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A 2-STEP HYBRID BLOCK BACKWARD DIFFERENTIATION FORMULA FOR THE APPROXIMATION OF INITIAL VALUE PROBLEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT
In this paper, a two-step hybrid backward differentiation formula, with two off-grid points incorporated at interpolation for the solution of initial value problems of ordinary differential equations is presented. The main method as well as the additional methods are obtained from the same continuous scheme formulated using interpolation and collocation methods with Legendre polynomial as the basis function. The method is seen to be of uniform order k=4. The stability of the proposed method is analyzed and discussed. Numerical experiments were carried out to ascertain the effectiveness of the method in comparison with the approximation produced by some existing methods.

Keywords: *BDF, Continuous Collocation, IVP, Legendre polynomial*.

1 INTRODUCTION

Wherever or whenever real-life problem is discussed, ordinary differential equations as a means of modelling the behavior of such reality cannot be overemphasized. Initial value problem of ordinary differential equations of the form

$$\frac{dy}{dx} = f(x), \quad y(x_0) = y_0 \tag{1}$$

occurs frequently in real life including but not limited to population dynamics, finance, engineering, to mention but a few. Over the years, several numerical methods had been adopted for the approximation of (1). One step methods (Euler methods, Runge-Kutta methods, …), Linear multistep methods (Adams Bashforth, Adams Moulton,…) (see Burden and Faires, ninth edition) have been found to be useful in providing approximation to problem (1). But the best is yet to come and this is the reason researchers over the years have not left any stone unturned in finding a better method, one that will be close enough to the exact solutions of some existing problems, so that minds can be at rest when addressing problems to which theoretical solution do not exist. According to Imanova and Ibrahimovic, (2013), Hybrid methods were introduced in the middle twentieth century which involve joining some one step and multistep methods while preserving their best properties. Advantages of hybrid methods include high accuracy as well as extended domain of stability. Ibijola et al (2012), Kumleng (2012), Yakusak and Adeniyi (2015), Olabode (2009) are some of the researchers who have developed hybrid methods for the solution of (1) with great success.

In this paper, we develop a two-step Hybrid Backward Differentiation Formula (2SHBDF) using interpolation and collocation techniques, incorporating two off-grid points at interpolation.

2 METHODOLOGY

It is assumed that the solution of (1) can be approximated by a polynomial of the form,

$$y(x) = \sum_{j=0}^{i+c-1} \alpha_j p_j(x), \tag{2}$$

where i and c are respectively, number of interpolation and collocation points and $\alpha_j's$ are coefficient to be determined. $p_j(x)$ can be any orthogonal polynomial. In this case, Legendre polynomial is used since, on inspection, it produces exactly the same continuous form as the popularly adopted power series. Incorporating k off-grid points for every k-step method here, requires that $i + c$ equations must be satisfied. That is,

$$\left.\begin{aligned} &y(x_n) = y_n, \\ &y(x_{n+j}) = y_{n+j}, j = 0, \left(\tfrac{1}{2}\right), 1, \dots, k - \tfrac{1}{2}, \\ &f(x_{n+k}) = f_{n+k}, \end{aligned}\right\} \tag{3}$$

where f implies the derivatives of y.

$k = 2$, $x \in [x_n, x_{n+2}]$, the continuous form of the method is as follows;

$$\varphi = \sum_{j=0}^{\frac{3}{2}} \alpha_j y_{n+j} + \beta_2 f_{n+2} \quad (4)$$

Which when evaluated in line with (3.1) produces the system of equations

$$Y_\omega = D\Psi_{\omega-n} \quad (5)$$

$$Y_\omega = (y_n, y_{n+\frac{1}{2}}, y_{n+1}, y_{n+\frac{3}{2}}, f_{n+2})^T,$$

$\Psi_\omega = (\alpha_0, \alpha_{\frac{1}{2}}, \alpha_1, \alpha_{\frac{3}{2}}, \beta_2)^T$ and D is a 5×5 matrix.

Using matrix inversion technique with the aid of maple software, the values of the unknown coefficients $\alpha_j's$ and β_2 were obtained and substituted appropriately to give the continuous form

$$\varphi = \alpha_0 y_n + \alpha_{\frac{1}{2}} y_{n+\frac{1}{2}} + \alpha_1 y_{n+1} + \alpha_{\frac{3}{2}} y_{n+\frac{3}{2}} + h\beta_2 f_{n+2} \quad (6)$$

Evaluating (3.13) at $x = x_n + 2h$ resulted in the main method

$$y_{n+2} = -\frac{3}{25} y_n + \frac{16}{25} y_{n+\frac{1}{2}} - \frac{36}{25} y_{n+1} + \frac{48}{25} y_{n+\frac{3}{2}} + \frac{6}{25} h f_{n+2} \quad (7)$$

To obtain the additional schemes that combine with the main method to form a block, the first derivative of (6) is obtained and evaluated at $x = x_{n+\frac{1}{2}}$, $x = x_{n+1}$ and $x = x_{n+\frac{3}{2}}$ which produced three other discrete schemes. Hence, the 2-step block Hybrid Differentiation Formula with 2 off-grid points is given as

$$\left.\begin{aligned}
f_{n+\frac{1}{2}} &= \frac{1}{25h}\begin{bmatrix} hf_{n+2} - 13y_n - 39y_{n+\frac{1}{2}} \\ +69y_{n+1} - 17y_{n+\frac{3}{2}} \end{bmatrix} \\
f_{n+1} &= -\frac{1}{75h}\begin{bmatrix} 3hf_{n+2} - 14y_n + 108y_{n+\frac{1}{2}} \\ -18y_{n+1} - 76y_{n+\frac{3}{2}} \end{bmatrix} \\
f_{n+\frac{3}{2}} &= \frac{1}{75h}\begin{bmatrix} 9hf_{n+2} - 17y_n + 99y_{n+\frac{1}{2}} \\ -279y_{n+1} + 197y_{n+\frac{3}{2}} \end{bmatrix} \\
y_{n+2} &= -\frac{3}{25} y_n + \frac{16}{25} y_{n+\frac{1}{2}} - \frac{36}{25} y_{n+1} \\
&\quad + \frac{48}{25} y_{n+\frac{3}{2}} + \frac{6}{25} h f_{n+2}
\end{aligned}\right\} \quad (8)$$

2.1 ANALYSIS OF THE METHOD

2.11 ORDER OF ACCURACY OF THE METHOD

In line with Endre (2014), let $y(x_{n+j})$, the solution to $y'(x_{n+j})$ be sufficiently differentiable, then $y(x_{n+j})$ and $y'(x_{n+j})$ can be expanded into a Taylor's series about point x_n to obtain

$$T_n = \frac{1}{h\sigma(1)}[C_0 y(x_n) + C_1 h y'(x_n) + C_2 h^2 y''(x_n) + \cdots] \quad (9)$$

Where

$$C_0 = \sum_{j=0}^{k} \alpha_j,$$

$$C_1 = \sum_{j=0}^{k} j\alpha_j - \sum_{j=0}^{k} \beta_j,$$

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$$C_q = \frac{1}{q!}\sum_{j=0}^{k} j^q \alpha_j - \frac{1}{(q-1)!}\sum_{j=0}^{k} j^{q-1}\beta_j,$$

Definition 1: A Linear multistep method is said to be of order of accuracy p if $C_0 = C_1 = . . . C_p = 0$, $C_{p+1} \neq 0$. The error constant being C_{p+1}.

In accordance with the definition 1 above, our method is of uniform order 4 with error constants in the table below.

METHOD	POINT OF EVALUATION	ORDER, P	ERROR CONSTANT, C_{p+1}
(8)	$x = x_{n+1/2}$	4	$-\frac{29}{320}$
	$x = x_{n+1}$	4	$-\frac{31}{160}$
	$x = x_{n+3/2}$	4	$-\frac{111}{320}$
	$x = x_{n+2}$	4	$-\frac{3}{40}$

2.12 CONSISTENCY OF THE METHOD

Definition 2: A linear multistep method is said to be consistent if

i. the order of accuracy $p > 1$,
ii. $\sum_{j=0}^{k} \alpha_j = 0$,
iii. $\rho'(1) = \sigma(1)$ where $p(r)$ and $\sigma(r)$ are first and second characteristic polynomials of the method.

Conditions i and ii were taken care of in 2.11 since the order $p > 1$ and $C_0 = \sum_{j=0}^{k} \alpha_j = 0$ in all cases.

For the third condition, the first and second characteristic polynomials are obtained and evaluated as follows;

Method (8),

Evaluated at $x = x_{n+1/2}$,

$$13y_n + 39y_{n+\frac{1}{2}} - 69y_{n+1} + 17y_{n+\frac{3}{2}} = -25hf_{n+\frac{1}{2}} + hf_{n+2}$$

$$\rho(r) = 17r^{\frac{3}{2}} - 69r^1 + 39r^{\frac{1}{2}} + 13,$$

$\sigma(r) = -25r^{\frac{1}{2}} + r^2$
$\rho'(1) = \sigma(1) = -24$
Evaluated at $x = x_{n+1}$,
$14y_n - 108y_{n+\frac{1}{2}} + 18y_{n+1} + 76y_{n+\frac{3}{2}} = 75hf_{n+1} + 3hf_{n+2}$
$\rho(r) = 76r^{\frac{3}{2}} + 18r^1 - 108r^{\frac{1}{2}} + 14,$
$\sigma(r) = 75r^1 + 3r^2$
$\rho'(1) = \sigma(1) = 78$
Evaluated at $x = x_{n+3/2}$,
$17y_n - 99y_{n+\frac{1}{2}} + 279y_{n+1} - 197y_{n+\frac{3}{2}} = -75hf_{n+\frac{3}{2}} + 9hf_{n+2}$
$\rho(r) = -197r^{\frac{3}{2}} + 279r^1 - 99r^{\frac{1}{2}} + 17,$
$\sigma(r) = -75r^{\frac{3}{2}} + 9r^2$
$\rho'(1) = \sigma(1) = -66$
Evaluated at $x = x_{n+2}$,
$3y_n - 16y_{n+\frac{1}{2}} + 36y_{n+1} - 48y_{n+\frac{3}{2}} + 25y_{n+2} = 6hf_{n+2}$
$\rho(r) = 25r^2 - 48r^{\frac{3}{2}} + 36r^1 - 16r^{\frac{1}{2}} + 3,$
$\sigma(r) = 6r^2$
$\rho'(1) = \sigma(1) = 6$
Hence, the method is consistent.

2.13 ZERO STABILITY

The derived Hybrid Backward Differentiation Formula can be written in a block form as follows.

$$IY_{\omega+1} = AY_{\omega+1} + BY_{\omega-1} + hCF_{\omega+1}. \qquad (10)$$

Where $I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$,

$$A = \begin{bmatrix} 0 & \frac{23}{13} & -\frac{17}{39} & 0 \\ 6 & 0 & -\frac{38}{9} & 0 \\ -\frac{99}{197} & \frac{279}{197} & 0 & 0 \\ \frac{16}{25} & -\frac{36}{25} & \frac{48}{25} & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & -\frac{7}{9} \\ 0 & 0 & 1 & \frac{17}{197} \\ 0 & 0 & 0 & -\frac{3}{25} \end{bmatrix} \text{ and}$$

$$C = \begin{bmatrix} -\frac{25}{39} & 0 & 0 & \frac{1}{39} \\ 0 & \frac{25}{6} & 0 & \frac{1}{6} \\ 0 & 0 & \frac{75}{197} & -\frac{9}{197} \\ 0 & 0 & 0 & \frac{6}{25} \end{bmatrix}$$

Applying the method to the test equation,

$$y' = \lambda y \qquad (11)$$

Amounts to solving the difference system,

$$[(I - zC) - A]Y_{\omega+j} - BY_{\omega-j} = 0,$$

$$z = h\lambda \qquad (12)$$

whose first characteristics polynomial is given as

$$\rho(R) = \det[R(I - A) - B]. \qquad (13)$$

Definition 3. (ZERO STABILITY): The block method (8) is said to be zero stable if the first characteristic polynomial $\rho(R) = \det[R(I - A) - B] = 0$ satisfies $|R_j| \leq 1$, $j = 1,2,3,\ldots$ and for those roots with $|R_j| = 1$, the multiplicity must not exceed 2.

The derived method is zero stable having satisfied the required condition with the roots of its first characteristic polynomial as $R = \{0, 0, 0, 1\}$.

2.14 CONVERGENCE

Convergence of a linear multistep method requires that it be consistent and zero stable. Our 2-step Hybrid Backward Differentiation Formula is convergent.

3 NUMERICAL EXPERIMENT

The problems below were used to establish the effectiveness of the method.

Note: the following notations were used to describe the results obtained.
$y(t_n)$ is the theoretical solution, y_n is the approximation using our method and Err(Sergey) signifies absolute error in the method used in Sergey et al (2016).

Problem 1;

$$y' = y - t^2 + 1,\ 0 \leq t \leq 2, y(0) = 0.5, h = 0.1.$$

Exact solution: $y(t) = t^2 + 2t + 1 - \frac{1}{2}e^t$
We selected problem 1 from [1] and we compared the result with the exact solution as well as the popular fourth order Runge-Kutta method since our method is also of order 4. The results in tables 1(a) and 1(b) show that our method gives better approximation with maximum error difference of 2.96×10^{-6}.

Problem 2;

$x'(t) = t + x(t), t \in [0,1], x(0) = 0.$
Exact solution: $x(t) = e^t - t - 1.$
This problem was solved in Sergey et al (2016). We solved it using our method and the result was compared with the exact solution on table 2(a) which shows a reasonable approximation. The error in our method was also compared with that of Sergey et al (2016) [7] in table 2(b) with step length h=0.1 and our method is way better.

Table 1(a): Comparison between Our method and the theoretical solution for problem 1.

t	$y(t_n)$	y_n	|Error|
0.1	0.65741454096	0.65741460349	6.25E-08
0.2	0.82929862091	0.82929868816	6.73E-08
0.3	1.01507059621	1.01507074691	1.51E-07
0.4	1.21408765117	1.21408781545	1.64E-07
0.5	1.42563936464	1.42563963949	2.75E-07
0.6	1.64894059980	1.64894090078	3.01E-07
0.7	1.88312364626	1.88312409284	4.47E-07
0.8	2.12722953575	2.12723002590	4.90E-07
0.9	2.38019844442	2.38019912529	6.81E-07
1.0	2.64085908577	2.64085983410	7.48E-07
1.1	2.90791698802	2.90791798505	9.97E-07
1.2	3.17994153863	3.17994263545	1.10E-06
1.3	3.45535166619	3.45535308599	1.42E-06
1.4	3.73240001657	3.73240157952	1.56E-06
1.5	4.00915546483	4.00915744574	1.98E-06
1.6	4.28348378780	4.28348596949	2.18E-06
1.7	4.55302630413	4.55302902502	2.72E-06
1.8	4.81517626779	4.81517926561	3.00E-06
1.9	5.06705277886	5.06705647028	3.69E-06
2.0	5.30547195053	5.30547601892	4.07E-06

Table 1(b): Comparison between RK4 and the theoretical solution problem 1.

t	$y(t_n)$	y_n	0
0.1	0.65741454096	0.65741437500	1.66E-07
0.2	0.82929862091	0.82929827599	3.45E-07
0.3	1.01507059621	1.01507005843	5.38E-07
0.4	1.21408765117	1.21408690570	7.45E-07
0.5	1.42563936464	1.42563839564	9.69E-07
0.6	1.64894059980	1.64893939041	1.21E-06
0.7	1.88312364626	1.88312217855	1.47E-06
0.8	2.12722953575	2.12722779067	1.75E-06
0.9	2.38019844442	2.38019640177	2.04E-06
1.0	2.64085908577	2.64085672418	2.36E-06
1.1	2.90791698802	2.90791428491	2.70E-06
1.2	3.17994153863	3.17993847018	3.07E-06
1.3	3.45535166619	3.45534820737	3.46E-06
1.4	3.73240001657	3.73239614113	3.88E-06
1.5	4.00915546483	4.00915114530	4.32E-06
1.6	4.28348378780	4.28347899554	4.79E-06
1.7	4.55302630413	4.55302100940	5.29E-06
1.8	4.81517626779	4.81517043981	5.83E-06
1.9	5.06705277886	5.06704638594	6.39E-06
2.0	5.30547195053	5.30546496022	6.99E-06

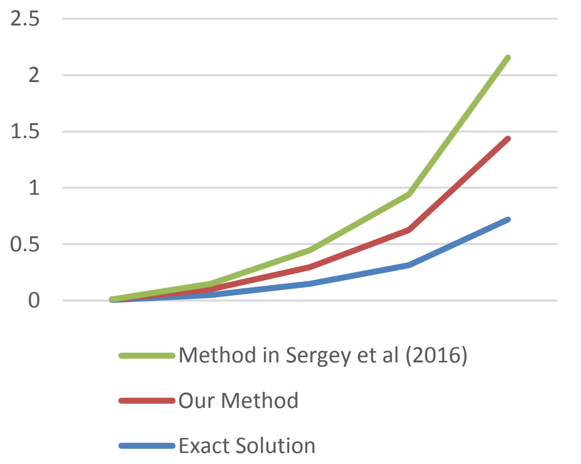
Table 2(a): Comparison between Our method and the theoretical solution for problem 2.

t	$y(t_n)$	y_n	|Error|
0	0	0	0
0.1	0.00517091808	0.00517079300	1.25E-07
0.2	0.02140275816	0.02140262366	1.34E-07
0.3	0.04985880758	0.04985850617	3.01E-07
0.4	0.09182469764	0.09182436908	3.29E-07
0.5	0.14872127070	0.14872072100	5.50E-07
0.6	0.22211880039	0.22211819844	6.02E-07
0.7	0.31375270747	0.31375181431	8.93E-07
0.8	0.42554092849	0.42553994819	9.80E-07
0.9	0.55960311116	0.55960174940	1.36E-06
1.0	0.71828182846	0.71828033178	1.50E-06

Table 2(b): Comparison between absolute error in our method and that of Sergey et al (2016) for problem 2.

t	Err(Sergey)	Absolute error in Our method
0.1	5.17E-03	1.25E-07
0.3	1.89E-02	3.01E-07
0.5	3.82E-02	5.50E-07
0.7	6.51E-02	8.93E-07
1	1.25E-01	1.50E-06

Fig. 1: Comparison among the exact solution, Approximation by Our method and Method [7]



3 CONCLUSION

A self-starting 2-step backward differentiation formula has been proposed and implemented for the approximation of solution of initial value problem of ordinary differential equation. The method is seen to be consistent and has a good stability property. Ongoing projects involve formation of higher steps using the same approach for the solution of certain classes of ordinary differential equations

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