)x - \sin(m-n)x]dx$

$= a_n/2 \ [-\cos(m+n)^x/m+n + \cos(m+n)^x/m-n]^{\pi}{}_{-\pi} = 0-0=0$

Therefore, $b_n \int^{\pi}{}_{-\pi} \sin nx \sin mx dx$

$= b_n \int^{\pi}{}_{-\pi} \sin^2 mx \ dx = b_n/2 \int^{\pi}{}_{-\pi} \ [1-\cos 2x]dx$

Assuming m = n

Therefore, $\int^{\pi}_{-\pi} f(x)\,\text{Sinmxdx} = b_n[x/2 - \text{Sin}2x/4]\,^{\pi}_{-\pi} = b_n\pi$

$$bn = 1/\pi \int^{\pi}_{-\pi} f(x)\,\text{Sinmxdx}$$

Example 3.4

Given that f(x) = (π-x), find bn in the interval [$-\pi$,π]

<u>Solution</u>

We know that,

$$bn = 1/\pi \int^{\pi}_{-\pi} f(x)\,\text{Sinmx dx}$$

Therefore, $bn = -1/\pi\int^{\pi}_{-\pi}\text{Sinmxdx} = 1/\pi\int^{\pi}_{-\pi} (\pi-x)\,d(-\text{Cos nx}/n)$

$= -1/\pi n[\pi-x]\ \text{Cos mx}\ |^{\pi}_{-\pi} - 1/\pi n \int^{\pi}_{-\pi} \text{Cos mx dx}$

$=1/\pi n * 2\pi\text{Cos}(-n\pi)-1/\pi^2\text{Sinmx}\ |^{\pi}_{-\pi} = 2/n\text{Cosn}\pi = 2(-1)^n/n$

But Cos$n\pi$ = 1 for n even and Cos$n\pi$ = -1 for odd

$b_n = 2(-1)^n/2$

Therefore the coefficient b_1, b_2,... b_n can then be determined. However, to generate a Fourier series for a function, the values of the coefficients. a_o, a_n and b_n must be determined. The values obtained determines the nature of the Fourier Series to be generated.

<u>Example 3.5</u>

Given that f(x) = x, express the function in form of Fourier Series.

Solution

We know that, $F(x) = a_o/2 + \Sigma^{\infty}{}_{n=1}a_n\text{Cosnx}$

$+ \Sigma^{\infty}{}_{n=1}b_n\text{sinmx}$

$X = a_o/2 + \Sigma^{\infty}{}_{n=1}b_n\text{Sinmx}$

From our formula,

$a_o = 1/\pi\int^{\pi}{}_{-\pi}f(x)dx = 1/\pi \int^{p}{}_{-p}xdx$

$=1/\pi[x^2/2]^{p}{}_{-p} = 1/\pi[P^2/2-P^2/2] = 0$

Also $a_n = 1/\pi\int^{p}{}_{-p}f(x)\text{Cosnxdx} = 1/\pi\int^{p}{}_{-p}x\text{Cosnxdx}$

Let u=x, du = dx

dv = Sinnx dx, v = -Cosnx/n

$= 1/\pi\int^{p}{}_{-p} x\text{Cosnxdx} = 1/\pi[x\text{Sinnx}/n]^{p}{}_{-p}$

$- \int^{p}{}_{-p}\text{Sinnx}/\text{ndx}$

$= 1/\pi\ [\text{Cosnx}/n^2]^{p}{}_{-p} = 1/\pi\{[\text{Cosnp}/n^2 - \text{Cosn}(-p)/n^2]$

$= 0$

Therefore,

$bn = 1/\pi\int^{p}{}_{-p}f(x)\text{Sinnx}\ dx = 1/\pi\int^{p}{}_{-p} x\text{Sinnxdx}$

$= 1/\pi[-x\text{Cosnx}/n + \int^{p}{}_{-p} \text{Cosnx}/\text{ndx}] = 1/\pi[-x\text{Cosnx}/n]^{p}{}_{-p}$

$= -1/\pi[p\text{Cosnp} + p\text{Cos}(-np)] = -1/\pi n*2p\text{Cosnp}$

Therefore, $b_n = -2/n\text{Cosnp}$

But $\text{Cosn}\pi = 1$ for n even and $\text{Cosn}\ \pi =-1$ for n odd.

We then have,

$n=1,\ b_1 = 2$

$n=2,\ b_2 = -2/2 = -1$

$n=3,\ b3_3 = 2/3$

$n=4,\ b_4 = -2/4$

Now, Since $a_o = 0$, $a_n = 0$ and $b_n = -2/n\text{Cos}np$

Then we have

$$X= 2/1\ \text{Sin}x -2/2\ \text{Sin}2x +2/3\ \text{Sin}3x - 2/4\ \text{Sin}4x +.....$$

$$X = 2[\text{Sin}x - 1/2\text{Sin}2x+ 1/3\text{Sin}3x -1/4\text{Sin}4x +1/5\text{Sin}5x...$$

ODD AND EVEN FUNCTION

The idea of odd and even function was introduced into Fourier Series to save unnecessary calculations. The significance of this will seen later.

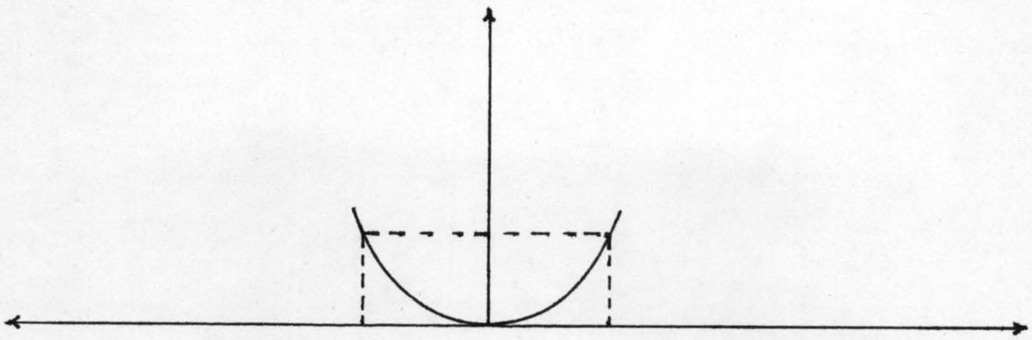
EVEN FUNCTION:- A function is said to be even if $f(-x) = f(x)$. That is the function value for a particular negative value of x is the same as that for the corresponding positive value·Example 3.6

Let $y = f(x) = x^2$

then

$$f(-x) = (-x)^2 = x^2$$

Therefore $y = x^2$ is an even function.The graph of this even function is as shown below. It is symmetrical about the y -axis



Graph of an even function, $f(x) = x^2$

Fig (3.1)

Example 3.7

Let $f(x) = x^4 \ 3x^2 - 5 \ \mathrm{Cos}x + 1$

Therefore, $f(x) = (-x)^4 + 3(-x)^2 - 5\mathrm{Cos}(-x) + 1$

$$= x^4 + 3x^2 - 5\mathrm{Cos}x + 1 = f(x)$$

Therefore, $f(x)$ is an even function.

Theorem 3.1

If $f(x)$ is defined over the interval $-\pi < x < \pi$ and $f(x)$ is even, then the Fourier Series for $f(x)$ contain Cosine terms only included in this is a_o which may be regarded as $a_n\mathrm{Cos}nx$ with $n=0$

Proof:-

Since $f(x)$ is even, $\int_{-\pi}^{o} f(x)dx = \int_{o}^{\pi} f(x)dx$

$a_o = 1/\pi\int_{-\pi}^{\pi} f(x)dx = 2/\pi\int_{o}^{\pi} f(x)dx$

Therefore,

$$a_o = 2/\pi\int_{o}^{\pi} f(x)dx$$

$a_n = 1/\pi \int^{\pi}_{-\pi} f(x)\text{Cosnxdx}$

But f(x) Cosnx is the product of two even function and therefore itself even

$= 1/\pi\int^{\pi}_{-\pi}f(x)\text{Cosnxdx} = 2/\pi\int^{\pi}_{o} f(x)\text{Cosnxdx}$

$b_n = 1/\pi\int^{\pi}_{-\pi}f(x)\text{Sinnxdx}$

Since f(x) Sinnx is the product of even function and odd function it is itself odd.

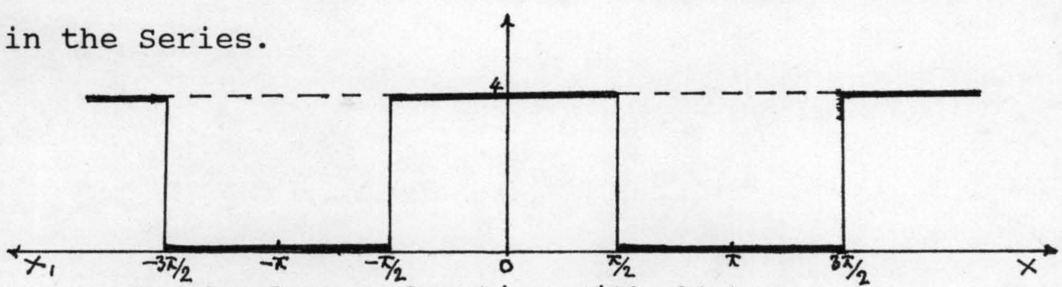
Therefore,

$bn = 1/\pi\int^{\pi}_{-\pi}f(x)\text{Sinnxdx} = 0$

$b_n = 0$. Since $b_n = 0$, Sine terms in the Fourier Series for f(x).

Example 3.8

The wave form shown below is symmetrical about the y axis. The function is therefore even and there will be no Sine terms in the Series.



Graph of even function with f(x) = 4
Figure (3.2)

Since it is an even function then,

$f(x) = \frac{1}{2}a_o + \Sigma^{\infty}_{n=1} a_n\text{Cosnx}$

Solution

$a_o = 1/\pi\int^{\pi}_{-\pi}f(x)dx = 2/\pi \int^{\pi}_{o}f(x)dx$

$= 2/\pi\int^{\frac{1}{2}\pi}_{o}4dx$

$= 2/\pi[4x]^{\pi/2} = \{[8\pi/2\pi - 8(0)/\pi \} = 4$

$a_n = 1/\pi\int^{\pi}_{-\pi}f(x)Cosnxdx = 2/\pi\int^{\pi}_{o}f(x)Cosnxdx$

$= 2/\pi\int^{\pi/2}_{0}Cosnxdx = 8/\pi[Sinnx/n]^{\pi/2}_{0}$

$= 8/\pi n \ Sinn\pi/2$

But $Sinn\pi/2 = 0$ for n even

$= 1$ for $n = 1, 5, 9, ...$

$= -1$ for $n = 3, 7, 11, ..$

and $a_1 = 8/\pi$, $a_3 = 8/3\pi$, $a_5 = 8/5\pi$, $= 8/7\pi$

Therefore, the required Series is

$f(x)=2+8/\pi \ [Cosx -1/3Cos3x +1/5Cos5x -1/7Cos7x+...$

ODD FUNCTION:-

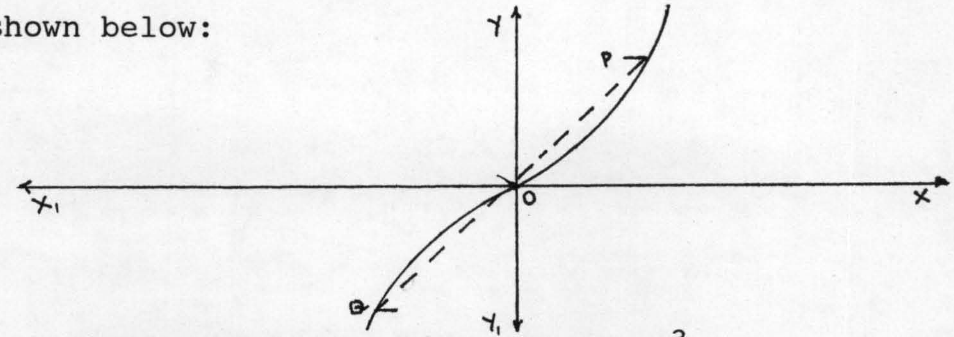
A function is said to be odd if $f(-x) = -f(x)$. That is, the function value for a particular negative value of x is numerically equal to that for the corresponding positive value of x but opposite in sign.

Example 3.9

Let $y = f(x) = x^3$

Then $f(-x) = (-x)^3 = -x^3$

Therefore, $y = x^3$ is an odd function,the graph of this odd function is as shown below. It is Symmetrical about the origin as shown below:



Graph of odd function, $f(x) = x^3$

Figure(3.3)

Example 3.10

Let $f(x) = x^5 + 3^x - \text{Sin}x + x$,

Then $f(-x) = (-x)^5 + 3(-x)^3 - \text{Sin}(-x) + (-x)$

$= (-x^5 - 3x^3 + \text{Sin } x - x)$

$= -(x^5 + 3x^3 - \text{Sin}x + x$

$= -f(x)$

Therefore, f(x) is and odd function.

Theorem 3.2

If f(x) is an odd function defined over the interval $-\pi < x < \pi$, Then the Fourier Series for f(x) contains Sine terms only.

Proof:-

Since f(x) is an odd function then $\int^{\pi}_{-\pi} f(x)dx = -\int^{\pi}_{o} f(x)dx$

therefore, $a_o = 1/\pi \int^{\pi}_{-\pi} f(x)dx$, But f(x) is odd

$a_o = 0$

$a_n = 1/\pi \int^{\pi}_{-\pi} f(x)\text{Cos}nxdx = 1/\pi \int^{\pi}_{-\pi} (\text{odd function})dx = 0$

$a_n = 0$

$b_n = 1/\pi\int^{\pi}_{-\pi} f(x)\,\mathrm{Sin}nx\,dx$

Since $f(x)$ is odd and Sin nx is also odd, the product $f(x)$ Sinnx is even.

$= 1/\pi\int^{\pi}_{o} f(x)\,\mathrm{Sin}nx\,dx = 1/\pi\int(\text{even function})\,dx$

Therefore, $b_n = 2/\pi\int f(x)\,\mathrm{Sin}nx\,dx$

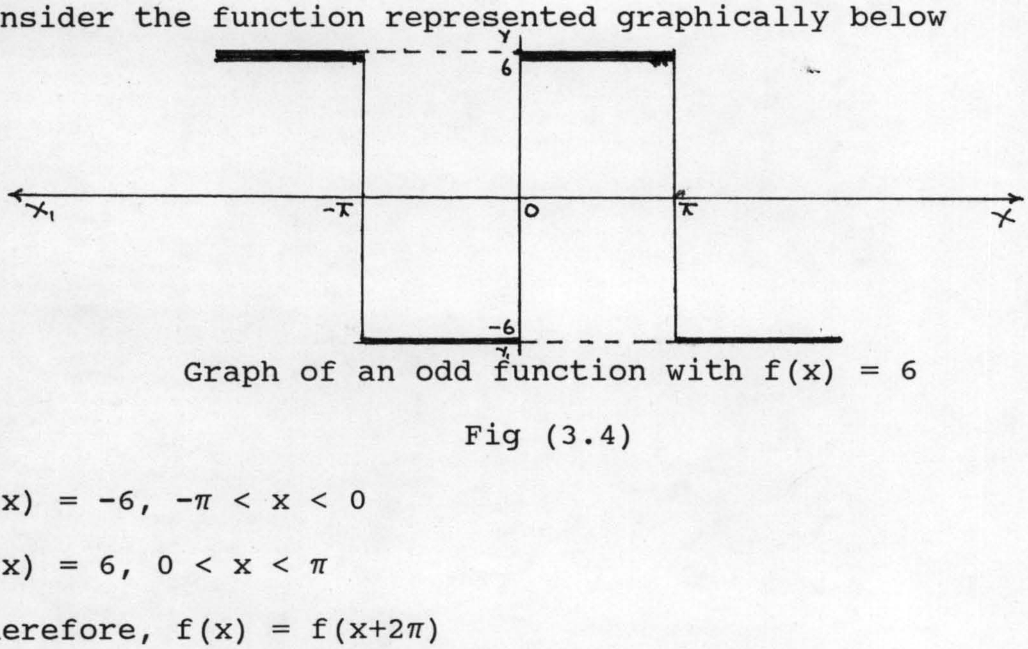
The Fourier Series in this case contains Sine terms only Since $a_n = 0$

So, if $f(x)$ is odd $a_o = 0$, $a_n = 0$

Therefore, $b_n = 2/\pi\int^{\pi}_{o} f(x)\,\mathrm{Sin}nx\,dx$

Example 3.11

Consider the function represented graphically below



Graph of an odd function with $f(x) = 6$

Fig (3.4)

$f(x) = -6, \ -\pi < x < 0$

$f(x) = 6, \ 0 < x < \pi$

therefore, $f(x) = f(x+2\pi)$

From the graph we can see that this is an odd function and therefore only sine terms are expected.

then $f(x) = \Sigma^{\infty}_{n=1} b_n \mathrm{Sin}nx$

$a_o = 0$ and $a_n = 0$

therefore, $b_n = 1/\pi\int^{\pi}_{-\pi}f(x)\text{Sin}nx dx$

f(x) Sin nx is a product of two odd functions and is therefore even $= 2/\pi\int^{\pi}_{o}f(x)\text{Sin}nx dx$

$= 2/\pi\int^{\pi}_{o} 6\text{Sin}nx dx = 12/\pi[-\text{Cos}nx/n]^{\pi}_{o} = 12/\pi n(1-\text{Cos}n\pi)$

$b_n \quad = 12/\pi n(1-\text{Cos}n\pi)$

But $b_n = 0$ for n even

and $b_n = 24/\pi n$ for n odd

Since $\text{Cos}n\pi = 1$ for even and $\text{Cos}n\pi = -1$ for odd

$b_1 = 24/\pi$, $b_3 = 24/3\pi$, $b_5 = 24/5\pi$,

So, the Fourier Series is

$$f(x) = 24/\pi\ [\text{Sin}x + 1/3\text{Sin}3x + 1/5\text{Sin}5x + -----]$$

There are situation whereby a function would be discovered not to belong to even nor odd functions. So, if f(x) is nether odd nor even function, then we must obtain expression for a_o, a_n and b_n in full.

This implies that, Given a function r(x) and

$r(x) \quad = x^3 + 2x + 1$

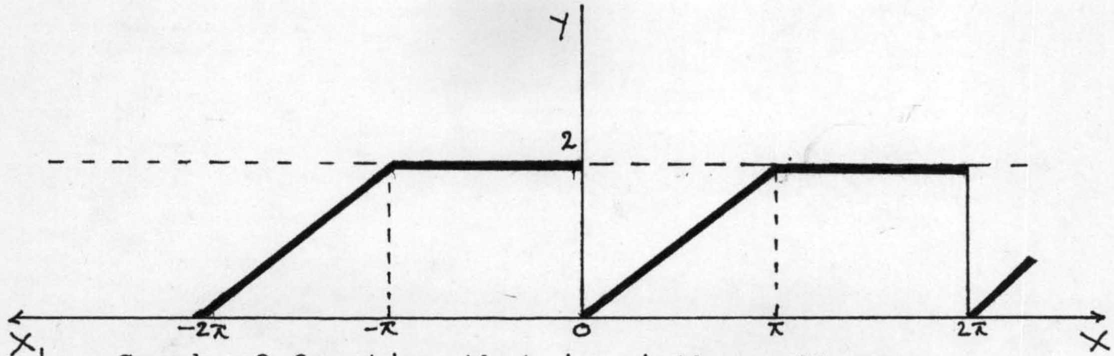
$r(-x) = (-x)^3 + 2(-x) + 1$

$\quad\quad = -x^3 - 2x + 1$

hence, the function r(x) is not even and not odd.

Example 3.12

Determine the Fourier Series for the function shown graphically below :-



Graph of function that is niether odd nor even

Fig(3.5)

<u>Solution</u>

We know that,

$f(x) = \frac{1}{2}a_o + \Sigma^{\infty}_{n=1} [a_n \text{ Cos}nx + b_n \text{ Sin}nx]$

$a_o = 1/\pi\int^{2\pi}_o f(x)dx = 1/\pi \{\int^{\pi}_o 2x/\pi dx + \int^{2\pi}_{\pi} 2dx\}$

$= 1/\pi\{[x^2/x]^{\pi}_o + [2x]\}^{2\pi}_{\pi} = 1/\pi[\pi+4\pi-2\pi] = 3$

Therefore, $a_o = 3$

$a_n = 1/\pi\int^{\pi}_o f(x)\text{Cos}nxdx = 1/\pi\{\int^{\pi}_o (2x/\pi)\text{Cos}nxdx$

$+ \int^{2\pi}_{\pi} 2\text{Cos}nxdx\}$

$= 2/\pi \{1/\pi[\pi\text{Sin } nx/n]^{\pi}_o - 1/\pi n\int^{2\pi}_0 \text{Sin}nxdx + \int^{2\pi}_{\pi} \text{Cos}nxdx\}$

$= 2/\pi\{\text{Sin}n\pi/n + 1/\pi n[\text{Cos}nx/n]^{\pi}_o + [\text{Sin}nx /n]^{2\pi}_{\pi}$

$=2/\pi \text{ Sin}n\pi + 2.1/\pi^2n^2\text{Cos}n\pi + 2/\pi n \text{ Sin}2n\pi - 2/\pi n \text{ Sin}n\pi$

$=2.1/\pi^2n^2 \text{ Cos}n\pi + 2/\pi n \text{ Sin } 2n\pi$

Therefore, $a_n = 0$ for even and $an = -4/\pi^2n^2$ for n odd

Then $b_n = 1/\pi\int^{\pi}_o f(x)\text{Sin}nx\, dx = 1/\pi\{\int^{\pi}_o (2x/\pi)\text{Sin}nxdx$

$+ \int^{2\pi}_{\pi} 2\text{Sin}nxdx\}$

$=2/\pi\{1/\pi[-x \text{ Cos}nx/n]^{\pi}_o + 1/\pi n\int^{\pi}_o \text{Cos}nxdx + \int^{2\pi}_{\pi} \text{Sin}nxdx\}$

$= 2/\pi\{1/\pi n\ (-\pi \cos n\pi) + 1/\pi n\ [\sin nx/n]_{o}^{\pi} + [-\cos nx/n]_{\pi}^{2\pi}$

$= 2/\pi\{ - 1/n \cos n\pi + (0-0) - 1/n (\cos 2\pi n - \cos n\pi)\}$

$= 2/\pi\{-1/n \cos 2n\pi\} = -2/\pi n \cos 2n\pi$

But $\cos 2n\pi = 1$

therefore, $bn = -2/\pi n$

Then, $b_1 = -2/\pi$, $b_2 = -2/2\pi$, $= b_3 = -2/3\pi$, $b_4 = -2/4\pi$

therefore,

$f(x) = 3/2 - 4/\pi^2 [\cos x + 1/9 \cos 3x + 1/25 \cos 5x + ...]$

$- 2/\pi[\sin x + 1/2 \sin 2x + 1/3 \sin 3x + 1/4 \sin 4x.....]$

HALF - RANGE SERIES

The idea of odd and even function is extended to half -range Series and this is really where the idea Saves time better and also points out a lot of difference.

Here functions of period 2π are defined over the range 0 to π instead of the normal 0 to 2π. The following graphs show different types of half range and tells the term to be expected in the Series.

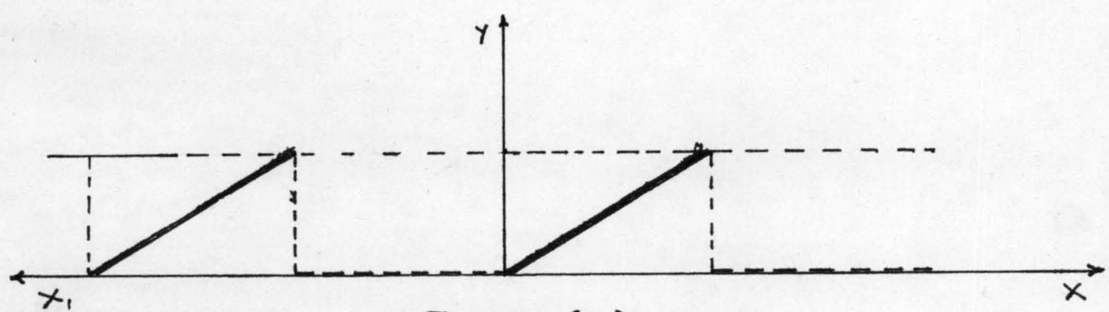


FIGURE (3.6)

Since the waveform is niether symmetrical about the y-axis nor the origin, then we say that f(x) here is niether odd nor even function.

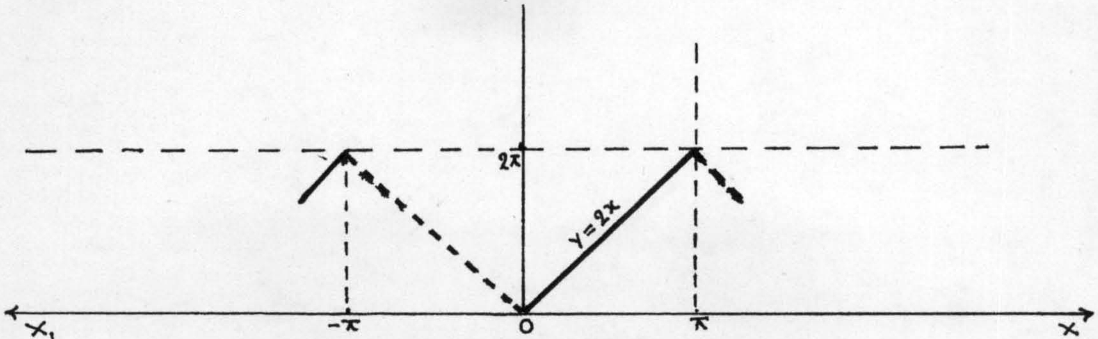


FIGURE (3·7)

Graph of an even function, f(x) = 2x

Here the waveform is symmetrical about the y - axis and the Fourier Series from here would have only Cosin terms.

Now we can find the Fourier Coefficient and Series here as usual .

From the graph above f(x) = 2x, i.e figure (3.7)

Therefore, $a_o = 2/\pi \int^{\pi}_{o} f(x)dx = 2/\pi \int^{\pi}_{o} 2xdx = 2/\pi$

$[x^2]^{\pi}_{o} = 2/\pi$

So, $a_o = 2/\pi$

then, $a_n = 2/\pi \int^{\pi}_{o} 2x\text{Cos}nx dx = 4/\pi \int^{\pi}_{o} x\text{Cos}nx dx$

$= 4/\pi[(\text{Sin}nx \cdot x/n)^{\pi}_{o} - 1/n \int^{\pi}_{o} \text{Sin}nx dx]$

$= 4/\pi\{(0-0) -1/n\ [-\text{Cos}nx/n]^{\pi}_{o}\} = 4/\pi n^2\ (\text{Cos}n\pi-1)$

but $\text{Cos}n\pi = 1$ for n even and -1 for n odd.

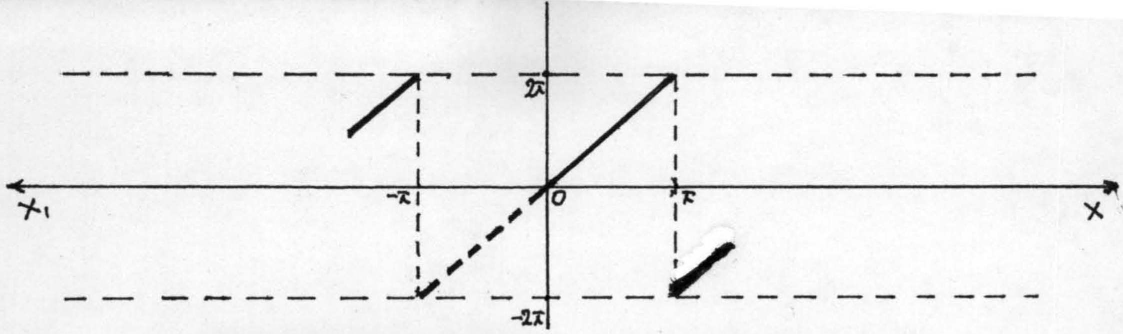
Therefore, $a_n = 0$ for n even and $a_n = -8/\pi n^2$ for n odd.

Then $a_1 = 8/\pi$, $a_3 = 8/9\pi$, $a_5 = 8/25\pi$,

For the case of bn, it will surely be equal to zero Since f(x) is an even function, which implies that $b_n = 0$

Therefore, the Series is:

$f(x) = \pi - 8/\pi\ [\text{Cos}x + 1/9\text{Cos}3x + 1/25\ \text{Cos}5x + \ldots$



Graph of an odd function, $f(x) = 1+x$.

FIGURE (3.8)

From the graph above we can see that it is symmetrical about the origin, and then should be expected to have only Sin terms in the Fourier Series

Now we can also find the Fourier Series and Coefficient here as usual.

From the above graph $f(x) = (1+x)$

Therefore, $a_o = 2/\pi \int^{\pi}_{o} f(x)dx -2/\pi\int^{\pi}_{o}(1+x)dx$

$$= 2/\pi[(1+x)]^{\pi}_{o}$$

$= 2/\pi\{[1+\pi] - [1+0]\} = 0$

For the a_n, it will surely be equal to zero Since $f(x)$ is an odd function

Therefore, $a_n = 0$

For b_n we have :-

$$b_n = 2/\pi\int^{\pi}_{o} (1+\pi)\text{Sinnxdx} = 2/\pi \{[(1+x) - \text{Cosnx}/n]$$

$$+ 1/\pi \int^{\pi}_{o} \text{Cosnxdx}\}$$

$$= 2/\pi \{-(1+\pi)/n \text{ Cos}n\pi + 1/n + \tfrac{1}{n}[\text{Sin } nx /n]^{\pi}_{o}\}$$

$$= 2/\pi \{1/n - (1+\pi)/n \text{ Cos}n\pi\} = 2/\pi n \{1-(1+\pi) \text{ Cos}n\pi\}$$

But $\text{Cos}n\pi = 1$ for even n and $\text{Cos}n\pi = -1$ for odd n

Therefore, $b_n = -2/2$ for n even

and $b_n = 4+2\pi$ for n odd

Then, the Series is :-

$f(x) = 4+2\pi/\pi \{\text{Sin } x + 1/3 \text{ Sin } 3x + 1/5 \text{ Sin } 5x + ...$

So, this shows that knowledge of odd and even functions and half range Series saves a great deal of unecessary work.

CHAPTER FOUR

In this chapter we shall attempt to introduce some examples of areas of applicability of the Fourier Series. In a nutshell, we present the following:

(i) In heating systems, the expression of the curve of solar radiation is done using Fourier Series.

(ii) It is used in wave and electronics to determine the value and nature of voltage and current used in linear circuit with time.

(iii) Fourier series is also used for representing the motion of vibrating string and when solving problems such as those involve in heat diffusion.

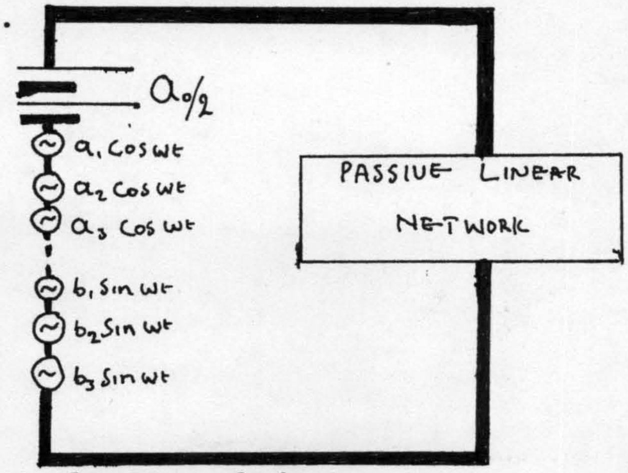
(iv) In transmission line in terms of natural, modes, Fourier Series analysis is also used.

(v) Another aspect that can not be left out is the boundary - value problems where Fourier Series is also extensively used.

However, we have in this chapter some good examples of the use of Fourier Series to obtain desired results in the applied mathematics earlier mentioned. The detailed aspect with examples are consider here below.

(4.1) ELECTRICAL CIRCUIT ANALYSIS

In the analysis of waveforms generated from electrical circuits, Fourier Series is employed Moreover, it was suggested that terms of a voltage Series could be applied to linear network and then obtain the corresponding harmonic terms of the current Series. Below is a diagram of a general circuit indicating harmonic terms.

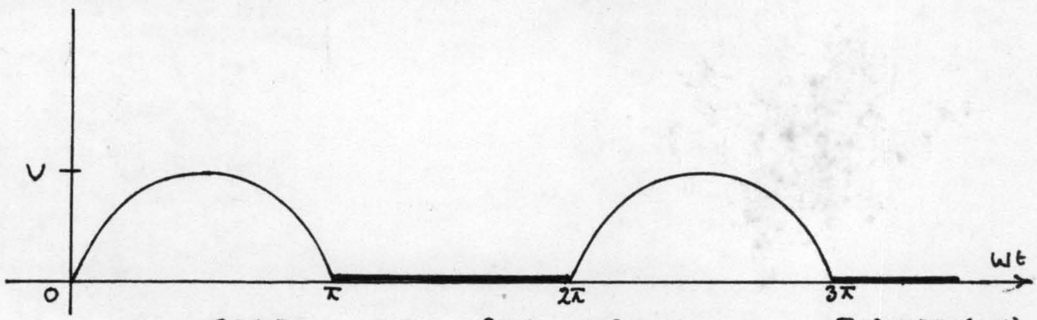


FIGURE(4.1)

A GENERAL CIRCUIT

Example 4.1

Find the Fourier Series for the half - wave rectified sine wave shown below



GRAPH OF $f(x) = V\sin wt$ FIGURE(4.2)

Solution

The wave above shows no symmetry and we therefore expect the Series to contain both Sine and cosine terms.

We know that

$$a_o = 1/\pi\int^{\pi}_{-\pi} f(x)dx$$

$$a_o = 1/\pi[\int^{\pi}_{o} v\mathrm{Sin}wtdwt$$

$$= v/\pi\int^{\pi}_{0}\mathrm{Sin}wtdwt$$

$$= v/\pi[-\mathrm{Cos}\ wt]^{\pi}_{0} = v/\pi\{[-\mathrm{Cos}\ \pi + \mathrm{Cos}\ 0]\} = 2v/\pi$$

Next we determine a_n, and we know that

$an = \frac{1}{2}\int^{\pi}{}_{\pi} f(x)Cosxdx$

$= 1/\pi[\int^{\pi}{}_{\pi} vSinwtCoswtdwt]$

$= v/\pi\ [(-SinwtSinnwt - CosnwtCoswt)/-n^2 + 1)]$

$= v/\pi\{[(-n\ Sin\pi Sin\pi - Cosn\pi Cos\pi)/-n^2 + 1]$

$- [(-nSinoSino -CosoCoso)/-n^2 + 1)]\}$

$a_n = v/\pi(1-n^2)(Cosn\pi+1)$

We know that $Cosn\pi = 1$ for even n and $Cosn\pi = -1$ for odd n

For n even $a_n = 2v/\pi(1-n^2)$

For n odd $a_n = 0$

Next we evaluate b_n and

$b_n = 1/\pi\int^{\pi}{}_{-\pi} f(x)Sinxdx$

$= 1/\pi\int^{\pi}{}_{o} vSinnwtSinwtdwt$

$= v/\pi[(nSinwtCosnwt - SinwtCoswt)/-n^2 + 1]^{\pi}{}_{o}$

$= 0$

But here the expression is indeterminate

For n =1 , and b_1 of evaluated Separately.

$b_1 = 1/\pi\int^{\pi}{}_{o} vSin^2wtdwt = v/\pi[wt/2 - Sin2wt/4]^{\pi}{}_{o}$

$= v/2$

Then the required Series is $f(x) = v/\pi[1+\pi/2Sin\ wt -2/3Cos\ wt$

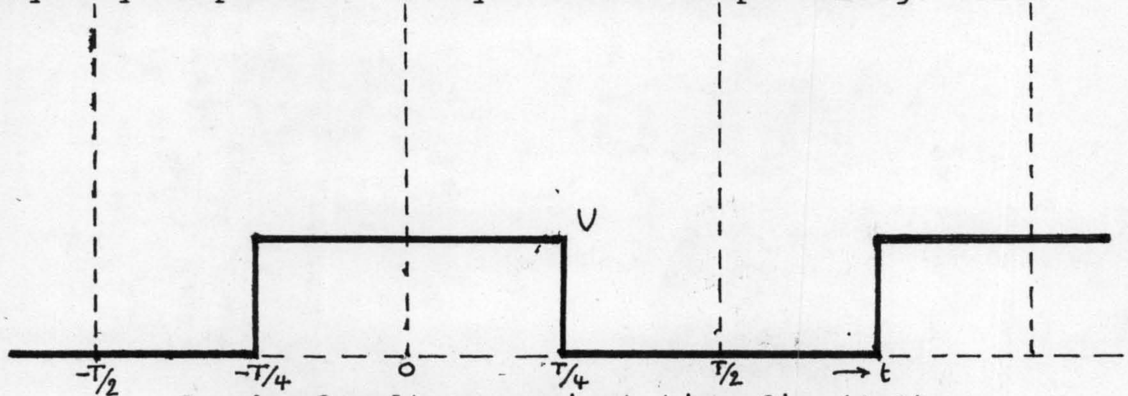
$- 2/15\ Coswt\ - 2/35\ Cos6wt....]$

4.2 ANALYSIS OF SQUARE WAVE SHAPE

The current that flows when certain voltage is applied to a linear circuit can be found by determining the current using the individual terms of the functions, say f(t) Fourier Series.

Example 4.2

Let us find by Fourier analysis the coefficients for the frequency Components in square wave shape voltage below -



Graph of voltage against time fig (4.3)

Voltage is U over half the period T and zero over the remaining half. The origin will be Selected arbitrarily in the center of the constant portion as shown, in other to make the function even. Voltage then drops to zero at t =T/4, or wt $=\pi/2$. The integral shows that the constant term a_o is

$$a_o = 1/2\pi \int_{-\pi}^{\pi} f(t)d(wt)$$

$$= 1/2\pi \int_{-\pi/2}^{\pi/2} vd(wt)$$

$$= v/2.$$

This is clearly the average value of the wave and the integral gives the coefficient as :-

$$b_n = 1/\pi \int_{-\pi}^{\pi} f(t)\ Sin(wt)d(wt)$$

$$= 1/\pi \int_{-\pi/2}^{\pi/2} vSinn(wt)dwt$$

$$= 0$$

This coefficient of Sine terms are zero as would be expected since Sine are odd functions, and we have selected the origin to make f(x) even.

Finally, the a_n terms are:

$$a_n = 1/\pi\int^{\pi}_{-\pi} f(t)\text{Cos}n(wt)dwt$$

$$= 1/\pi\int^{\pi/2}_{-\pi/2} v\text{Cos}n(wt)dt$$

$$a_n = v/n\pi[\text{Sin}n(wt)]^{\pi/2}_{-\pi/2}$$

Therefore the value of a_n is zero if n is even, and is $v/n\pi$ if n is 1,5,9 .... And is $-2v/n\pi$ if n is 3,7,11,... There fore, the Series expansion of the square wave voltage may be written as :-

$f(t) = v/2 + 2v/\pi[\text{Cos}wt - \text{Cos}3wt/3 + \text{Cos}5wt/5 - \text{Cos}7wt/7+...]$

Then the current that flows when such voltage is applied to a linear circuit is found by determining the circuits using the individual terms and then superposing them

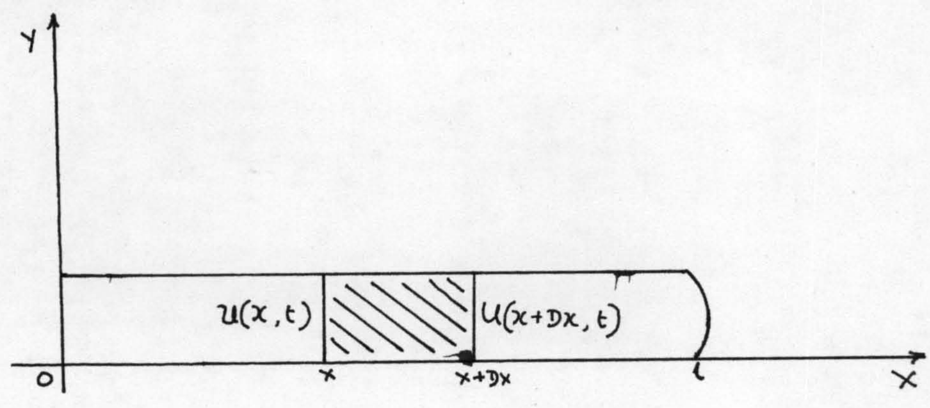
4.3 HEAT FLOW

The objective of any heat transfer analysis is usually to predict heat flow or temperature which results from certain heat flow. So, considering a material the total heat flow vector is directed so that it is perpendicular to the lines of constant temperature in the material. And if the temperature distribution in the material is known, It is at this point that Fourier Series is employed to help do the perfect prediction of the heat flow or temperature.

This idea of predicting heat flow is employed in the deduction of heat equation in one dimension. The heat equation obtained is then analyzed unsing Fourier Series methods so that the heat flow could be well predicted.

Example 4.3

Given a bar of length l of uniform cross-sectional area A deduce the heat equation and then obtain a Fourier Series from the equation Let us consider figure (4.4)



A BAR , FIGURE (4.4)

Solution

Take the origin at one end of x -axis along the bar.

Let u(x,t) be the temperature of the shaded section at x and time t.

We know that heat flows from a higher temperature to a lower temperature. Therefore, the amount of heat flow per unit area per sec = Rate of heat flow = -kDU/DX

Where k is the normal conductivity of the material. At limits as Du, Dt $\rightarrow$ 0

Rate of heat flow = $-k\delta v/\delta t$

Therefore volume of material between x and x+Dx = ADx

Then mass of material between x and x+Dx = $\int$ADx

Now, let c be the heat capacity of the material, then the quantity of heat in the material is $\int$ACDX$\bar{U}$(x,t)

where $\bar{U}$(x,t) is the mean temperature. The increase in the amount of heat in the time interval (t,t+Dt) is

$C\int ADX\bar{U}(x,t+Dt) - C\int ADx\bar{U}(x,t)$

= Dt * Rate of heat flow

$= KADt\bar{U}_x(x+Dx,t) - KADt\bar{U}(x,t)$

where $\bar{U}_x$ is the mean value of U_x in the time interval (t,t+Dt) therefore, we have

$C\int ADX\bar{U}(x,t+Dt) - C\int ADX\bar{U}(x,t) = KADt\bar{U}_x\ (x+Dx,t) - KADt\bar{U}_x(x,t)$

Dividing through by ADXDt

$C\int[\bar{U}(X,+Dt)-\bar{U}(x,t)]/Dt$

$= K[\bar{U}_x(x+Dx,t)-\bar{U}_x(x,t)]/Dx$

At limits Dx,Dt $\rightarrow$ 0

$$c\int U_t = KU_{xx}$$

$\delta u/\delta t = K/C\int \delta^2 u/\delta x^2$

Since $\alpha^2 = k/c\int \Longrightarrow \delta u/\delta t = \alpha\delta u^2/\delta x^2$ ----------(3.3)

Now, to obtain a complete solution to the equation (3.3) above some boundary conditions would have to be satisfied. This is what would then take us into our required Fourier Series

Conditions :-

i) $U_x(0,t) = 0$

ii) $U_x(1,t) = 0$

iii) $U(x,0) = f(x)$

With the condition above, the solutions to the partial differential equations may be obtained

i.e $\delta u/\delta t = \alpha^2 \delta^2 u/\delta x^2$

which is the same as

$U_t = \alpha^2 U_{xx}$.

We can now proceed to solve the equation by variable Separable method.

Let $U(x,t) = X(x)\,T(t)$

$\delta u/\delta t = X\dot{T}$ and $\delta^2 u/\delta x^2 = X''T$

$\Longrightarrow U_t = X\dot{T}$ and $U_{xx} = X''T$

$\Longrightarrow$ Substituting into main equation (3.3) we have

$X\dot{T} = \alpha^2 X''T$ ------ (3.4)

Dividing by XT we get

$\dot{T}/T = \alpha^2 X''/X = -P^2$

where P is a constant

Therefore,

$X''/X + P^2/\alpha^2 = 0 \Longrightarrow X'' + P^2 x/\alpha^2 = 0$

Then $X = A\cos px/\alpha + B\sin px/\alpha$ ---- (3.5)

Also $\dot{T}/T = -P^2$ =====> $dt/dt = -P^2$

Therefore, $\dot{T} = e^{-P^2t} = 1/e^{P^2t}$

This is the same as

$$\int dT/T = \int - P^2 dt$$

==> $\ln t = -P^2t$ and $T = e^{-P^2t}$ -----(3.6)

The solution to equation 3.3 is

$U(x,t) = (ACospx/\alpha + BSinPx/\alpha)\ e^{-P^2t}$

Now applying condition (1) we get

$U_x(x,t) = e^{-p^2t}\ [-Ap/\alpha Sinpx/\alpha + Bp\alpha/Cospx/\alpha]$

$U_x(0,t) = e^{-p^2t}\ [0 + Bp/\alpha] = 0$

==> $B = 0$

Hence $U(x,t) = e^{-p^2t}\ [ACospx/\alpha]$

Applying second condition we get

$U_x(l,t) = e^{-p^2t}[Ap/\alpha Sinpl/\alpha] = 0$

==> $Sinpl/\alpha = 0$ and $pl/\alpha = n\pi$

$n = 1,2,3$ ....

==> $p = \alpha n\pi/l$

Therefore, $U(x,t) = e^{-(\alpha n\pi/l)^2t}\ [AnCosn\pi x/l\ ]$ -----------3.7

considering the principle of superposition equation, (3.7) may be summed up and then rewritten as

$U(x,t) = \Sigma^{\infty}_{n=1}\ e^{-(\alpha n\pi/l)^2t}\ [A\ cosn\pi x/l]$ --- 3.8

Finally, applying condition 3 to equation (3.8) we get

$U(x,0) = \Sigma\ An\ Cosn\pi x/l = f(x)$ ------ (3.9)

Since $e^{-(\alpha n\pi/l)^2t} = 0$, Since $t = 0$

Expanding equation (3.9) we have

A_1 Cosπx/l + A_2 Cos2πx/l + A_3 Cos3πx/l +.... = f(x)

This is a Fourier Cosine Series and the coefficients A_1, A_2, A_3 ... can be easily determined in the interval 0<x<l

==> An = $2/l\int^{\pi}_0$ Cosnπx/l*f(x)dx.

Now suppose that f(x) = x and 2l = π

==> l = $\pi/2$

Therefore An = $2/\pi\int^{\pi}_0$xCosnxdx

Using differentiation by parts

Let u = x the du = dx

dv = Cosnx then v = Sinnx/n

then

$2/\pi[x \text{ Sinnx}/n + \text{Cosnx}/n^2]^{\pi}_0 = 2/n^2\pi\ [\text{Cosn}\pi - 1]$

But, for even n, Cosnπ = 1 and Cosnπ = - 1 for odd n.

$A_{1,}$ = $2/\pi$ * -2 = $-4/\pi$, A_2 = 0, A_3 = $4/3^2\pi$ ...

Hence

f(x) = $-4/\pi$ [Cos x + Cos3x/3^2 + Cos5x/5^2 + Cos7x/7^2 + -- = x.

Now let X = π, we get $-4/\pi[1+1/3^2+1/5^2+1/7^2+1/7^2+ ---]$

= x

Since Cosnπ = 1

Finally, $[1+1/3^2+1/5^2+1/7^2+...] = \pi 2/4$

(4.4) <u>VIBRATION</u> <u>STRING</u>

We know that Fourier Series is a mathematical expression that is important both in mathematics itself and in a wide variety of applications in the physical Sciences, especially in the theory of wave motion and vibrations

Therefore, in this section we would derive and solve the equation of the vibrating string. this equation is the wave equation, which occurs throughout many branches of mathematical physics. Meanwhile, let us consider the example below.

Example 4.4

Given an elastic string stretched between two support O and B along x axis, find the equation of the vibrating string and obtain Fourier Series from the equation derived.

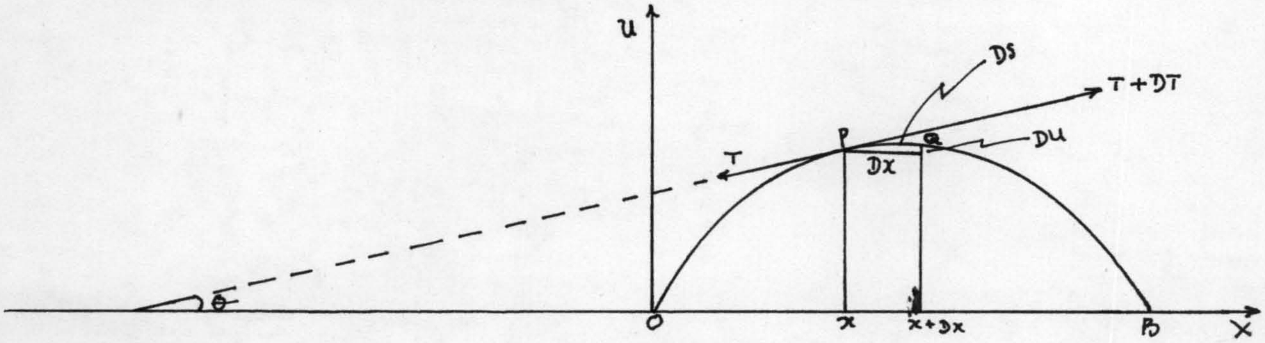


Fig (4.5)

<u>Solution</u>

Let U(x,t) be the Vatical distance of the points P of the string at a distance x from O and at time, t.

Let the distance PQ = Ds and ʃ = density of the string.

We know that the mass of the string Ds is = ʃ Ds

where ʃ = density of string. Now, the complement of the vertical force of P is $T\text{Sin}\theta\,|_{x+Dx} - T\text{Sin}\theta\,|_{x}$

But, vertical acceleration = $\delta^2 u/\delta t^2$

Since force = mass * acceleration, we then have that

(3.11) $\quad -ʃDS\ \delta^2 u/\delta t^2 = T\text{Sin}\theta|_{x+dx} - T\ \text{Sin}\theta\ |_x$

Clearly, we know that $\text{Sin}\theta = \tan\theta\text{Cos}\theta$

$= Sin\Theta = \tan\Theta / Sec\,\Theta$

Also we know that

$$Sec^2\Theta = 1 + \tan^2\Theta$$

$$Sec\Theta = (1+\tan^2\Theta)^{\frac{1}{2}}$$

Therefore $Sin\Theta = \tan\Theta\,[1+\tan^2\Theta]^{-\frac{1}{2}}$

Now,

If U is small and T is approximately constant and Θ is small then $\tan\Theta = Du/Dx$

And at limits $\delta u/\delta x = U_x$

$\Longrightarrow$ that $Sin\Theta = \tan\Theta(1+\tan^2\Theta)^{-\frac{1}{2}}$

$$= U_x(1+U^2{}_x)^{-\frac{1}{2}} = U_x$$

Hence, by dividing equation 3.11 by Dx we then have

$$\int Ds/Dx\ \delta u^2/\delta t^2 = \frac{TSin\Theta|_{x+Dx} - TSin\Theta\ |_x}{Dx}$$

But, as Dx, Ds $\rightarrow$ 0 we get

$$\int ds/dx\ \delta s^2/dt^2 = T\delta Sin\Theta/dx = T\delta u_x/\delta x \text{ ---- } (3.12)$$

Clearly, ds/dx equal to a constant at a limit

Hence, From (3.12) we get

$$\int \delta^2 u/\delta t^2 = T\delta^2 u/\delta x^2 = \alpha^2 du^2/\delta x^2$$

Therefore $U_{tt} = \alpha U_{xx}$. -------------(3.13)

Now, to obtain a complete solution to the equation(3.13) above, Some boundary conditions must be satisfied. This is what would then take us into our required Fourier Series.

Conditions

i) $U_x(0,t) = 0$

ii) $U_x(l,t) = 0$

iii) $U_x(x,0) = f(x)$.

With the conditions above, the solutions to the partial differential equation may be obtained, that is

$\delta^2 u/\delta t^2 = \alpha^2 \delta u^2/\delta x^2$

This may be written as

$$U_{tt} = \alpha U_{xx}$$

Using the method of variable separable we have:

Let $U(x,t) = X(x)\, T(x)$ which is the solution of equation (3.13)

$\delta u/\delta t = X\dot{T}$ and then $\delta u^2/\delta t^2 = X\ddot{T}$

$\delta^2 u/\delta x^2 = X''T$

$\Longrightarrow U_{tt} = X\ddot{T}$ and $U_{xx} = X''T$

$\Longrightarrow$ Substituting into main equation 3.13

we have $XT = \alpha^2 X''T$ -------- (3.14)

Dividing through by XT we get

$\ddot{T}/T = \alpha^2 X''/X = -P^2$

where P is a constant

therefore,

$$X''/X + P^2/\alpha^2 = 0 \Longrightarrow X'' + P^2x/\alpha^2 = 0$$

Then, $X = A\ \mathrm{Cos}p/\alpha^{x} + ß\mathrm{Sin}p/\alpha x$ ------(3.15)

Also

$\ddot{T}/T = -p^2 \Longrightarrow \ddot{T}/T + P^2 = 0$

$\Longrightarrow \ddot{T} + PT = 0$

Therefore, $T = (C\mathrm{Cos}pt + B\mathrm{Sin}pt)$ -------------(3.16)

Then the solution to the equation 3.13 is

$U(x,t) = (A\mathrm{Cos}p/\alpha x + B\mathrm{Sin}p/\alpha x)\ (C\mathrm{Cos}pt + B\mathrm{Sin}pt)$

Now applying condition (1) we get

$U_x(x,t) = -(Ap/\alpha Sinpx/\alpha + Bp/\alpha Cos\ px/\alpha)(-CSinpt+DCospt)$

$U_x(0,t) = (0+Bp/\alpha)(-CSinpt + DCospt) = 0$

$U(x,t) = (ACospx/\alpha\ (CCospt+BSinpt)$

Again, applying Condition (2) we get

$U_x(l,t) = -\ (Ap/\alpha Sinpl/\alpha)(CCospt + DSinpt) = 0$

$\Longrightarrow Sinpl/\alpha = 0$ and $pl/\alpha = n\pi \qquad n=1,2,3...$

$\Longrightarrow\ p = \alpha n\pi/l$

Therefore, $U(x,t) = (ACosn\pi x/l)(CCos(\alpha n\pi/l)t +\ DSin(\alpha n\pi/l)t)$

$= 0$

Now Applying condition 3 we get

$U(x,0) = (ACosn\pi x/l)[CCos(\alpha n\pi/l)*0 + DSin(\alpha n\pi/l)*0) = f(x)$

$\Longrightarrow U(x,t) = ACosn\pi x/l = f(x)$

From the principle of superposition

$U(x,t) = \Sigma^{\infty}_{n=1}\ AnCos\ n\pi x/l = f(x)$ --- (3.17)

Expanding equation (3.17) we have

$A_1 Cos\pi x/l + A_2 Cos2\pi x/l + A_3 Cos3\pi x/l + ...$

This is a Fourier Cosine Series and the coefficients A_1, A_2, A_3,... can be easily determined in the interval $0< x < l$

$\Longrightarrow An = 2/l \int^{\pi}_{0} Cosn\pi x/l*f(x)dx$

Now, Suppose that $f(x) =x$ and $2l = \pi \quad \Longrightarrow l =\pi/2$

Therefore, $An = 2/\pi \int^{\pi}_{0} xCosnxdx.$

Using differentiation by part, we have :-

Let $u = x$ Then $du = dx$

$dv = Cosnx$ then $v = Sinnx/n$

Then,

$$2/\pi[x\mathrm{Sin}nx/n + \mathrm{Cos}nx/n^2]^{\pi}_{o} = 2/\pi n^2[\mathrm{Cos}n\pi - 1]$$

But, for even n $\mathrm{Cos}n\pi = 1$ and $\mathrm{Cos}n\pi = -1$ for odd n

Therefore,

$$A_1 = 2/\pi*-2 = -4/\pi,\ A_2 = 0,\ A_3 = -4/3^2\pi$$

Hence,

$$f(x) = -4/\pi[\mathrm{Cos}x + \mathrm{Cos}3x/3^2 + \mathrm{Cos}5x/5^2 + \ldots]$$

Example 4.5

Find Fourier Series to represent the displacement of the string when it is pulled aside by Y_o at the point $x = l/4$

$Y(x) = 4Y_ox/l,\ 0< x< l/4$

$= 4Y_o/3(1-x/l),\ \ l/4 \le x \le l$

Solution

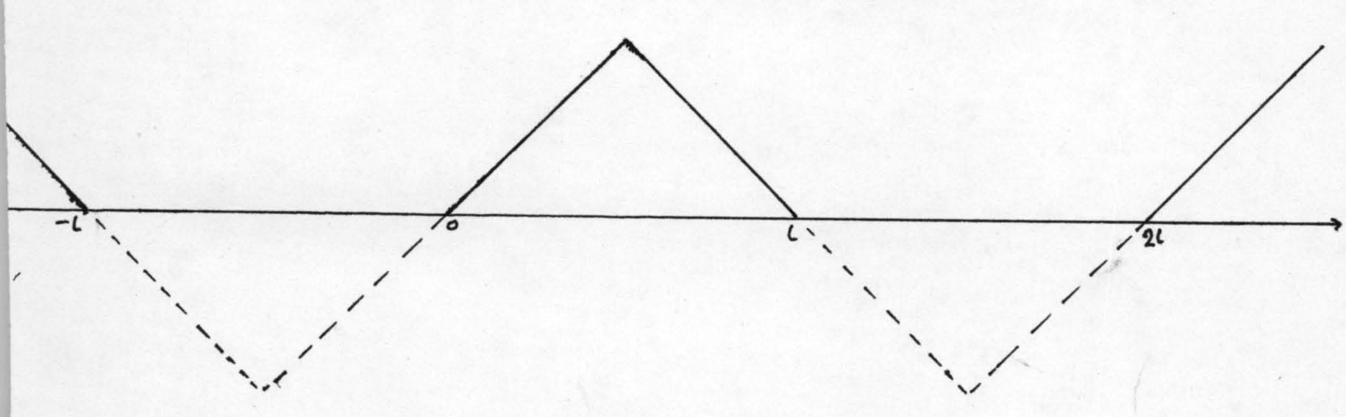


Fig (4.6)

$Y(x) = \Sigma^{\infty}_{n=1} B_n\mathrm{Sin}(nk'x)$ where $k' = 2\pi/2l$

$B_n = 2/2l\int_{-l}^{l} Y(x)\mathrm{Sin}(nk'x)dx$

$$= 2l/l\int_0^l Y(x)\,\mathrm{Sin}(nk'x)\,dx$$

Since the integrand is symmetric about x = 0

$$B_n = 32Y_o\ \mathrm{Sin}[(n\pi/l)/2n^2\pi^2]$$

$$\text{Therefore,}\quad Y(x) = 32Y_o/3\pi^2[(2)^{-\frac{1}{2}}\ \mathrm{Sin}(\pi x/l) + \tfrac{1}{4}\mathrm{Sin}(2\pi x/l) + (2)^{-\frac{1}{2}}/9\ \mathrm{Sin}(3\pi x/l) + \ldots$$

CHAPTER FIVE

CONCLUSION

Soon after the recognition of the significance of the coefficient of Fourier Series, by Joseph Fourier, it became of great importance and virtually the only way of expressing heat flow, motion of vibrating string, voltage and currents in series forms.

Today, in the field of engineering and physics, the practical use of Fourier Series is seen. This practical aspect include the process of predicting heat flow and analysis of wave.

In the practical use of Fourier Series, the idea of one of its properties known as "odd and even function" is seen to be of great importance, that is, once a function is described as either odd or even, it is then dealt with depending on the kind of coefficients associated with such a description, thereby saving the time for calculating coefficients not needed in the problems.

In chapter four (4) of this project, the manipulations involved in the practical use of Fourier Series are given with some good examples. To then prove that the series obtained in every problem are credible, a computer solution of one of the problems was also obtained.

The result of the heat problem in chapter 4, example (4.2) in comparison to that obtained on computer turned out to be the same. The only difference so far observed is the fact that the fractions in the series generated can only be expressed in decimal forms. However, the procedure to the result of the heat flow problem is as expressed in chapter 4, example (4.2). The procedure to result on computer is not different. That is the program was designed and coded to follow the same pattern to compute the Fourier Series to the problem depending on the number of terms demanded. The various terms in the series are then summed up to give the exact value. The procedural design and codes are as shown in the appendix.

RECOMMENDATION

The computer language, TURBO BASIC was used in coding the design of the heat problems. The language was chosen in preference to others due to its capability to handle mathematical manipulations. It is also user friendly and flexible. However, it is successfully proved that the Fourier Series generated as the results to the heat problem is credible.

Even though this project work covered alot of aspect of Fourier Series, there are still other aspect that if research is done further and included, this project would bring more of achievement than even expected. One of these aspects is the area that deals with the Convergence of Fourier Series. However, during this further research, much light should be thrown on the procedure to obtaining the final approximate value of any given Fourier Series.

In conclusion, Fourier series is extensively in use and would with time gain more ground in the area of technology.

REFERENCES

Encyclopedia Americana, Vol. II **Falstagg to Francke**

Encyclopedia Britanica, Vol. IV **(Excom to Francke)**

Fields and Waves in Communncation Electronics By **Simon Ramo, John R. Whinnery, Theodore Van Duzer**

Further Engineering Mathematics By **K. A. Stroud**

Fundamentals of Mathematics and Statistics By **C. I. Brookes, I. G. betteley, S. M. Loxston**

Heating and Cooling Load calculation By **P. G. Dowin**

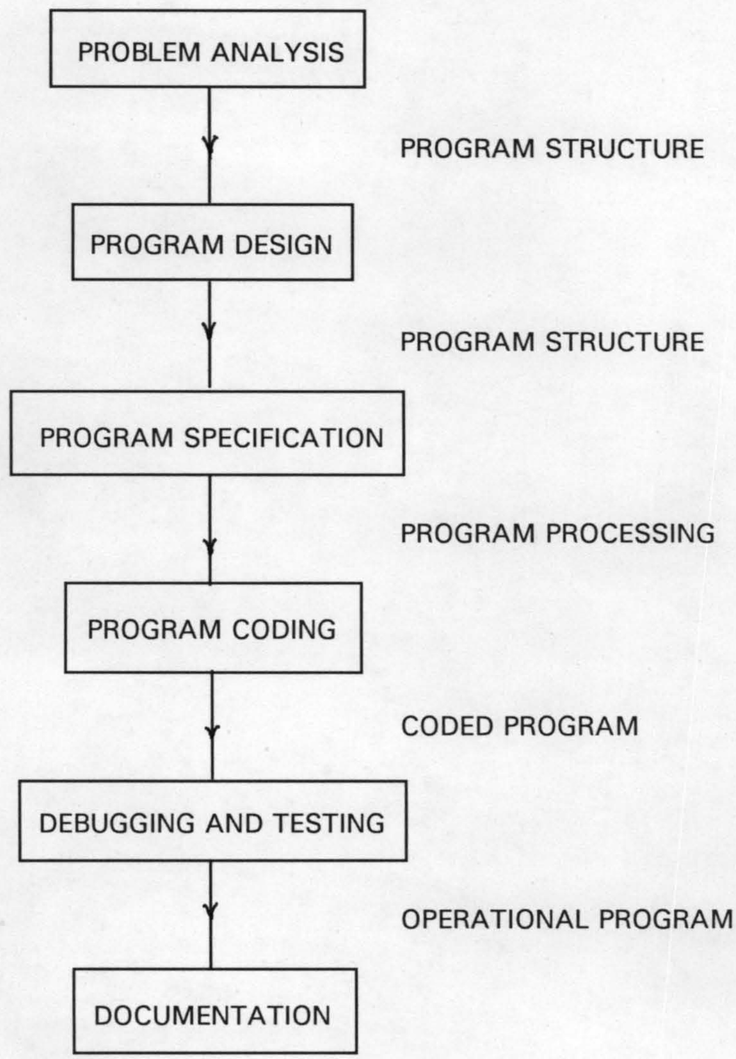
Heat Transfer By **Alan I. Chapman**

Multiple Integrals, Field Theory By **B.M. Budak and S. V. Fomin**

Theory and Problems of Electric Circuit By **Joseph A. Edminister**

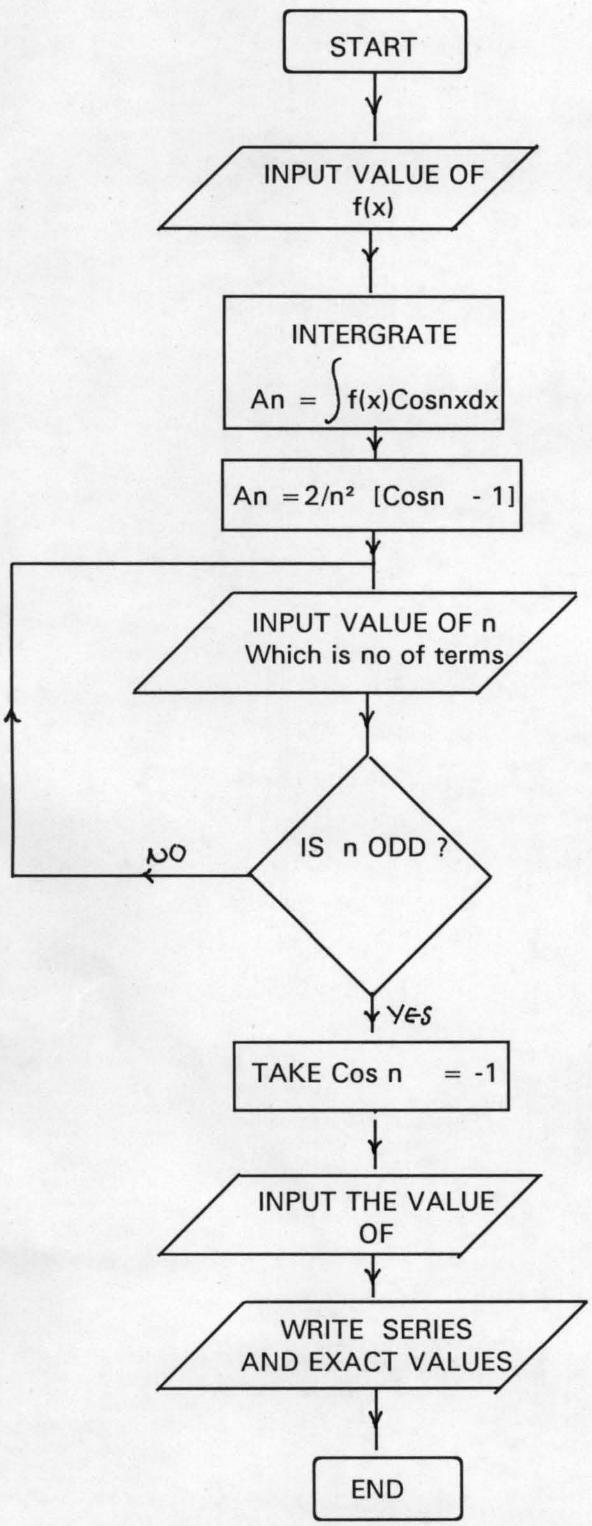
APPENDIX A

PROGRAM DEVELOPMENT PROGRESS



APPENDIX B

FLOW CHART



```
:lude "a:fourier.txt"
›r 15,1

=1
 i=1 to 32
.d k$
:ate row,1:print k$
/=row+1
 row>22 then
 row=1
 goto presskey
l if
ıext i
lelay 0.8
'oto 30

;skey:
›cate 23,25:print"Press any key to continue . . ."
;=input$(1)
:s
› 10

-- solving the heat equation using the fourier
-- series method.
-- conditions are u(0,t)=u(x,0)=0 and u(x,0)=f(x)=x
-- in the interval 0 to +Đ

:ls
›i=22/7
.nput"Enter the number of terms of series to be generated:",nt%

lim a(nt%),b(nt%),p(nt%)
----- initialize a(n) values
for i=1 to nt%
 a(i)=0
ıext i

----- compute values for cosine
----- here we've assumed that f(x)=x and l=Đ
----- the costant values for the fourier series is given by
-----          2 ¶
----- A(n)= - °cos(nĐ/l).f(x)dx        [0,l] integration interval
-----          l ¾

for n=1 to nt%
  q=pi*n
  p(n)=cos(q)-1
next n

----- compute the fourier constants---------

for n=1 to nt%
   pie=pi*n*n
   a(n)=2*p(n)/(pie)
next n

-- the general solution for the eqation is sum[exp(anp/l) An Cos(np/l)x]
-- at t=0 we have a fourier cosine series
```

```
print
input"Enter the value of x: ",x
print
print"The series obtain is given below . . ."
print
for j=1 to nt% step 2
  UU=a(j)*cos(j*x)
  pp=pp+uu
 print using "###.####";uu;:print" +";
next j
print". . . = ";x
print:print
color 18,1
print"We see that as the number of terms of increases the"
print"sum approaches the exact value Đ."
color 7
End
```