FEDERAL UNIVERSITY OF TECHNOLOGY MINNA, NIGER STATE, NIGERIA

CENTRE FOR OPEN DISTANCE AND e-LEARNING (CODeL)

B.TECH. COMPUTER SCIENCE PROGRAMME

COURSE TITLE

LINEAR ALGEBRA 2

COURSE CODE

MAT 222

COURSE CODE

MAT 222

COURSE UNIT

2

Course Coordinator
Bashir MOHAMMAD (Ph.D.)
Department of Computer Science
Federal University of Technology, (FUT) Minna
Minna, Niger State, Nigeria.

Course Development Team

MAT 222: Linear Algebra 2

Subject Matter Experts	Mr. Salihu O. Nasiru Department of Mathematics FUT Minna, Nigeria.
	K.RAUF (Ph.D.) Department of Mathematics & Statistics FUT Minna, Nigeria..
Course Coordinator	Bashir MOHAMMAD (Ph.D.) Department of Computer Science FUT Minna, Nigeria.
ODL Experts	Amosa Isiaka GAMBARI (Ph.D.) Nicholas Ehikioya ESEZOBOR Ezra Zhiri KOLO
Instructional System Designers	Oluwole Caleb FALODE (Ph.D.) Bushrah Temitope OJOYE (Mrs.)
Language Editors	Chinenye Priscilla UZOCHUKWU (Mrs.) Mubarak Jamiu ALABEDE
Centre Director	Abiodun Musa AIBINU (Ph.D.) Centre for Open Distance & e-Learning FUT Minna, Nigeria.

MAT 222 Study Guide

Introduction

MAT 222 Linear Algebra is a 2- credit unit course for students studying towards acquiring a Bachelor of Science in any field. The course is divided into 4 modules and 14 study units. It will first introduce system of linear equation, change of basis and equivalence and similarity. Next, eigenvalues and eigenvectors, minimum and characteristic polynomials of a linear transformation (matrix) and Caley-Hamilton theorems. Finally, the student is introduced to bilinear and quadratic forms and also to orthogonal diagonalization canonical forms.

The course guide therefore gives you an overview of what MAT 222 is all about, the textbooks and other materials to be referenced, what you expect to know in each unit, and how to work through the course material.

Recommended Study Time

This course is a 2-credit unit course having 14 study units. You are therefore enjoined to spend at least 2 hours in studying the content of each study unit.

What You Are About to Learn in This Course

The overall aim of this course, give you a good foundation in Linear Algebra, a course you might be taking in depth later. At the end of this course you will:

1. Understand the important concepts in linear algebra
2. Have understood systems of linear equations and their solutions, matrices and their properties
3. Learned about determinants and their properties, vector spaces, subspaces, bases and dimension of vector spaces
4. Know how to apply these concepts to such real world phenomena and also improve your ability to think and reason logically.

Course Aims

This course aims to introduce students to linear algebra and systems of linear equations. It is expected that the knowledge will enable the reader to communicate mathematics both orally and in writing.

Course Objectives

It is important to note that each unit has specific objectives. Students should study them carefully before proceeding to subsequent units. Therefore, it may be useful to refer to these objectives in the course of your study of the unit to assess your progress. You should always look at the unit objectives after completing a unit. In this way, you can be sure that you have done what is required of you by the end of the unit.

However, below are overall objectives of this course. On completing this course, you should be able to discuss:

i. Solution of system of linear equation
ii. Eigenvalues and eigenvectors
iii. Basis and dimension
iv. Diagonalization of a matrix
v. Condition of Diagonalizability
vi. Characteristic Polynomial
vii. Cayley Hamilton theorem and some useful terms Discuss the cell cycle and mitosis
viii. Minimum Polynomial of a Matrix
ix. Bilinear form of Matrix
x. Transformation of Matrix from one Basis to the other
xi. Symmetric and Anti-Symmetric Bilinear Forms
xii. Quadratic Forms
xiii. Real Symmetric Bilinear Form
xiv. Canonical and Triangular Form.

Working Through This Course

In order to have a thorough understanding of the course units, you will need to read and understand the contents.

This course is designed to cover approximately sixteen weeks, and it will require your devoted attention. You should do the exercises in the Tutor-Marked Assignments and submit to your tutors.

Course Materials

The major components of the course are:

1. Course Guide
2. Study Units
3. Text Books
4. Assignment File
5. Presentation Schedule

Study Units

There are 14 study units and 4 Modules in this course. They are:

Module One	Unit 1: Solution of System of Linear Equation Unit 2: Eigenvalues and Eigenvectors Unit 3: Basis and Dimension Unit 4: Diagonalization of a matrix

Module Two	Unit 1: Condition of Diagonalizability Unit 2: Characteristic Polynomial Unit 3: Cayley Hamilton theorem and some useful terms Unit 4: Minimum Polynomial of a Matrix
Module Three	Unit 1: Bilinear form of Matrix Unit 2: Transformation of Matrix from one Basis to the other Unit 3: Symmetric and Anti-Symmetric Bilinear Forms
Module Four	Unit 1: Quadratic Forms Unit 2: Real Symmetric Bilinear Form Unit 3: Canonical and Triangular Form

Recommended Texts

The following texts and Internet resource links will be of enormous benefit to you in learning this course:

1. Blake, J., *Intermediate Pure Mathematics.* (5th Edition). MacMillan Press Limited. 1977 London.
2. Bunday, J., *Pure Mathematics for Advanced Level.* (2nd Edition). Heinemann Educational Books Limited, 1988. London.
3. Clarke, L. H., *Pure Mathematics at Advanced Level.* Metric Edition. Heinemann Educational Books Limited, 1977. London.
4. STROUD, K.A., *Engineering Mathematics*. (4th Edition). MacMillan Press Limited, 1995. London.
5. STROUD, K.A., *Further Engineering Mathematics*. (3rd Edition). MacMillan Press Limited. 1995. London.
6. TRANTER, C. J. & LAMBE, C. G., *Advanced Level Mathematics*, *Pure and Applied*. (4th Edition). Holder & Stoughton. 1979. Great Britain.

Assignment File

The assignment file will be given to you in due course. In this file, you will find all the details of the work you must submit to your tutor for marking. The marks you obtain for these assignments will count towards the final mark for the course. Altogether, there are Tutor marked Assignments (TMAs)s for this course.

Presentation Schedule

The presentation schedule included in this course guide provides you with important dates for completion of each Tutor marked Assignments (TMAs). You should therefore endeavour to meet the deadlines.

Assessment

There are two aspects to the assessment of this course. First, there are Tutor marked Assignments (TMAs)s; and second, the written examination. Therefore, you are expected to take note of the facts, information and problem solving gathered during the course. The Tutor marked Assignments (TMAs)s must be submitted to your tutor for formal assessment, in accordance to the deadline given. The work submitted will count for 40% of your total course mark.

At the end of the course, you will need to sit for a final written examination. This examination will account for 60% of your total score. You will be required to submit some assignments by uploading them to MAT 222 Page on the ulearn portal.

Tutor-Marked Assignment (TMA)

There are TMAs in this course. You need to submit all the TMAs. The best 10 will therefore be counted. When you have completed each assignment, send them to your tutor as soon as possible and make certain that it gets to your tutor on or before the stipulated deadline. If for any reason you cannot complete your assignment on time, contact your tutor before the assignment is due to discuss the possibility of extension. Extension will not be granted after the deadline, unless on extraordinary cases.

Final Examination and Grading

The final examination for MAT 222 will last for a period of 2 hours and has a value of 60% of the total course grade. The examination will consist of questions which reflect the self-assessment questions and Tutor marked Assignments (TMAs)s that you have previously encountered. Furthermore, all areas of the course will be examined. It would be better to use the time between finishing the last unit and sitting for the examination, to revise the entire course. You might find it useful to review your TMAs and comment on them before the examination. The final examination covers information from all parts of the course.

Practical Strategies for Working Through This Course

1. Read the course guide thoroughly
2. Organize a study schedule. Refer to the course overview for more details. Note the time you are expected to spend on each unit and how the assignment relates to the units. Important details, e.g. details of your tutorials and the date of the first day of the semester are available. You need to gather together all this information in one place such as a diary, a wall chart calendar or an organizer. Whatever method you choose, you should decide on and write in your own dates for working on each unit.
3. Once you have created your own study schedule, do everything you can to stick to it. The major reason that students fail is that they get behind with their course works. If you get into difficulties with your schedule, please let your tutor know before it is too late for help.
4. Turn to Unit 1 and read the introduction and the learning outcomes for the unit.
5. Assemble the study materials. Information about what you need for a unit is given in the table of content at the beginning of each unit. You will almost always need

both the study unit you are working on and one of the materials recommended for further readings, on your desk at the same time.

6. Work through the unit, the content of the unit itself has been arranged to provide a sequence for you to follow. As you work through the unit, you will be encouraged to read from your set books
7. Keep in mind that you will learn a lot by doing all your assignments carefully. They have been designed to help you meet the objectives of the course and will help you pass the examination.
8. Review the objectives of each study unit to confirm that you have achieved them. If you are not certain about any of the objectives, review the study material and consult your tutor.
9. When you are confident that you have achieved a unit's objectives, you can start on the next unit. Proceed unit by unit through the course and try to pace your study so that you can keep yourself on schedule.
10. When you have submitted an assignment to your tutor for marking, do not wait for its return before starting on the next unit. Keep to your schedule. When the assignment is returned, pay particular attention to your tutor's comments, both on the Tutor marked Assignments (TMAs) form and also written on the assignment. Consult you tutor as soon as possible if you have any questions or problems.
11. After completing the last unit, review the course and prepare yourself for the final examination. Check that you have achieved the unit objectives (listed at the beginning of each unit) and the course objectives (listed in this course guide).

Tutors and Tutorials

There are few hours of tutorial provided in support of this course. You will be notified of the dates, time and location together with the name and phone number of your tutor as soon as you are allocated a tutorial group. Your tutor will mark and comment on your assignments, keep a close watch on your progress and on any difficulties you might encounter and provide assistance to you during the course. You must mail your Tutor marked Assignments (TMAs) to your tutor well before the due date. At least two working days are required for this purpose. They will be marked by your tutor and returned to you as soon as possible.

Do not hesitate to contact your tutor by telephone, e-mail or discussion board if you need help. The following might be circumstances in which you would find help necessary: contact your tutor if:

i. You do not understand any part of the study units or the assigned readings.
ii. You have difficulty with the self-test or exercise.
iii. You have questions or problems with an assignment, with your tutor's comments on an assignment or with the grading of an assignment.

You should endeavour to attend the tutorials. This is the only opportunity to have face to face contact with your tutor and ask questions which are answered instantly. You can raise any problem encountered in the course of your study. To gain the maximum benefit from the course tutorials, have some questions handy before attending them. You will learn a lot from participating actively in discussions.

GOODLUCK!

Table of Contents

Module 1

Unit 1

Solution of a System of a Linear Equations

Content

1.0 Introduction

To solve a system means to find all values of the variables that satisfy all the equations in the system simultaneously. For example, consider the following system, which consists of two linear equations in two unknowns:

$$x_1 + x_2 = 3$$

$$3x_1 + x_2 = 4$$

Although there are infinitely many solutions to each equation separately, there is only one pair of numbers x_1 and x_2 which satisfies both equations at the same time. This ordered pair, (x_1, x_2) = (2, 1), is called the solution to the system.

There are various ways of solving a system of linear equations. But two of such methods will be considered in this unit.

2.0 Learning Outcomes

At the end of this unit, you should be able to how to solve a system of linear equations by Gaussian Elimination Method and Gauss-Jordan Method, and also know how to represent the system of m linear equations in the n unknowns.

3.0 Leaning Content

3.1 Gaussian Elimination Method:

The system of m linear equations in the n unknowns is represented below

$$\begin{vmatrix} a_{11} & a_{12} & a_{13}\ldots\ldots\ldots\ldots a_{1n} \\ a_{21} & a_{22} & a_{23}\ldots\ldots\ldots\ldots a_{2n} \\ \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3}\ldots\ldots\ldots\ldots a_{nn} \end{vmatrix} \; 0 \; \begin{vmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{vmatrix} = \begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{vmatrix}$$

i.e A.X = b

All the information for solving the set of equations is provided by the matrix co-efficient A and the column matrix b. If the elements of b is written within the matrix A; an augmented matrix B of the given set of equation is obtained

$$B = \left|\begin{array}{ccc|c} a_{11} & a_{12} & a_{13}\ldots\ldots\ldots\ldots\ldots a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23}\ldots\ldots\ldots\ldots\ldots a_{2n} & b_2 \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & a_{n3}\ldots\ldots\ldots\ldots\ldots a_{nn} & b_n \end{array}\right|$$

We then perform row operations on B to reduce the matrix co-efficient s of x to a Triangular Matrix.

The right-hand column is then detached back to its original position and by the idea of back substitution; we obtain the values for matrix co-efficient x.

Example 1.

Solve the equations below by Gaussian Elimination Method

$2x_1 – x_2 + 4x_3 + x_4 =11$

$x_1 + x_2 + x_3 + x_4 = 5$

$- 5x_1 + 3x_2 + x_3 + x_4 = 3$

$- x_1 + 2x_2 + x_3 – x_4 = 7$

Solution

The Augmented matrix is A

$$\begin{pmatrix} 2 & -1 & 4 & 1 & 11 \\ 1 & 1 & 1 & 1 & 5 \\ -5 & 3 & 1 & 1 & 3 \\ -1 & 2 & 1 & -1 & 7 \end{pmatrix}$$

To have a leading 1 in the first row; divide row 1 by the first number (2)

$$\begin{pmatrix} 1 & -½ & 2 & ½ & {}^{11}/_{2} \\ 1 & 1 & 1 & 1 & 5 \\ -5 & 3 & 1 & 1 & 3 \\ -1 & 2 & 1 & -1 & 7 \end{pmatrix}$$

Using elementary row-operation to obtain Zeros below the leading 1 in the first row; we have

$R_2^1 = R_2 + R_4$

$R_3^1 = 5R_2 + R_3$ =

$R_4^1 = R_1 + R_4$

$$\begin{pmatrix} 1 & -\frac{1}{2} & 2 & \frac{1}{2} & \frac{11}{2} \\ 0 & 3 & 2 & 0 & 12 \\ 0 & 8 & 6 & 6 & 28 \\ 0 & \frac{3}{2} & 3 & -\frac{1}{2} & \frac{25}{2} \end{pmatrix}$$

Also to obtain Zero below 3 in the second column; we have

$R_3^1=8R_2-3R_3$

$R_4^1=R_2-2R_4$;

The matrix is now;

$$\begin{pmatrix} 1 & -\frac{1}{2} & 2 & \frac{1}{2} & \frac{11}{2} \\ 0 & 3 & 2 & 0 & 12 \\ 0 & 0 & -2 & -18 & 12 \\ 0 & 0 & -4 & 1 & -13 \end{pmatrix}$$

Also $R_4^1=2R_3-R_4$ will give Zero below -2 in the 3rd column; we have

$$\begin{pmatrix} 1 & \frac{1}{2} & 2 & \frac{1}{2} & \frac{11}{2} \\ 0 & 3 & 2 & 0 & 12 \\ 0 & 0 & -2 & -18 & 12 \\ 0 & 0 & 0 & -37 & 37 \end{pmatrix}$$

To obtain a leading 1 in the 2nd, 3rd and 4th row; divide each row by the first number in that row; the matrix is now;

$$\begin{pmatrix} 1 & -\frac{1}{2} & 2 & \frac{1}{2} & \frac{11}{2} \\ 0 & 1 & \frac{2}{3} & 0 & 4 \\ 0 & 0 & 1 & 9 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} \text{------------------®}$$

The corresponding equations are:

$x_1-x_2+ 2x_3 + \underline{x_4} = 11/2$

$$2$$

$$x_2+{}^{2}/_{3}\,x_3= \quad 4$$

$$x_3+9x_4= \quad -6$$

$$x_4 = -1$$

Expressing the leading variables in terms of the free variables, we have

$$x_1 = {}^{11}/_2 + \tfrac{1}{2} x_2 - 2x_3 - \tfrac{1}{2} x_4$$

$$x_2 = 4 - {}^{2}/_{3}\, x_3$$

$$x_3 = -6 - 9x_4$$

$$x_4 = -1$$

Using back our substitution, we have

$x_4=-1$

$x_3=-6 \;\; -9(-1) \;\; =-6+9 \;\; =3$

$x_2= 4 \;\; - \frac{2}{3}(3) \;\; =4-2= \;\; 2$

$x_1=\frac{11}{2} \;\; +\tfrac{1}{2}(2)-2(3) \;\; -\tfrac{1}{2}(-1)$

$\quad =\frac{11}{2}+1 \;\; -6+\tfrac{1}{2} \;\; =1$

Hence, $x_1=1,\ x_2= 2,\ x_3= 3,\ x_4= -1$

3.2 Gauss-Jordan Elimination Method

The matrix ® obtained in the first method is in reduced row-echelon form. From the reduced row-echelon matrix, we perform further row operations to obtain only 1 in the leading diagonal of the co-efficient matrix. The right hand column is then detached to its original position; and we obtain the values of x directly. For this method we obtain two triangular matrixes, one above and the other below. An example is illustrated below:

Example

$$\begin{pmatrix} 1 & -\tfrac{1}{2} & 2 & \tfrac{1}{2} & {}^{11}/_{2} \\ 0 & 1 & {}^{2}/_{3} & 0 & 4 \\ 0 & 0 & 1 & 9 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

From the row-echelon matrix

We perform further row operations

$R_3^1 = 9R_4 - R_3 =$

$$\begin{pmatrix} 1 & -\frac{1}{2} & 2 & \frac{1}{2} & \frac{11}{2} \\ 0 & 1 & \frac{2}{3} & 0 & 4 \\ 0 & 0 & -1 & 0 & -3 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

Also,

$R_2^1 = \frac{2}{3}R_3 + R_2 =$

$$\begin{vmatrix} 1 & -\frac{1}{2} & 2 & \frac{1}{2} & \frac{11}{2} \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & -1 & 0 & -3 \\ 0 & 0 & 0 & 1 & -1 \end{vmatrix}$$

$R_1^1 = R_1 + \frac{1}{2} R_2 =$

$$\begin{vmatrix} 1 & 0 & 2 & \frac{1}{2} & \frac{13}{2} \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & -1 & 0 & -3 \\ 0 & 0 & 0 & 1 & -1 \end{vmatrix}$$

$R_1^1 = R_1 + 2R_3,$

$R_1^1 = R_1 - \frac{1}{2} R_4 =$

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & -1 & 0 & -3 \\ 0 & 0 & 0 & 1 & -1 \end{vmatrix}$$

Multiply row 3 by -1, the matrix becomes,

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -1 \end{vmatrix}$$

The corresponding equations are:

x_1=1

x_2=2

x_3=3

x_4=-1

Hence,

$x_1 = 1$, $x_2 = 2$, $x_3 = 3$, $x_4 = -1$

4.0 Conclusion

We have observed that a system of linear equations is a collection of m equations in the variable quantities x1; x2; x3; : : : ; xn of the form,

a11x1 + a12x2 + a13x3 + _ _ _ + a1nxn = b1

a21x1 + a22x2 + a23x3 + _ _ _ + a2nxn = b2

a31x1 + a32x2 + a33x3 + _ _ _ + a3nxn = b3

...

am1x1 + am2x2 + am3x3 + _ _ _ + amnxn = bm

Where the values of aij , bi and xj are from the set of complex numbers, C.

We can begin to describe our strategy for solving linear systems. Given a system of linear equations that looks difficult to solve,we would like to have an equivalent system that is easy to solve. Since the systems will have equal solution sets, we can solve the easy system and get the solution set to the difficult system.

5.0 Summary

You have learnt in this unit, how to use Gaussian Elimination method and Gauss-Jordan method in solving a system of linear equation. And that all the information for solving the set of equations is provided by the matrix co-efficients A and the column matrix b. If the elements of b is written within the matrix A; an augmented matrix B of the set of equation is obtained.

6.0 Tutor Marked Assignments (TMAs)

Find the augmented matrix for the following equations:

x1 - x2 + 2x3 = 1

2x1 + x2 + x3 = 8

x1 + x2 = 5

7.0 References/Further Reading

BLAKEY, J Intermediate Pure Mathematics, 5th Edition. MacMillan Press Limited.1977 London

BUNDAY, B.D Pure Mathematics for Advanced Level, Second Edition. Heinemann Educational Books Limited, 1988. London

CLARKE, L. Pure Mathematics at Advanced Level, Metric Edition. Heinemann Educational Books Limited, 1977.London

Unit 2

Eigenvalues and Eigenvectors

Content

1.0 Introduction

The eigenvalue problem is a problem of considerable theoretical interest and wide-ranging application. For example, this problem is crucial in solving systems of differential equations, analyzing population growth models, and calculating powers of matrices (in order to define the exponential matrix). Other areas such as physics, sociology, biology, economics and statistics have focused considerable attention on "eigenvalues" and "eigenvectors"-their applications and their computations.

2.0 Learning Outcomes

At the end of this unit, you should be able define the word eigenvectors and eigenvalues. Also, you should be able to find the eigenvalues and the corresponding eigenvectors for the co-efficient of matrix and determine their characteristic determinant and equations.

3.0 Learning Content

3.1 Definition of Eigenvalues and Eigenvectors

Let T be a linear operator on a vector space V over a field K, A scalar $\lambda \in K$ is called an Eigenvalue of T if there exists a non-zero vector $v \in V$ for which $T(v)= \lambda v$.

Every vector satisfying this relation is then called an Eigenvector of T belonging to an Eigenvalue λ.

In many applications of matrices to problems, equation of the form

$A \cdot X = \lambda x$ which occur where $A = [aij]$ is a square matrix

λ is equal to a number (scalar).

Clearly $x = 0$ is a solution for any value of λ and is not normally useful for non-trivial solutions i.e $X \neq 0$; the values of λ are called the Eigenvalues or characteristics values of the matrix A and the corresponding solutions of the given equations $A \cdot X = \lambda X$ are called the Eigenvectors or characteristic vectors of A. Expressed as a set of separate equation we have,

$A . X = \lambda X$

$$\begin{vmatrix} a_{11} & a_{12}\ldots\ldots\ldots\ldots a_{1n} \\ a_{21} & a_{22}\ldots\ldots\ldots\ldots a_{2n} \\ . & \\ . & \\ . & \end{vmatrix} \cdot \begin{vmatrix} x_1 \\ x_2 \\ . \\ . \\ . \end{vmatrix} = \lambda \begin{vmatrix} x_1 \\ x_2 \\ . \\ . \\ . \end{vmatrix}$$

$$\begin{vmatrix} a_{n1} & a_{n2} \ldots\ldots\ldots\ldots\ldots a_{nn} \end{vmatrix} \quad \begin{vmatrix} x_n \end{vmatrix} \quad \begin{vmatrix} x_n \end{vmatrix}$$

$a_{11}\,x_1 + a_{12}\,x_2 + \ldots\ldots\ldots a_{1n}\,x_n = \lambda x_1$

$a_{21}\,x_1 + a_{22}\,x_2 + \ldots\ldots\ldots a_{2n}\,x_n = \lambda x_2$

$a_{n1}\,x_1 + a_{n2}\,x_2 + \ldots\ldots\ldots a_{nn}\,x_n = \lambda x_n$

Bring right-hand side to left-hand side; we have

$(a_{11} - \lambda)\,x_1 + a_{12}\,x_2 + \ldots\ldots\ldots\ldots a_{1n}x_n = 0$

$A_{21}\,x_1 + (a_{22} - \lambda)x_2 + \ldots\ldots\ldots a_{2n}\,x_2 = 0$

.

.

$a_{n1}\,x_1 + a_{n2}\,x_2 + \ldots\ldots\ldots(a_{nn} - \lambda)x_n = 0$

That is

$$\begin{vmatrix} (a_{11} - \lambda) & a_{12}\ldots\ldots\ldots\ldots\ldots a_{1n} \\ a_{21} & (a_{22} - \lambda)\ \ldots\ldots\ldots a_{2n} \\ . & \\ . & \\ . & \\ a_{n1} & a_{n2}\ldots\ldots\ldots\ldots\ldots(a_{nn} - \lambda) \end{vmatrix} \cdot \begin{vmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ . \\ . \\ 0 \\ 0 \end{vmatrix}$$

$A \cdot X = \lambda x$ now becomes

$A \cdot X - \lambda x = 0$

$(A - \lambda I)X = 0$

The identity matrix I is introduced in other to convert the scalar λ to a matrix because a matrix can only be subtracted from another matrix. For this set of homogeneous linear equations to have a non-trivial solution, $[A - \lambda I]$ must be zero.

$[\lambda - \lambda I]$ is called the characteristic determinant of A.

$[A - \lambda I] = 0$ is called the characteristic equation.

On expanding the determinant we get a polynomial of degree n and the solution of the characteristic equation gives the values of λ i.e. Eigen values.

Self-Assessment Exercise (SAE)

Self-Assessment Answer (SAA)

3.2 How to find the Eigenvalues and the Eigenvectors

Example 1

Find the Eigenvalue and the associated non-zero Eigenvectors of the matrix

$$A = \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix}$$

SOLUTION

$A \cdot X \quad = \quad \lambda x$…………………(1)

$$\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \lambda$$

$\Rightarrow x_1 + 2x_2 = \lambda x_1$

$3x_1 + 2x_2 = \lambda x_2$

$\Rightarrow (1 - \lambda)\, x_1 + 2x_2 = 0$

$= 3x_1 + (2 - \lambda)\, x_2 = 0$

The characteristic determinant $[A - \lambda I]$

$$\begin{vmatrix} (1 - \lambda) & 2 \\ 3 & (2 - \lambda) \end{vmatrix}$$

The characteristic equation is $[A - \lambda I] = 0$

$(1 - \lambda)(2 - \lambda) - 6 = 0$

$2 - \lambda - 2\lambda + \lambda^2 - 6 = 0$

$\lambda^2 - 3\lambda - 4 = 0$

$\lambda^2 + \lambda - 4\lambda - 4 = 0$

$\lambda(\lambda + 1) - 4(\lambda + 1) = 0$

$(\lambda - 4)(\lambda + 1) = 0$

$\lambda (\lambda + 1) - 4(\lambda + 1) = 0$

$\Rightarrow \lambda_1 = 4, \lambda_2 = -1$

Each Eigenvalue obtained has corresponding to it a solution of X called an Eigenvector.

In matrices, the term vector corresponds to line matrix or column matrix

That is,

line matrix – [ ]

Column matrix - $\begin{pmatrix} \\ \end{pmatrix}$

Now; to obtain the Eigenvectors that corresponds to the Eigenvalues λ_1 & λ_2 we recall our initial equation (1)

$\Rightarrow A \cdot X = \lambda x$

For $\lambda = -1$; we have

$$\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = -1 \begin{vmatrix} x_1 \\ x_2 \end{vmatrix}$$

$| =-1 |$

$x_1 + 2x_2 = -x_1 \Rightarrow 2x_1 = -2x_2$……………….(a)

$3x_1 + 2x_2 = -x_2 \Rightarrow 3x_1 = -3x_2$……………....(b)

From both (a) & (b); we have

$x_1 = -x_2$

∴ For $\lambda = -1$ the Eigenvectors are

$$X = \begin{vmatrix} x_1 \\ -x_2 \end{vmatrix} = \begin{vmatrix} 1 \\ -1 \end{vmatrix}$$

The simplest of such vectors is when they are assigned the value of 1. Therefore,

For $\lambda = -1$; the Eigenvectors $X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

For $\lambda = 4$

$A \cdot X = \lambda x$

= 4

$\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix}$ $\begin{matrix} x_1 \;+ \\ 3x_1 \end{matrix}$ $\begin{vmatrix} x_1 \\ x_2 \end{vmatrix}$

$2x_2 = 4x_1 \Rightarrow 3x_1 = 2x_2$……………….(a)

$+ 2x_2 = 4x_2 \Rightarrow 3x_1 = 2x_2$…………………(b)

From (a) & (b) ; we have

$x_1 = {}^{2}/_{3}x_2$ or $3x_1 = 2x_2$

∴ The Eigenvectors are:

$X = \begin{vmatrix} x_1 \\ {}^{2}/_{3}x_2 \end{vmatrix}$ or $X = \begin{vmatrix} 3x_1 \\ 2x_2 \end{vmatrix}$

The simplest of such eigenvectors is when the x's are assigned the value of 1

∴ For λ = 4; we have

$$X = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Hence the solution.

Example 2

Determine the eigenvalue and the corresponding eigenvectors for co-efficient matrix A given

$$A = \begin{vmatrix} 2 & 0 & 1 \\ -1 & 4 & -1 \\ -1 & 2 & 0 \end{vmatrix}$$

Solution

$A \cdot X = \lambda x$

$[A - \lambda I]\, x = 0$

The characteristic equation is $A - \lambda I = 0$

$$\begin{vmatrix} 2 & 0 & 1 \\ -1 & 4 & -1 \\ -1 & 2 & 0 \end{vmatrix} - = \begin{vmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{vmatrix} \; 0$$

$$\begin{vmatrix} (2-\lambda) & 0 & 1 \\ -1 & (4-\lambda) & -1 \\ -1 & 2 & -\lambda \end{vmatrix}$$

The characteristic determinant is

$$\begin{vmatrix} (2-\lambda) & 0 & 1 \\ -1 & (4-\lambda) & -1 \\ -1 & 2 & -\lambda \end{vmatrix}$$

For the characteristic equation; we have

$$\begin{vmatrix} (2-\lambda) & 0 & 1 \\ -1 & (4-\lambda) & -1 \\ -1 & 2 & -\lambda \end{vmatrix} = 0$$

$(2 - \lambda) \; - \lambda\,(4 - \lambda(4 - \lambda) + 2 \; - 0 \; -1\{(-\lambda) - 1 \; + 1\} - 2 + \{1(4 - \lambda) \; = 0\}$

$\Rightarrow (2 - \{\lambda)(\lambda^2 - 4\lambda + 2) + (2\} - \lambda) = 0$

$(2 - \lambda) \; \{\lambda^2 \; - \; 4\lambda + 2 + 1\} = 0$

$(2 - \lambda)\{\lambda^2 \; - 4\lambda + 3\} = 0$

$(2 - \lambda) \; \lambda^2 \; - \; \lambda - 3\lambda + 3 \; = 0$

$(2 - \lambda)\{\lambda \; (\lambda - 1) - 3\,(\lambda - 1)\} = 0$

$(2 - \lambda)(\lambda - 3)(\lambda - 1) = 0$

$\Rightarrow \lambda_1 = 1$

$\lambda 2 = 2$

$\lambda_3 = 3$

To determine the corresponding eigenvector we have for $\lambda = 1$; recall

$A \cdot X = \lambda x$

$(A - \lambda I)\, x = \; 0$

$$\begin{vmatrix} (2-\lambda) & 0 & 1 \\ -1 & (4-\lambda) & -1 \\ -1 & 2 & -\lambda \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & 1 \\ -1 & 3 & -1 \\ -1 & 2 & -1 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}$$

$\Rightarrow x_1 + 0x_2 + x_3 = 0 \quad \Rightarrow x_1 = -x_3$ or $x_3 = -x_1$..............................i

$- x_1 + 3x_2 - x_3 = 0 \quad \Rightarrow -x_1 + 3x_2 + x_1 = 0$……………ii

$-x_1 + 2x_2 - x_3 = 0 \quad \Rightarrow -x_1 + 2x_2 + x_1 = 0$……………iii

From the ii and iii equation, we have,

$3x_2 = 0 \quad \Rightarrow x_2 = 0$

From the equation i we have, $x_1 = -x_3$

∴ The simplest of such values x can take is 1; that's the corresponding eigenvectors for eigenvalue λ = 1 is

$$X \quad \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} -x_1 \\ 0 \\ x_3 \end{vmatrix} \quad \begin{vmatrix} -1 \\ 0 \\ 1 \end{vmatrix}$$

\- =

For λ = 2; substitute value of λ in (A – λI) x = 0. The characteristic equation becomes,

$$\begin{vmatrix} 0 & 0 & 1 \\ -1 & 2 & -1 \\ -1 & 2 & -2 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}$$

$\Rightarrow x_3 = 0$

$-x_1 + 2x_2 - x_3 = 0$ but $x_3 = 0$

∴ $x_1 = 2x_2$

Similarly,

$-x_1 + 2x_2 - 2x_3 = 0$

$-x_1 + 2x_2 = 0 \Rightarrow x_1 = 2x_2$

For λ = 2; the simplest of such values is

$$\begin{vmatrix} x_1 \\ x_2 \end{vmatrix}$$

$$X = \begin{vmatrix} x_3 \end{vmatrix} = \begin{vmatrix} 2 \\ 1 \\ 0 \end{vmatrix}$$

For $\lambda = 3$

$$\begin{vmatrix} -1 & 0 & 1 \\ -1 & 1 & -1 \\ -1 & 2 & -3 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}$$

$-x_1+x_3 = 0$

$-x_1+x_2 -x_3= 0$

$-x_1+2x_2-3x_3 = 0$

Substitute value for $x_1 = x_3$ in the 2nd and 3rd equation

$- x_3 + x_2 - x_3 =0 \Rightarrow x_2 = 2x_3$

$- x_3 +2x_2 - 3x_3 = 0$

$-4x_3= 2x_2 \Rightarrow x_2 = 2x_3$

For $\lambda=3$, the corresponding eigenvectors are:

$$X = \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} 1 \\ 2 \\ 1 \end{vmatrix}$$

Self-Assessment Exercise (SAE)

Self-Assessment Answer (SAA)

4.0 Conclusion

Eigenvectors make understanding of Linear transformations easy. They are the "axes" (directions) along which a linear transformation act simply by "stretching/compressing" and/or "flipping". Eigenvalues give you the factors by which this compression occurs.

The more directions you have along which you understand the behavior of a linear transformation, the easier it is to understand the linear transformation; so you want to have as many as linearly independent Eigenvectors as possible associated to a single linear transformation.

5.0 Summary

You have learnt in this unit how to define the eigenvectors and eigenvalues. Also, learnt how to find eigenvectors, eigenvalues and their characteristic function and equations.

6.0 Tutor marked Assignments (TMAs)

Find the eigenvalues and the corresponding eigenvectors for the co-efficient of the matrix A and B

$$A = \begin{vmatrix} 2 & 1 & 1 \\ 1` & 3 & 2 \\ -1 & 1 & 2 \end{vmatrix}$$

$$B = \begin{vmatrix} 1 & 2 & 2 \\ 1` & 3 & 1 \\ 2 & 2 & 1 \end{vmatrix}$$

7.0 References/Further Reading

BLAKEY, J Intermediate Pure Mathematics, 5th Edition. MacMillan Press Limited.1977 London

BUNDAY, B.D Pure Mathematics for Advanced Level, Second Edition. Heinemann Educational Books Limited, 1988. London

CLARKE, L. Pure Mathematics at Advanced Level, Metric Edition. Heinemann Educational Books Limited, 1977.London

Unit 3

Basis and Dimensions and Definition of Some Terms

Content

1.0 Introduction

It is very important to define terms in mathematics. As a language of science the major concepts involves using terms correctly. This is why you must learn to use the definitions in this unit and this course.

2.0 Learning Outcomes

At the end of this unit you should be able to define Vector, Scalar, Field, Vector Space, Basis and Dimensions, linear Dependence and Independence. Also, you should be able to show how Vectors are Dependent or Independence.

3.0 Learning Content

3.1 Vector

A vector is a quantity that has both magnitude and direction e.g Velocity, Force etc.

3.2 Salar:

A Scalar is a quantity that has only a magnitude without a direction e.g Temperature etc

3.3 Field:

A commutative ring R with a unit element is called a field if every non-zero element $a \in R$ has a multiplicative inverse that is, there exist an element $a^{-1} \in R$ such that $aa^{-1} = a^{-1}a = 1$.

3.4 Vector Space:

Let K be a given field and V be a non-empty set with rules addition and scalar multiplication which assigns to any $u,v \in V$; a sum $u+v \in V$ and to any $u \in V$, $k \in K$; a product $ku \in V$. Then V is called a Vector Space over K if the following conditions hold:

C_1: $\forall$ u,v,w $\in$ V; (u + v) + w = u + (v+w)

C_2: There is a zero vector o$\in$V for which u + o = U $\forall$ u$\in$V

C_3: $\forall$ u$\in$V $\ni$ a vector – u$\in$V $\ni$ U + (- u) = 0

C_4: $\forall$ u,v$\in$V ; u + v = v + u

C_5: $\forall$ scalars k$\in$K and vectors u,v$\in$V k(u + v) = ku + kv

C_6: $\forall$ scalars a,b$\in$K and any vector U$\in$V, (a+b)u=au+bu

C_7: $\forall$ scalars a,b$\in$K and any vector U $\in$V (ab) v = a (bu)

C_8: For the unit scalar 1$\in$k 1·u = u $\forall$ U$\in$V

3.5 Basis

A Vector V Is Said to be of finite dimension n or to be n-dimensional, written as

dim V = n if ∃ linearly independent vectors e_1, e_2.........e_n which span V. The sequence [e_1, e_2.........e_n] is then called a Basis V

3.6 Theorem

Let V be a finite dimensional vector space, then every basis of V has the same number of elements.

The vector space [0] is said to have dimension O.

3.7 Linear Dependence and Independence:

Let V be a vector space over a field K. The vectors v_1,.......vm $\in$ V are said to be linearly dependent over K if there exist scalars a_1, ... am $\in$ k not all of them are zero, such that $a_1v_1 + a_2v_2 + ... + a_mv_m = 0$..........(1) otherwise, the vectors are said to be linearly independent over K. The relation (1) will always hold if all the a's are zero. If the relation holds only in this case i.e $a_1v_1 + a_2v_2 + ...a_mv_m = 0$ only if $a_1 = 0$, $a_2 = 0$, ...$a_m=0$; then the vectors are linearly independent. If the relation holds when one of the a's$\neq$0, then the vectors are linearly dependent.

Example 1

The vectors u = (1,-1,0)

V = (1,3,-1) and

W = (5,3,-2) are dependent since

$$3u + 2v - w = 0$$

$$3(1,-1,0) + 2(1,3,-1) - (5,3,-2) = 0$$

$$\Rightarrow (3,-3,0) + (2,6,-2) - (5,3,-2) = 0$$

$$3 + 2 - 5 = 0$$

$$-3 + 6 - 3 = 0$$

$$0 = 0$$

Self-Assessment Exercise (SAE)

Self-Assessment Answer (SAA)

4.0 Conclusion

In order to have a comprehensive understanding of Basis and Dimensions, you need to define certain terms like Vector, Scalar, Field etc. Each of this definition will come in a useful way in your further studies of Basis and Dimensions.

5.0 Summary

You have learnt in this unit the definition of the following terms: Vector, Scalar, Field, Vector Space, Basis and Dimensions, Linear Dependence and Independence. Also, how to show that vectors are either dependence or Independence.

6.0 Tutor marked Assignments (TMAs)

Show that the following vectors are independent:
u = (6,2,3,4), v = (0,5,-3,1) and w = (0,0,7, 2).

7.0 Reference/Further Reading

BLAKEY, J Intermediate Pure Mathematics, 5th Edition. MacMillan Press Limited.1977 London

BUNDAY, B.D Pure Mathematics for Advanced Level, Second Edition. Heinemann Educational Books Limited, 1988. London

CLARKE, L. Pure Mathematics at Advanced Level, Metric Edition. Heinemann Educational Books Limited, 1977.London

Unit 4

Diagonalization of a Matrix

Content

1.0 Introduction

It is very important to know how a square matrix A is diagonalizable and that is the basic concept of this unit.

2.0 Learning Outcomes

At the end of this unit you should be able to know the steps involved in the process of diagonalizing a diagonalizable $n \times n$ matrix A and also the conditions necessary for diagonalizing a matrix.

3.0 Learning Content

3.1 Process of Diagonalizing a Matrix

Let $T : V \rightarrow V$ be a linear operator on a vector space V with finite dimension n. Then T can be represented by a diagonal matrix.

$$\begin{pmatrix} K_1 & 0 \ldots\ldots\ldots\ldots 0 \\ 0 & K_2 \ldots\ldots\ldots\ldots 0 \\ 0 & 0 \ldots\ldots\ldots\ldots K_n \end{pmatrix}$$

If and only if there exists a basis $(V_1, \ldots\ldots\ldots V_n)$ for which

$$T(V_1) = K_1 V_1$$

$$T(V_2) = K_2 V_2$$

$$.$$

$$T(V_n) = K_2 V_2$$

That is, such that the vectors $V_1, \ldots\ldots. V_n$ are Eigen vectors of T belonging respectively to eigenvalues $K_1, \ldots\ldots. K_n$.

A square matrix A is called diagonalizable if there is an invertible matrix P ∋ P^1AP is a diagonal matrix. P is said to diagonalize A.

THEOREM 1:

If A is an n×n matrix, then the following are equivalent.

a. A is diagonalizable.

b. A has n linearly independent eigen-vectors.

THEOREM 2:

An n-square matrix A is similar to a diagonal matrix B if and only if A has n linearly independent e-vectors. In this case, the diagonal elements of B are the corresponding eigen-values. If we let P be the matrix whose columns are the n- independent e-vectors of A. Then,

$B = P^{-1}AP$

The following are the steps involved in the process of diagonalizing a diagonalizable nxn matrix A.

STEP 1:

Find n linearly independent eigen-vectors of A; say $p_1, p_2 \ldots\ldots\ldots\ldots p_n$

STEP 2:

Form the marix P having $p_1, p_2 \ldots\ldots\ldots\ldots p_n$ as its column vectors

STEP 3:

The matrix $P^{-1}AP$ will then be diagonal with $\lambda_1, \lambda_2 \ldots\ldots\ldots\ldots \lambda_n$ as its successive diagonal entries, where λi is the eigen-value corresponding to Pi, i = 1, 2……….n.

Example 1

Find a matrix P that diagonalizes

$$A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$$

Solution

STEP 1:

$$|\lambda I - A1| = \begin{pmatrix} \lambda & 0 & 2 \\ -1 & \lambda-2 & -1 \\ -1 & 0 & \lambda-3 \end{pmatrix} = \lambda[(\lambda-2)(\lambda-3)] + 2(\lambda-2)$$

$$\lambda(\lambda^2 - 5\lambda + 6) + 2\lambda - 4$$

$= \lambda^3 - 5\lambda^2 + 6\lambda + 2\lambda - 4$

$\Rightarrow \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$

$\lambda = 1$ and $\lambda = 2$ are solutions

i.e $(\lambda - 1)$ and $(\lambda - 2)$ are factors

$\Rightarrow (\lambda-1)(\lambda-2) = \lambda^2 - 3\lambda + 2$

To obtain the remaining factors, we use long division.

$$\lambda - 2$$

$$\lambda^2 - 3\lambda + 2 \sqrt{\lambda^3 - 5\lambda^2 + 8\lambda - 4}$$

$$\lambda^3 - 3\lambda^2 + 2\lambda$$

$$-2\lambda^2 + 6\lambda - 4$$

$$-2\lambda^2 + 6\lambda - 4$$

$$0$$

The eigen-values of A are $\lambda=2$; $\lambda = 1$

To determine the eigen-vectors

$(\lambda I - A)x = 0$

For $\lambda = 2$

$$\begin{pmatrix} \lambda & 0 & 2 \\ -1 & \lambda-2 & -1 \\ -1 & 0 & \lambda-3 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x_1+2x_3=0$

$-x_1-x_3=0$

$-x_1-x_3 =0$

$\Rightarrow -x_1=x_3$

i.e $x_1 = -x_3$

Let $x_3=S$

$x_1=-S$

Let $x_2=t$

$\Rightarrow x_1 = -S, \quad x_2= t, x_3= S$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -S \\ 0 \\ S \end{bmatrix} + \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = S \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The eigen-vectors corresponding to $\lambda = 2$ are:

$$P_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{ and } P_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

For $\lambda = 1$

$$\begin{pmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{pmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1+ 2x_3= 0$

$-x_1-x_2-x_3=0$

$-x_1-2x_2=0$

$\Rightarrow x_1=-2x_3$

$-(-2x_2) \quad -x_2-x_3 \;=0$

$2x_3-x_2-x_3=0 \quad \Rightarrow -x_2+x_3=0$

$x_3=x_2$

Let $\quad x_3= S$

$x_2= S$

$x_1= 2S$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2S \\ S \\ S \end{pmatrix} =S\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

i.e. The eigen-vector corresponding to λ = 1 is

$$P_3 \quad = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

We show that [P_1, P_2, P_3] is linearly in the pendent

i.e $a_1P_1+a_2P_2+a_3P_3=0$

$$\begin{pmatrix} a_1 \\ 0 \\ 1 \end{pmatrix} \begin{matrix} -1 \\ + a_2 \\ \end{matrix} \begin{pmatrix} \\ 1 \\ 0 \end{pmatrix} \begin{matrix} 0 \\ + a_3 \\ \end{matrix} \begin{pmatrix} \\ 1 \\ 1 \end{pmatrix} = \begin{matrix} -2 \\ \\ \end{matrix} \begin{pmatrix} \\ 0 \\ 0 \end{pmatrix} \quad 0$$

$\Rightarrow -a_1-2a_3=0 \quad \Rightarrow -(a_1)-2a_3 = 0$ a

$a_2+ a_3=0 \quad \Rightarrow a_2 = - a_3$

$a_1+ a_3=0 \quad \Rightarrow a_1 = -a_3$

$a_1 = -a_3$

$a_2 = -a_3$

$-(-a_1) -2a_3=0$

i.e. $-a_3 = 0 \quad \Rightarrow a_3 =0$

$a_2 = -a_3 \Rightarrow a_2= 0$

$a_1 = -a_3 \quad \Rightarrow a_1= 0$

Since $a_1= 0$, $a_2=0$ and $a_3= 0$

This has shown that P_1, P_2 and P_3 are linearly independent

STEP 2:

$$P = \begin{pmatrix} p_1 & p_2 & p_3 \end{pmatrix} = \begin{pmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} \begin{pmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Self-Assessment Exercise (SAE)

Self-Assessment Answer (SAA)

EXAMPLE 2

Find a matrix P that diagonalizes

$$A = \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix}$$

Solution

The characteristic equation

$$|\lambda I - A| = \begin{vmatrix} \lambda+3 & -2 \\ 2 & \lambda-1 \end{vmatrix} = 0$$

$(\lambda - 1)(\lambda+3) + 4 = 0$

$\lambda^2 + 2\lambda - 3 + 4 = 0$; $\lambda^2 + 2\lambda + 1 = 0$

$\lambda^2 + \lambda + \lambda + 1 = 0$
$\lambda(\lambda + 1) + 1(\lambda + 1) = 0$; $(\lambda + 1)^2 = 0$
$\lambda = -1$ is the only eigen-value of A
$\lambda = -1$

To find corresponding eigen-vector

$$\begin{matrix} 2 \\ 2 \end{matrix} \begin{pmatrix} -2 & x_1 \\ -2 & x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} \ \\ \ \end{pmatrix}$$

$2x_1 - 2x_2 = 0 \Rightarrow x_1 = x_2$
$2x_1 - 2x_2 = 0 \Rightarrow x_1 = x_2$
Let $x_2 = S$
$x_1 = S$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} S \\ S \end{pmatrix} - S \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The vector $P_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The eigenspace is 1-dimensional. A does not have two linearly independent vectors and is therefore not diagonalizable.

Self-Assessment Exercise (SAE)

Self-Assessment Answer (SAA)

3.2 Conditions for Diagonalizability

Theorem 1:
If $v_1, v_2, \ldots V_n$ are eigen-vectors of A, corresponding to distinct eigen-values $X_1, X_2 \ldots X_n$; then $\{v_1 v_2, \ldots v_n\}$ is a linearly independent set.
As a consequence of this theorem ; the second theorem follows:
Theorem 2:
If an nxn matrix A has n distinct eigenvalue; then A is diagonalizable.

4.0 Conclusion

A square matrix A is called diagonalizable if there is an invertible matrix P $\in$ $P^{-1}AP$ is a diagonal matrix. P is said to diagonalixe A.

5.0 Summary

You have learnt in this unit how to diagonalize a matrix and the steps and conditions involved in the process of diagonalizing a matrix. Also, some theorem was introduced in this unit to authenticate the diagonalization process.

6.0 Tutor marked Assignments (TMAs)

$$\text{Let } A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{pmatrix}$$

Verify that A is diagonalizable.
Hence, find the diagonal matrix.

7.0 References/Further Reading

BLAKEY, J Intermediate Pure Mathematics, 5th Edition. MacMillan Press Limited.1977 London

BUNDAY, B.D Pure Mathematics for Advanced Level, Second Edition. Heinemann Educational Books Limited, 1988. London

CLARKE, L. Pure Mathematics at Advanced Level, Metric Edition. Heinemann Educational Books Limited, 1977.London

Module 2

Unit 1

Computing Powers of a Matrix

Content

1.0 Introduction

Square matrices can be multiplied by themselves repeatedly in the same way as ordinary numbers, because they always have the same number of rows and columns. This repeated multiplication can be described as a power of the matrix. This is why you have to learn how to compute powers of a matrix in this unit.

2.0 Learning Outcomes

At the end of this unit, you will learn how to compute powers of matrices and illustrate them with examples.

3.0 Learning Content

3.1 Powers of a Matrix

If A is an $n \times n$ matrix and P is an invertible matrix, then

$$(P^{-1}AP)^2 = P^{-1}APP^{-1}AP$$
$$= P^{-1}AIAP = P^{-1}AAP$$
$$= P^{-1}A^2P$$

For any positive integer n;

$$(P^{-1}AP)^n = P^{-1}A^nP$$

$$\Rightarrow P^{-1}A^nP = (P^{-1}AP)^n = D^n$$

$$\Rightarrow A^n = PD^nP^{-1}$$

Example

$$\text{If } A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$$

Find A^{13}

Solution:

P = matrix for eigenvectors of A

$$P = \begin{pmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$D = P^{-1}AP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$D^{13} = \begin{pmatrix} 2^{13} & 0 & 0 \\ 0 & 2^{13} & 0 \\ 0 & 0 & 1^{13} \end{pmatrix}$$

$$A^{13} = PD^{13}P^{-1}$$

$$A^{13} = \begin{pmatrix} -1 & 0 & -2 \\ 0 & 1 & 1\ 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2^{13} & 0 & 0 \\ 2^{13}\ 0 & & 1 \\ 0 & 0 & 1^{13} \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & \\ 1 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 8190 & 0 & -16382 \\ 8191 & 8192 & 8191 \\ 8191 & 0 & 16383 \end{pmatrix}$$

Self-Assessment Exercise (SAE)

Self-Assessment Answer (SAA)

4.0 Conclusion

The need to compute the powers of a matrix arise in many practical problems. It is easy to raise a diagonal matrix to a power but when raising an arbitrary matrix (not necessary a diagonal matrix) to a power, it is often helpful to exploit this property by diagonalizing the matrix.

5.0 Summary

You have learnt in this unit how to compute powers of a matrix and also being able to learnt that rectangular matrices do not have the same number of rows and columns so they can never be raised to a power.

6.0 Tutor marked Assignments (TMAs)

If $A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$

Find A^{14}

7.0 References/Further Reading

BLAKEY, J Intermediate Pure Mathematics, 5th Edition. MacMillan Press Limited.1977 London

BUNDAY, B.D Pure Mathematics for Advanced Level, Second Edition. Heinemann Educational Books Limited, 1988. London

CLARKE, L. Pure Mathematics at Advanced Level, Metric Edition. Heinemann Educational Books Limited, 1977.London

Unit 2

Characteristic Polynomial and Equation

Content

1.0 Introduction

In linear algebra, every square matrix is associated with a characteristic polynomial. This polynomial encodes several important properties of matrix, most notably its eigenvalues, its determinant and its trace.

2.0 Objectives

At the end of this unit you should be able to explain the word characteristic polynomial and be able to find the characteristic polynomial of any given matrix

3.0 Learning Content

3.1 Characteristic Polynomial

Consider an n-square matrix A over a field K

$$A = \begin{pmatrix} a_{11} & a_{12} \dots\dots\dots & a_{1n} \\ a_{21} & a_{22} \dots\dots\dots & a_{2n} \\ a_{n1} & a_{n2} \dots\dots\dots & a_{nn} \end{pmatrix}$$

The matrix $tI_n - A$ where I_n is the n-square identity matrix and t is an indeterminant called the characteristic matrix of A.

$$tI_n - A = \begin{pmatrix} t & 0 \dots\dots & 0 \\ 0 & t \dots\dots & 0 \\ & \dots\dots\dots\dots & \\ 0 & 0 \dots\dots & t \end{pmatrix} - \begin{pmatrix} a_{11} & a_{12} \dots\dots\dots & a_{1n} \\ a_{21} & a_{22} \dots\dots\dots & a_{2n} \\ & \dots\dots\dots\dots\dots\dots & \\ a_{n1} & a_{n2} \dots\dots\dots & a_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} t- a_{11} & - a_{12} \dots\dots\dots & a_{1n} \\ - a_{21} & t- a_{22} \dots\dots\dots & a_{2n} \\ - a_{n1} & - a_{n2} \dots\dots\dots & t- a_{nn} \end{pmatrix}$$

Its determinant,

$\Delta_A(t) = \det(tI_n - A)$ which is a polynomial in t is called the characteristic polynomial of A.

Also, $\Delta_A(t) = \det(tI_n - A) = 0$ is called the characteristic equation of A.

Each term in the determinant contains one and only entry from each row and from each column; hence the above characteristic polynomial is of the form

$\Delta_A(t) = (t - a_{11})\ (t - a_{22})\ldots\ldots\ldots\ldots(t - a_{nn})$with at most n – 2 factors of the form t-a_{ii}.

Thus, the characteristic polynomial $\Delta_A(t) = \det(tI_n - A)$ of A is a Monic Polynomial of degree n. A polynomial is monic if it is leading co-efficient is 1.

EXAMPLE

The characteristic polynomial of the matrix

$$A = \begin{pmatrix} 1 & 3 & 0 \\ -2 & 2 & -1 \\ 4 & 0 & -2 \end{pmatrix} \quad \text{is}$$

$$\Delta(t) = |tI - A| = \begin{pmatrix} t-1 & -3 & 0 \\ 2 & t-2 & 1 \\ -4 & 0 & t+2 \end{pmatrix}$$

$= \ (t - 1)\ (t - 2)\ (t + 2) + 3[2(t + 2) + 4]\ + 0$

$(t - 1)\ (t - 2)\ (t + 2) + 6\ (t + 2) + 12$

$= t^3 - t^3 + 2t + 28$

$\Delta(t)$ is a Monic Polynomial of degree 3

Self-Assessment Exercise (SAE)

Self-Assessment Answer (SAA)

4.0 Conclusion

The characteristic polynomial can as well be extended to a graph g which can be defined as the characteristic polynomial of its adjacency matrix and can be computed in mathematical form using characteristic polynomial [adjacency matrix[g], x].

5.0 Summary

You have learnt in this unit the meaning of characteristic polynomial and how to find the characteristic polynomial of a matrix.

6.0 Tutor marked Assignments (TMAs)

Find the characteristic polynomial of

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$$

7.0 References/Further Reading

BLAKEY, J Intermediate Pure Mathematics, 5^{th} Edition. MacMillan Press Limited.1977 London

BUNDAY, B.D Pure Mathematics for Advanced Level, Second Edition. Heinemann Educational Books Limited, 1988. London

CLARKE, L. Pure Mathematics at Advanced Level, Metric Edition. Heinemann Educational Books Limited, 1977.London

Unit 3

Cayley Hamilton Theorem and Some Useful Lemma

Content

1.0 Introduction

For a better understanding of linear algebra, there is need for you to have an in-depth knowledge of Cayley Hamilton Theorem because it is one of the most useful theorems in mathematics that is what this unit is all about.

2.0 Learning Outcomes

At the end of this unit you should be able to state the Cayley Hamilton theorem and some other useful lemma for the theorem.

3.0 Learning Content

3.1 Statement of Cayley Hamilton Theorem

The Cayley Hamilton theorem states that “Every matrix is a zero of its characteristic polynomial or every matrix satisfies its characteristic equa-

tion, that equation define by pA (t) = 0.

3.2 Useful Lemma to Proof Cayley Hamilton Theorem

Lemma 1:

Suppose for all j¸j large enough,

A0 + A1¸ + ¢ ¢ ¢ + Am¸m = 0;

where the Ai are n £ n matrices. Then each Ai = 0:

Proof: Multiply by ¸¡m to obtain

A0¸¡m + A1¸¡m+1 + ¢ ¢ ¢ + Am¡1¸¡1 + Am = 0:

Now let j¸j ! 1 to obtain Am = 0: With this, multiply by ¸ to obtain

A0¸¡m+1 + A1¸¡m+2 + ¢ ¢ ¢ + Am¡1 = 0:

Now let j¸j ! 1 to obtain Am¡1 = 0: Continue multiplying by ¸ and letting ¸ ! 1 to

obtain that all the Ai = 0: This proves the lemma.

With the lemma, here is a simple corollary.

Corollary 3.4.3 Let Ai and Bi be n £ n matrices and suppose

A0 + A1¸ + ¢ ¢ ¢ + Am¸m = B0 + B1¸ + ¢ ¢ ¢ + Bm¸m

for all j¸j large enough. Then Ai = Bi for all i: Consequently if ¸ is replaced by any n £ n

matrix, the two sides will be equal. That is, for C any n £ n matrix,

A0 + A1C + ¢ ¢ ¢ + AmCm = B0 + B1C + ¢ ¢ ¢ + BmCm:

Proof: Subtract and use the result of the lemma.

With this preparation, here is a relatively easy proof of the Cayley Hamilton theorem.

Theorem 3.4.4 Let A be an n£n matrix and let p (¸) ´ det (¸I ¡ A) be the characteristic

polynomial. Then p (A) = 0:

1A special case was ¯rst proved by Hamilton in 1853. The general case was announced by Cayley some

time later and a proof was given by Frobenius in 1878.

96 DETERMINANTS

Proof: Let C (¸) equal the transpose of the cofactor matrix of (¸I ¡ A) for j¸j large.

(If j¸j is large enough, then ¸ cannot be in the ¯nite list of eigenvalues of A and so for such

¸; (¸I ¡ A)¡1 exists.) Therefore, by Theorem 3.3.15

C (¸) = p (¸) (¸I ¡ A)¡1 :

Note that each entry in C (¸) is a polynomial in ¸ having degree no more than n ¡ 1:

Therefore, collecting the terms,

C (¸) = C0 + C1¸ + ¢ ¢ ¢ + Cn¡1¸n¡1

for Cj some n £ n matrix. It follows that for all j¸j large enough,

(¸I ¡ A)

¡

C0 + C1¸ + ¢ ¢ ¢ + Cn¡1¸n¡1¢

= p (¸) I

and so Corollary 3.4.3 may be used. It follows the matrix coefficients corresponding to equal

powers of ¸ are equal on both sides of this equation. Therefore, if ¸ is replaced with A; the

two sides will be equal. Thus

0 = (A ¡ A)

¡

C0 + C1A + ¢ ¢ ¢ + Cn¡1An¡1¢

= p (A) I = p (A) :

This proves the Cayley Hamilton theorem.

Self-Assessment Exercise (SAE)

Self-Assessment Answer (SAA)

4.0 Conclusion

You need to have a better understanding of Cayley Hamilton theorem in this unit in order to solve any problem in minimum polynomial that will be introduced in the next unit

5.0 Summary

You have learnt in this unit how to state the theorem of Cayley Hamilton and also some useful lemma that is needed for better understanding of the theorem.

6.0 Tutor marked Assignments (TMAs)

Apply the Cayley Hamilton theorem to solve the matrix below

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$$

7.0 References/Further Reading

BLAKEY, J Intermediate Pure Mathematics, 5^{th} Edition. MacMillan Press Limited.1977 London

BUNDAY, B.D Pure Mathematics for Advanced Level, Second Edition. Heinemann Educational Books Limited, 1988. London

CLARKE, L. Pure Mathematics at Advanced Level, Metric Edition. Heinemann Educational Books Limited, 1977.London

Unit 4

Minimum Polynomial of a Matrix

Content

1.0 Introduction

The minimum polynomial is often the same as the characteristic polynomial but not always. Minimum polynomials are useful for constructing and analyzing field extensions.

2.0 Learning Outcome

At the end of this unit you should be able to define minimum polynomial and be conversant with some theorems on minimum polynomial. Also, you should be able to find the minimum polynomial of a given matrix

3.0 Learning Content

3.1 Definition of Minimum Polynomial

If A is an n-square matrix over a field F then, there exists non-zero polynomialsf(λ) ∋ f (A) = 0 for example, the characteristic polynomial of A. Among these polynomials we consider the lowest degree polynomial whose leading co-efficient is 1 (i.e. monic). Such a polynomial m(λ) exists and is unique. It is called the minimum polynomial of A.

3.2 Minimum Polynomial Theorems

Theorem 1:

Every polynomial which has A as a zero is divisible by the minimum polynomial m(λ) of A. In particular, the characteristic polynomial p(λ) of A is divisible by m(λ).

Proof:

1. Let f(x) be a polynomial $\in$ f(A) = 0

2. By the division of algorithm given polynomials q(λ) and r(λ)

$$f(\lambda) = m(\lambda)q(\lambda)+r(\lambda) \text{ where } r(\lambda) = 0 \text{ or } \deg r(\lambda) < \deg m(\lambda)$$

3. Let λ= A in the equation above, it implies

$$f(A) = m(A)\, q(A) + r(A)$$

But f(A) = 0 = m(A)

$\Rightarrow$0 = 0 + r(A)

$\Rightarrow$r(A) = 0

4. If r((λ) ≠ 0 $\Rightarrow$ r (λ) is a polynomial of degree less than m(λ) which has A as a zero but this contradicts the definition of a minimum polynomial

Hence, r(λ)= 0

5. It implies that f(λ) = m(λ)q(λ)

i.e f(λ) is divisible by m(λ)

Theorem 2:

The characteristic and minimum polynomials of a matrix A have same irreducible factors.

This does not imply that m(λ) = P(λ) only that any irreducible factor of one must be divisible by the other. In particular, since a linear factor is irreducible m(λ) and P(λ) have the same linear factors, hence they have the same roots.

Theorem 3:

A scalar λ as an eigenvalue for a matrix A if and only if λ is a root of the minium polynomial of A.

Example 1

Find the minimum polynomial m(λ) of the matrix

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

Solution:

The characteristic polynomial of A is

$$P(\lambda) = |\lambda I - A|$$

$$= \begin{vmatrix} \lambda-2 & -1 & 0 & 0 \\ 0 & \lambda-2 & 0 & 0 \\ 0 & 0 & \lambda-2 & 0 \\ 0 & 0 & 0 & \lambda-5 \end{vmatrix}$$

$P(\lambda) = (\lambda-2)^3 (\lambda-5)$

The minimum, polynomial m(λ) must divide the characteristic polynomial P (λ)

Also each irreducible factor of P (λ) i.e. λ - 2 and λ - 5 must be factors of m (λ).

Thus m(λ) must be one of the following polynomials.

$M_1(\lambda) = (\lambda - 2)(\lambda - 5)$

$M_2(\lambda) = (\lambda - 2)^2 (\lambda - 5)$

$M_3(\lambda) = (\lambda - 2)^3 (\lambda - 5)$

But

$M_1(A) = (A - 2I)\ (A - 5I) \neq 0$

$M_2(A) = (A - 2I)^2\ (A - 5I)$

$$= \left\{ \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \right\}^2 \left\{ \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix} - \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix} \right\} = 0$$

$M_3(A) = (A - 2I)^3\ (A - 5I) = 0$

By Cayley Hamilton theorem, $M_3(A) = P(A) = 0$.

However, the degree of $M_2(\lambda)$ is less than the degree of $M_3(\lambda)$.

Hence, $M_2(\lambda)$ is the minimum polynomial of A.

Self-Assessment Exercise (SAE)

Self-Assessment Answer (SAA)

EXAMPLE 2

Let A be a 3 X3 matrix over the real field R, show that A cannot be a zero of the polynomial $f(\lambda) = \lambda^3 + 3$

SOLUTION:

By Cayley Hamilton theorem; A is a zero of its characteristic polynomial P(λ). Since P(λ) is of degree 3, it implies that P(λ) has at least one real root. Assume A is a zero of f(λ). Since f(λ) is irreducible over R, f(λ) must be the minimum polynomial of A. But f(λ) has no real roots. This contradicts the fact that m(λ) and P(λ) have the same roots. Thus, A is not a zero of f(λ).

EXAMPLE 3

Find the minimum polynomial m(λ) of

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 4 \end{pmatrix}$$

SOLUTION:

The characteristic polynomial

$P(\lambda) = |\lambda I - A|$

$$= \begin{vmatrix} \lambda-2 & -1 & 0 & 0 \\ 0 & \lambda-2 & 0 & 0 \\ 0 & 0 & \lambda-1 & -1 \\ 0 & 0 & 2 & \lambda-4 \end{vmatrix}$$

$$= \begin{vmatrix} \lambda-2 & -1 \\ 0 & \lambda-2 \end{vmatrix} \begin{vmatrix} \lambda-1 & -1 \\ 2 & \lambda-4 \end{vmatrix}$$

$$= (\lambda^2 - 4\lambda + 4)(\lambda^2 - 5\lambda + 4 + 2)$$

$$= (\lambda - 2)^2 (\lambda - 3)$$

$$P(\lambda) = (\lambda - 3)(\lambda - 2)^3$$

The minimum polynomial m(λ) must divide P(λ). Also, each irreducible factor of P(λ) i.e. (λ – 3) and (λ - 2) must be a factor of m(λ)

Thus m(λ) is one of the following polynomials:

$m_1(\lambda) = (\lambda - 3)(\lambda - 2)$

$m_2(\lambda) = (\lambda - 3)(\lambda - 2)^2$

$m_3(\lambda) = (\lambda - 3)(\lambda - 2)^3$

$m_1(A) = (A - 3I)(A-2I) \neq 0$

$m_2(A) = (A-3I)(A-2I)^2$

$$= \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -2 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

By Cayley Hamilton theorem, $m_3(A) = (A-3I)(A-2I)^3 = 0$

However, the deg $m_2(\lambda) <$ deg $m_3(\lambda)$.

Hence, $m_2(\lambda) = m(\lambda)$ which is the minimum polynomial of A.

Self-Assessment Exercise (SAE)

Self-Assessment Answer (SAA)

4.0 Conclusion

In order to solve any question on minimum polynomial, you need to have a better understanding of Cayley Hamilton theorem. So do ensure to study this unit carefully.

5.0 Summary

You have learnt in this unit the definition of minimum polynomial and its theorems and also how to find a minimum polynomial of a given matrix.

6.0 Tutor marked Assignments (TMAs)

Find the minimal polynomial m(λ) of the matrix

$$D = \begin{pmatrix} a & b & 0 \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix}$$

7.0 References/Further Reading

BLAKEY, J Intermediate Pure Mathematics, 5th Edition. MacMillan Press Limited.1977 London

BUNDAY, B.D Pure Mathematics for Advanced Level, Second Edition. Heinemann Educational Books Limited, 1988. London

CLARKE, L. Pure Mathematics at Advanced Level, Metric Edition. Heinemann Educational Books Limited, 1977.London

Module 3

Unit 1: Bilinear form of Matrix
Unit 2: Transformation of Matrix from one Basis to the other
Unit 3: Symmetric and Anti-Symmetric Bilinear Forms

Unit 1

Bilinear Form of Matrices

Content

1.0 Introduction

The definition of bilinear form can easily be extended to include modules over a commutative ring, with linear maps replaced by module homomorphism. When F is the field of complex numbers C, one is often more interested in sesquilinear form which are similar to bilinear form but are conjugate linear in one argument

2.0 Learning Outcome

At the end of this unit you should be able to know how bilinear polynomial corresponding to matrix is form, bilinear forms and matrices and the definition of bilinear forms.

3.0 Learning Contents

3.1 Definition of Bilinear Forms

Let V be a vector space of finite dimension over a field K. A bilinear form on V is a mapping F:VxV$\rightarrow$K which satisfies the following:

(i). $f(au_1+bu_2, v) = af(u,v)+bf(u_2v)$

(ii). $f(u,av_1+bv_2) = af(u,v_1) + bf(u,v_2)$

for all a,b$\in$k and all u_i,$v_i$$\in$v. We express condition (i) by saying f is linear in the first variable and condition (ii) by saying f is linear in the second variable.

Example 1:

Let α and β be arbitrary linear functional on v. Let f:v x v$\rightarrow$k be defined by f(u,v)=α(u)β(v). Then f is bilinear because α and β are each linear.

Example 2:

Let f be a dot product on R^n. That is f(u,v) = u.v =$a_1+b_2+......+a_n b_n$ where u=(a_i) and v = b_i. Then F is a bilinear form on R^n.

Example 3:

Let A = (aij) be any nxn matrix over K. Then A may be viewed as a bilinear form on K^n by defining

$f(x, y) = X^t AY =$

$$(x_1, x_2 \ldots\ldots\ldots x_n) \begin{vmatrix} a_{11} & a_{12}\ldots\ldots a_{1n} \\ a_{21} & a_{22}\ldots\ldots a_{2n} \\ \vdots & \\ a_{n1} & a_{n2}\ldots\ldots a_{nn} \end{vmatrix} \begin{vmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{vmatrix}$$

$\Rightarrow \sum_{i,J=1}^{n} aij = a_{11}\, x_1\, y_1 + a_{12}\, x_2\, y_2 + \ldots\ldots\ldots\ldots a_{nn}\, x_{12}\, y_n$

The above expression in variables Xi, Yi is termed bilinear polynomial corresponding to the matrix A.

3.2 Bilinear forms and Matrices

Let F be a bilinear form on V and let $\{e_1, \ldots\ldots\ldots e_n\}$ be a basis of V. suppose u,v $\in$ V and suppose

$u = a_1\, e_1 + a_2 e_2 + \ldots\ldots\ldots\ldots + a_n e_n$

$v = b_1\, e_1 + b_2\, e_2 + \ldots\ldots\ldots\ldots + b_n e_n$

Then

$F(u, v) = f(a_1\, e_1 + \ldots\ldots\ldots a_n\, e_n,\ b_1\, e_1 + \ldots\ldots\ldots\ldots b_n e_n)$

$\qquad = a_1\, b_1\, f(e_1, e_1) + a_1\, b_2\, f(e_1, e_2) + \ldots\ldots\ldots\ldots + a_n b_n\, f(e_n, e_n)$

$$= \sum_{i,j=1}^{n} aibj\ f(e_{i,}, e_j)$$

Then f is completely determined by $f(e_i, e_j)$. The matrix A= (aij) where aij $=f(v_1,v_2)$ is called the matrix representation of f relative to the basis $\{e_i\}$ or simply the matrix of f in $\{e_i\}$. It represents f in the sense that

$f(u,v) = \sum\ a_i\, b_j\, f(e_i, e_j)$

$$= (a_1, a_2 \ldots\ldots\ldots a_n)\ A \begin{pmatrix} b_1 \\ b_2 \\ \\ b_n \end{pmatrix}$$

$$= [U]_e^t\ A\ [V]_e$$

For all $u,v \in V$

$[U]_e$ denotes the coordinates (column) vector of $u \in V$ in the basis $\{e_i\}$

Self-Assessment Exercise (SAE)

Self-Assessment Answer (SAA)

4.0 Conclusion

In order to appreciate the knowledge of bilinear forms, you must have a better understanding of matrix, vector space and some other terms. Each of these terms will come in very useful in further studies of Bilinaer forms

5.0 Summary

You have learnt in this unit the definition of bilinear forms, bilinear forms and matrix. Also learnt how bilinear polynomial corresponding to matrix is formed

6.0 Tutor marked Assignments (TMAs)

Define bilinear forms and state the condition necessary for bilinear polynomial corresponding to matrix is form.

7.0 References/Further Reading

BLAKEY, J Intermediate Pure Mathematics, 5th Edition. MacMillan Press Limited.1977 London

BUNDAY, B.D Pure Mathematics for Advanced Level, Second Edition. Heinemann Educational Books Limited, 1988. London

CLARKE, L. Pure Mathematics at Advanced Level, Metric Edition. Heinemann Educational Books Limited, 1977.London

Unit 2

Transformation of Matrix from One Basis to The Other

Content

1.0 Introduction

Since it is often desirable to work with more than one basis for a linear vector space, it is of fundamental importance in linear algebra to be able to easily transform coordinate-wise representations of vectors and linear transformations taken with respect to one basis to their equivalent representations with respect to another basis. Such a transformation is called a change of basis in matrix.

2.0 Learning Outcome

At the end of this unit you should be able to understand how a matrix is transform from one basis to other and how does a matrix representing a bilinear form transform when a new basis is selected. Also, some theorems will be introduced for better understanding of this unit.

3.0 Learning Content

3.1 Transformation of matrix theorem

Theorem

Let P be the transition matrix from one basis to the other. If A is the matrix of F in the original basis, then we have $B = P^t AP$ as the matrix of F in the new basis.

EXAMPLE 1:

Let F be the bilinear form defined by $f(x_1, x_2), (y_1, y_2) = 2x_1y_1 – 3x_1y_2 + x_2y_2$

i. Find the matrix A of F in the basis B $=\{u_1 = (1,0), u_2 = (1,1)\}$
ii. Find a matrix C of F in the basis $B^1 = \{v_1 = (2,1) , v_2 = (1,1)\}$
iii. Find the transition matrix P from the basis B^1 to the basis B and verify that $C = P^t A P$

SOLUTION

i. Set A = (aij) where aij $= f(u_i, u_j)$,

$a_{11} = f(u_1, u_1) = f(1,0), (1,0)$

$a_{11} = f(u_1, u_1) = 2(1) (1) – 3 (1) (0) + 0$

$(0)= 2 – 0 + 0 = 2$

$a_{12} = f(u_1, u_2) = f\ (1,0), (1,1) = 2 – 3 + 0 = -1$

$a_{21} = f(u_2, u_1) = f(1,1) (1,0) = 2 – 0 + 0 =2$

$a_{22} = f(u_2, u_2) = f(0(1), (1,1) = 2) – 3 + 1 = 0$

Thus,

$$A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 2 & 0 \end{pmatrix}$$

Is the matrix of F in the basis $B^1 = \{u_1, u_2\}$

ii. Set $C = (c_{ij})$ where $c_{ij} = f(v_i, v_j)$

$C_{11} = f(v_1, v_1) = f(2,1), (2,1) = 8 - 6 + 1 = 3$

$C_{12} = f(v_1, v_2) = 4 + 6 - 1 = 9$

$C_{21} = f(v_2, v_1) = 4 - 3 - 1 = 0$

$C_{22} = f(v_2, v_2) = f(1,-1), (1,-1) = 2 + 3 + 1 = 6$

Thus, $C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} 3 & 9 \\ 0 & 6 \end{pmatrix}$

Is the matrix of F in the basis $B^1 = \{v_1, v_2\}$

iii. We must find v_1 & v_2 in terms of u_i,

$V_1 = (2,1)$

$(2,1) = au_1 + bu_2$

$\Rightarrow a(1,0) + b(1,1) = (2,1)$

$a+b=2$

$b = 1$

$\Rightarrow a = 2 - 1 = 1$

But

$V_1 = au_1 + bu_2$

Substitute values obtained for a and b

$V_1 = u_1 + u_2$

$= [V_1]_B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$V_2 = (1,-1) = au_1 + bu_2$

$au_1 + bu_2 = (1, -1)$

$a(1,0) + b(1,1) = (1,-1)$

$a + b = 1$

$b = -1 \Rightarrow a = 1 - (-1) = 2$

$\therefore (1,-1) = 2u_1 - u_2$

$\therefore [V_2]_B = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

Then,

$P = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$ $P^t = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$

$P^t AP = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 9 \\ 0 & 6 \end{pmatrix} = C$

Self-Assessment Exercise (SAE)

Self-Assessment Answer (SAA)

EXAMPLE 2

Let A be an nxn matrix over K. show that the following mapping f is a bilinear form on K^n : $f(x, y) = X^t A Y$

SOLUTION:

For any $a,b \in k$ and any $x_i y_i \in k^n$

$f(ax_1 + bx_2, y) = (ax_1 + bx_2)^t A Y$

$= a x_1^t AY + b x_2^t AY$

$= af(x_1 y) + bf(x_2, y)$

Hence, f is linear in the first variable

Also,

$f(x_1 ay_1 + by_2) = x^t A(ay_1 + by_2) = ax^t Ay_1 + bx^t Ay_2$

$\Rightarrow f(x_1 ay_1 + by_2) = af(x_1 y_1) + bf(x_2, y_2)$

Hence, f is linear in the second variable and so f is a bilinear form on K^n

Self-Assessment Exercise (SAE)

Self-Assessment Answer (SAA)

4.0 Conclusion

Understand that there is nothing extremely special about the standard basis vector [1,0] and [0,1]. All 2D vectors may be represented as linear combinations of these vectors. You may take linear combinations of any other reasonable (linearly independent) set of vectors, and still be able to express the same as linear combinations of those.

5.0 Summary

You have learnt in this unit how a matrix is transform from one basis to other and how does a matrix representing a bilinear form transform when a new basis is selected.

6.0 Tutor marked Assignments (TMAs)

Let $U = (x_1, x_2, x_3)$ and $V = (y_1, y_2, y_3)$ and let
$f(u,v) = 3x_1y_1 – 2x_1y_2 + 5x_2y_1 + 7x_2y_2 – 8x_2y_3 + 4x\,y_2 – x_3y_3$
Express F in matrix notation.

7.0 Reference/Further Reading

BLAKEY, J Intermediate Pure Mathematics, 5th Edition. MacMillan Press Limited.1977 London

BUNDAY, B.D Pure Mathematics for Advanced Level, Second Edition. Heinemann Educational Books Limited, 1988. London

CLARKE, L. Pure Mathematics at Advanced Level, Metric Edition. Heinemann Educational Books Limited, 1977.London

Unit 3

Symmetric and Anti-Symmetric Bilinear Forms

Content

1.0 Introduction

Symmetric bilinear form is of great importance in the study of orthogonal polarity and quadrics. They are also more briefly referred to as just symmetric forms when bilinear is understood.

2.0 Learning Outcome

At the end of this unit you should be able to know when a bilinear form is symmetric and also when it is Anti-Symmetric.

3.0 Learning Content

3.1 Definition of Symmetric Bilinear Forms

A bilinear form F on V is said to be symmetric if F(u, v) = F(v, u) for every $u,v \in v$. If A is a matrix representation of F we can write

$$F(x, y) = X^t AY = (X^t AY)^t = Y^t A^t X \ldots\ldots\ldots\ldots\ldots\ldots(i)$$

(We use the fact that X^tAY is scalar and therefore equals its transpose). If F is symmetric, then

$$F(y, x) = Y^tAX \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(ii)$$

i.e. $Y^tA^tX = Y^tAX$ this follows from (i) and (ii) and it is due to the definition above.

Now; Since this is true for all vectors x, y it follows that $A = A^t$ or A is symmetric. Conversely, if A is symmetric then F is also symmetric.

THEOREM

Let A be a symmetric matrix over K (in which $1+1 \neq 0$). Then such an invertible (or non–singular) matrix P ∋ P^tAP is diagonal

EXAMPLE 1

For the symmetric matrix A given below, find a non–singular matrix P ∋ P^tAP that is diagonal.

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 5 & -4 \\ -3 & -4 & 8 \end{pmatrix}$$

SOLUTION

Since an invertible matrix P is a product of elementary matrices; one way of obtaining the diagonal from P^t A P is by a sequence of elementary row operations and the same sequence of elementary column operations. These same elementary row operations on I will yield P^t . Therefore, it is convenient to form the block matrix (A, I).

$$(A, I) = \begin{pmatrix} 1 & 2 & -3 & 1 & 0 & 0 \\ 2 & 5 & -4 & 0 & 1 & 0 \\ -3 & -4 & 8 & 0 & 0 & 1 \end{pmatrix}$$

We apply the operations to (A, I)

$R_2^1 = -2R_1+R_2$

$R_3^1 = 3R_1 +R_3$

And then the corresponding operation to A

$C_2^1 = -2C_1+C_2$

$C_3^1 = 3C_1+C_3$

To obtain,

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 2 & -1 & 3 & 0 & 1 \end{array}\right) \text{ and}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ 0 & 2 & -1 & 3 & 0 & 1 \end{array}\right) \text{ respectively}$$

We now apply the operations

$R_3= -2R_2 + R_3$ to (A, I) and

$C_3 = -2C_2+ C_3$ to A to obtain

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ 0 & 0 & -5 & 7 & -2 & 1 \end{array}\right) \text{ and}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & -5 & 7 & -2 & 1 \end{array}\right) \text{ respectively}$$

Now A has been diagonalized. We set,

$$P^t AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -5 \end{pmatrix} \text{ and}$$

$$P = \begin{pmatrix} 1 & -2 & 7 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

Self-Assessment Exercise (SAE)

Self-Assessment Answer (SAA)

3.2 Definition of Anti-Symmetric Bilinear Form

A bilinear form F on V is said to be alternating if

(i) $F(v,v) = 0 \ \forall \ v \ E \ V$

If F is alternating then,

$0 = F(u + v, u+v) = F(u, u) + F(u, v) + F(v, u) + F(v, v)$

(ii) $F(u, v) = - F(v, u)$ for every $u,v \in V$.

A bilinear form which satisfies condition (ii) is said to be anti-symmetric (skew – symmetric).

4.0 Conclusion

A symmetric bilinear form is a bilinear form on a vector space that is symmetric. Conversely, an alternating bilinear form is anti-symmetric.

5.0 Summary

You have learnt in this unit the definition of symmetric and anti-symmetric bilinear forms and also the conditions necessary for a bilinear forms to be symmetry or anti-symmetric.

6.0 Tutor Marked Assignment

1. Define symmetric and anti-symmetric.
2. State the conditions necessary for a bilinear form to be anti-symmetric

7.0 References/Further Reading

BLAKEY, J Intermediate Pure Mathematics, 5^{th} Edition. MacMillan Press Limited.1977 London

BUNDAY, B.D Pure Mathematics for Advanced Level, Second Edition. Heinemann Educational Books Limited, 1988. London

CLARKE, L. Pure Mathematics at Advanced Level, Metric Edition. Heinemann Educational Books Limited, 1977.London

Module 4

Unit 1

Quadratic Forms

Content

1.0 Introduction

In mathematics, a quadratic form is a homogeneous polynomial of degree two in a number of variables. Quadratic forms are homogeneous quadratic polynomials in a variables. In the cases of one, two, and three variables, they are called unary, binary, and ternary.

2.0 Learning Outcomes

At the end of this unit you should be able to know what quadratic forms in linear algebra is all about.

3.0 Learning Content

3.1 Definition of Quadratic Forms

A mapping $q: v \rightarrow k$ is called a quadratic form if $q(v) = f(v, v)$ for some symmetric bilinear form f on V.

We call q the quadratic form associated with the symmetric bilinear form f. If $1+1 \neq 0$ in k; then f is obtainable from q according to the identity.

$$f(u, v) = \tfrac{1}{2}\left(q(u+v) - q(u) - q(v)\right)$$

The above formula is called the Polar form of f.

Now; if the symmetric matrix A = (aij) is the matrix representation of f,q can be represented in the form.

$q(x) = f(x,x) = X^tAX$

$q(x) = X^tAX$

$$= (x_1, x_2 \ldots\ldots\ldots x_n)\begin{pmatrix} a_{11} & a_{12} \ldots\ldots\ldots\ldots a_n \\ \vdots & \vdots \\ a_{n1} \ldots\ldots\ldots\ldots & \ldots a_{nn} \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$= \sum_{ij=1} aij\, x_i x_j = a_{11} x_1^2 + a_{22} x_2^2 + \ldots\ldots a_{nn} x_n^2 + 2 \sum_{kj} aij\, x_i x_j$$

The above formula expression is called the quadratic polynomial corresponding to the symmetric matrix A. Note that if A is diagonal; then q has the diagonal representation.

$$q(x) = a_{11} x_1^2 + a_{22} x_2^2 + \ldots\ldots\ldots\ldots + a_{nn} x_n^2$$

By the foregoing theorem, every quadratic form has the above representation (when $1+1 \neq 0$)

EXAMPLES

Find the symmetric matrix which corresponds to each of the following quadratic polynomials.

a. $q(x, y) = 2x^2 – 6xy – 8y^2$
b. $q(x, y, z) = 3x^2 + 4xy – y^2 + 8xz – 6yz + z^2$
c. $q (x, y, z) = x^2 – 2yz + xz$

SOLUTION

The symmetric matrix A = (aij) has the diagonal entry a_{ii} equal to the co–efficient x_i^2 and has the entries aij and aji each equal to half the co-efficient of $x_i x_j$. Thus,

(a) $\begin{pmatrix} 2 & -3 \\ -3 & -8 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & 2 & 4 \\ 2 & -1 & -3 \\ 4 & -3 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 0 & ½ \\ 0 & 0 & -1 \\ ½ & -1 & 0 \end{pmatrix}$

Self-Assessment Exercise (SAE)

Self-Assessment Answer (SAA)

4.0 Conclusion

Quadratic forms occupy a central place in various branches of mathematics, not only in linear algebra it also includes, number, theory, group theory etc.

5.0 Summary

You have learnt in this unit how to define the quadratic form of linear algebra and also how to find symmetric matrix that corresponds to quadratic polynomials.

6.0 Tutor marked Assignments (TMAs)

Find the symmetric matrix which corresponds the following quadratic polynomials.

a. $q(x, y, z) = 3x^2 + 4xy – y^2 + 8xz – 6yz + z^2$

7.0 References/Further Reading

BLAKEY, J Intermediate Pure Mathematics, 5th Edition. MacMillan Press Limited.1977 London

BUNDAY, B.D Pure Mathematics for Advanced Level, Second Edition. Heinemann Educational Books Limited, 1988. London

CLARKE, L. Pure Mathematics at Advanced Level, Metric Edition. Heinemann Educational Books Limited, 1977.London

Unit 2

Real Symmetric Bilinear Form

Content

1.0 Introduction

With the help of real symmetry bilinear forms, we can easily find the signature of a matrix in any linear algebra problem which this unit is all about.

2.0 Learning Outcomes

At the end of this unit you should be able to define and find the signature of a matrix in.

3.0 Learning Content

3.1 Nature of Real Symmetric Bilinear Forms

Theorem:

Let f be a symmetric bilinear form on V over R. Then there is a basis of V in which f is represented by a diagonal matrix; every other diagonal representation has the same number P of Positive entries and the same number N of Negative entries. The difference S= P –N is called the signature of f.

EXAMPLE:

For the real symmetric matrix A, find a non-singular matrix P such that P^tAP is diagonal and find its signature

$$A = \begin{pmatrix} 1 & -3 & 2 \\ -3 & 7 & -5 \\ 2 & -5 & 8 \end{pmatrix}$$

SOLUTION:

First form the block matrix (A, I)

$$(A, I) = \left(\begin{array}{ccc:ccc} 1 & -3 & 2 & 1 & 0 & 0 \\ -3 & 7 & -5 & 0 & 1 & 0 \\ 2 & -5 & 8 & 0 & 0 & 1 \end{array}\right)$$

Apply the row operations

$R_2^1 = 3R_1 + R_2$ and $R_3^1 = -$ [illegible] $+ R_3$ to (A, I) and

Corresponding column operations

$C_2^1 = 3C_1 + C_2$

$C_3^1 = -2C_1 + C_3$ to A to obtain

$$\left(\begin{array}{ccc|ccc} 1 & -3 & 2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 3 & 1 & 0 \\ 0 & 0 & 4 & -2 & 0 & 1 \end{array}\right) \text{ and } \left(\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 1 & 0 \\ 0 & 1 & 4 & 0 & 1 \end{array}\right)$$

Respectively

Let apply the row operations

$R_2^1 = R_2 + 2R_3$ and corresponding column operations

$C_3^1 = C_2 + 2C_3$ to obtain

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & . & 0 & 0 \\ 0 & -2 & 1 & 3 & 1 & 0 \\ 0 & 0 & 9 & - & 1 & 2 \end{array}\right) \text{ and then}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 3 & 1 & 0 \\ 0 & 0 & 18 & -1 & 1 & 2 \end{array}\right)$$

Now A has been diagonalized with

$$P^tAP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 18 \end{pmatrix} \text{ and } P = \begin{pmatrix} 1 & 3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

The signature S of A is

S = P – N = 2 – 1 = 1

Self-Assessment Exercise (SAE)

Self-Assessment Answer (SAA)

4.0 Conclusion

The signature of a matrix can be found easily when you know the number of the positive and negative entries.

5.0 Summary

You have learnt in this unit how to find the signature of a matrix haven known the positive and negative entries.

6.0 Tutor marked Assignments (TMAs)

What is signature of a matrix

7.0 References/Further Reading

BLAKEY, J Intermediate Pure Mathematics, 5th Edition. MacMillan Press Limited.1977 London

BUNDAY, B.D Pure Mathematics for Advanced Level, Second Edition. Heinemann Educational Books Limited, 1988. London

CLARKE, L. Pure Mathematics at Advanced Level, Metric Edition. Heinemann Educational Books Limited, 1977.London

Unit 3

Canonical and Triangular Forms

Content

1.0 Introduction

Generally, in mathematics, a canonical form (often called normal form or standard form) of an object is a standard way of presenting that object. Canonical form can also mean a differential form that is defined in a natural (canonical) way. Finding a Canonical form is called canonization.

2.0 Learning Outcomes

At the end of this unit you should be able to have details knowledge of what canonical form and triangular form in linear algebra is all about.

3.0 Learning Content

3.1 Canonical Forms

Let T be a linear operator on a vector space of finite dimension. T may not have a diagonal matrix representation. But T can be considered in three forms namely: the Triangular, Jordan and Rational Canonical forms.

The triangular and Jordan Canonical forms exist for T if and only if the characteristic polynomial Δ(t) of T has its entire root in the base field K. This is always true if K is the complex field C but may not be true if K is the real field R.

3.2 Triangular Forms

Let T be a linear operator on an n-dimensional vector space V. Suppose T can be represented by the triangular matrix.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{1n} \\ a_{22} \ldots\ldots\ldots\ldots\ldots\ldots\ldots & & a_{2n} \\ . & & \\ ._{n} & & \end{pmatrix}$$

Then the characteristic polynomial of T is

$\Delta(t) = |t\,I - A| = (t - a_{11})\,(t - a_{22})\ldots\ldots\ldots.(t - a_{nn})$ is a product of linear factors. The converse is also true.

Theorem:

Let A be a square matrix whose characteristic polynomial factors into linear polynomials. Then A is similar to a triangular matrix i.e. there exist an invertible matrix P such that $P^{-1}AP$ is triangular.

We say that an operator T can be brought into triangular form if it a be represented by a triangular matrix. Note that in this case, the Eigen-value of T are precisely those entries appearing on the main diagonal.

Self-Assessment Exercise (SAE)

Self-Assessment Answer (SAA)

4.0 Conclusion

In the mathematical discipline of linear algebra, a triangular matrix is a special kind of square matrix. A square matrix is called lower triangular if all the entires above the main diagonal are zero. Conversely, a square marix is called upper triangular if all the entries below the main diagonal are zero. A triangular matrix is one that is either lower triangular or upper triangular. A matrix that is both upper and lower triangular is a diagonal matrix

5.0 Summary

You have learnt in this unit the understanding of canonical and triangular forms of linear algebra and the major differences between canonical and triangular forms of linear algebra.

6.0 Tutor marked Assignments (TMAs):

Let A be a square matrix over the complex field C. Show that $\sqrt{\lambda}$ or - $\sqrt{\lambda}$ is an eigen-value of A.

7.0 Reference/Further Reading

BLAKEY, J Intermediate Pure Mathematics, 5th Edition. MacMillan Press Limited.1977 London

BUNDAY, B.D Pure Mathematics for Advanced Level, Second Edition. Heinemann Educational Books Limited, 1988. London

CLARKE, L. Pure Mathematics at Advanced Level, Metric Edition. Heinemann Educational Books Limited, 1977.London