MODELLING MAGNETOHYDRODYNAMICS FLOW OF INCOMPRESSIBLE FLUID THROUGH PARALLEL PLATES IN INCLINED MAGNETIC FIELD IN THE PRESENCE OF VISCOUS DISSIPATION ENERGY

BY

YUSUF, Yusha'u
MTech/SPS/2019/10474

DEPARTMENT OF MATHEMATICS
FEDERAL UNIVERSITY OF TECHNOLOGY
MINNA

NOVEMBER, 2023

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A THESIS SUBMITTED TO THE POSTGRADUATE SCHOOL

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IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE AWARD

OF THE DEGREE OF MASTER OF TECHNOLOGY (MTech)

IN MATHEMATICS

NOVEMBER, 2023

DECLARATION

I hereby declare that this thesis titled: **"Modelling Magnetohydrodynamics Flow of Incompressible Fluid through Parallel Plates in Inclined Magnetic Field in the Presence of Viscous Dissipation Energy"** is a collection of my original research work and it has not been presented for any other qualification anywhere. Information from other sources (published or unpublished) and their contributions has been duly acknowledged.

YUSUF, Yusha'u
MTech/SPS/2019/10474
FEDERAL UNIVERSITY OF TECHNOLOGY
MINNA, NIGERIA.

…………………….
Signature & Date

CERTIFICATION

This thesis titled: **"Modelling Magnetohydrodynamics Flow of Incompressible Fluid through Parallel Plates in Inclined Magnetic Field in the Presence of Viscous Dissipation Energy"** by YUSUF, Yusha'u MTech/SPS/2019/10474 meets the regulations governing the award of the degree of Master of Technology (MTech) of the Federal University of Technology, Minna and it is approved for its contribution to scientific knowledge and literary presentation.

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DEDICATION

This research work is dedicated to Almighty God, the Creator of earth and heaven, and the most beneficent and merciful.

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ABSTRACT

Magnetohydrodynamics (MHD) finds its application in solar physics, geophysics and meteorology. This thesis presents a mathematical model describing the flow of electrically conducting incompressible fluid through two parallel plates in inclined magnetic field in the presence of viscous dissipation energy. The dimensionless coupled non-linear partial differential equations governing magnetohydrodynamics flow of incompressible fluid through parallel plates in inclined magnetic field in the presence of viscous dissipation energy were solved analytically using polynomial approximation method. The effects of the various physical parameters on the velocity, concentration and temperature of the flow were shown graphically and discussed. It is observed that Reynolds number and Nusselt number reduce the velocity of the fluid, whereas Solutal Grashof number, Thermal Grashof number and Kinematic viscosity number enhance the velocity of the fluid. A considerable effect was also observed on the concentration and temperature profiles. Kinematic viscosity number enhances the concentration of the fluid while Nusselt number and Peclet energy number reduce the temperature of the fluid.

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CHAPTER ONE

1.0 INTRODUCTION

1.1 Background to the Study

The word magnetohydrodynamics (MHD) is derived from the words magneto (implies magnetic field), hydro (meaning water) and dynamics refer to movement. It is also known as magneto-fluid dynamics or hydromagnetic is the study of magnetic properties and behaviour of electrically conducting fluids. Examples of such magneto-fluids are plasma, liquid metal (such as mercury), saltwater and electrolytes. The fundamental concept of MHD is that the magnetic field stimulates currents in a flowing conductive fluid and causes the magnetic field to change. The application of electromagnetic fields in controlling the heat transfer as in aerodynamic heating leads to the study of MHD heat transfer. This MHD heat transfer has gained significance owing to recent advancement of space technology. The MHD heat transfer can be divided in two parts. One contains problems in which the heating is an incidental by product of electromagnetic fields as in MHD generators and pumps, and the second consists of problems in which the primary use of electromagnetic fields is to control the heat transfer (Tagawa *et al.*, 2002).

Heat transfer in channels partially filled with porous media has gained considerable attention in recent years because of its various applications in contemporary technology. These applications include nuclear reactors, blood flow in lungs or in arteries, porous journal bearing, porous flat plate collectors, packed bed thermal storage solidification of concentrated alloys, fibrous and granular insulation, grain storage and drying, paper drying, and food storage. Besides, the use of porous subtracts to improve heat transfer in channels, which is considered as porous layers, finds applications in heat exchangers, electronic cooling, heat pipes, filtration and chemical reactors. In these applications engineers avoid filling entire channel with a solid matrix to reduce the pressure drop. The flow between parallel plates is a classical problem that has important applications in MHD power generators and pumps, accelerators, aerodynamic heating, electrostatic precipitation, polymer technology, petroleum industry, purification of crude oil and fluid droplets and spays.

1.2 Statement of the Research Problem

The need for the study of magnetohydrodynamic (MHD) flow of an incompressible and electrically conducting fluid through various cross sections have rapidly increased in recent years as the efficiency of the devices used in engineering and industries depends on the particles suspended in the fluid under the effect of magnetic field. Particularly, the flow and heat transfer of electrically conducting fluids in channels under the effect of a transverse magnetic field occur in MHD pumps, accelerators, and generators. Channels, in particular narrow channels, are common parts of many MHD devices. Therefore, investigation of MHD phenomena in plane layers and channels with conducting fluids is important for understanding their basic mechanisms, improving the existing industrial processes and for developing new MHD devices.

1.3 Scope and Limitation

The thesis focuses on the mathematical model for analysing magnetohydrodynamics flow of incompressible fluid through parallel plates in inclined magnetic field in the presence of viscous dissipation energy. This research work is limited to the mathematical modelling of the phenomenon.

1.4 Aim and Objectives

The aim of this research work is to establish an approximate analytical solution capable of analysing magnetohydrodynamics flow of incompressible fluid through parallel plates in inclined magnetic field in the presence of viscous dissipation energy.

The objectives of this study are to:

1. Formulate mathematical model governing magnetohydrodynamics flow of incompressible fluid.
2. Obtain the approximate analytical solution of the model using polynomial approximation method.
3. Provide the graphical representation of the solutions obtained.
4. Analyse the solution obtained.

1.5 Significance of the Study

This study focuses on the magnetohydrodynamics (MHD) flow of incompressible fluid through parallel plates in inclined magnetic field in the presence of viscous dissipation energy. Its findings will be of great significance in Applied Mathematics and Engineering. The findings will assist in further development of plasma physics and geophysics. The study will also assist in improvement of the current devices which employ MHD flow that is MHD power generators, cooling systems aerodynamics, heating polymer technology, MHD pumps and electromagnetic flow meter. Thermal radiation effects on flow and heat transfer processes are also of major importance in the space technology and high temperature processes.

1.6 Defination of Terms

Differential Equation (DE): An equation containing derivatives of a dependent variable with respect to one or more independent variables.

Eckert Number (Ec): The kinetic energy of the flow relative to the boundary layer enthalpy difference.

Fluid: **A**ny substance that yields readily to external pressure and has no fixed shape.

Grashof Number (Gr): The ratio of buoyancy to viscous forces.

Heat Transfer: **T**he exchange of thermal energy between physical systems. The rate of heat transfer is dependent on the temperature of the system and the properties of the intervening medium through which the heat is transferred.

Incompressible Fluid: An incompressible fluid is defined as the fluid whose volume or density does not change with pressure.

Magnetic Field: A magnetic field is a vector field, meaning it has a specified magnitude and direction at any point.

Modeling: The mathematical representation of a real phenomenon that is difficult to observe directly.

Nusselt Number (Nu): The dimensionless temperature gradient at the surface.

Ordinary Differential Equation (ODE): An equation containing a single independent variable.

Partial Differential Equation (PDE): An equation containing two or more independent variables.

Peclet Mass Number (Pe_m): The dimensionless independent mass transfer parameter.

Peclet Energy Number (Pe): The dimensionless independent heat transfer parameter.

Reynolds Number (Re): The ratio of inertia force to the viscous force present in the fluid.

Sherwood Number (Sh): The dimensionless concentration gradient at the surface.

Viscous: Having a thick, sticky consistency between solid and liquid.

Viscosity: A measure of how thick a fluid is.

CHAPTER TWO

2.0 LITERATURE REVIEW

2.1 Review of Related Literature

Some of the important literatures available about magnetohydrodynamic (MHD) flow of incompressible fluid through parallel plates were reviewed. Mhone and Makinde (2006) carried out an investigation on unsteady MHD flow with heat transfer in a divergence channel. The non-linear governing equations were obtained and solved analytically using perturbation technique. Results showed that the effect of increasing values of heat on steady flow was to dampen the velocity profile. This is known as Hartmann flow. Moreover for a channel of varying across different sections of the channel, the dampening was more pronounced in the centre of the channel. This creates a stagnation point and consequently fluid was pushed to the walls of the channel, thereby increasing the velocity in the boundary layer. Singh and Kumar (2009) investigated the heat and mass transfer MHD flow through porous medium. Palani and Srikanth (2009) studied the MHD flow of an electrically conducting fluid over a semi-infinite vertical plate under the influence of the transversely applied magnetic field. Ahmed (2009) studied heat and mass transfer effects on free convective three dimensional unsteady flows over a porous vertical plate.

Rajesh (2010) studied radiation effects on MHD free convection flow near a vertical plate with ramped wall temperature. Radiation and mass transfer effects on MHD free convection fluid flow embedded in a porous medium with heat generation/absorption was studied by Shankar *et al.* (2010). The problem of dissipation effects on MHD nonlinear flow and heat transfer past a porous surface with prescribed heat flux have been studied by Devi and Ganga (2010). Kwanza *et al.* (2010) studied the unsteady free convection MHD flow past a semiinfinite vertical porous plate in the presence of strong magnetic field. A finite difference method was used to solve the non-linear partial differential equations, after non-dimensionalizing the equations. The effect of Hall and ion- slip currents together with that of viscous dissipation and radiation absorption among other parameters on velocity, temperature and concentration profiles were presented graphically. It was found that in the presence of heating of the plate by free convection current, the velocity boundary layer thickness decreased. The results also showed that increase in mass diffusion parameter increased the primary velocity profiles and decreased the secondary velocity profiles. Shyam *et al*. (2010) examined the Soret and Dufour effects on the MHD natural convection over a vertical surface embedded in a Darcy porous medium in the presence of thermal radiation. Ali-Chamkha and Mansour (2011) examined the effect of chemical reaction, thermal radiation, and heat generation or absorption on the unsteady MHD free convective heat and mass transfer along an infinite vertical plate.

Poonia and Chaudhary (2012) studied the effects of heat transfer on MHD free convective flow through porous medium with viscous dissipation. Jain *et al*. (2012) presented an unsteady three dimensional free convection flow with combined heat and mass transfer over a vertical plate embedded in a porous medium with time dependent suction velocity and transverse sinusoidal permeability. Sharma *et al*. (2012) investigated the flow of a viscous incompressible electrically conducting fluid along a porous vertical isothermal non-conducting plate with variable suction and internal heat generation in the presence of transverse magnetic field. Ravikumar *et al*. (2012) investigated the heat and mass transfer effects on MHD flow of viscous incompressible and electrically conducting fluid through a non-homogeneous porous medium in the presence of heat source and oscillatory suction velocity. Sandeep and Sugunamma (2013) analyzed the effects of inclined magnetic field and radiation on free convective flow of dissipative fluid past a vertical plate through porous medium in presence of heat source. Ashokkunar *et al*. (2013) studied closed form solutions of heat and mass transfer in the flow of a MHD viscous-elastic fluid over a porous stretching sheet. It was found that the heat and mass transfer distribution decreased with the increasing values of the viscous-elastic parameter. Sreekala *et al*. (2014) investigated the unsteady hydromagnetic flow of an electrically conducting Maxwell fluid in a parallel plate channel bounded by porous medium under the influence of a uniform magnetic field of strength Ho inclined at an angle of inclination with the normal to the boundaries. The perturbations were created by a constant pressure gradient along the plates. The time required for the transient state to decay and the ultimate steady state solution are discussed in detail. The exact solutions for the velocity of the Maxwell fluid consists of steady state were analytically derived, its behaviour computationally discussed with reference to the various governing parameters with the help of graphs. The shear stresses on the boundaries were also obtained analytically and their behaviour was computationally discussed in detail.

Choudhury and Das (2016) studied unsteady, two-dimensional free convective MHD visco-elastic flow with heat and mass transfer past a semi-infinite moving vertical porous plate with variable suction in presence of homogeneous first-order chemical reaction and temperature dependent heat generation was presented. The equations governing the flow field were solved by perturbation technique. Expressions for velocity, temperature, mass concentration and skin friction coefficient are obtained. The velocity field and the skin friction coefficient were illustrated graphically to observe the visco-elastic effects in combination with other flow parameters involved in the solution. It was observed that the flow field is significantly affected by the visco-elastic parameter. Olayiwola (2016) worked on an analytical method of studying chemically reacting flow in a laminar premixed flame of carbon monoxide/oxygen mixture in the region of the stagnation point, its results showed that velocity increased as Prandtl number increased, Biot number decreased the fluid velocity and enhanced the species concentration and flame temperature. Babu *et al.* (2017) studied the effect of unsteady MHD free convective flow of a viscoelastic incompressible electrically conducting fluid past a moving vertical plate through a porous medium with time dependent oscillatory permeability and suction in presence of a uniform transverse magnetic field and heat source and chemical reaction along with heat and mass transfer were reported. A uniform magnetic field acted perpendicular to the porous surface, which absorbed the fluid with a suction velocity varying with time. The governing equations of the fluid flow, heat and mass transfer are solved by applying multi parameter perturbation technique. Comparison with previously published work had been conducted and the results were found to be in concordance with the previous study. A parametric study was performed on the influence of the visco-elastic fluid parameter, the magnetic field parameter, the permeability parameter, on the fluid velocity. The expressions for transient velocity, temperature, species concentration and non-dimensional skin friction at the plate were illustrated through tables to observe the visco-elastic effect in combination of other flow parameters involved in the solution. Dwivedi *et al.* (2018) presented MHD flow of fluids through vertical porous channel having porous medium placed in magnetic field was of industrial importance, therefore in this paper study had been made on the flow of a viscous incompressible, electrically conducting fluid through a vertical channel filled with porous channel, the field was applied perpendicular to the direction of flow. After forming the governing equation under suitable boundary conditions, a solution for velocity has been derived and taking different values of parameters, a graph had been plotted between velocity and applied magnetic field. Interference had been drawn to check the obtained results with physical nature of the problem. The study is useful in applying the result in industrial field. Jha and Yusuf (2020) studied the role of magnetic field on fully developed natural convection flow in an annulus due to symmetric of surfaces. The transport equations concerned with the model under consideration were rendered non-dimensional and transformed into the ordinary differential equation using Laplace transform technique. The solution obtained was then transformed to time domain using the Riemann-sum approximation approach. The governing equations were also solved using implicit finite difference method so as to establish the accuracy of the Riemann-sum approximation approach at transient as well as at steady state solution. The solutions obtained were graphically represented and the effects of pertinent parameters on the flow formation were investigated in detail. The Hartmann number (M), was seen to have a retarding effect on the velocity, skin-frictions and the mass flow rate. Skin-friction at both surfaces and the mass flow rate within the annulus were found to be directly proportional to the radii ratio (λ).

Ziya and Manoj (2011) studied MHD heat and mass transfer free convection flow near the lower stagnation point of an isothermal cylinder imbedded in porous domain with the presence of radiation. The equations of conservation of mass, momentum, energy and concentration which governed the study were obtained and transformed into a system of non-linear ordinary differential equations. They were solved by Runge-Kutta and shooting methods. Velocity profiles, temperature distribution and concentration distribution for the flow were presented for various values of radiation parameter, viscosity variation parameter, thermal conductivity variation, Prandtl number and Schmidt number. The skin friction, local Nusselt number and Sherwood number were also calculated for all parameters involved in the problem. The findings showed that increase in Schmidt number lead to decrease in skin friction and Nusselt number but it lead to increase in Sherwood number.

Hanvey *et al.* (2017) presented MHD flow of incompressible fluid through parallel plates in inclined magnetic field having porous medium with heat and mass transfer. Their model equations are:

$$\frac{\partial u^*}{\partial t^*} = -v_0 \frac{\partial u^*}{\partial y^*} - \frac{\partial p^*}{\partial x^*} + \mu \frac{\partial^2 u^*}{\partial y^{*2}} - \frac{\sigma}{\rho} \mathrm{B}_0^2 u^* \sin^2\theta + \rho g \beta (T^* - T_\infty^*) + \rho g \beta (C^* - C_\infty^*) \tag{2.1}$$

$$\frac{\partial T^*}{\partial y^*} = \frac{K}{\rho Cp} \frac{\partial^2 T^*}{\partial y^{*2}} \tag{2.2}$$

where,

u^* is the component of velocity along x^* - axis,

μ is the viscosity of the fluid,

σ is the electrical conductivity of the fluid,

ρ is the density of the fluid,

g is the acceleration due to gravity,

T^* is the fluid temperature,

B_0 is the magnetic field strength component,

v_0 is the characteristic velocity,

C_p is the specific heat at constant pressure,

K is the thermal conductivity.

2.2 Summary of Review and Gaps to Fill

In reviewing the above literature, it has been discovered that several works had been carried out on magnetohydrodynamics flow of incompressible fluid through parallel plates. Some authors considered magnetohydrodynamics flow of incompressible fluid through parallel plates in inclined magnetic field without considering viscous dissipation energy. In review of the above, this research work seeks to consider magnetohydrodynamics flow of incompressible fluid through parallel plates in inclined magnetic field with viscous dissipation energy, thereby extended the work of Hanvey *et al.* (2017) by incorporating viscous energy dissipation term to the energy equation and also introduced concentration equation to the set of model equations.

CHAPTER THREE

3.0 MATERIALS AND METHODS

3.1 Mathematical Formulation

An electrically conducting, unsteady, viscous, incompressible Newtonian fluid moving between two infinite parallel plates kept at a distance of 2h apart are placed in inclined magnetic field. The lower plate at $x = -L$ is being subjected to both heat and mass flux and the upper plate at $x = L$ is being subjected to both convective heat and mass transfer. Consider one dimensional flow so that the axis of the channel formed by two plates is x-axis and the flow is in this direction. The equations governing the flow field are as follows:

$$\rho\left(\frac{\partial u^*}{\partial t^*} + v_0\frac{\partial u^*}{\partial y^*}\right) = -\frac{\partial p^*}{\partial x^*} + \mu\frac{\partial^2 u^*}{\partial y^{*2}} + \sigma B_0^2 \sin^2\theta u^* + \rho g\beta(T^* - T_\infty^*) + \rho g\beta(C^* - C_\infty^*) \tag{3.1}$$

$$\rho Cp\left(\frac{\partial T^*}{\partial t^*} + v_0\frac{\partial T^*}{\partial y^*}\right) = K\frac{\partial^2 T^*}{\partial y^{*2}} + \mu\left(\frac{\partial u^*}{\partial y^*}\right)^2 \tag{3.2}$$

$$\rho\left(\frac{\partial C^*}{\partial t^*} + v_0\frac{\partial C^*}{\partial y^*}\right) = D\frac{\partial^2 C^*}{\partial y^{*2}} \tag{3.3}$$

where,

u^* is the component of velocity along x^* - axis,

g is the acceleration due to gravity,

μ is the viscosity of the fluid,

σ is the electrical conductivity of the fluid,

ρ is the density of the fluid,

B_0 is the magnetic field strength component,

v_0 is the characteristic velocity,

C_p is the specific heat at constant pressure,

∞ is the angle of inclination,

T^* is the fluid temperature,

C^* is the species concentration,

T_∞^* is the far field temperature,

C_∞^* is the far field concentration,

β is the volume expansion coefficient for the heat transfer,

K is the thermal conductivity,

D is the diffusion coefficient.

The initial and boundary conditions are formulated as:

$$\left.\begin{aligned}
&u^*(y^*,0)=0,\quad u^*(-L,t)=U,\quad \left.\frac{\partial u^*}{\partial y^*}\right|_{y^*=L}=0\\
&T^*(y^*,0)=T^*_\omega,\quad -K^*\left.\frac{\partial T^*}{\partial y^*}\right|_{y^*=-L}=q_h^{II},\quad -K^*\left.\frac{\partial T^*}{\partial y^*}\right|_{y^*=L}=h\left(T^*-T^*_\infty\right)\\
&C^*(y^*,0)=C^*_\omega,\quad -D^*\left.\frac{\partial C^*}{\partial y^*}\right|_{y^*=-L}=q_m^{II},\quad -D^*\left.\frac{\partial C^*}{\partial y^*}\right|_{y^*=L}=h_m\left(C^*-C^*_\infty\right)
\end{aligned}\right\} \tag{3.4}$$

3.2 Non-dimensionalisation

Equations (3.1) to (3.4) were non-dimensionalised using the following dimensionless variables

$$x=\frac{x^*}{L},\ t=\frac{Ut^*}{L},\ u=\frac{u^*}{U},\ \theta=\frac{T^*-T^*_\infty}{T^*_\omega-T^*_\infty},\ \phi=\frac{C^*-C^*_\infty}{C^*_\omega-C^*_\infty},\ y=\frac{y^*}{L},\ p=\frac{p^*}{\rho U^2},\ v=\frac{v_0}{U} \tag{3.5}$$

From equation (3.5), the following equation was obtained

$$\left.\begin{aligned}
&u^*=Uu,\ x^*=Lx,\ y^*=Ly,\ v_0=Uv,\ p^*=\rho pU^2,\ t^*=\frac{Lt}{U},\\
&T^*=(T^*_\omega-T^*_\infty)\theta+T^*_\infty,\ C^*=(C^*_\omega-C^*_\infty)\phi+C^*_\infty,\ \partial u^*=U\partial u,\ \partial x^*=L\partial x,\\
&\partial y^*=L\partial y,\ \partial p^*=\rho U^2\partial p,\ \partial t^*=\frac{L}{U}\partial t,\ \partial T^*=(T^*_\omega-T^*_\infty)\partial\theta,\ \partial C^*=(C^*_\omega-C^*_\infty)\partial\phi,\\
&\partial^2u^*=U\partial^2u,\ \partial y^{*2}=L^2\partial y^2,\ \partial^2T=(T^*_\omega-T^*_\infty)\partial^2\theta,\ \partial^2C=(C^*_\omega-C^*_\infty)\partial^2\phi
\end{aligned}\right\} \tag{3.6}$$

Put equation (3.6) in equation (3.1), equation (3.7) was obtained

$$\left.\begin{aligned}
\rho\left(\frac{U^2}{L}\frac{\partial u}{\partial t}+\frac{U^2v}{L}\frac{\partial u}{\partial y}\right)=-\frac{\rho U^2}{L}\frac{\partial p}{\partial x}+\frac{\mu U}{L^2}\frac{\partial^2u}{\partial y^2}+\sigma B_0^2\sin^2\theta Uu+\rho g\beta(T^*_\omega-T^*_\infty)\theta+\\
\rho g\beta(C^*_\omega-C^*_\infty)\phi
\end{aligned}\right\} \tag{3.7}$$

Multiply through equation (3.7) by $\frac{L}{\rho U^2}$ have,

$$\left.\frac{\partial u}{\partial t}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{\mu}{\rho UL}\frac{\partial^2 u}{\partial y^2}+\frac{\sigma L\mathrm{B}_0^2\sin^2\theta u}{\rho U}+\frac{g\beta L(T_\omega^*-T_\infty^*)\theta}{U^2}+\frac{g\beta L(C_\omega^*-C_\infty^*)\phi}{U^2}\right\} \tag{3.8}$$

So,

$$\frac{\partial u}{\partial t}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{1}{\mathrm{Re}}\frac{\partial^2 u}{\partial y^2}+M^2u+G_{r\theta}\theta+G_{r\phi}\phi \tag{3.9}$$

where,

$$\left.\begin{aligned}
&M=N\sin\theta,\quad N=\sqrt{\frac{\sigma\mathrm{B}_0^2L}{\rho U}},\quad \mathrm{Re}=\frac{\rho UL}{\mu}=\mathrm{Re}\,ynolds\ number,\\
&G_{r\theta}=\frac{g\beta L(T_\omega^*-T_\infty^*)\theta}{U^2}=Thermal\,Grashof\ number,\\
&G_{r\phi}=\frac{g\beta L(C_\omega^*-C_\infty^*)\phi}{U^2}=Solutal\ Grashof\ number
\end{aligned}\right\} \tag{3.10}$$

Put equation (3.6) in equation (3.2) have,

$$\rho Cp\left(\frac{U(T_\omega^*-T_\infty^*)}{L}\frac{\partial\theta}{\partial t}+\frac{Uv(T_\omega^*-T_\infty^*)}{L}\frac{\partial\theta}{\partial y}\right)=\frac{K(T_\omega^*-T_\infty^*)}{L}\frac{\partial^2\theta}{\partial y^2}+\mu\left(\frac{U}{L}\frac{\partial u}{\partial y}\right)^2 \tag{3.11}$$

Multiply through equation (3.11) by $\dfrac{L}{\rho CpU(T_\omega^*-T_\infty^*)}$ have,

$$\frac{\partial\theta}{\partial t}+v\frac{\partial\theta}{\partial y}=\frac{K}{\rho CpU}\frac{\partial^2\theta}{\partial y^2}+\frac{\mu U^2}{\rho CpUL(T_\omega^*-T_\infty^*)}\left(\frac{\partial u}{\partial y}\right)^2 \tag{3.12}$$

$$\frac{\partial\theta}{\partial t}+v\frac{\partial\theta}{\partial y}=\frac{1}{Pe}\frac{\partial^2\theta}{\partial y^2}+\frac{Ec}{\mathrm{Re}}\left(\frac{\partial u}{\partial y}\right)^2 \tag{3.13}$$

where,

$$Pe = \frac{\rho CpU}{K} = Peclet\ Energy\ number,$$

$$Ec = \frac{U^2}{Cp(T_\omega^* - T_\infty^*)} = Eckert\ number \tag{3.14}$$

Put equation (3.6) in equation (3.3) have,

$$\rho\left(\frac{U(C_\omega^* - C_\infty^*)}{L}\frac{\partial\phi}{\partial t} + \frac{Uv(C_\omega^* - C_\infty^*)}{L}\frac{\partial\phi}{\partial y}\right) = \frac{D(C_\omega^* - C_\infty^*)}{L}\frac{\partial^2\phi}{\partial y^2} \tag{3.15}$$

Multiply through equation (3.15) by $\frac{L}{\rho U(C_\omega^* - C_\infty^*)}$ have,

$$\frac{\partial\phi}{\partial t} + v\frac{\partial\phi}{\partial y} = \frac{D}{\rho U}\frac{\partial^2\phi}{\partial y^2} \tag{3.16}$$

$$\frac{\partial\phi}{\partial t} + v\frac{\partial\phi}{\partial y} = \frac{1}{Pem}\frac{\partial^2\phi}{\partial y^2} \tag{3.17}$$

where,

$$P_{em} = \frac{\rho U}{D} = Peclet\ mass\ number$$

Put equation (3.6) in equation (3.4) have,

$$
\left.
\begin{array}{l}
u^*(y^*,0)=0 \Rightarrow\ Uu(Ly,0)=0\ \ \Rightarrow u(y,0)=0 \\
u^*(-L,t)=U \Rightarrow\ Uu(-1,t)=U \Rightarrow\ u\big|_{y=-1}=1 \\
\left.\dfrac{\partial u^*}{\partial y^*}\right|_{y^*=L}=0 \Rightarrow\ \dfrac{U}{L}\left.\dfrac{\partial u}{\partial y}\right|_{Ly=L}=0 \Rightarrow\ \left.\dfrac{\partial u}{\partial y}\right|_{y=1}=0 \\
T^*(y^*,0)=T^*_\omega \Rightarrow\ \theta(y,0)(T^*_\omega-T^*_\infty)+T^*_\infty=T^*_\omega\ \Rightarrow\ \theta(y,0)=\dfrac{T^*_\omega-T^*_\infty}{T^*_\omega-T^*_\infty}=1 \\
-K^*\left.\dfrac{\partial T^*}{\partial y^*}\right|_{y^*=-L}=q_h^{II}\ \Rightarrow -\dfrac{K^*(T^*_\omega-T^*_\infty)}{L}\left.\dfrac{\partial \theta}{\partial y}\right|_{Ly=-L}=q_h^{II}\ \Rightarrow\ \left.\dfrac{\partial \theta}{\partial y}\right|_{y=-1}=-q_1 \\
-K^*\left.\dfrac{\partial T^*}{\partial y^*}\right|_{y^*=L}=h(T^*-T^*_\infty)\Rightarrow\ -\dfrac{K^*(T^*_\omega-T^*_\infty)}{L}\left.\dfrac{\partial \theta}{\partial y}\right|_{Ly=L}=h(T^*-T^*_\infty)\ \Rightarrow \\
\left.\dfrac{\partial \theta}{\partial y}\right|_{y=1}=-\dfrac{hL(T^*-T^*_\infty)}{K^*(T^*_\omega-T^*_\infty)}=-Nu\,\theta\big|_{y=1}\ \Rightarrow \left.\dfrac{\partial \theta}{\partial y}\right|_{y=1}=-Nu\,\theta\big|_{y=1} \\
C^*(y^*,0)=C^*_\omega\ \Rightarrow\ \phi(y,0)(C^*_\omega-C^*_\infty)+C^*_\infty=C^*_\omega\ \Rightarrow \phi(y,0)=\dfrac{C^*_\omega-C^*_\infty}{C^*_\omega-C^*_\infty}=1 \\
-D^*\left.\dfrac{\partial C^*}{\partial y^*}\right|_{y^*=-L}=q_m^{II}\ \Rightarrow -\dfrac{D^*(C^*_\omega-C^*_\infty)}{L}\left.\dfrac{\partial \phi}{\partial y}\right|_{Ly=-L}=q_m^{II}\ \Rightarrow \left.\dfrac{\partial \phi}{\partial y}\right|_{y=-1}=-q_2 \\
-D^*\left.\dfrac{\partial C^*}{\partial y^*}\right|_{y^*=L}=h_m(C^*-C^*_\infty)\ \Rightarrow -\dfrac{D^*(C^*_\omega-C^*_\infty)}{L}\left.\dfrac{\partial \phi}{\partial y}\right|_{Ly=L}=h_m(C^*-C^*_\infty)\ \Rightarrow \\
\left.\dfrac{\partial \phi}{\partial y}\right|_{y=1}=-\dfrac{Lh_m(C^*-C^*_\infty)}{D^*(C^*_\omega-C^*_\infty)}=-Sh\,\phi\big|_{y=1}
\end{array}
\right\}
\tag{3.18}
$$

where,

$$Nu=\frac{hL}{K^*}=Nusselt\ number\ ,\ \ Sh=\frac{Lh_m}{D^*}=Sherwood\ number$$

Therefore, the dimensionless equations with the initial and boundary conditions are:

$$\left.\begin{aligned}&\frac{\partial u}{\partial t}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{1}{\mathrm{Re}}\frac{\partial^2 u}{\partial y^2}+M^2u+G_{r\theta}\theta+G_{r\phi}\phi\\&u(y,0)=0,\quad u\big|_{y=-1}=1,\quad \frac{\partial u}{\partial y}\bigg|_{y=1}=0\end{aligned}\right\} \tag{3.19}$$

$$\left.\begin{aligned}&\frac{\partial \theta}{\partial t}+v\frac{\partial \theta}{\partial y}=\frac{1}{Pe}\frac{\partial^2 \theta}{\partial y^2}+\frac{Ec}{\mathrm{Re}}\left(\frac{\partial u}{\partial y}\right)^2\\&\theta(y,0)=1,\quad \frac{\partial \theta}{\partial y}\bigg|_{y=-1}=-q_1,\quad \frac{\partial \theta}{\partial y}\bigg|_{y=1}=-Nu\,\theta\big|_{y=1}\end{aligned}\right\} \tag{3.20}$$

$$\left.\begin{aligned}&\frac{\partial \phi}{\partial t}+v\frac{\partial \phi}{\partial y}=\frac{1}{Pem}\frac{\partial^2 \phi}{\partial y^2}\\&\phi(y,0)=1,\quad \frac{\partial \phi}{\partial y}\bigg|_{y=-1}=-q_2,\quad \frac{\partial \phi}{\partial y}\bigg|_{y=1}=-Sh\,\phi\big|_{y=1}\end{aligned}\right\} \tag{3.21}$$

3.3 Method of Solution

Solution Via Polynomial Approximation Method (PAM)

Here, let

$$0<G_{r\theta}<<1 \ \ such\ that$$

$$\left.\begin{aligned}u(y,t)&=u_0(y,t)+G_{r\theta}u_1(y,t)\\\theta(y,t)&=\theta_0(y,t)+G_{r\theta}\theta_1(y,t)\\\phi(y,t)&=\phi_0(y,t)+G_{r\theta}\phi_1(y,t)\end{aligned}\right\} \tag{3.22}$$

Substituting equation (3.22) into equations (3.19) through (3.21) and simplify, have:

$G_{r\theta}^0$:

$$\left.\begin{aligned}&\frac{\partial u_0}{\partial t}+v\frac{\partial u_0}{\partial y}=-\frac{\partial p}{\partial x}+\frac{1}{\mathrm{Re}}\frac{\partial^2 u_0}{\partial y^2}+M^2u_0+G_{r\phi}\phi_0\\&u_0(y,0)=0,\quad u_0\big|_{y=-1}=1,\quad \left.\frac{\partial u_0}{\partial y}\right|_{y=1}=0\end{aligned}\right\} \tag{3.23}$$

$$\left.\begin{aligned}&\frac{\partial \theta_0}{\partial t}+v\frac{\partial \theta_0}{\partial y}=\frac{1}{Pe}\frac{\partial^2 \theta_0}{\partial y^2}+\frac{Ec}{\mathrm{Re}}\left(\frac{\partial u_0}{\partial y}\right)^2\\&\theta_0(y,0)=1,\quad \left.\frac{\partial \theta_0}{\partial y}\right|_{y=-1}=-q_1,\quad \left.\frac{\partial \theta_0}{\partial y}\right|_{y=1}=-Nu\,\theta_0\big|_{y=1}\end{aligned}\right\} \tag{3.24}$$

$$\left.\begin{aligned}&\frac{\partial \phi_0}{\partial t}+v\frac{\partial \phi_0}{\partial y}=\frac{1}{Pem}\frac{\partial^2 \phi_0}{\partial y^2}\\&\phi_0(y,0)=1,\quad \left.\frac{\partial \phi_0}{\partial y}\right|_{y=-1}=-q_2,\quad \left.\frac{\partial \phi_0}{\partial y}\right|_{y=1}=-Sh\,\phi\big|_{y=1}\end{aligned}\right\} \tag{3.25}$$

$G_{r\theta}^1$:

$$\left.\begin{aligned}&\frac{\partial u_1}{\partial t}+v\frac{\partial u_1}{\partial y}=-\frac{\partial p}{\partial x}+\frac{1}{\mathrm{Re}}\frac{\partial^2 u_1}{\partial y^2}+M^2u_1+\theta_0+G_{r\phi}\phi_1\\&u_1(y,0)=0,\quad u_1\big|_{y=-1}=0,\quad \left.\frac{\partial u_1}{\partial y}\right|_{y=1}=0\end{aligned}\right\} \tag{3.26}$$

$$\left.\begin{aligned}&\frac{\partial \theta_1}{\partial t}+v\frac{\partial \theta_1}{\partial y}=\frac{1}{Pe}\frac{\partial^2 \theta_1}{\partial y^2}+\frac{2Ec}{\mathrm{Re}}\left(\frac{\partial u_0}{\partial y}\frac{\partial u_1}{\partial y}\right)\\&\theta_1(y,0)=0,\quad \left.\frac{\partial \theta_1}{\partial y}\right|_{y=-1}=0,\quad \left.\frac{\partial \theta_1}{\partial y}\right|_{y=1}=-Nu\,\theta_1\big|_{y=1}\end{aligned}\right\} \tag{3.27}$$

$$\left.\begin{aligned}&\frac{\partial\phi_1}{\partial t}+v\frac{\partial\phi_1}{\partial y}=\frac{1}{Pem}\frac{\partial^2\phi_1}{\partial y^2}\\&\phi_1(y,0)=0,\quad \left.\frac{\partial\phi_1}{\partial y}\right|_{y=-1}=0,\quad \left.\frac{\partial\phi_1}{\partial y}\right|_{y=1}=-Sh\,\phi_1\big|_{y=1}\end{aligned}\right\} \tag{3.28}$$

Assume Polynomial Solutions:

$$\left.\begin{aligned}u_0(y,t)&=a_0(t)+a_1(t)y+a_2(t)y^2\\\theta_0(y,t)&=b_0(t)+b_1(t)y+b_2(t)y^2\\\phi_0(y,t)&=c_0(t)+c_1(t)y+c_2(t)y^2\\u_1(y,t)&=d_0(t)+d_1(t)y+d_2(t)y^2\\\theta_1(y,t)&=e_0(t)+e_1(t)y+e_2(t)y^2\\\phi_1(y,t)&=f_0(t)+f_1(t)y+f_2(t)y^2\end{aligned}\right\} \tag{3.29}$$

$$\left.\begin{aligned}\frac{\partial u_0}{\partial y}&=a_1+2a_2y\\\frac{\partial\theta_0}{\partial y}&=b_1+2b_2y\\\frac{\partial\phi_0}{\partial y}&=c_1+2c_2y\\\frac{\partial u_1}{\partial y}&=d_1+2d_2y\\\frac{\partial\theta_1}{\partial y}&=e_1+2e_2y\\\frac{\partial\phi_1}{\partial y}&=f_1+2f_2y\end{aligned}\right\} \tag{3.30}$$

$$u_0\big|_{y=-1}=a_0-a_1+a_2=1 \tag{3.31}$$

$$a_1=a_0+a_2-1 \tag{3.32}$$

$$\left.\frac{\partial u_0}{\partial y}\right|_{y=1} = a_1 + 2a_2 = 0 \tag{3.33}$$

$$\Rightarrow a_2 = -\frac{a_1}{2} \tag{3.34}$$

Substituting equation (3.34) into equation (3.32) have,

$$a_1 = a_0 - \frac{a_1}{2} - 1 \tag{3.35}$$

$$a_1 = \frac{2}{3}(a_0 - 1) \tag{3.36}$$

Substituting equation (3.36) into equation (3.34) have,

$$a_2 = -\frac{1}{3}(a_0 - 1) \tag{3.37}$$

$$u_0\big|_{y=1} = a_0 + a_1 + a_2 = a_0 + \frac{2}{3}(a_0 - 1) - \frac{1}{3}(a_0 - 1) \tag{3.38}$$

$$u_0\big|_{y=1} = \frac{4}{3}a_0 - \frac{1}{3} \tag{3.39}$$

$$a_0 = \frac{3}{4}\left(u_0\big|_{y=1} + \frac{1}{3}\right) \tag{3.40}$$

That is,

$$a_0 = \frac{1}{4}\left(3u_0\big|_{y=1} + 1\right) \tag{3.41}$$

Substituting equation (3.41) into equation (3.36) have,

$$a_1 = \frac{2}{3}\left(\frac{1}{4}(3u_0\big|_{y=1} + 1) - 1\right) \tag{3.42}$$

$$a_1 = \frac{1}{2}\left(u_0\big|_{y=1} - 1\right) \tag{3.43}$$

Substituting equation (3.41) into equation (3.37) have,

$$a_2 = -\frac{1}{3}\left(\frac{1}{4}(3u_0\big|_{y=1} + 1) - 1\right) \tag{3.44}$$

$$a_2 = -\frac{1}{4}\left(u_0\big|_{y=1} - 1\right) \tag{3.45}$$

Also,

$$\left.\frac{\partial \theta_0}{\partial y}\right|_{y=-1} = b_1 - 2b_2 = -q_1 \tag{3.46}$$

$$\Rightarrow b_2 = \frac{b_1 + q_1}{2} \tag{3.47}$$

$$\left.\frac{\partial \theta_0}{\partial y}\right|_{y=1} = b_1 + 2b_2 = -Nu\,\theta_0\big|_{y=1} \tag{3.48}$$

$$\Rightarrow b_1 = -\left(2b_2 + Nu\,\theta_0\big|_{y=1}\right) \tag{3.49}$$

Substituting equation (3.49) into equation (3.47) have,

$$b_2 = \frac{q_1 - 2b_2 - Nu\,\theta_0\big|_{y=1}}{2} \tag{3.50}$$

$$2b_2 = \frac{q_1 - Nu\,\theta_0\big|_{y=1}}{2} \tag{3.51}$$

$$b_2 = \frac{1}{4}\left(q_1 - Nu\,\theta_0\big|_{y=1}\right) \tag{3.52}$$

Substituting equation (3.52) into equation (3.49) have,

$$b_1 = -\left(\frac{1}{2}\left(q_1 - Nu\,\theta_0\big|_{y=1}\right) + Nu\,\theta_0\big|_{y=1}\right) = -\frac{1}{2}\left(q_1 + Nu\,\theta_0\big|_{y=1}\right) \tag{3.53}$$

$$\theta_0\big|_{y=1} = b_0 - \frac{1}{2}\left(q_1 + Nu\,\theta_0\big|_{y=1}\right) + \frac{1}{4}\left(q_1 - Nu\,\theta_0\big|_{y=1}\right) \tag{3.54}$$

$$\theta_0\big|_{y=1} = b_0 - \frac{q_1}{4} - \frac{3Nu}{4}\,\theta_0\big|_{y=1} \tag{3.55}$$

That is,

$$b_0 = \frac{1}{4}\left((4+3Nu)\,\theta_0\big|_{y=1} + q_1\right) \tag{3.56}$$

Similarly,

$$c_0 = \frac{1}{4}\left((4+3Sh)\,\phi_0\big|_{y=1} + q_2\right) \tag{3.57}$$

$$c_1 = -\frac{1}{2}\left(q_2 + Sh\,\phi_0\big|_{y=1}\right) \tag{3.58}$$

$$c_2 = \frac{1}{4}\left(q_2 - Sh\,\phi_0\big|_{y=1}\right) \tag{3.59}$$

Also,

$$u_1\big|_{y=-1} = d_0 - d_1 + d_2 = 0 \tag{3.60}$$

$$\Rightarrow d_1 = d_0 + d_2 \tag{3.61}$$

$$\left.\frac{\partial u_1}{\partial y}\right|_{y=1} = d_1 + 2d_2 = 0 \tag{3.62}$$

$$\Rightarrow d_2 = -\frac{d_1}{2} \tag{3.63}$$

Substituting equation (3.63) into equation (3.61) have,

$$d_1 = d_0 - \frac{d_1}{2} \tag{3.64}$$

$$d_1 = \frac{2}{3} d_0 \tag{3.65}$$

Put equation (3.65) into equation (3.63) have,

$$d_2 = -\frac{1}{3} d_0 \tag{3.66}$$

$$u_1\big|_{y=1} = d_0 + d_1 + d_2 = d_0 + \frac{2}{3} d_0 - \frac{1}{3} d_0 = \frac{4}{3} d_0 \tag{3.67}$$

$$d_0 = \frac{3}{4} u_1\big|_{y=1} \tag{3.68}$$

Substituting equation (3.68) into equation (3.65) have,

$$d_1 = \frac{1}{2} u_1\big|_{y=1} \tag{3.69}$$

Substituting equation (3.68) into equation (3.66) have,

$$d_2 = -\frac{1}{4} u_1\big|_{y=1} \tag{3.70}$$

Also,

$$\left.\frac{\partial \theta_1}{\partial y}\right|_{y=-1} = e_1 - 2e_2 = 0 \tag{3.71}$$

$$\Rightarrow e_2 = \frac{e_1}{2} \tag{3.72}$$

$$\left.\frac{\partial \theta_1}{\partial y}\right|_{y=1} = e_1 + 2e_2 = -Nu\,\theta_1\big|_{y=1} \tag{3.73}$$

$$\Rightarrow e_1 = -\left(2e_2 + Nu\,\theta_1\big|_{y=1}\right) \tag{3.74}$$

Substituting equation (3.74) into equation (3.72) have,

$$e_2 = -\frac{\left(2e_2 + Nu\,\theta_1\big|_{y=1}\right)}{2} \tag{3.75}$$

$$e_2 = -\frac{Nu}{4}\theta_1\big|_{y=1} \tag{3.76}$$

Put equation (3.76) into equation (3.74) have,

$$e_1 = -\frac{Nu}{2}\theta_1\big|_{y=1} \tag{3.77}$$

$$\theta_1\big|_{y=1} = e_0 - \frac{Nu}{2}\theta_1\big|_{y=1} - \frac{Nu}{4}\theta_1\big|_{y=1} \tag{3.78}$$

$$e_0 = \frac{1}{4}(4 + 3Nu)\theta_1\big|_{y=1} \tag{3.79}$$

Similarly,

$$f_0 = \frac{1}{4}(4 + 3Sh)\phi_1\big|_{y=1} \tag{3.80}$$

$$f_1 = -\frac{Sh}{2}\phi_1\big|_{y=1} \tag{3.81}$$

$$f_2 = -\frac{Sh}{4}\phi_1\big|_{y=1} \tag{3.82}$$

Then, equation (3.29) becomes,

$$\left.\begin{aligned}
u_0(y,t) &= \frac{1}{4}\left(3u_0\big|_{y=1} + 1\right) + \frac{1}{2}\left(u_0\big|_{y=1} - 1\right)y - \frac{1}{4}\left(u_0\big|_{y=1} - 1\right)y^2 \\
\theta_0(y,t) &= \frac{1}{4}\left((4+3Nu)\,\theta_0\big|_{y=1} + q_1\right) - \frac{1}{2}\left(q_1 + Nu\,\theta_0\big|_{y=1}\right)y + \frac{1}{4}\left(q_1 - Nu\,\theta_0\big|_{y=1}\right)y^2 \\
\phi_0(y,t) &= \frac{1}{4}\left((4+3Sh)\,\phi_0\big|_{y=1} + q_2\right) - \frac{1}{2}\left(q_2 + Sh\,\phi_0\big|_{y=1}\right)y + \frac{1}{4}\left(q_2 - Sh\,\phi_0\big|_{y=1}\right)y^2 \\
u_1(y,t) &= \frac{3}{4}u_1\big|_{y=1} + \frac{1}{2}u_1\big|_{y=1}\,y - \frac{1}{4}u_1\big|_{y=1}\,y^2 \\
\theta_1(y,t) &= \frac{1}{4}(4+3Nu)\,\theta_1\big|_{y=1} - \frac{Nu}{2}\theta_1\big|_{y=1}\,y - \frac{Nu}{4}\theta_1\big|_{y=1}\,y^2 \\
\phi_1(y,t) &= \frac{1}{4}(4+3Sh)\,\phi_1\big|_{y=1} - \frac{Sh}{2}\phi_1\big|_{y=1}\,y - \frac{Sh}{4}\phi_1\big|_{y=1}\,y^2
\end{aligned}\right\} \tag{3.83}$$

For a long geometry,

$$\left.\begin{aligned}
\bar{u}_0 &= \frac{1}{2}\int_{-1}^{1} u_0\, dy \quad and \quad \frac{\partial \bar{u}_0}{\partial t} = \frac{1}{2}\int_{-1}^{1} \frac{\partial u_0}{\partial t}\, dy \\
\bar{\theta}_0 &= \frac{1}{2}\int_{-1}^{1} \theta_0\, dy \quad and \quad \frac{\partial \bar{\theta}_0}{\partial t} = \frac{1}{2}\int_{-1}^{1} \frac{\partial \theta_0}{\partial t}\, dy \\
\bar{\phi}_0 &= \frac{1}{2}\int_{-1}^{1} \phi_0\, dy \quad and \quad \frac{\partial \bar{\phi}_0}{\partial t} = \frac{1}{2}\int_{-1}^{1} \frac{\partial \phi_0}{\partial t}\, dy \\
\bar{u}_1 &= \frac{1}{2}\int_{-1}^{1} u_1\, dy \quad and \quad \frac{\partial \bar{u}_1}{\partial t} = \frac{1}{2}\int_{-1}^{1} \frac{\partial u_1}{\partial t}\, dy \\
\bar{\theta}_1 &= \frac{1}{2}\int_{-1}^{1} \theta_1\, dy \quad and \quad \frac{\partial \bar{\theta}_1}{\partial t} = \frac{1}{2}\int_{-1}^{1} \frac{\partial \theta_1}{\partial t}\, dy \\
\bar{\phi}_1 &= \frac{1}{2}\int_{-1}^{1} \phi_1\, dy \quad and \quad \frac{\partial \bar{\phi}_1}{\partial t} = \frac{1}{2}\int_{-1}^{1} \frac{\partial \phi_1}{\partial t}\, dy
\end{aligned}\right\} \tag{3.84}$$

where,

$\bar{u}_0,\ \bar{\theta}_0,\ \bar{\phi}_0,\ \bar{u}_1,\ \bar{\theta}_1$ *and* $\bar{\phi}_1$ *are the average of* $u_0,\ \theta_0,\ \phi_0,\ u_1,\ \theta_1$ *and* ϕ_1 *respectively.*

That is,

$$\bar{u}_0 = \frac{1}{2}\int_{-1}^{1} u_0\, dy \tag{3.85}$$

$$\bar{u}_0 = \frac{1}{2}\int_{-1}^{1}\left(\frac{1}{4}\left(3u_0\big|_{y=1}+1\right)+\frac{1}{2}\left(u_0\big|_{y=1}-1\right)y-\frac{1}{4}\left(u_0\big|_{y=1}-1\right)y^2\right)dy \tag{3.86}$$

$$\bar{u}_0 = \frac{1}{4}\left(3u_0\big|_{y=1}+1\right)+0-\frac{1}{12}\left(u_0\big|_{y=1}-1\right) \tag{3.87}$$

$$\bar{u}_0 = \frac{2}{3}u_0\big|_{y=1}+\frac{1}{3} \tag{3.88}$$

Also,

$$\bar{\theta}_0 = \frac{1}{2}\int_{-1}^{1} \theta_0\, dy \tag{3.89}$$

$$\bar{\theta}_0 = \frac{1}{2}\int_{-1}^{1}\left(\frac{1}{4}\left((4+3Nu)\,\theta_0\big|_{y=1}+q_1\right)-\frac{1}{2}\left(q_1+Nu\,\theta_0\big|_{y=1}\right)y+\frac{1}{4}\left(q_1-Nu\,\theta_0\big|_{y=1}\right)y^2\right)dy \qquad (3.90)$$

$$\bar{\theta}_0 = \frac{1}{4}\left((4+3Nu)\,\theta_0\big|_{y=1}+q_1\right)-0+\frac{1}{12}\left(q_1-Nu\,\theta_0\big|_{y=1}\right) \qquad (3.91)$$

$$\bar{\theta}_0 = \left(\frac{12+8Nu}{12}\right)\theta_0\big|_{y=1}+\frac{q_1}{3} \qquad (3.92)$$

Similarly,

$$\bar{\phi}_0 = \left(\frac{12+8Sh}{12}\right)\phi_0\big|_{y=1}+\frac{q_2}{3} \qquad (3.93)$$

$$\bar{u}_1 = \frac{2}{3}u_1\big|_{y=1} \qquad (3.94)$$

$$\bar{\theta}_1 = \left(\frac{12+8Nu}{12}\right)\theta_1\big|_{y=1} \qquad (3.95)$$

$$\bar{\phi}_1 = \left(\frac{12+8Sh}{12}\right)\phi_1\big|_{y=1} \qquad (3.96)$$

and,

$$\left.\begin{aligned}
\frac{\partial \bar{u}_0}{\partial t} &= \frac{2}{3}\frac{\partial u_0}{\partial t}\bigg|_{y=1} \\
\frac{\partial \bar{\theta}_0}{\partial t} &= \left(\frac{12+8Nu}{12}\right)\frac{\partial \theta_0}{\partial t}\bigg|_{y=1} \\
\frac{\partial \bar{\phi}_0}{\partial t} &= \left(\frac{12+8Sh}{12}\right)\frac{\partial \phi_0}{\partial t}\bigg|_{y=1} \\
\frac{\partial \bar{u}_1}{\partial t} &= \frac{2}{3}\frac{\partial u_1}{\partial t}\bigg|_{y=1} \\
\frac{\partial \bar{\theta}_1}{\partial t} &= \left(\frac{12+8Nu}{12}\right)\frac{\partial \theta_1}{\partial t}\bigg|_{y=1} \\
\frac{\partial \bar{\phi}_1}{\partial t} &= \left(\frac{12+8Sh}{12}\right)\frac{\partial \phi_1}{\partial t}\bigg|_{y=1}
\end{aligned}\right\} \tag{3.97}$$

Integrating the governing equations with respect to y have,

$$\left.\begin{aligned}
\frac{1}{2}\int_{-1}^{1}\frac{\partial u_0}{\partial t}dy = \frac{1}{2\,\mathrm{Re}}\int_{-1}^{1}\frac{\partial}{\partial y}\left(\frac{\partial u_0}{\partial y}\right)dy - \frac{v}{2}\int_{-1}^{1}\frac{\partial u_0}{\partial y}dy + \frac{M^2}{2}\int_{-1}^{1}u_0 dy + \\
\frac{G_{r\phi}}{2}\int_{-1}^{1}\phi_0 dy + -\frac{1}{2}\frac{\partial p}{\partial x}\int_{-1}^{1}dy
\end{aligned}\right\} \tag{3.98}$$

$$\left.\begin{aligned}
\frac{\partial \bar{u}_0}{\partial t} = -\frac{1}{2\,\mathrm{Re}}\int_{-1}^{1}\left(\frac{1}{2}(u_0|_{y=1}-1)\right)dy - \frac{v}{2}\int_{-1}^{1}\left(\frac{1}{2}\left(u_0|_{y=1}-1\right) - \frac{1}{2}\left(u_0|_{y=1}-1\right)y\right)dy + \\
\frac{M^2}{2}\int_{-1}^{1}\left(\frac{1}{4}\left(3u_0|_{y=1}+1\right) + \frac{1}{2}\left(u_0|_{y=1}-1\right)y - \frac{1}{4}\left(u_0|_{y=1}-1\right)y^2\right)dy + \\
\frac{G_{r\phi}}{2}\int_{-1}^{1}\left(\frac{1}{4}\left((4+3Sh)\,\phi_0|_{y=1}+q_2\right) - \frac{1}{2}\left(q_2 + Sh\,\phi_0|_{y=1}\right)y + \frac{1}{4}\left(q_2 - Sh\,\phi_0|_{y=1}\right)y^2\right)dy - \\
\frac{1}{2}\frac{\partial p}{\partial x}\int_{-1}^{1}dy
\end{aligned}\right\} \tag{3.99}$$

$$\left.\begin{aligned}
\frac{\partial \bar{u}_0}{\partial t} = -\frac{1}{2\,\mathrm{Re}}\left(u_0|_{y=1}-1\right) - \frac{v}{2}\left(u_0|_{y=1}-1\right) + \frac{M^2}{2}\left(\frac{1}{2}\left(3u_0|_{y=1}+1\right) - \frac{1}{6}\left(u_0|_{y=1}-1\right)\right) + \\
\frac{G_{r\phi}}{2}\left(\frac{1}{2}\left((4+3Sh)\phi_0|_{y=1}+q_2\right) + \frac{1}{6}\left(q_2 - Sh\,\phi_0|_{y=1}\right)\right) - \frac{\partial p}{\partial x}
\end{aligned}\right\} \tag{3.100}$$

$$\left.\begin{aligned}\frac{2}{3}\frac{\partial u_0}{\partial t}\bigg|_{y=1} &= -\frac{1}{2\,\mathrm{Re}}\left(u_0\big|_{y=1}-1\right)-\frac{v}{2}\left(u_0\big|_{y=1}-1\right)+\frac{M^2}{2}\left(\frac{4}{3}u_0\big|_{y=1}+\frac{2}{3}\right)+\\ &\frac{G_{r\phi}}{2}\left(\frac{1}{2}\left(\left(4+3Sh\right)\phi_0\big|_{y=1}+q_2\right)+\frac{1}{6}\left(q_2-Sh\,\phi_0\big|_{y=1}\right)\right)-\frac{\partial p}{\partial x}\end{aligned}\right\}\tag{3.101}$$

$$\left.\begin{aligned}\frac{2}{3}\frac{\partial u_0}{\partial t}\bigg|_{y=1} &= -\frac{1}{2}\left(\frac{1}{\mathrm{Re}}+v-\frac{4M^2}{3}\right)u_0\big|_{y=1}+\frac{G_{r\phi}}{2}\left(\frac{(4+3Sh)}{2}-\frac{Sh}{6}\right)\phi_0\big|_{y=1}+\\ &\left(\frac{1}{2\,\mathrm{Re}}+\frac{v}{2}+\frac{M^2}{3}+\frac{G_{r\phi}}{2}\left(\frac{q_2}{2}+\frac{q_2}{6}\right)\right)-\frac{\partial p}{\partial x}\end{aligned}\right\}\tag{3.102}$$

Take,

$$\frac{\partial p}{\partial x}=\lambda e^{-\sigma t}\tag{3.103}$$

That is,

$$\left.\begin{aligned}\frac{\partial u_0}{\partial t}\bigg|_{y=1}+\frac{3}{4}\left(\frac{1}{\mathrm{Re}}+v-\frac{4M^2}{3}\right)u_0\big|_{y=1} &= \frac{3}{4}G_{r\phi}\left(\frac{12+8Sh}{6}\right)\phi_0\big|_{y=1}+\\ &\frac{3}{2}\left(\frac{1}{2\,\mathrm{Re}}+\frac{v}{2}+\frac{M^2}{3}+\frac{G_{r\phi}}{2}\left(\frac{2}{3}q_2\right)\right)-\frac{3\lambda}{2}e^{-\sigma t}\end{aligned}\right\}\tag{3.104}$$

$$\frac{\partial u_0}{\partial t}\bigg|_{y=1}+p(t)\,u_0\big|_{y=1}=q(t)\tag{3.105}$$

where,

$$\left.\begin{aligned}&\mathrm{A}=\frac{3}{4}G_{r\phi}\left(\frac{12+8Sh}{6}\right),\ \mathrm{B}=\frac{3}{2}\left(\frac{1}{2\,\mathrm{Re}}+\frac{v}{2}+\frac{M^2}{3}+\frac{G_{r\phi}}{2}\left(\frac{2}{3}q_2\right)\right),\\ &C=\frac{3\lambda}{2},\ p(t)=\frac{3}{4}\left(\frac{1}{\mathrm{Re}}+v-\frac{4M^2}{3}\right),\ q(t)=\mathrm{A}\,\phi_0\big|_{y=1}+\mathrm{B}-Ce^{-\sigma t}\end{aligned}\right\}\tag{3.106}$$

Also,

$$\frac{1}{2}\int_{-1}^{1}\frac{\partial\theta_0}{\partial t}\,dy=\frac{1}{2Pe}\int_{-1}^{1}\frac{\partial}{\partial y}\left(\frac{\partial\theta_0}{\partial y}\right)dy-\frac{v}{2}\int_{-1}^{1}\frac{\partial\theta_0}{\partial y}\,dy+\frac{Ec}{2Re}\int_{-1}^{1}\left(\frac{\partial u_0}{\partial y}\right)^2 dy \tag{3.107}$$

$$\left.\begin{aligned}\frac{1}{2}\int_{-1}^{1}\frac{\partial\theta_0}{\partial t}\,dy=\frac{1}{2Pe}\int_{-1}^{1}\frac{1}{2}\left(q_1-Nu\,\theta_0\big|_{y=1}\right)dy+\\ \frac{v}{2}\int_{-1}^{1}\left(\frac{1}{2}\left(q_1+Nu\,\theta_0\big|_{y=1}\right)+\frac{1}{2}\left(q_1-Nu\,\theta_0\big|_{y=1}\right)y\right)dy+\\ \frac{Ec}{2Re}\int_{-1}^{1}\left(\frac{1}{4}\left(1-2u_0\big|_{y=1}\right)-\frac{1}{2}\left(1-2u_0\big|_{y=1}\right)y+\frac{1}{4}\left(1-2u_0\big|_{y=1}\right)y^2\right)dy\end{aligned}\right\} \tag{3.108}$$

$$\left.\begin{aligned}\frac{\partial\bar{\theta}_0}{\partial t}=\frac{1}{2Pe}\left(q_1-Nu\,\theta_0\big|_{y=1}\right)+\frac{v}{2}\left(q_1+Nu\,\theta_0\big|_{y=1}\right)+\\ \frac{Ec}{2Re}\left(\frac{1}{2}\left(1-2u_0\big|_{y=1}\right)+\frac{1}{6}\left(1-2u_0\big|_{y=1}\right)\right)\end{aligned}\right\} \tag{3.109}$$

$$\frac{\partial\bar{\theta}_0}{\partial t}=\left(\frac{vNu}{2}-\frac{Nu}{2Pe}\right)\theta_0\big|_{y=1}-\left(\frac{2Ec}{3\mathrm{Re}}\right)u_0\big|_{y=1}+\left(\left(\frac{1}{2Pe}+\frac{v}{2}\right)q_1+\frac{Ec}{3\mathrm{Re}}\right) \tag{3.110}$$

$$\left(\frac{12+8Nu}{12}\right)\frac{\partial\theta_0}{\partial t}\bigg|_{y=1}=\left(\frac{(vPe-1)Nu}{2Pe}\right)\theta_0\big|_{y=1}-\left(\frac{2Ec}{3\mathrm{Re}}\right)u_0\big|_{y=1}+\left(\left(\frac{1+vPe}{2Pe}\right)q_1+\frac{Ec}{3\mathrm{Re}}\right) \tag{3.111}$$

$$\left.\begin{aligned}\frac{\partial\theta_0}{\partial t}\bigg|_{y=1}+\left(\frac{12}{12+8Nu}\right)\left(\frac{(1-vPe)Nu}{2Pe}\right)\theta_0\big|_{y=1}=\\ \left(\frac{12}{12+8Nu}\right)\left(\left(\frac{1+vPe}{2Pe}\right)q_1+\frac{Ec}{3\mathrm{Re}}\right)-\left(\frac{12}{12+8Nu}\right)\left(\frac{2Ec}{3\mathrm{Re}}\right)u_0\big|_{y=1}\end{aligned}\right\} \tag{3.112}$$

$$\frac{\partial\theta_0}{\partial t}\bigg|_{y=1}+p_1(t)\,\theta_0\big|_{y=1}=q_1(t) \tag{3.113}$$

where,

$$D=\left(\frac{12}{12+8Nu}\right)\left(\left(\frac{1+vPe}{2Pe}\right)q_1+\frac{Ec}{3\mathrm{Re}}\right),\quad \mathrm{E}=\left(\frac{12}{12+8Nu}\right)\left(\frac{2Ec}{3\mathrm{Re}}\right),$$
$$p_1(t)=\left(\frac{12}{12+8Nu}\right)\left(\frac{(1-vPe)Nu}{2Pe}\right),\quad q_1(t)=D-\mathrm{E}\,u_0\big|_{y=1} \tag{3.114}$$

Also,

$$\frac{1}{2}\int_{-1}^{1}\frac{\partial\phi_0}{\partial t}\,dy=\frac{1}{2Pem}\int_{-1}^{1}\frac{\partial}{\partial y}\left(\frac{\partial\phi_0}{\partial y}\right)dy-\frac{v}{2}\int_{-1}^{1}\frac{\partial\phi_0}{\partial y}\,dy \tag{3.115}$$

$$\frac{\partial\overline{\phi}_0}{\partial t}=\frac{1}{2Pem}\int_{-1}^{1}\left(\frac{1}{2}\left(q_2-Sh\,\phi_0\big|_{y=1}\right)\right)dy+$$
$$\frac{v}{2}\int_{-1}^{1}\left(\frac{1}{2}\left(q_2+Sh\,\phi_0\big|_{y=1}\right)+\frac{1}{2}\left(q_2-Sh\,\phi_0\big|_{y=1}\right)y\right)dy \tag{3.116}$$

$$\left(\frac{12+8Sh}{12}\right)\frac{\partial\phi_0}{\partial t}\bigg|_{y=1}=\frac{1}{2Pem}\left(q_2-Sh\,\phi_0\big|_{y=1}\right)+\frac{v}{2}\left(q_2+Sh\,\phi_0\big|_{y=1}\right) \tag{3.117}$$

$$\left(\frac{12+8Sh}{12}\right)\frac{\partial\phi_0}{\partial t}\bigg|_{y=1}=-\left(\frac{Sh}{2Pem}-\frac{vSh}{2}\right)\phi_0\big|_{y=1}+\left(\frac{1}{2Pem}+\frac{v}{2}\right)q_2 \tag{3.118}$$

$$\frac{\partial\phi_0}{\partial t}\bigg|_{y=1}+\left(\frac{12}{12+8Sh}\right)\left(\frac{(1-vPem)Sh}{2Pem}\right)\phi_0\big|_{y=1}=\left(\frac{12}{12+8Sh}\right)\left(\frac{1+vPem}{2Pem}\right)q_2 \tag{3.119}$$

$$\frac{\partial\phi_0}{\partial t}\bigg|_{y=1}+p_2(t)\,\phi_0\big|_{y=1}=q_2(t) \tag{3.120}$$

where,

$$p_2(t)=\left(\frac{12}{12+8Sh}\right)\left(\frac{(1-vPem)Sh}{2Pem}\right),\quad q_2(t)=\left(\frac{12}{12+8Sh}\right)\left(\frac{1+vPem}{2Pem}\right)q_2 \tag{3.121}$$

Also,

$$\left.\begin{array}{l}\frac{1}{2}\int_{-1}^{1}\frac{\partial u_1}{\partial t}\,dy=\frac{1}{2\,\mathrm{Re}}\int_{-1}^{1}\frac{\partial}{\partial y}\left(\frac{\partial u_1}{\partial y}\right)dy-\frac{v}{2}\int_{-1}^{1}\frac{\partial u_1}{\partial y}\,dy+\frac{M^2}{2}\int_{-1}^{1}u_1\,dy+\\ \qquad\qquad \frac{1}{2}\int_{-1}^{1}\theta_0\,dy+\frac{G_{r\phi}}{2}\int_{-1}^{1}\phi_1\,dy\end{array}\right\} \tag{3.122}$$

$$\left.\begin{array}{l}\frac{\partial\overline{u}_1}{\partial t}=-\frac{1}{4\,\mathrm{Re}}\int_{-1}^{1}u_1\big|_{y=1}\,dy-\frac{v}{2}\int_{-1}^{1}\left(\frac{1}{2}u_1\big|_{y=1}-\frac{1}{2}u_1\big|_{y=1}\,y\right)dy+\\ \quad\frac{M^2}{2}\int_{-1}^{1}\left(\frac{3}{4}u_1\big|_{y=1}+\frac{1}{2}u_1\big|_{y=1}\,y-\frac{1}{4}u_1\big|_{y=1}\,y^2\right)dy+\\ \quad\frac{1}{2}\int_{-1}^{1}\left(\begin{array}{l}\frac{1}{4}\Big(\big(4+3Nu\big)\theta_0\big|_{y=1}+q_1\Big)-\frac{1}{2}\Big(q_1+Nu\,\theta_0\big|_{y=1}\Big)y+\\ \frac{1}{4}\Big(q_1-Nu\,\theta_0\big|_{y=1}\Big)y^2\end{array}\right)dy+\\ \quad\frac{G_{r\phi}}{2}\int_{-1}^{1}\left(\frac{1}{4}\big(4+3Sh\big)\phi_1\big|_{y=1}-\frac{Sh}{2}\phi_1\big|_{y=1}\,y-\frac{Sh}{2}\phi_1\big|_{y=1}\,y^2\right)dy\end{array}\right\} \tag{3.123}$$

$$\left.\begin{array}{l}\frac{\partial\overline{u}_1}{\partial t}=-\frac{1}{2\,\mathrm{Re}}u_1\big|_{y=1}-\frac{v}{2}\Big(u_1\big|_{y=1}\Big)+\frac{M^2}{2}\left(\frac{3}{2}u_1\big|_{y=1}-\frac{1}{6}u_1\big|_{y=1}\right)+\\ \quad\frac{1}{2}\left(\frac{1}{2}\Big(\big(4+3Nu\big)\theta_0\big|_{y=1}+q_1\Big)+\frac{1}{6}\Big(q_1-Nu\,\theta_0\big|_{y=1}\Big)\right)+\\ \quad\frac{G_{r\phi}}{2}\left(\frac{1}{2}\big(4+3Sh\big)\phi_1\big|_{y=1}-\frac{Sh}{6}\phi_1\big|_{y=1}\right)\end{array}\right\} \tag{3.124}$$

$$\left.\begin{array}{l}\frac{2}{3}\frac{\partial u_1}{\partial t}\bigg|_{y=1}=\left(-\frac{1}{2\,\mathrm{Re}}-\frac{v}{2}+\frac{2M^2}{3}\right)u_1\big|_{y=1}+\left(\frac{12+8Nu}{12}\right)\theta_0\big|_{y=1}+\\ \qquad\qquad\qquad\qquad G_{r\phi}\left(\frac{12+8Sh}{12}\right)\phi_1\big|_{y=1}+\frac{q_1}{3}\end{array}\right\} \tag{3.125}$$

$$\frac{\partial u_1}{\partial t}\bigg|_{y=1}+p_3(t)\,u_1\big|_{y=1}=q_3(t) \tag{3.126}$$

where,

$$\left.\begin{aligned}&F=\frac{3}{2}\left(\frac{12+8Nu}{12}\right),\ F_1=G_{r\phi}\left(\frac{12+8Sh}{12}\right),\ G=\frac{q_1}{2},\\&p_3(t)=\frac{3}{2}\left(\frac{1}{2\,\mathrm{Re}}+\frac{v}{2}-\frac{2M^2}{3}\right),\ q_3(t)=F\,\theta_0\big|_{y=1}+F_1\,\phi_1\big|_{y=1}+G\end{aligned}\right\} \tag{3.127}$$

Also,

$$\frac{1}{2}\int_{-1}^{1}\frac{\partial\theta_1}{\partial t}\,dy=\frac{1}{2Pe}\int_{-1}^{1}\frac{\partial}{\partial y}\left(\frac{\partial\theta_1}{\partial y}\right)dy-\frac{v}{2}\int_{-1}^{1}\frac{\partial\theta_1}{\partial y}\,dy+\frac{Ec}{Re}\int_{-1}^{1}\left(\frac{\partial u_0}{\partial y}.\frac{\partial u_1}{\partial y}\right)dy \tag{3.128}$$

$$\left.\begin{aligned}\frac{\partial\overline{\theta}_1}{\partial t}&=\frac{1}{2Pe}\int_{-1}^{1}\left(-\frac{Nu}{2}\,\theta_1\big|_{y=1}\right)dy-\frac{v}{2}\int_{-1}^{1}\left(-\frac{Nu}{2}\,\theta_1\big|_{y=1}-\frac{Nu}{2}\,\theta_1\big|_{y=1}\,y\right)dy+\\&\quad\frac{Ec}{Re}\int_{-1}^{1}\left(\frac{1}{2}\left(u_0\big|_{y=1}-1\right)-\frac{1}{2}\left(u_0\big|_{y=1}-1\right)y\right).\left(\frac{1}{2}u_1\big|_{y=1}-\frac{1}{2}u_1\big|_{y=1}\,y\right)dy\end{aligned}\right\} \tag{3.129}$$

$$\left.\begin{aligned}\frac{\partial\overline{\theta}_1}{\partial t}&=\frac{1}{2Pe}\int_{-1}^{1}\left(-\frac{Nu}{2}\,\theta_1\big|_{y=1}\right)dy-\frac{v}{2}\int_{-1}^{1}\left(-\frac{Nu}{2}\,\theta_1\big|_{y=1}-\frac{Nu}{2}\,\theta_1\big|_{y=1}\,y\right)dy+\\&\quad\frac{Ec}{Re}\int_{-1}^{1}\frac{1}{4}\left(u_1\big|_{y=1}\left(u_0\big|_{y=1}-1\right)-2\,u_1\big|_{y=1}\left(u_0\big|_{y=1}-1\right)y+u_1\big|_{y=1}\left(u_0\big|_{y=1}-1\right)y^2\right)dy\end{aligned}\right\} \tag{3.130}$$

$$\left.\begin{aligned}\left(\frac{12+8Nu}{12}\right)\frac{\partial\theta_1}{\partial t}\bigg|_{y=1}&=-\frac{Nu}{2Pe}\,\theta_1\big|_{y=1}-\frac{v}{2}\left(-Nu\,\theta_1\big|_{y=1}\right)+\\&\quad\frac{Ec}{4Re}\left(2u_1\big|_{y=1}\left(u_0\big|_{y=1}-1\right)+\frac{2}{3}u_1\big|_{y=1}\left(u_0\big|_{y=1}-1\right)\right)\end{aligned}\right\} \tag{3.131}$$

$$\left(\frac{12+8Nu}{12}\right)\frac{\partial\theta_1}{\partial t}\bigg|_{y=1}=-\left(\frac{Nu}{2Pe}-\frac{vNu}{2}\right)\theta_1\big|_{y=1}+\frac{Ec}{4Re}\left(\frac{8}{3}u_1\big|_{y=1}\left(u_0\big|_{y=1}-1\right)\right) \tag{3.132}$$

$$\frac{\partial\theta_1}{\partial t}\bigg|_{y=1}+\left(\frac{12}{12+8Nu}\right)\left(\frac{(1-vPe)Nu}{2Pe}\right)\theta_1\big|_{y=1}=\frac{2Ec}{3Re}\left(\frac{12}{12+8Nu}\right)u_1\big|_{y=1}\left(u_0\big|_{y=1}-1\right) \tag{3.133}$$

$$\left.\frac{\partial \theta_1}{\partial t}\right|_{y=1} + p_4(t)\,\theta_1\big|_{y=1} = q_4(t) \tag{3.134}$$

where,

$$\left.\begin{aligned} &H = \frac{2Ec}{3Re}\left(\frac{12}{12+8Nu}\right), \quad p_4(t) = \left(\frac{12}{12+8Nu}\right)\left(\frac{(1-vPe)Nu}{2Pe}\right), \\ &q_4(t) = H\, u_1\big|_{y=1}\left(u_0\big|_{y=1} - 1\right) \end{aligned}\right\} \tag{3.135}$$

Also,

$$\frac{1}{2}\int_{-1}^{1} \frac{\partial \phi_1}{\partial t}\, dy = \frac{1}{2Pem}\int_{-1}^{1} \frac{\partial}{\partial y}\left(\frac{\partial \phi_1}{\partial y}\right) dy - \frac{v}{2}\int_{-1}^{1} \frac{\partial \phi_1}{\partial y}\, dy \tag{3.136}$$

$$\frac{\partial \overline{\phi}_1}{\partial t} = -\frac{Sh}{4Pem}\int_{-1}^{1} \phi_1\big|_{y=1}\, dy - \frac{v}{2}\int_{-1}^{1}\left(-\frac{Sh}{2}\,\phi_1\big|_{y=1} - \frac{Sh}{2}\,\phi_1\big|_{y=1}\, y\right) dy \tag{3.137}$$

$$\left(\frac{12+8Sh}{12}\right)\left.\frac{\partial \phi_1}{\partial t}\right|_{y=1} = -\frac{Sh}{2Pem}\,\phi_1\big|_{y=1} + \frac{vSh}{2}\,\phi_1\big|_{y=1} \tag{3.138}$$

$$\left(\frac{12+8Sh}{12}\right)\left.\frac{\partial \phi_1}{\partial t}\right|_{y=1} = -\left(\frac{Sh}{2Pem} - \frac{vSh}{2}\right)\phi_1\big|_{y=1} \tag{3.139}$$

$$\left.\frac{\partial \phi_1}{\partial t}\right|_{y=1} + \left(\frac{12}{12+8Sh}\right)\left(\frac{(1-vPem)Sh}{2Pem}\right)\phi_1\big|_{y=1} = 0 \tag{3.140}$$

$$\left.\frac{\partial \phi_1}{\partial t}\right|_{y=1} + p_5(t)\phi_1 = q_5(t) \tag{3.141}$$

where,

$$p_5(t)=\left(\frac{12}{12+8Sh}\right)\left(\frac{(1-vPem)Sh}{2Pem}\right),\ q_5(t)=0 \tag{3.142}$$

Consider equation (3.120). Then,

$$\phi_0\big|_{y=1}=e^{-p_2(t)t}\int_0^t q_2(x)\,e^{p_2(x)x}dx+f(y)e^{-p_2(t)t} \tag{3.143}$$

$$\phi_0\big|_{y=1}=e^{-p_2(t)t}\left(\frac{q_2(x)}{p_2(x)}\bigg|_0^t\right)+f(y)e^{-p_2(t)t} \tag{3.144}$$

$$\phi_0\big|_{y=1}=e^{-p_2(t)t}\left(\frac{q_2(t)}{p_2(t)}\left(e^{p_2(t)t}-1\right)\right)+f(y)e^{-p_2(t)t} \tag{3.145}$$

$$\phi_0\big|_{y=1}=\frac{q_2(t)}{p_2(t)}\left(1-e^{-p_2(t)t}\right)+f(y)e^{-p_2(t)t} \tag{3.146}$$

At,

$$t=0,\ \ f(y)=1 \tag{3.147}$$

$$\phi_0\big|_{y=1}=\frac{q_2(t)}{p_2(t)}+\left(1-\frac{q_2(t)}{p_2(t)}\right)e^{-p_2(t)t} \tag{3.148}$$

$$\phi_0\big|_{y=1}=r+r_1e^{-p_2(t)t} \tag{3.149}$$

where,

$$r=\frac{q_2(t)}{p_2(t)},\ r_1=\left(1-\frac{q_2(t)}{p_2(t)}\right) \tag{3.150}$$

Consider equation (3.105). Then,

$$u_0\big|_{y=1} = e^{-p(t)t}\int_0^t q(x)\,e^{p(x)x}dx + f(y)e^{-p(t)t} \tag{3.151}$$

$$u_0\big|_{y=1} = e^{-p(t)t}\int_0^t \left(\left(\mathrm{A}r + \mathrm{A}r_1 e^{-p_2(x)x}\right) + \mathrm{B} - Ce^{-\sigma x}\right)e^{p(x)x}dx + f(y)e^{-p(t)t} \tag{3.152}$$

$$u_0\big|_{y=1} = e^{-p(t)t}\int_0^t \left((\mathrm{A}r + \mathrm{B})e^{p(x)x} + \mathrm{A}r_1 e^{(p(x)-p_2(x))x} - Ce^{(p(x)-\sigma)x}\right)dx + f(y)e^{-p(t)t} \tag{3.153}$$

$$\left.\begin{aligned} u_0\big|_{y=1} = \frac{\mathrm{A}r + \mathrm{B}}{p(t)}\left(1 - e^{-p(t)t}\right) + \frac{\mathrm{A}r_1}{p(t) - p_2(t)}\left(e^{-p_2(t)t} - e^{-p(t)t}\right) - \\ \frac{C}{p(t) - \sigma}\left(e^{-\sigma t} - e^{-p(t)t}\right) + f(y)e^{-p(t)t} \end{aligned}\right\} \tag{3.154}$$

At,

$$t = 0, \quad f(y) = 0 \tag{3.155}$$

$$\left.\begin{aligned} u_0\big|_{y=1} = \frac{\mathrm{A}r + \mathrm{B}}{p(t)} - \left(\frac{\mathrm{A}r + \mathrm{B}}{p(t)} + \frac{\mathrm{A}r_1}{p(t) - p_2(t)} - \frac{C}{p(t) - \sigma}\right)e^{-p(t)t} + \\ \frac{\mathrm{A}r_1}{p(t) - p_2(t)}e^{-p_2(t)t} - \frac{C}{p(t) - \sigma}e^{-\sigma t} \end{aligned}\right\} \tag{3.156}$$

$$u_0\big|_{y=1} = r_2 - r_3 e^{-p(t)t} + r_4 e^{-p_2(t)t} - r_5 e^{-\sigma t} \tag{3.157}$$

where,

$$r_2 = \frac{\mathrm{A}r + \mathrm{B}}{p(t)}, \quad r_3 = \left(\frac{\mathrm{A}r + \mathrm{B}}{p(t)} + \frac{\mathrm{A}r_1}{p(t) - p_2(t)} - \frac{C}{p(t) - \sigma} \right), \quad r_4 = \frac{\mathrm{A}r_1}{p(t) - p_2(t)} \\ r_5 = \frac{C}{p(t) - \sigma} \tag{3.158}$$

Consider equation (3.113). Then,

$$\theta_0\big|_{y=1} = e^{-p_1(t)t} \int_0^t \left(D - \mathrm{E}\left(r_2 - r_3 e^{-p(x)x} + r_4 e^{-p_2(x)x} - r_5 e^{-\sigma x} \right) \right) e^{-p_1(x)x} dx + f(y) e^{-p_1(t)t} \tag{3.159}$$

$$\theta_0\big|_{y=1} = \frac{D - \mathrm{E}r_2}{p_1(t)} \left(1 - e^{-p_1(t)t} \right) + \frac{\mathrm{E}r_3}{p_1(t) - p(t)} \left(e^{-p(t)t} - e^{-p_1(t)t} \right) - \\ \frac{\mathrm{E}r_4}{p_1(t) - p_2(t)} \left(e^{-p_2(t)t} - e^{-p_1(t)t} \right) + \frac{\mathrm{E}r_5}{p_1(t) - \sigma} \left(e^{-\sigma t} - e^{-p_1(t)t} \right) + f(y) e^{-p_1(t)t} \tag{3.160}$$

At,

$$t = 0, \quad f(y) = 1 \tag{3.161}$$

$$\theta_0\big|_{y=1} = \frac{D - \mathrm{E}r_2}{p_1(t)} + \begin{pmatrix} 1 - \left(\frac{D - \mathrm{E}r_2}{p_1(t)} \right) - \frac{\mathrm{E}r_3}{p_1(t) - p(t)} + \frac{\mathrm{E}r_4}{p_1(t) - p_2(t)} - \\ \frac{\mathrm{E}r_5}{p_1(t) - \sigma} \end{pmatrix} e^{-p_1(t)t} + \\ \frac{\mathrm{E}r_3}{p_1(t) - p(t)} e^{-p(t)t} - \frac{\mathrm{E}r_4}{p_1(t) - p_2(t)} e^{-p_2(t)t} + \frac{\mathrm{E}r_5}{p_1(t) - \sigma} e^{-\sigma t} + \tag{3.162}$$

$$\theta_0\big|_{y=1} = r_6 + r_7 e^{-p_1(t)t} + r_8 e^{-p(t)t} - r_9 e^{-p_2(t)t} + r_{10} e^{-\sigma t} \tag{3.163}$$

where,

$$\left.\begin{aligned} &r_6 = \frac{D - \mathrm{E}r_2}{p_1(t)},\ r_7 = \left(1 - \left(\frac{D - \mathrm{E}r_2}{p_1(t)}\right) - \frac{\mathrm{E}r_3}{p_1(t) - p(t)} + \frac{\mathrm{E}r_4}{p_1(t) - p_2(t)} - \frac{\mathrm{E}r_5}{p_1(t) - \sigma}\right), \\ &r_8 = \frac{\mathrm{E}r_3}{p_1(t) - p(t)},\ r_9 = \frac{\mathrm{E}r_4}{p_1(t) - p_2(t)},\ r_{10} = \frac{\mathrm{E}r_5}{p_1(t) - \sigma} \end{aligned}\right\} \tag{3.164}$$

Consider equation (3.141). Then,

$$\phi_1\big|_{y=1} = e^{-p_5(t)t} \int_0^t 0.e^{p_5(x)x}\, dx + f(y) e^{-p_5(t)t} \tag{3.165}$$

$$\phi_1\big|_{y=1} = f(y) e^{-p_5(t)t} \tag{3.166}$$

At,

$$t = 0,\quad f(y) = 0 \tag{3.167}$$

$$\phi_1\big|_{y=1} = 0 \tag{3.168}$$

Consider equation (3.126). Then,

$$\left. u_1\big|_{y=1} = e^{-p_3(t)t} \int_0^t \left(F \left(\begin{aligned} &r_6 + r_7 e^{-p_1(x)x} + r_8 e^{-p(x)x} - \\ &r_9 e^{-p_2(x)x} + r_{10} e^{-\sigma x} \end{aligned} \right) + G \right) e^{p_3(x)x}\, dx + f(y) e^{-p_3(t)t} \right\} \tag{3.169}$$

$$\left. \begin{aligned} u_1\big|_{y=1} = &\frac{Fr_6 + G}{p_3(t)}\left(1 - e^{-p_3(t)t}\right) + \frac{Fr_7}{p_3(t) - p_1(t)}\left(e^{-p_1(t)t} - e^{-p_3(t)t}\right) + \\ &\frac{Fr_8}{p_3(t) - p(t)}\left(e^{-p(t)t} - e^{-p_3(t)t}\right) - \frac{Fr_9}{p_3(t) - p_2(t)}\left(e^{-p_2(t)t} - e^{-p_3(t)t}\right) + \\ &\frac{Fr_{10}}{p_3(t) - \sigma}\left(e^{-\sigma t} - e^{-p_3(t)t}\right) + f(y) e^{-p_3(t)t} \end{aligned} \right\} \tag{3.170}$$

At,

$$t = 0, \quad f(y) = 0 \tag{3.171}$$

$$\left. u_1 \right|_{y=1} = \frac{Fr_6 + G}{p_3(t)} + \left(\begin{array}{l} \dfrac{Fr_9}{p_3(t) - p_2(t)} - \dfrac{Fr_8}{p_3(t) - p(t)} - \dfrac{Fr_7}{p_3(t) - p_1(t)} - \\ \dfrac{Fr_{10}}{p_3(t) - \sigma} - \left(\dfrac{Fr_6 + G}{p_3(t)} \right) \end{array} \right) e^{-p_3(t)t} + \\ \frac{Fr_7}{p_3(t) - p_1(t)} e^{-p_1(t)t} + \frac{Fr_8}{p_3(t) - p(t)} e^{-p(t)t} - \frac{Fr_9}{p_3(t) - p_2(t)} e^{-p_2(t)t} + \\ \frac{Fr_{10}}{p_3(t) - \sigma} e^{-\sigma t} \tag{3.172}$$

$$\left. u_1 \right|_{y=1} = r_{11} + r_{12} e^{-p_3(t)t} + r_{13} e^{-p_1(t)t} + r_{14} e^{-p(t)t} - r_{15} e^{-p_2(t)t} + r_{16} e^{-\sigma t} \tag{3.173}$$

where,

$$r_{11} = \frac{Fr_6 + G}{p_3(t)}, \quad r_{12} = \left(\begin{array}{l} \dfrac{Fr_9}{p_3(t) - p_2(t)} - \dfrac{Fr_8}{p_3(t) - p(t)} - \dfrac{Fr_7}{p_3(t) - p_1(t)} - \\ \dfrac{Fr_{10}}{p_3(t) - \sigma} - \left(\dfrac{Fr_6 + G}{p_3(t)} \right) \end{array} \right), \\ r_{13} = \frac{Fr_7}{p_3(t) - p_1(t)}, \quad r_{14} = \frac{Fr_8}{p_3(t) - p(t)}, \quad r_{15} = \frac{Fr_9}{p_3(t) - p_2(t)}, \quad r_{16} = \frac{Fr_{10}}{p_3(t) - \sigma} \tag{3.174}$$

Consider equation (3.134). Then,

$$\theta_1\Big|_{y=1} = e^{-p_4(t)t}\int_0^t \begin{pmatrix} Hr_{11}\left(r_2-1\right)e^{p_4(x)x} + \\ Hr_{12}\left(r_2-1\right)e^{\left(p_4(x)-p_3(x)\right)x} + \\ Hr_{13}\left(r_2-1\right)e^{\left(p_4(x)-p_1(x)\right)x} + \\ H\left(r_{14}\left(r_2-1\right)-r_3r_{11}\right)e^{\left(p_4(x)-p(x)\right)x} - \\ H\left(r_{15}\left(r_2-1\right)+r_4r_{11}\right)e^{\left(p_4(x)-p_2(x)\right)x} + \\ H\left(r_{16}\left(r_2-1\right)-r_5r_{11}\right)e^{\left(p_4(x)-\sigma\right)x} - \\ Hr_3r_{12}e^{\left(p_4(x)-\left(p(x)+p_3(x)\right)\right)x} - \\ Hr_3r_{13}e^{\left(p_4(x)-\left(p(x)+p_1(x)\right)\right)x} - \\ Hr_3r_{14}e^{\left(p_4(x)-2p(x)\right)x} + \\ H\left(r_3r_{15}+r_4r_{14}\right)e^{\left(p_4(x)-\left(p(x)+p_2(x)\right)\right)x} - \\ H\left(r_3r_{16}+r_5r_{14}\right)e^{\left(p_4(x)-\left(p(x)+\sigma\right)\right)x} + \\ Hr_4r_{12}e^{\left(p_4(x)-\left(p_2(x)+p_3(x)\right)\right)x} + \\ H\left(r_4r_{16}+r_5r_{15}\right)e^{\left(p_4(x)-\left(p_2(x)+\sigma\right)\right)x} - \\ Hr_5r_{13}e^{\left(p_4(x)-\left(p_1(x)+\sigma\right)\right)x} - \\ Hr_5r_{12}e^{\left(p_4(x)-\left(p_3(x)+\sigma\right)\right)x} + \\ Hr_4r_{13}e^{\left(p_4(x)-\left(p_1(x)+p_2(x)\right)\right)x} - \\ Hr_4r_{15}e^{\left(p_4(x)-2p_2(x)\right)x} - Hr_5r_{16}e^{\left(p_4(x)-2\sigma\right)x} \end{pmatrix} dx + f(y)e^{-p_4(t)t} \tag{3.175}$$

$$\theta_1\big|_{y=1} = \left(\begin{array}{l} \frac{Hr_{11}\left(r_2-1\right)}{p_4(t)}\left(1-e^{-p_4(t)t}\right)+\frac{Hr_{12}\left(r_2-1\right)}{p_4(t)-p_3(t)}\left(e^{-p_3(t)t}-e^{-p_4(t)t}\right)+ \\ \frac{Hr_{13}\left(r_2-1\right)}{p_4(t)-p_1(t)}\left(e^{-p_1(t)t}-e^{-p_4(t)t}\right)+ \\ \frac{H\left(r_{14}\left(r_2-1\right)-r_3r_{11}\right)}{p_4(t)-p(t)}\left(e^{-p(t)t}-e^{-p_4(t)t}\right)- \\ \frac{H\left(r_{15}\left(r_2-1\right)+r_4r_{11}\right)}{p_4(t)-p_2(t)}\left(e^{-p_2(t)t}-e^{-p_4(t)t}\right)+ \\ \frac{H\left(r_{16}\left(r_2-1\right)-r_5r_{11}\right)}{p_4(t)-\sigma}\left(e^{-\sigma t}-e^{-p_4(t)t}\right)- \\ \frac{Hr_3r_{12}}{p_4(t)-\left(p(t)+p_3(t)\right)}\left(e^{-\left(p(t)+p_3(t)\right)t}-e^{-p_4(t)t}\right)- \\ \frac{Hr_3r_{13}}{p_4(t)-\left(p(t)+p_1(t)\right)}\left(e^{-\left(p(t)+p_1(t)\right)t}-e^{-p_4(t)t}\right)- \\ \frac{Hr_3r_{14}}{p_4(t)-2p(t)}\left(e^{-2p(t)t}-e^{-p_4(t)t}\right)+ \\ \frac{H\left(r_3r_{15}+r_4r_{14}\right)}{p_4(t)-\left(p(t)+p_2(t)\right)}\left(e^{-\left(p(t)+p_2(t)\right)t}-e^{-p_4(t)t}\right)- \\ \frac{H\left(r_3r_{16}+r_5r_{14}\right)}{p_4(t)-\left(p(t)+\sigma\right)}\left(e^{-\left(p(t)+\sigma\right)t}-e^{-p_4(t)t}\right)+ \\ \frac{Hr_4r_{12}}{p_4(t)-\left(p_2(t)+p_3(t)\right)}\left(e^{-\left(p_2(t)+p_3(t)\right)t}-e^{-p_4(t)t}\right)+ \\ \frac{H\left(r_4r_{16}+r_5r_{15}\right)}{p_4(t)-\left(p_2(t)+\sigma\right)}\left(e^{-\left(p_2(t)+\sigma\right)t}-e^{-p_4(t)t}\right)- \\ \frac{Hr_5r_{13}}{p_4(t)-\left(p_1(t)+\sigma\right)}\left(e^{-\left(p_1(t)+\sigma\right)t}-e^{-p_4(t)t}\right)- \\ \frac{Hr_5r_{12}}{p_4(t)-\left(p_3(t)+\sigma\right)}\left(e^{-\left(p_3(t)+\sigma\right)t}-e^{-p_4(t)t}\right)+ \\ \frac{Hr_4r_{13}}{p_4(t)-\left(p_1(t)+p_2(t)\right)}\left(e^{-\left(p_1(t)+p_2(t)\right)t}-e^{-p_4(t)t}\right)- \\ \frac{Hr_4r_{15}}{p_4(t)-2p_2(t)}\left(e^{-2p_2(t)t}-e^{-p_4(t)t}\right)-\frac{Hr_5r_{16}}{p_4(t)-2\sigma}\left(e^{-2\sigma t}-e^{-p_4(t)t}\right) \end{array} \right) \tag{3.176}$$

The solution for dimensionless equations (3.19) – (3.21) are shown in (3.177)

$$\left.\begin{aligned}
u_0(y,t) &= \frac{1}{4}\left(3u_0\big|_{y=1}+1\right)+\frac{1}{2}\left(u_0\big|_{y=1}-1\right)y-\frac{1}{4}\left(u_0\big|_{y=1}-1\right)y^2\\
\theta_0(y,t) &= \frac{1}{4}\left((4+3Nu)\,\theta_0\big|_{y=1}+q_1\right)-\frac{1}{2}\left(q_1+Nu\,\theta_0\big|_{y=1}\right)y+\frac{1}{4}\left(q_1-Nu\,\theta_0\big|_{y=1}\right)y^2\\
\phi_0(y,t) &= \frac{1}{4}\left((4+3Sh)\,\phi_0\big|_{y=1}+q_2\right)-\frac{1}{2}\left(q_2+Sh\,\phi_0\big|_{y=1}\right)y+\frac{1}{4}\left(q_2-Sh\,\phi_0\big|_{y=1}\right)y^2\\
u_1(y,t) &= \frac{3}{4}u_1\big|_{y=1}+\frac{1}{2}u_1\big|_{y=1}\,y-\frac{1}{4}u_1\big|_{y=1}\,y^2\\
\theta_1(y,t) &= \frac{1}{4}\left(4+3Nu\right)\theta_1\big|_{y=1}-\frac{Nu}{2}\theta_1\big|_{y=1}\,y-\frac{Nu}{4}\theta_1\big|_{y=1}\,y^2\\
\phi_1(y,t) &= \frac{1}{4}\left(4+3Sh\right)\phi_1\big|_{y=1}-\frac{Sh}{2}\phi_1\big|_{y=1}\,y-\frac{Sh}{4}\phi_1\big|_{y=1}\,y^2
\end{aligned}\right\} \tag{3.177}$$

where,

$$u_0\big|_{y=1} = r_2 - r_3e^{-p(t)t} + r_4e^{-p_2(t)t} - r_5e^{-\sigma t} \tag{3.178}$$

$$\theta_0\big|_{y=1} = r_6 + r_7e^{-p_1(t)t} + r_8e^{-p(t)t} - r_9e^{-p_2(t)t} + r_{10}e^{-\sigma t} \tag{3.179}$$

$$\phi_0\big|_{y=1} = r + r_1e^{-p_2(t)t} \tag{3.180}$$

$$u_1\big|_{y=1} = r_{11} + r_{12}e^{-p_3(t)t} + r_{13}e^{-p_1(t)t} + r_{14}e^{-p(t)t} - r_{15}e^{-p_2(t)t} + r_{16}e^{-\sigma t} \tag{3.181}$$

$$
\theta_1\big|_{y=1} = \left(\begin{array}{l}
\frac{Hr_{11}\left(r_2-1\right)}{p_4(t)}\left(1-e^{-p_4(t)t}\right)+\frac{Hr_{12}\left(r_2-1\right)}{p_4(t)-p_3(t)}\left(e^{-p_3(t)t}-e^{-p_4(t)t}\right)+ \\
\frac{Hr_{13}\left(r_2-1\right)}{p_4(t)-p_1(t)}\left(e^{-p_1(t)t}-e^{-p_4(t)t}\right)+ \\
\frac{H\left(r_{14}\left(r_2-1\right)-r_3r_{11}\right)}{p_4(t)-p(t)}\left(e^{-p(t)t}-e^{-p_4(t)t}\right)- \\
\frac{H\left(r_{15}\left(r_2-1\right)+r_4r_{11}\right)}{p_4(t)-p_2(t)}\left(e^{-p_2(t)t}-e^{-p_4(t)t}\right)+ \\
\frac{H\left(r_{16}\left(r_2-1\right)-r_5r_{11}\right)}{p_4(t)-\sigma}\left(e^{-\sigma t}-e^{-p_4(t)t}\right)- \\
\frac{Hr_3r_{12}}{p_4(t)-\left(p(t)+p_3(t)\right)}\left(e^{-\left(p(t)+p_3(t)\right)t}-e^{-p_4(t)t}\right)- \\
\frac{Hr_3r_{13}}{p_4(t)-\left(p(t)+p_1(t)\right)}\left(e^{-\left(p(t)+p_1(t)\right)t}-e^{-p_4(t)t}\right)- \\
\quad\frac{Hr_3r_{14}}{p_4(t)-2p(t)}\left(e^{-2p(t)t}-e^{-p_4(t)t}\right)+ \\
\quad\frac{H\left(r_3r_{15}+r_4r_{14}\right)}{p_4(t)-\left(p(t)+p_2(t)\right)}\left(e^{-\left(p(t)+p_2(t)\right)t}-e^{-p_4(t)t}\right)- \\
\quad\frac{H\left(r_3r_{16}+r_5r_{14}\right)}{p_4(t)-\left(p(t)+\sigma\right)}\left(e^{-\left(p(t)+\sigma\right)t}-e^{-p_4(t)t}\right)+ \\
\quad\frac{Hr_4r_{12}}{p_4(t)-\left(p_2(t)+p_3(t)\right)}\left(e^{-\left(p_2(t)+p_3(t)\right)t}-e^{-p_4(t)t}\right)+ \\
\quad\frac{H\left(r_4r_{16}+r_5r_{15}\right)}{p_4(t)-\left(p_2(t)+\sigma\right)}\left(e^{-\left(p_2(t)+\sigma\right)t}-e^{-p_4(t)t}\right)- \\
\quad\frac{Hr_5r_{13}}{p_4(t)-\left(p_1(t)+\sigma\right)}\left(e^{-\left(p_1(t)+\sigma\right)t}-e^{-p_4(t)t}\right)- \\
\quad\frac{Hr_5r_{12}}{p_4(t)-\left(p_3(t)+\sigma\right)}\left(e^{-\left(p_3(t)+\sigma\right)t}-e^{-p_4(t)t}\right)+ \\
\quad\frac{Hr_4r_{13}}{p_4(t)-\left(p_1(t)+p_2(t)\right)}\left(e^{-\left(p_1(t)+p_2(t)\right)t}-e^{-p_4(t)t}\right)- \\
\quad\frac{Hr_4r_{15}}{p_4(t)-2p_2(t)}\left(e^{-2p_2(t)t}-e^{-p_4(t)t}\right)-\frac{Hr_5r_{16}}{p_4(t)-2\sigma}\left(e^{-2\sigma t}-e^{-p_4(t)t}\right)
\end{array}\right) \tag{3.182}
$$

$$
\phi_1\big|_{y=1} = 0 \tag{3.183}
$$

The computations were done using Maple 17 version and the graphs generated were shown and discussed in Chapter four.

CHAPTER FOUR

4.0 RESULTS AND DISCUSSION

4.1 Results

In this analysis we examine the effects of Reynolds number $\left(\mathrm{R}_e\right)$, Solutal Grashof number $\left(G_{r\phi}\right)$, Thermal Grashof number $\left(G_{r\theta}\right)$, Peclet mass number $\left(P_{em}\right)$, Nusselt number $\left(Nu\right)$, Peclet energy number $\left(P_e\right)$, Kinematic viscosity number $\left(v\right)$, Eckert number $\left(E_c\right)$, Sherwood number $\left(Sh\right)$ on the velocity $u\left(y,t\right)$ of the fluid, temperature of the fluid $\theta\left(y,t\right)$ and concentration of the fluid $\phi\left(y,t\right)$. Analytical solutions given by equations (3.177), (3.178), (3.179), (3.180), (3.181), (3.182) and (3.183) were computed using computer symbolic algebraic package MAPLE 17.

The results obtained from the solutions are shown in Figure 4.1 to 4.16. The effect of Reynolds number $\left(\mathrm{R}_e\right)$ on velocity $u\left(y,t\right)$ against distance y is depicted in figure 4.1. The effect of Reynolds number $\left(\mathrm{R}_e\right)$ on velocity $u\left(y,t\right)$ against time t is depicted in figure 4.2. The effect of Solutal Grashof number $\left(G_{r\phi}\right)$ on velocity $u\left(y,t\right)$ against distance y is depicted in figure 4.3. The effect of Solutal Grashof number $\left(G_{r\phi}\right)$ on velocity $u\left(y,t\right)$ against time t is depicted in figure 4.4. The effect of Thermal Grashof number $\left(G_{r\theta}\right)$ on velocity $u\left(y,t\right)$ against distance y is depicted in figure 4.5. The effect of Thermal Grashof number $\left(G_{r\theta}\right)$ on velocity $u\left(y,t\right)$ against time t is depicted in figure 4.6.

The effect of Peclet mass number $\left(P_{em}\right)$ on concentration $\phi(y,t)$ against distance y is depicted in figure 4.7. The effect of Peclet mass number $\left(P_{em}\right)$ on concentration $\phi(y,t)$ against time t is depicted in figure 4.8. The effect of Thermal Grashof number $\left(G_{r\theta}\right)$ on temperature $\theta(y,t)$ against distance y is depicted in figure 4.9. The effect of Peclet energy number $\left(P_e\right)$ on temperature $\theta(y,t)$ against distance y is depicted in figure 4.10. The effect of Kinematic viscosity number (v) on velocity $u(y,t)$ against distance y is depicted in figure 4.11. The effect of Kinematic viscosity number (v) on velocity $u(y,t)$ against time t is depicted in figure 4.12. The effect of Kinematic viscosity number (v) on concentration $\phi(y,t)$ against distance y is depicted in figure 4.13. The effect of Kinematic viscosity number (v) on concentration $\phi(y,t)$ against time t is depicted in figure 4.14. The effect of Eckert number $\left(E_c\right)$ on temperature $\theta(y,t)$ against distance y is depicted in figure 4.15. The effect of Sherwood number (Sh) on concentration $\phi(y,t)$ against time t is depicted in figure 4.16.

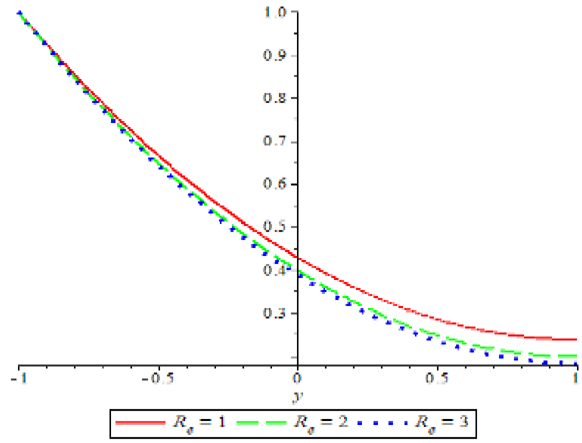

Figure 4.1: Effect of Reynolds Number (R_e) on Velocity $u(y,t)$ against Distance y

It is observed that the velocity of the fluid decreases along distance as Reynolds number increases.

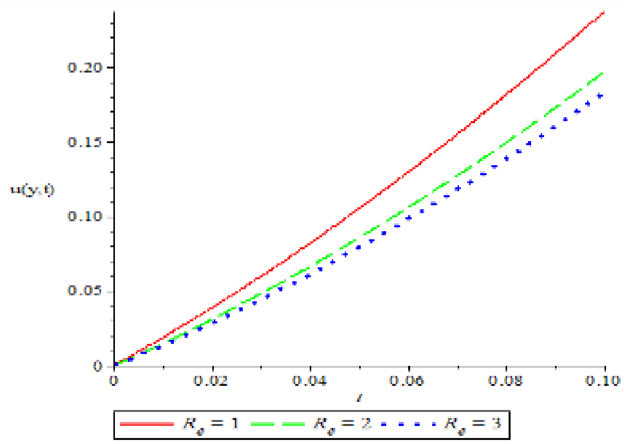

Figure 4.2: Effect of Reynolds Number (R_e) on Velocity $u(y,t)$ against Time t

It is observed that the velocity of the fluid increases with time as Reynolds number increases.

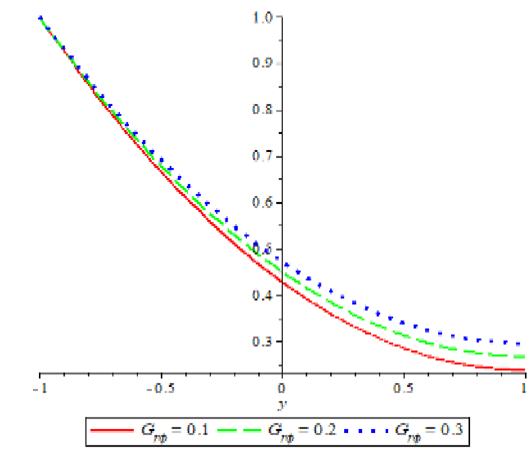

Figure 4.3: Effect of Solutal Grashof Number $(G_{r\phi})$ on Velocity $u(y,t)$ against Distance

It is observed that the velocity of the fluid decreases along distance as Solutal Grashof number increases.

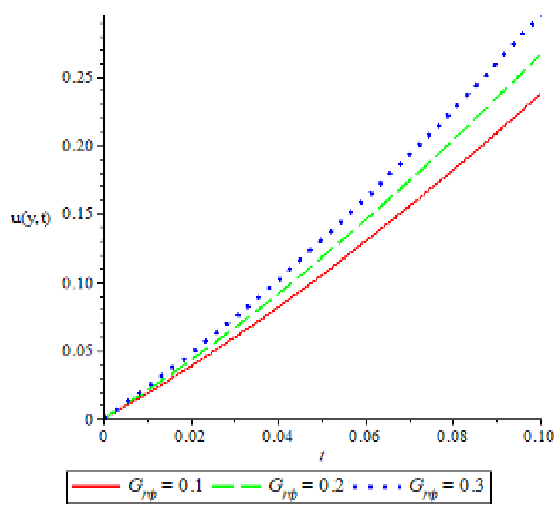

Figure 4.4: Effect of Solutal Grashof Number $(G_{r\phi})$ on Velocity $u(y,t)$ against Time t

It is observed that the velocity of the fluid increases with time as Solutal Grashof number increases.

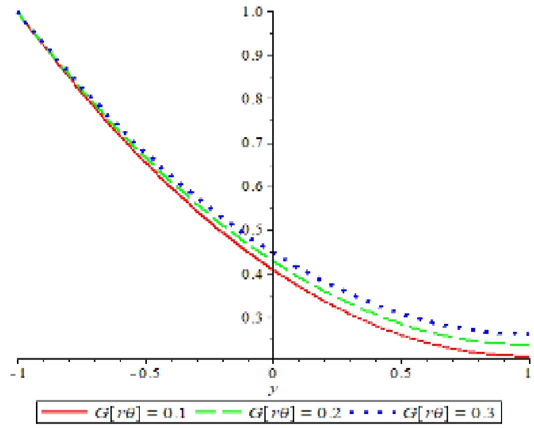

Figure 4.5: Effect of Thermal Grashof Number $(G_{r\theta})$ on Velocity $u(y,t)$ against Distance

It is observed that the velocity of the fluid decreases along distance as Thermal Grashof number increases.

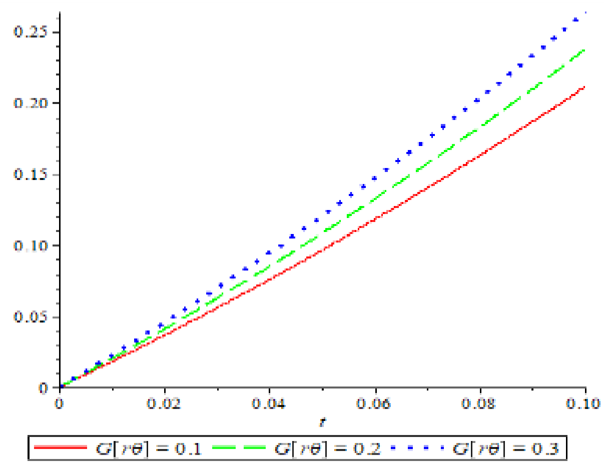

Figure 4.6: Effect of Thermal Grashof Number $(G_{r\theta})$ on Velocity $u(y,t)$ against Time t

It is observed that the velocity of the fluid increases with time as Thermal Grashof number increases.

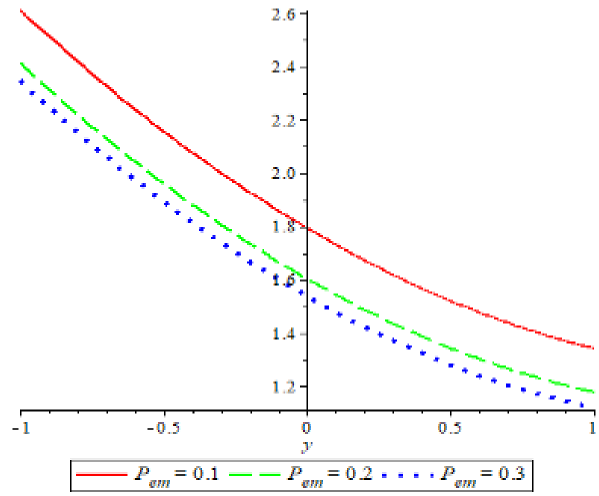

Figure 4.7: Effect of Peclet Mass Number (P_{em}) on Concentration $\phi(y,t)$ against Distance

It is observed that the concentration of the fluid decreases along distance as Peclet mass number increases.

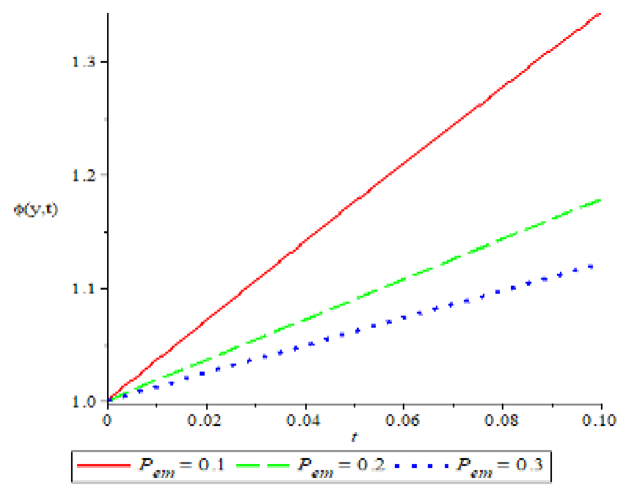

Figure 4.8: Effect of Peclet Mass Number (P_{em}) on Concentration $\phi(y,t)$ against Time t

It is observed that the concentration of the fluid increases with time as Peclet mass number increases.

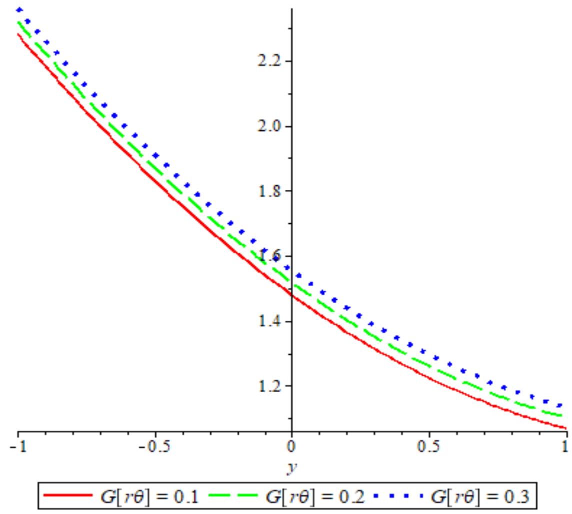

Figure 4.9: Effect of Thermal Grashof Number $(G_{r\theta})$ on Temperature $\theta(y,t)$ against Distance y

It is observed that the temperature of the fluid decreases along distance as Thermal Grashof number increases.

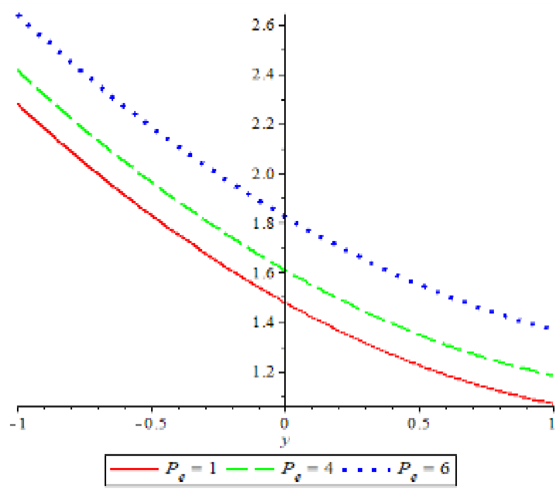

Figure 4.10: Effect of Peclet Energy Number (P_e) on Temperature $\theta(y,t)$ against Distance y

It is observed that the temperature of the fluid decreases along distance as Peclet energy number increases.

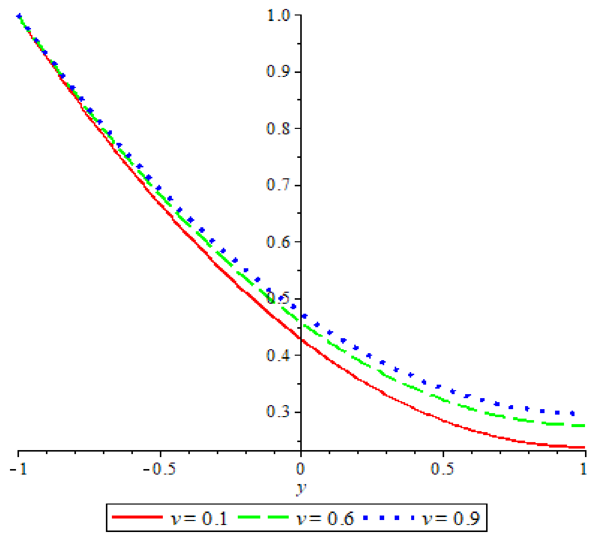

Figure 4.11: Effect of Kinematic Viscosity Number (v) on Velocity $u(y,t)$ against Distance y

It is observed that the velocity of the fluid decreases along distance as Kinematic viscosity number increases.

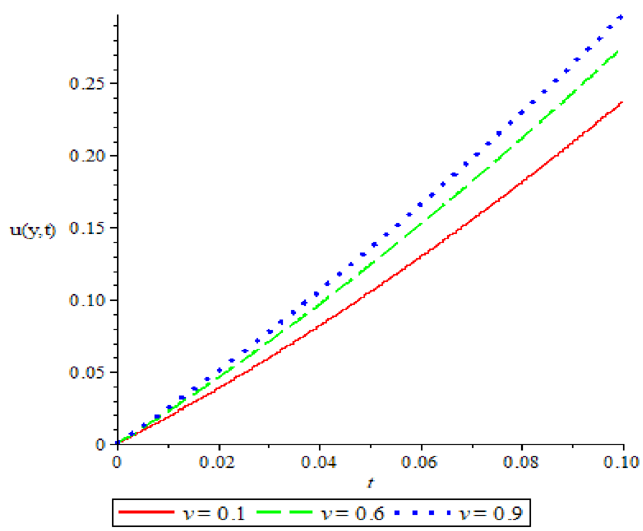

Figure 4.12: Effect of Kinematic Viscosity Number (v) on Velocity $u(y,t)$ against Time t

It is observed that the velocity of the fluid increases with time as Kinematic viscosity number increases.

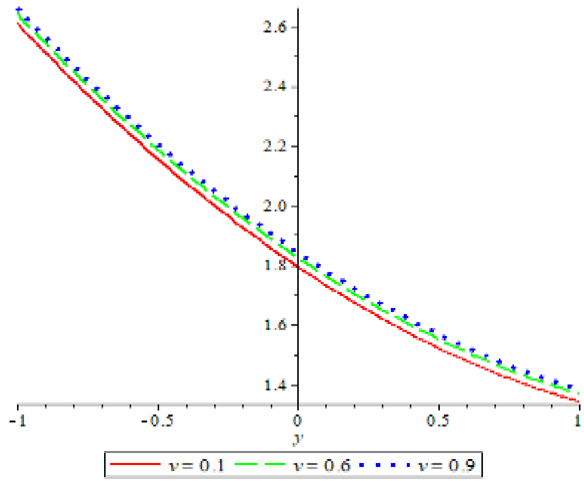

Figure 4.13: Effect of Kinematic Viscosity Number (v) on Concentration $\phi(y,t)$ against Distance y

It is observed that the concentration of the fluid decreases along distance as Kinematic viscosity number increases.

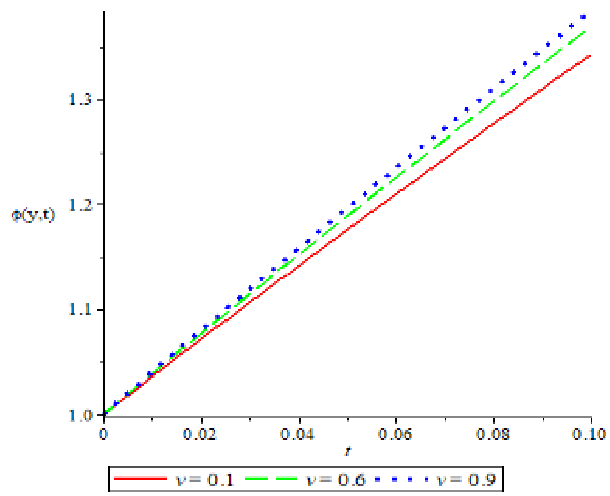

Figure 4.14: Effect of Kinematic Viscosity Number (v) on Concentration $\phi(y,t)$ against Time t

It is observed that the concentration of the fluid increases with time as Kinematic viscosity number increases.

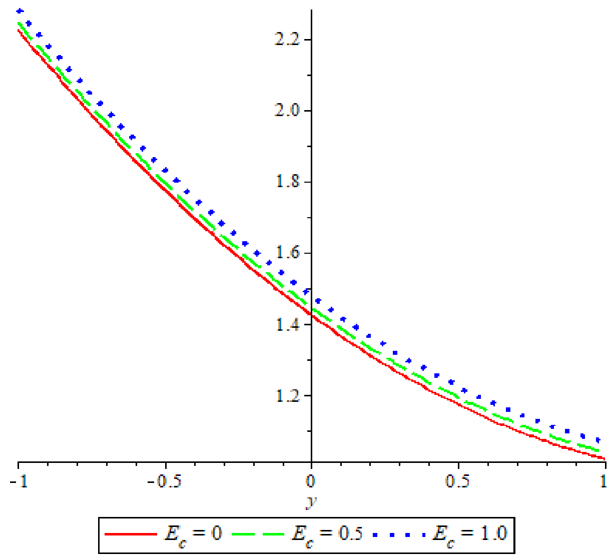

Figure 4.15: Effect of Eckert Number (E_c) on Temperature $\theta(y,t)$ against Distance y

It is observed that the temperature of the fluid decreases along distance as Eckert number increases.

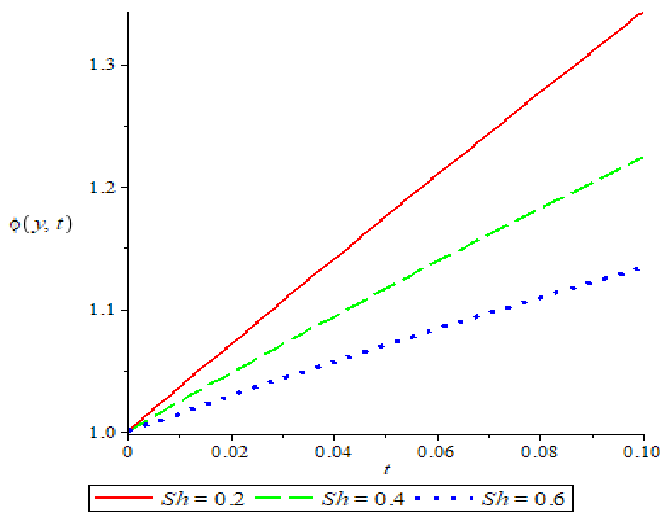

Figure 4.16: Effect of Sherwood Number (Sh) on Concentration $\phi(y,t)$ against Time t

It is observed that the concentration of the fluid increases with time as Sherwood number increases.

4.2 Discussion of the Results

Figure 4.1: shows the effect of Reynolds number $\left(\mathrm{R}_e\right)$ on velocity $u(y,t)$ against distance. It is observed that the velocity of the fluid decreases along distance and this velocity decreases as Reynolds number increases.

Figure 4.2: depicts the effect of Reynolds number $\left(\mathrm{R}_e\right)$ on velocity $u(y,t)$ against time. It is observed that the velocity of the fluid increases with time and this velocity decreases as Reynolds number increases.

Figure 4.3: displays the effect of Solutal Grashof number $\left(G_{r\phi}\right)$ on velocity $u(y,t)$ against distance. It is observed that the velocity of the fluid decreases along distance and this velocity increases as Solutal Grashof number increases.

Figure 4.4: depicts the effect of Solutal Grashof number $\left(G_{r\phi}\right)$ on velocity $u(y,t)$ against time**.** It is observed that the velocity of the fluid increases with time and this velocity increases as Solutal Grashof number increases.

Figure 4.5: presents the effect of Thermal Grashof number $\left(G_{r\theta}\right)$ on velocity $u(y,t)$ against distance. It is observed that the velocity of the fluid decreases along distance and this velocity increases as Thermal Grashof number increases.

Figure 4.6: shows the effect of Thermal Grashof number $\left(G_{r\theta}\right)$ on velocity $u(y,t)$ against time. It is observed that the velocity of the fluid increases with time and this velocity increases as Thermal Grashof number increases.

Figure 4.7: displays the effect of peclet mass number $\left(P_{em}\right)$ on concentration $\phi(y,t)$ against distance. It is observed that the concentration of the fluid decreases along distance and this concentration decreases as Peclet mass number increases.

Figure 4.8: presents the effect of peclet mass number $\left(P_{em}\right)$ on concentration $\phi\left(y,t\right)$ against time. It is observed that the concentration of the fluid increases with time and this concentration decreases as Peclet mass number increases.

Figure 4.9: shows the effect of Thermal Grashof number $\left(G_{r\theta}\right)$ on temperature $\theta\left(y,t\right)$ against distance. It is observed that the temperature of the fluid decreases along distance and this temperature increases as Thermal Grashof number $\left(G_{r\theta}\right)$ increases.

Figure 4.10: shows the effect of Peclet energy number $\left(P_{e}\right)$ on temperature $\theta\left(y,t\right)$ against distance. It is observed that the temperature of the fluid decreases along distance and this temperature increases as Peclet energy number increases.

Figure 4.11: shows the effect of Kinematic viscosity number $\left(v\right)$ on velocity $u\left(y,t\right)$ against distance. It is observed that the velocity of the fluid decreases along distance and this velocity increases as Kinematic viscosity number increases.

Figure 4.12: presents the effect of Kinematic viscosity number $\left(v\right)$ on velocity $u\left(y,t\right)$ against time. It is observed that the velocity of the fluid increases with time and this velocity increases as Kinematic viscosity number increases.

Figure 4.13: depicts the effect of Kinematic viscosity number $\left(v\right)$ on concentration $\phi\left(y,t\right)$ against distance. It is observed that the concentration of the fluid decreases along distance and this concentration increases as Kinematic viscosity number increases.

Figure 4.14: depicts the effect of Kinematic viscosity number (v) on concentration $\phi(y,t)$ against time. It is observed that the concentration of the fluid increases with time and this concentration increases as Kinematic viscosity number increases.

Figure 4.15: shows the effect of Eckert number (E_c) on temperature $\theta(y,t)$ against distance. It is observed that the temperature of the fluid decreases along distance and this temperature increases as Eckert number increases.

Figure 4.16: depicts the effect of Sherwood number (Sh) on concentration $\phi(y,t)$ against time. It is observed that the concentration of the fluid increases with time and this concentration decreases as Sherwood number increases.

CHAPTER FIVE

5.0 CONCLUSION AND RECOMENDATION

5.1 Conclusion

A mathematical analysis has been carried out to model magnetohydrodynamics flow of incompressible fluid through parallel plates in inclined magnetic field in the presence of viscous dissipation energy. The dimensionless governing coupled non-linear partial differential equations were solved analytically using polynomial approximation method. The effects of the dimensionless parameters as shown on the graphs were analyzed. From the results obtained, we can conclude that:

(i) Reynolds number reduces the velocity of the fluid.

(ii) Solutal Grashof number, Thermal Grashof number and Kinematic viscosity number enhance the velocity of the fluid.

(iii) Peclet mass number and Sherwood number reduce the concentration of the fluid.

(iv) Thermal Grashof number, Peclet energy number and Eckert number enhance the temperature of the fluid.

5.2 Recommendations

Further work can be carried out on magnetohydrodynamics flow of incompressible fluid through parallel plates in inclined magnetic field in the presence of viscous dissipation energy using other analytical methods to ascertain how best the results can be obtained.

5.3 Contributions to Knowledge

In this study, the following contributions were made:

(i) This research work extended the work of Hanvey *et al.* (2017) by incorporating viscous energy dissipation term in the heat process and also introduced concentration equation to the set of model equations.

(ii) Magnetohydrodynamics flow of incompressible fluid through parallel plates in inclined magnetic field in the presence of viscous dissipation energy was solved using polynomial approximation method.

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