An Order 2k Hybrid Backward Differentiation Formula for Stiff System of Ordinary Differential Equations Using Legendre Polynomial as Basis Function

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Abstract

In this paper, a k-step, (k=2, 3, 4), Block Hybrid Backward Differentiation Formula for the solution of Stiff systems of Ordinary Differential Equation have been formulated through continuous collocation approach. k off - grid points were incorporated at interpolation in order to retain the single function evaluation characteristic, which is peculiar to Backward Differentiation Formula. The basic properties of numerical methods were analyzed and the methods were found to be consistent with a uniform order $2k$, zero stable and as such, convergent. The region of absolute stability of the methods were analyzed using the general linear method (GLM), and found to be stable over a large region. The methods compute the solution of Stiff systems in a block-by-block way by some discrete schemes obtained from the associated continuous scheme which are combined and implemented as a set of block formulae. Numerical experiments were carried out and the results obtained, were compare with the exact or analytical solutions and some methods found in literatures.

Keywords*: Continuous Collocation, Hybrid Block Backward Differentiation Formula, Ordinary Differential Equation, Stiff systems, Legendre polynomial.*

1. Introduction

In the study of vibrations, chemical reactions, and electrical circuits, initial-value problems of ordinary differential equation arise in the form,

$$\left.\begin{aligned} y_1' &= f_1(t, y_1, y_2, \ldots, y_n) \\ &\vdots \\ y_n' &= f_n(t, y_1, y_2, \ldots, y_n) \end{aligned}\right\} \qquad (1)$$

which is usually treated in tandem with an initial condition

$$y_n(x_{no}) = y_{no} \qquad (2)$$

There exist certain classes of ordinary differential equations to which some numerical methods are not applicable. One of such classes is stiff system of ordinary differential equations. Stiff systems are characterized by the presence of transient and steady component. This characteristic makes the numerical solution unstable unless the step size is extremely small. Due to this restriction placed on the choice of step size, numerical solution of stiff system has been of great concern to researchers, most of who were able to come up with various formulations. Cooper (1969) and Baraff et al. (1997) described the results given by explicit methods as “consistently unsatisfactory” and “don’t do a very good job” respectively. Both of them recommended implicit multistep methods for the problem. Baraff et al. (1997) even suggested that where possible, one should change one’s formulation of problem to avoid solving stiff ordinary differential equation.

A number of researchers have developed various implicit methods for the approximation of stiff system of ordinary differential equations (Abhulimen and Ukpebor, 2018; Akinfenwa, 2017; Biala, 2015, Mehrkanoon et al.,2010; Chollom et al., 2014).

While Curtiss and Hirschfelder (1952) pioneered the use of Backward Differentiation Formula for the solution of stiff differential equation due to the restriction that A-stability puts on the choice of suitable methods for stiff systems. Several successful efforts have been made by various researchersin formulating various BDF based methods, including its higher derivatives, for its approximation (Akinfenwa et al., 2013; Babangida et al., 2016; Bakari et al., 2018; Ehigie et al., 2013; Nwachukwu and Okor, 2018).

Hybrid methods are obtained by incorporating off-grid (off-step) points in the derivation process in order to overcome Dahlquist Barrier theorem. (The order of LMM cannot exceed $k + 1$ if k is odd or $k + 2$ if is even). A k – Step continuous hybrid formula Special mention was made of hybrid methods in Akinfenwa et al (2011). They are obtained by incorporating off-grid (off-step) points in the derivation process in order to overcome Dahlquist Barrier theorem. (The order of LMM cannot exceed $k + 1$ if k is odd or $k + 2$ if is even).

This research is aimed to formulate an order 2k hybrid Backward Differentiation Formulae (BDF) with single function evaluation which is peculiar to BDF as main method. The Dahlquist Barrier theorem was overcome by incorporating the k-step point in the derivation process.

A k–Step continuous hybrid formula is of the type,

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h\sum_{j=0}^{k}\beta_k f_{n+j} + h\beta_v f_{n+v} \quad (3)$$

see Akinfenwa *et al.*, (2011). Where k is the step size, α_k =1, $\alpha_j,(j=0,1,\ldots k-1)$ and β_j, are unknown constants which are to be uniquely determined. Hybrid methods are characterized by their high accuracy and extended domain of stability.

2. Derivation of the Method

Here, it is assumed that the analytical solution of (1) can be approximated by a polynomial of the form,

$$y(x)=\sum_{j=0}^{i+c-1}\alpha_j p_j(x) \tag{4}$$

where i and c are respectively, number of interpolation and collocation points, $\alpha_j{}'s$ are coefficient to be determined and $p_j(x)$ can be any orthogonal polynomial. In this case, Legendre polynomial is used which, on inspection, produces exactly the same continuous form as the popularly adopted power series.

Incorporating k off-grid points for every k-step method requires that the following conditions must be satisfied:

$$y(x_n)=y_n \tag{5}$$

$$y(x_{n+j})=y_{n+j},\ \ j=0,\left(\frac{1}{2}\right),1,\ldots,k-\frac{1}{2} \tag{6}$$

$$f(x_{n+k})=f_{n+k} \tag{7}$$

where f implies the derivative of y.

(5), (6)and (7)result in $(i+c)$ system of equations which is solved through matrix inversion algorithm. This is with an intention to obtain values for α_j such that the continuous form of the method can be expressed as;

$$y(x)=\sum_{j=0}^{k-\frac{1}{2}}\alpha_j(x)y_{n+j}+h\beta_k(x)f_k \tag{8}$$

2.1 *2-Step Block Hybrid Backward Differentiation formula with 2 Off-grid Points (2SBHBDF).*

To derive a 2-step backward differentiation formula with two off-grid points, the following specifications were considered; $k=2$, $i=4$, $c=1$ and $x\in[x_n, x_{n+2}]$. This results in a system of equations

$$Y_\omega = D\Psi_{\omega-n} \tag{9}$$

where $Y_\omega = (y_n, y_{n+\frac{1}{2}}, y_{n+1}, y_{n+\frac{3}{2}}, f_{n+2})^T$, $\Psi_\omega = (\alpha_0, \alpha_{\frac{1}{2}}, \alpha_1, \alpha_{\frac{3}{2}}, \beta_2)^T$ and

$$D = \begin{pmatrix} 1 & x_n & \frac{1}{2}\left(3x_n^2-1\right) & \frac{1}{2}\left(x_n^3-3x_n\right) & \frac{1}{8}\left(35x_n^4-30x_n^2+3\right) \\ 1 & x_{n+\frac{1}{2}} & \frac{1}{2}\left(3x_{n+\frac{1}{2}}^2-1\right) & \frac{1}{2}\left(x_{n+\frac{1}{2}}^3-3x_{n+\frac{1}{2}}\right) & \frac{1}{8}\left(35x_{n+\frac{1}{2}}^4-30x_{n+\frac{1}{2}}^2+3\right) \\ 1 & x_{n+1} & \frac{1}{2}\left(3x_{n+1}^2-1\right) & \frac{1}{2}\left(x_{n+1}^3-3x_{n+1}\right) & \frac{1}{8}\left(35x_{n+1}^4-30x_{n+1}^2+3\right) \\ 1 & x_{n+\frac{3}{2}} & \frac{1}{2}\left(3x_{n+\frac{3}{2}}^2-1\right) & \frac{1}{2}\left(x_{n+\frac{3}{2}}^3-3x_{n+\frac{3}{2}}\right) & \frac{1}{8}\left(35x_{n+\frac{3}{2}}^4-30x_{n+\frac{3}{2}}^2+3\right) \\ 0 & 1 & 3x_{n+1} & \frac{1}{2}\left(x_{n+2}^2-3\right) & \frac{1}{8}\left(140x_{n+2}^3-60x_{n+2}\right) \end{pmatrix}$$

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Using matrix inversion technique with the aid of maple software, the values of α_0, $\alpha_{\frac{1}{2}}$, α_1, $\alpha_{\frac{3}{2}}$ and β_2 were obtained

substituted into (8) and setting $k = x - x_n$ and evaluating at $x = x_n + 2h$ resulted in the main method

$$y_{n+2} = -\frac{3}{25}y_n + \frac{16}{25}y_{n+\frac{1}{2}} - \frac{36}{25}y_{n+1} + \frac{48}{25}y_{n+\frac{3}{2}} + \frac{6}{25}hf_{n+2} \tag{10}$$

To obtain the additional schemes that combine with the main method to form a block, the first derivative of (8) was obtained and evaluated at $x = x_{n+\frac{1}{2}}$, $x = x_{n+1}$ and $x = x_{n+\frac{3}{2}}$ which produced three other discrete schemes given as

$$f_{n+\frac{3}{2}} = \frac{1}{75h}\left[9hf_{n+2} - 17y_n + 99y_{n+\frac{1}{2}} - 279y_{n+1} + 197y_{n+\frac{3}{2}}\right] \tag{11}$$

$$f_{n+1} = -\frac{1}{75h}\left[3hf_{n+2} - 14y_n + 108y_{n+\frac{1}{2}} - 18y_{n+1} - 76y_{n+\frac{3}{2}}\right] \tag{12}$$

$$f_{n+\frac{1}{2}} = \frac{1}{25h}\left[hf_{n+2} - 13y_n - 39y_{n+\frac{1}{2}} + 69y_{n+1} - 17y_{n+\frac{3}{2}} \right] \quad (13)$$

2.2 *3-Step Block Hybrid Backward Differentiation formula with 3 off-grid points (3SBHBDF)*

In this case, $k = 3$, $i = 6$, $c = 1$ and $x \in [x_n, x_{n+3}]$. Evaluating (1.8) at $x = x_n + 3h$, the main method below was obtained.

$$y_{n+3} = -\frac{10}{147}y_n + \frac{72}{147}y_{n+\frac{1}{2}} - \frac{225}{147}y_{n+1} + \frac{400}{147}y_{n+\frac{3}{2}} - \frac{450}{147}y_{n+2} + \frac{360}{147}y_{n+\frac{5}{2}} + \frac{30}{147}hf_{n+3} \quad (14)$$

and additional schemes were obtained in order to provide for the available number of unknown as

$$f_{n+\frac{3}{2}} = \frac{1}{4410h}\left[300hf_{n+3} - 394y_n + 2925y_{n+\frac{1}{2}} - 9600y_{n+1} + 18700y_{n+\frac{3}{2}} - 26550y_{n+2} + 14919y_{n+\frac{5}{2}} \right]$$
(15)

$$f_{n+2} = -\frac{1}{4410h}\left[60hf_{n+3} - 167y_n + 1320y_{n+\frac{1}{2}} - 4860y_{n+1} + 12560y_{n+\frac{3}{2}} - 6045y_{n+2} - 2808y_{n+\frac{5}{2}} \right]$$
(16)

$$f_{n+\frac{3}{2}} = \frac{1}{4410h}\left[30hf_{n+3} - 157y_n + 1395y_{n+\frac{1}{2}} - 6840y_{n+1} + 400y_{n+\frac{3}{2}} + 6165y_{n+2} - 963y_{n+\frac{5}{2}} \right] \quad (17)$$

$$f_{n+1} = -\frac{1}{2205h}\left[15hf_{n+3} - 152y_n + 1800y_{n+\frac{1}{2}} + 2460y_{n+1} - 5680y_{n+\frac{3}{2}} + 1980y_{n+2} - 408y_{n+\frac{5}{2}} \right]$$
(18)

$$f_{n+\frac{1}{2}} = \frac{1}{882}\left[12hf_{n+3} - 298y_n - 2235y_{n+\frac{1}{2}} + 4320y_{n+1} - 2780y_{n+\frac{3}{2}} + 1290y_{n+2} - 297y_{n+\frac{5}{2}} \right]$$
(19)

2.3 4-Step Block Hybrid Backward Differentiation formula with 4 off-grid point(4SBHBDF)

In a similar way as in cases of $k = 2$ and $k = 3$ above, setting $k = 4$, $i = 8$, $c = 1$ and

$x \in [x_n, x_{n+4}]$, we obtained the block

$$f_{n+\frac{1}{2}} = \frac{1}{22830h}\left[150hf_{n+4} - 5745y_n - 72387y_{n+\frac{1}{2}} + 158410y_{n+1} - 156450y_{n+\frac{3}{2}} - 127925y_{n+2} - 74305y_{n+\frac{5}{2}} + 27762y_{n+3} - 5210y_{n+\frac{7}{2}}\right]$$
(20)

$$f_{n+1} = \frac{1}{-479430h}\left[1050hf_{n+4} - 17385y_n + 276360y_{n+\frac{1}{2}} + 901117y_{n+1} - 1894200y_{n+\frac{3}{2}} + 1161825y_{n+2} - 600040y_{n+\frac{5}{2}} + 210315y_{n+3} - 37992y_{n+\frac{7}{2}}\right]$$
(21)

$$f_{n+\frac{3}{2}} = \frac{1}{31962h}\left[42hf_{n+4} - 391y_n + 4662y_{n+\frac{1}{2}} - 32354y_{n+1} - 27825y_{n+\frac{3}{2}} + 78435y_{n+2} - 30394y_{n+\frac{5}{2}} + 9478y_{n+3} - 1611y_{n+\frac{7}{2}}\right]$$
(22)

$$f_{n+2} = -\frac{1}{79905h}\left[105hf_{n+4} - 597y_n + 6328y_{n+\frac{1}{2}} - 32942y_{n+1} + 130200y_{n+\frac{3}{2}} - 3675y_{n+2} - 123928y_{n+\frac{5}{2}} + 29033y_{n+3} - 4408y_{n+\frac{7}{2}}\right]$$
(23)

$$f_{n+\frac{5}{2}} = \frac{1}{479430h}\left[1050hf_{n+4} - 368y_n + 36645y_{n+\frac{1}{2}} - 169610y_{n+1} + 502950y_{n+\frac{3}{2}} - 1235325y_{n+2} + 470687y_{n+\frac{5}{2}} + 450030y_{n+3} - 51690y_{n+\frac{7}{2}}\right]$$
(24)

$$f_{n+3} = -\frac{1}{159810h}\left[1050hf_{n+4} - 2165y_n + 20664y_{n+\frac{1}{2}} - 89705y_{n+1} + 236600y_{n+\frac{3}{2}} - 436275y_{n+2} + 678440y_{n+\frac{5}{2}} - 333039y_{n+3} - 74520y_{n+\frac{7}{2}}\right]$$
(25)

$$f_{n+\frac{7}{2}} = \frac{1}{159810h}\left[7350hf_{n+4} - 7545y_n + 70070y_{n+\frac{1}{2}} - 292334y_{n+1} + 723975y_{n+\frac{3}{2}} - 1189475y_{n+2} + 1393070y_{n+\frac{5}{2}} - 1324470y_{n+3} + 626709y_{n+\frac{7}{2}}\right]$$
(26)

$$y_{n+4} = -\frac{35}{761}y_n + \frac{320}{761}y_{n+\frac{1}{2}} - \frac{3920}{2283}y_{n+1} + \frac{3136}{761}y_{n+\frac{3}{2}} - \frac{4900}{761}y_{n+2} + \frac{15680}{761}y_{n+\frac{5}{2}} - \frac{3920}{761}y_{n+3} + \frac{2240}{761}y_{n+\frac{7}{2}} + \frac{140}{761}hf_{n+3}$$
(27)

3. Analysis of the methods

3.1 Order of accuracy and Error constant

Following S$\ddot{u}$li (2014), let $y\left(x_{n+j}\right)$, the solution to $y'\left(x_{n+j}\right)$ be sufficiently differentiable, then $y\left(x_{n+j}\right)$ and $y'\left(x_{n+j}\right)$ can be expanded into a Taylor's series about point x_n to obtain

$$T_n = \frac{1}{h\sigma(1)}\left[C_0 y\left(x_n\right) + C_1 h y'\left(x_n\right) + C_2 h^2 y''\left(x_n\right) + \ldots\right] \quad (28)$$

Where

$$\left.\begin{aligned} C_0 &= \sum_{j=0}^{k}\alpha_j \\ C_1 &= \sum_{j=0}^{k} j\alpha_j - \sum_{j=0}^{k}\beta_j, \\ &. \\ &. \\ &. \\ C_q &= \frac{1}{q!}\sum_{j=0}^{k} j^q\alpha_j - \frac{1}{(q-1)!}\sum_{j=0}^{k} j^{q-1}\beta_j \end{aligned}\right\} \quad (29)$$

Definition 3.21: A Linear multistep method is said to be of order of accuracy p if $C_0 = C_1 = .\ .\ .C_p = 0$, $C_{p+1} \neq 0$, C_{p+1} is called the error constants.

From our calculations, we have that the block methods of step number k has uniform order $2k$ and the error constants are shown in tables 1, 2 and 3 below.

Table 1. Order and Error constants for the proposed 2-step Block Hybrid Backward Differentiation Formula

Method	Order, P	Error constant, C_{p+1}
(10)	4	$-\frac{29}{320}$
(12)	4	$-\frac{31}{160}$
(12)	4	$-\frac{111}{320}$
(13)	4	$-\frac{3}{40}$

Table 2. Order and Error constants for the proposed 3-step Block Hybrid Backward Differentiation Formula

Method	Order, P	Error constant, C_{p+1}
(19)	6	$-\frac{159}{448}$
(18)	6	$-\frac{81}{224}$

(17)	6	$-\frac{501}{896}$
(16)	6	$-\frac{177}{224}$
(15)	6	$-\frac{1035}{448}$
(14)	6	$-\frac{15}{224}$

Table 3. Order and Error constants for the proposed 4-step Block Hybrid Backward Differentiation Formula

Method	Order, P	Error constant, C_{p+1}
(20)	8	$-\frac{1335}{1024}$
(21)	8	$-\frac{12115}{1536}$
(22)	8	$-\frac{817}{3072}$
(23)	8	$-\frac{277}{512}$
(24)	8	$-\frac{12815}{3072}$
(25)	8	$-\frac{405}{1536}$
(26)	8	$-\frac{12145}{1024}$
(27)	8	$-\frac{35}{192}$

3.2 Consistency

Definition: A linear multistep method is said to be consistent if the following conditions are satisfied.

i. the order of accuracy $p > 1$,

ii. $\sum_{j=0}^{k} \alpha_j = 0$,

iii. $\rho'(1) = \sigma(1)$, where $\rho(r)$ and $\sigma(r)$ are respectively, first and second characteristic polynomials of the methods.

Conditions i and ii were taken care of in section 3.1 since the order $p > 1$ and $C_0 = \sum_{j=0}^{k} \alpha_j = 0$ in all cases.

For the third condition, the first and second characteristic polynomials were obtained and evaluated in what follows.

For all the methods, conditions for consistency are satisfied. Hence, they are consistent with uniform order of accuracy, $p = 2k > 0$.

The summary of order of accuracy, error constants as well as the parameter for measuring consistency as obtained above are presented in Tables 4, 5 and 6.

Table 4. Parameters for determining consistency of 2-step Block Hybrid Backward Differentiation Formula

Method	Order, P	$\sum \alpha_j$	$\rho'(1)$	$\sigma(1)$
(13)	4	0	-24	-24
(12)	4	0	78	78
(11)	4	0	-66	-66
(10)	4	0	6	6

Table 5. Parameters for determining consistency of 3-step Block Hybrid Backward Differentiation Formula

Method	Order, P	$\sum \alpha_j$	$\rho'(1)$	$\sigma(1)$
(19)	6	0	-870	-870
(18)	6	0	2220	2220
(17)	6	0	-4380	-4380
(16)	6	0	4470	4470
(15)	6	0	-4110	-4110
(14)	6	0	30	30

Table 6. Parameters for determining consistency of 4-step Block Hybrid Backward Differentiation Formula

Method	Order, P	$\sum \alpha_j$	$\rho'(1)$	$\sigma(1)$
(20)	8	0	-23680	-23680
(21)	8	0	480480	480480
(22)	8	0	-31920	-31920
(23)	8	0	80010	80010
(24)	8	0	-478380	-478380
(25)	8	0	160860	160860
(26)	8	0	-152460	-152460
(27)	8	0	420	420

3.3 Zero stability

The derived Hybrid Backward Differentiation Formula can be written in a block form as follows.

$$A^{(1)}Y_{\omega+1} = A^{(0)}Y_{\omega-1} + hBF_{\omega+1} \qquad (30)$$

whose first characteristics polynomial is given as

$$\rho(R) = \det\left[RA^{(1)} - A^{(0)}\right] \qquad (31)$$

Definition (ZERO STABILITY): The block method (30) is said to be zero stable if no root of the first characteristic polynomial $\rho(R)$ satisfies $|R_j| \leq 1,\ j = 1, 2, 3, \ldots$ and for those roots with $|R_j| = 1,$ the multiplicity must not exceed 2.

3.3.1 Zero stability of 2-step block hybrid backward differentiation formula with 2 off grid points.

Expressing methods (10), (11), (12) and (13) in the form (30),

$$A^{(1)} = \begin{pmatrix} 1 & -\frac{23}{13} & \frac{17}{39} & 0 \\ -6 & 1 & \frac{38}{9} & 0 \\ \frac{99}{197} & -\frac{279}{197} & 1 & 0 \\ -\frac{16}{25} & \frac{36}{25} & -\frac{48}{25} & 1 \end{pmatrix}, \quad A^{(0)} = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{3} \\ 0 & 0 & 0 & -\frac{7}{9} \\ 0 & 0 & 0 & \frac{17}{197} \\ 0 & 0 & 0 & -\frac{3}{25} \end{pmatrix} \text{ and } B = \begin{pmatrix} -\frac{25}{39} & 0 & 0 & \frac{1}{39} \\ 0 & \frac{25}{6} & 0 & \frac{1}{6} \\ 0 & 0 & \frac{75}{197} & -\frac{9}{197} \\ 0 & 0 & 0 & \frac{6}{25} \end{pmatrix}$$

$$\rho(R) = -\frac{1000}{2561}R^3(R-1) = 0$$

$R = \{0,0,0,1\}$.

The method is zero stable since it satisfies $|R_j| \le 1$.

3.3.2 Zero stability of 3-step Block Hybrid Backward Differentiation Formula with 3 off grid points.

Expressing methods (14), (15), (16), (17), (18) and (19) in the form (30),

$$A^{(1)} = \begin{pmatrix} 1 & -\frac{288}{149} & \frac{556}{447} & -\frac{86}{149} & \frac{99}{745} & 0 \\ \frac{30}{41} & 1 & -\frac{284}{123} & \frac{33}{123} & -\frac{34}{205} & 0 \\ \frac{279}{80} & -\frac{171}{10} & 1 & \frac{1233}{80} & -\frac{963}{400} & 0 \\ -\frac{88}{403} & \frac{324}{403} & -\frac{2512}{1209} & 1 & \frac{72}{155} & 0 \\ \frac{975}{4973} & -\frac{3200}{4973} & \frac{18700}{14919} & -\frac{8850}{4973} & 1 & 0 \\ -\frac{24}{49} & \frac{75}{49} & -\frac{400}{147} & \frac{150}{49} & -\frac{120}{497} & 1 \end{pmatrix},$$

$$A^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -\frac{2}{15} \\ 0 & 0 & 0 & 0 & 0 & \frac{38}{615} \\ 0 & 0 & 0 & 0 & 0 & \frac{157}{400} \\ 0 & 0 & 0 & 0 & 0 & -\frac{167}{6045} \\ 0 & 0 & 0 & 0 & 0 & \frac{394}{14919} \\ 0 & 0 & 0 & 0 & 0 & -\frac{10}{147} \end{pmatrix} \text{ and}$$

$$B=\begin{pmatrix} -\frac{294}{745} & 0 & 0 & 0 & 0 & \frac{4}{745} \\ 0 & -\frac{147}{164} & 0 & 0 & 0 & -\frac{1}{164} \\ 0 & 0 & \frac{441}{40} & 0 & 0 & -\frac{3}{40} \\ 0 & 0 & 0 & \frac{294}{403} & 0 & \frac{4}{403} \\ 0 & 0 & 0 & 0 & \frac{1470}{4973} & -\frac{100}{4973} \\ 0 & 0 & 0 & 0 & 0 & \frac{10}{49} \end{pmatrix}$$

$$\rho(R)=-\frac{134481277728}{12243162971}R^5(R-1)=0$$

$$R=\{0,0,0,0,0,1\}$$

The method is zero stable having it satisfied $|R_j|\le 1$.

3.3.3 Zero stability of 4-step block hybrid backward differentiation formula with 4 off grid points

Expressing methods (20), (21), (22), (23), (24), (25) (26) and (27) in the form of (30),

$$\begin{pmatrix} 1 & -\frac{22630}{10341} & \frac{7450}{3447} & -\frac{18275}{10341} & \frac{10615}{10341} & -\frac{1322}{3447} & \frac{5210}{72387} & 0 \\ \frac{39480}{128731} & 1 & -\frac{270600}{128731} & \frac{165975}{128731} & -\frac{85720}{128731} & \frac{30045}{128731} & -\frac{37992}{901117} & 0 \\ -\frac{222}{1325} & \frac{4622}{3975} & 1 & -\frac{747}{265} & \frac{4342}{3975} & -\frac{1354}{3975} & \frac{537}{9275} & 0 \\ -\frac{904}{525} & \frac{3706}{525} & -\frac{248}{7} & 1 & \frac{17704}{525} & -\frac{1382}{175} & \frac{4408}{3675} & 0 \\ \frac{5235}{67241} & -\frac{24230}{67241} & \frac{71850}{67241} & -\frac{176475}{67241} & 1 & \frac{64290}{67241} & -\frac{51690}{470687} & 0 \\ -\frac{984}{15859} & \frac{12815}{47577} & -\frac{33800}{47577} & \frac{20775}{15859} & -\frac{96920}{67241} & 1 & \frac{24840}{111013} & 0 \\ \frac{70070}{626709} & -\frac{292334}{626709} & \frac{241325}{208903} & -\frac{1189475}{626709} & \frac{1393070}{626709} & -\frac{441490}{208903} & 1 & 0 \\ -\frac{320}{761} & \frac{3920}{2283} & -\frac{3136}{761} & \frac{4900}{761} & -\frac{15680}{2283} & \frac{3920}{761} & -\frac{2240}{761} & 1 \end{pmatrix}$$

$$A^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{5}{63} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{17385}{901117} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{391}{27825} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{199}{1225} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3687}{470687} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{2165}{333903} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2515}{208903} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{35}{761} \end{pmatrix}$$

$$B = \begin{pmatrix} -\frac{7610}{24129} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{50}{24129} \\ 0 & -\frac{68490}{128731} & 0 & 0 & 0 & 0 & 0 & -\frac{150}{128731} \\ 0 & 0 & -\frac{1522}{1325} & 0 & 0 & 0 & 0 & \frac{2}{1325} \\ 0 & 0 & 0 & \frac{761}{35} & 0 & 0 & 0 & \frac{1}{35} \\ 0 & 0 & 0 & 0 & \frac{68490}{67241} & 0 & 0 & -\frac{150}{67241} \\ 0 & 0 & 0 & 0 & 0 & -\frac{7610}{15859} & 0 & \frac{50}{15859} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{53270}{208903} & -\frac{2450}{208903} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{140}{761} \end{pmatrix}$$

$$\rho(R) = -\frac{143199134699167502254080000}{58211987311115247963453333} R^7 (R-1) = 0$$

$R = \{0,0,0,0,0,0,0,1\}$.

Having satisfied $|R_j| \leq 1$, the method is zero stable.

3.4 Convergence

Here, the convergence of the hybrid backward differentiation formula developed, is considered in agreement with the fundamental theorem of Dahlquist which states that, “The necessary and sufficient condition for LMM to be convergent is for it to be consistent and zero stable”. (see Henrici, 1962). Following this theorem, the methods developed are convergent having satisfied the necessary and sufficient conditions of consistency and zero stability.

3.5 Region of Absolute Stability of the Method

Definition: The stability domain, otherwise known as stability region, of a numerical method is the set $S = \{z \in C : |R(z)| \le 1\}$

The region of absolute stability is obtained using the general linear method (GLM), which is described as generalization of Runge-Kutta (multistage) methods and linear multistep (multi-value) methods.

The derived methods are written in the form

$$\begin{bmatrix} Y \\ y_{i+1} \end{bmatrix} = \begin{bmatrix} A & U \\ B & V \end{bmatrix} \begin{bmatrix} hf(Y) \\ y_{i+1} \end{bmatrix} \quad (32)$$

Where $A = \begin{bmatrix} a_{11} & . & . & . & a_{1s} \\ . & . & & & . \\ . & & . & & . \\ . & & & . & . \\ a_{s1} & . & . & . & a_{ss} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & . & . & . & b_{1s} \\ . & . & & & . \\ . & & . & & . \\ . & & & . & . \\ b_{s1} & . & . & . & b_{ss} \end{bmatrix}$, $Y = \begin{bmatrix} y_n \\ y_{n+\frac{1}{2}} \\ \cdot \\ \cdot \\ \cdot \\ y_{n+k} \end{bmatrix}$ and $y_{n+1} = \begin{bmatrix} y_{n+k} \\ . \\ . \\ . \\ . \\ y_{n+k-1} \end{bmatrix}$

Definition: For a general linear method (A, B, U, V), stability matrix $M(z)$ I defined by

$$M(z) = V + zB(I - zA)^{-1}U \quad (33)$$

and the characteristic polynomial is given by

$$\varphi(\mu, z) = \det[\mu I - M(z)] \quad (34)$$

Definition: A general linear method (A, B, U, V), is said to be A-stable if for all $z \in C^-$, $I - zA$ is non-singular and $M(z)$ is the stability polynomial.

Definition: A general linear method (A, B, U, V), is said to be L-stable if it is A-stable and $\rho(M(\infty)) = 0$ or the stronger condition, $M(\infty) = 0$.

To obtain and plot region of absolute stability (also known as domain of absolute stability)

Elements of the matrices A, B, U and V were obtained from interpolation and collocation points and then substituted into the stability matrix (33) and the stability function (34).

3.5.1 Region of Absolute Stability for 2-step Hybrid Backward Differentiation Formula

The method, in block form, has coefficients

$$\left[\begin{array}{ccccc|cc} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -\frac{25}{39} & 0 & 0 & 0 & \frac{23}{13} & -\frac{1}{3} \\ 0 & 0 & \frac{25}{6} & 0 & 0 & 0 & -\frac{7}{9} \\ 0 & 0 & 0 & \frac{75}{197} & -\frac{9}{197} & 0 & \frac{17}{197} \\ 0 & 0 & 0 & 0 & \frac{6}{25} & -\frac{36}{25} & -\frac{3}{25} \\ \hline 0 & 0 & 0 & 0 & \frac{6}{25} & -\frac{36}{25} & -\frac{3}{25} \\ 0 & 0 & -\frac{25}{6} & 0 & \frac{1}{6} & 0 & -\frac{7}{9} \end{array}\right]$$

With stability polynomial,

$$\varphi(\mu,z)=\frac{1}{3}\left[450\mu^2z^2-60857\mu z-1767\mu^2z-450\mu^2-2775\mu z-298\mu+504\right] \qquad (35)$$

The plot of region of absolute stability is shown in figure (3.1) where it is found that the method is stiffly stable with stiffness criteria, $D=0.43$.

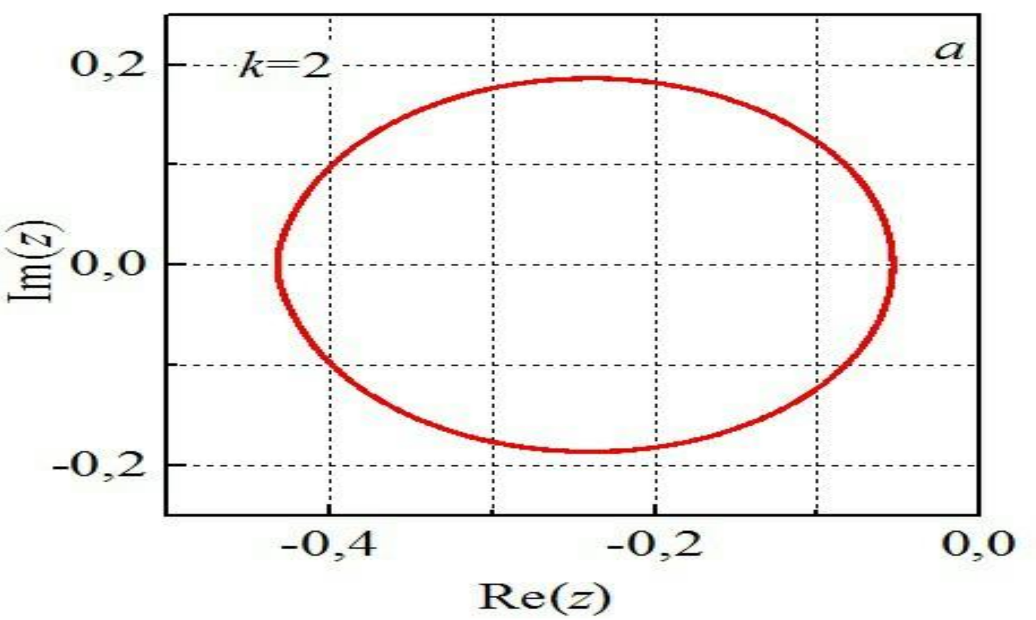

Fig. 1. Region of Absolute Stability of 2-Step Hybrid Backward Differentiation Formula

3.5.2 Region of Absolute Stability for 3-step Block Hybrid Backward Differentiation Formula

The method, in block form, has coefficients

$$\left(\begin{array}{cccccc|ccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-\frac{294}{745} & 0 & 0 & 0 & 0 & \frac{4}{745} & -\frac{288}{149} & -\frac{86}{149} & -\frac{2}{15} \\
0 & -\frac{147}{164} & 0 & 0 & 0 & -\frac{1}{164} & 0 & \frac{33}{123} & \frac{38}{615} \\
0 & 0 & \frac{441}{40} & 0 & 0 & -\frac{3}{40} & -\frac{177}{10} & \frac{1233}{80} & \frac{157}{400} \\
0 & 0 & 0 & \frac{294}{403} & 0 & \frac{4}{403} & \frac{324}{403} & 0 & -\frac{167}{6045} \\
0 & 0 & 0 & 0 & \frac{1470}{4973} & -\frac{100}{4973} & -\frac{3200}{4973} & -\frac{8850}{4973} & \frac{394}{14919} \\
0 & 0 & 0 & 0 & 0 & \frac{10}{49} & \frac{75}{49} & \frac{150}{49} & -\frac{10}{147} \\
- & - & - & - & - & - & - & - & - \\
0 & 0 & 0 & 0 & 0 & \frac{10}{49} & \frac{75}{49} & \frac{150}{49} & -\frac{10}{147} \\
0 & 0 & 0 & \frac{294}{403} & 0 & -\frac{100}{4973} & \frac{324}{403} & 0 & -\frac{167}{6045} \\
0 & -\frac{147}{164} & 0 & 0 & 0 & -\frac{1}{164} & 0 & \frac{33}{123} & \frac{38}{615}
\end{array}\right)$$

The stability polynomial was obtained and the plot of region of absolute stability is shown in figure 2 where it is found that the method is stiffly stable with stiffness criteria, $D = 10$

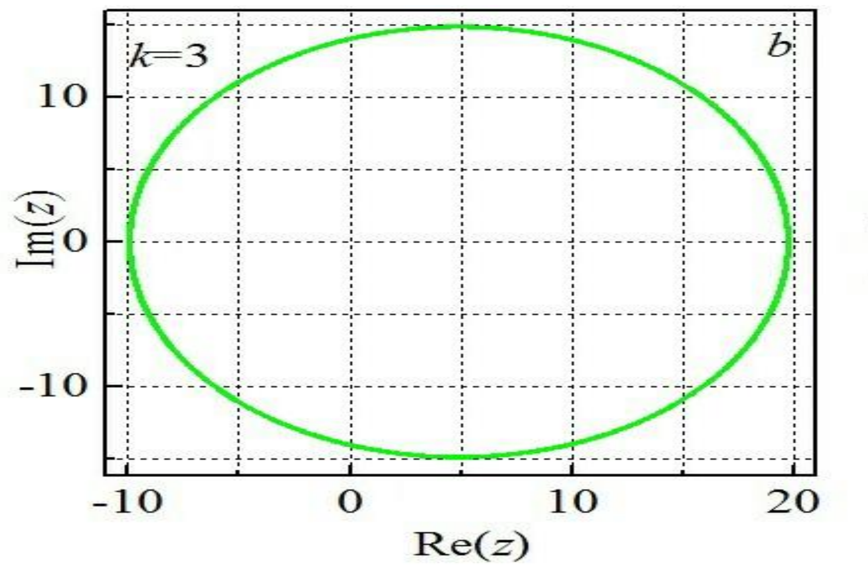

Fig. 2. Region of Absolute Stability of 3-Step Hybrid Backward Differentiation Formula

3.2.5.3 Region of Absolute Stability for 4-step Hybrid Backward Differentiation Formula

The coefficients of the method, is expressed as

$$\begin{bmatrix} A & U \\ B & V \end{bmatrix}$$

$$\text{Where } A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{7610}{24129} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{50}{24129} \\ 0 & -\frac{68490}{128731} & 0 & 0 & 0 & 0 & 0 & -\frac{150}{128731} \\ 0 & 0 & -\frac{1522}{1325} & 0 & 0 & 0 & 0 & \frac{2}{1325} \\ 0 & 0 & 0 & \frac{761}{35} & 0 & 0 & 0 & \frac{1}{35} \\ 0 & 0 & 0 & 0 & \frac{68490}{67241} & 0 & 0 & -\frac{150}{67241} \\ 0 & 0 & 0 & 0 & 0 & \frac{7610}{15859} & 0 & \frac{50}{15859} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{53270}{208903} & -\frac{2450}{208903} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{140}{761} \end{pmatrix},$$

$$U = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1322}{3447} & \frac{18275}{10341} & \frac{22630}{10341} & -\frac{5}{63} \\ -\frac{30045}{128731} & -\frac{165975}{128731} & 0 & -\frac{17385}{901117} \\ \frac{1354}{13975} & \frac{747}{265} & -\frac{4622}{3975} & -\frac{391}{27825} \\ \frac{1382}{175} & 0 & \frac{4706}{525} & \frac{199}{1225} \\ -\frac{64290}{67241} & \frac{176475}{67241} & \frac{24230}{67241} & \frac{3687}{470687} \\ 0 & -\frac{20775}{15859} & -\frac{12815}{47577} & -\frac{2165}{333039} \\ \frac{441490}{208903} & \frac{1189475}{626709} & \frac{292334}{626709} & \frac{2515}{208903} \\ \frac{3920}{761} & -\frac{4900}{761} & -\frac{3920}{2283} & -\frac{35}{761} \end{pmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{140}{761} \\ 0 & 0 & 0 & 0 & 0 & \frac{7610}{15859} & 0 & \frac{50}{15859} \\ 0 & 0 & 0 & \frac{761}{35} & 0 & 0 & 0 & \frac{1}{35} \\ 0 & -\frac{68490}{128731} & 0 & 0 & 0 & 0 & 0 & -\frac{150}{128731} \end{bmatrix}$$

$$\text{and } V = \begin{bmatrix} -\frac{3920}{2283} & -\frac{4900}{761} & -\frac{3920}{2283} & -\frac{35}{761} \\ 0 & -\frac{20775}{15859} & -\frac{12815}{47577} & -\frac{2165}{333039} \\ \frac{297}{197} & 0 & -\frac{4706}{525} & -\frac{199}{1225} \\ -\frac{30045}{128731} & -\frac{165975}{128731} & 0 & -\frac{17385}{901117} \end{bmatrix}$$

The stability polynomial was obtained and the plot of region of absolute stability is shown in Figure.3 below where it is found that the methods is stiffly stable with stiffness criteria, $D = 20$.

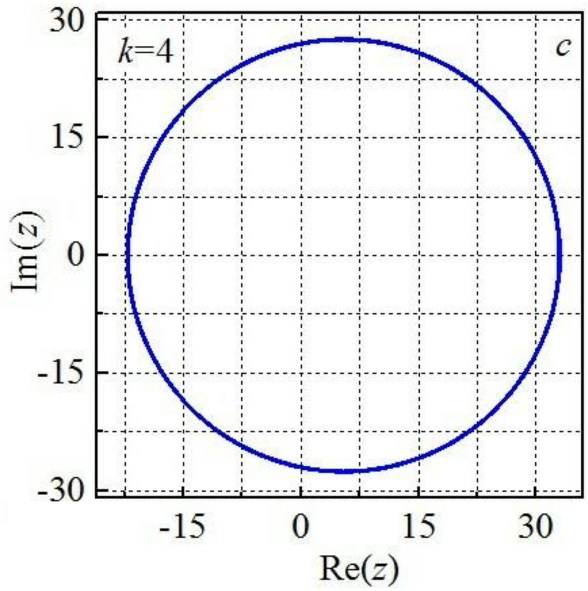

Figure 3: Region of Absolute Stability of 4-Step Hybrid Backward Differentiation Formula

4. Numerical Experiments

In this section, the efficiency of the hybrid method formulated in section 2 is tested on some problems of stiff system of ordinary differential equations. The self-starting method is implemented efficiently by combining the methods as simultaneous numerical integrator for IVPs for example, the method (17) - (30) are combined to obtain the initial conditions at x_{n+2}, $n(mod 2) \neq 0$ and $0 \leq n \leq N$ using computed values $y(x_{n+2})$ over sub-interval $[x_0, x_2]$.

Problems on Stiff System

4.1 $y' = -y + 95z$, $y(0) = 1$

$z' = -y - 97z$, $z(0) = 1$, $t \in [0,1]$, $h = 0.0625,\ 0.03125$

Exact solution: $y(t) = \frac{95}{47}e^{-2t} - \frac{48}{47}e^{-96t}$

$$z(t)=\frac{48}{47}e^{-96t}-\frac{1}{47}e^{-t}$$

This problem was solved in Biala et al. (2015), Ehigie et al. (2013) and Sahi et al. (2012).The absolute error in the results obtained with the new method for $h=0.0625$ and $h=0.3125$ are shown in Figures4.1a and 4.1b while comparison between the proposed method and existing methods is shown in Table 4.1.

4.2 $\quad y_1{}^{'}=998y_1+1998y_2\,,\; y(0)=1$

$$y_2{}^{'}=-999y_1-1999y_2\,,\; z(0)=1$$

Exact solution: $y_1(t)=4e^{-t}-3e^{-1000t}$

$$y_2(t)=-2e^{-t}+3e^{-t}t\in[0,],\, h=0.1$$

This was solved in Akinfenwa et al. (2011) and Ehigie et al. (2013). The absolute error in the results obtained with the new method is shown in figure 4.2 while comparison between the proposed method and existing methods is shown in Table 4.2.

4.3 $\quad y''+1001+1000y$

Reduced to:

$$y'=z\,,\; y(0)=1$$

$$z'{}'=-1000y-1001z\,, z(0)=1$$

Exact solution: $y(t)=4e^{-t}-3e^{-1000t}$

$$z(t)=-2e^{-t}+3e^{-t},\quad t\in[0,],\quad h=0.1$$

This was solved in Abhulimen and Omeike (2011), Akinfenwa et al. (2014) and Ehigie et al. (2013). The absolute error in the results obtained with the new method is shown in figure 4.3while comparison between the proposed method and existing methods is shown in Table 4.3.

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4.4 $\quad y_1{}^{'}=-1002y_1+1000y_2{}^{2}\,,\; y_1(0)=1$

$$y_2{}^{'}=y_1-y_2(\,1+y_2\,),\; y_2(0)=1$$

Exact solution: $y_1(x)=e^{-2x}$

$$y_2(x) = e^{-x}, x \in [0,1], h = 0.02.$$

This was solved in Akinfenwa et al. (2013). The absolute error in the results obtained with the new method is shown in figure 4.4while comparison between the proposed method and existing methods is shown in Table 4.4.

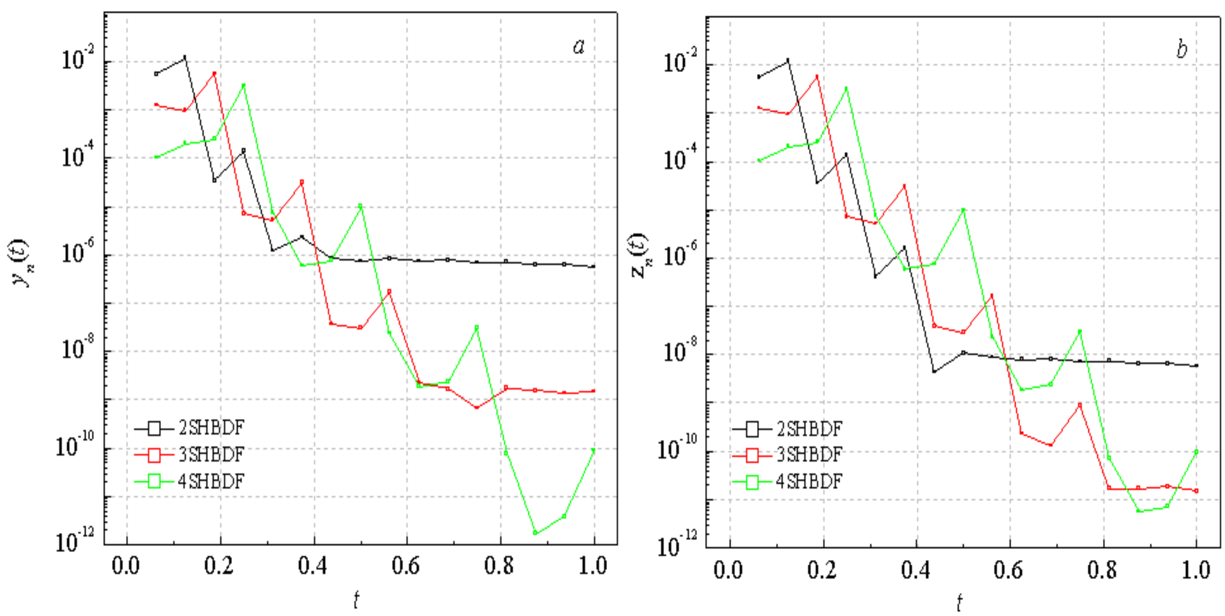

Fig. 4. Absolute Error in the Proposed Methods for Problem 4.1 with $\boldsymbol{h} = 0.0625$

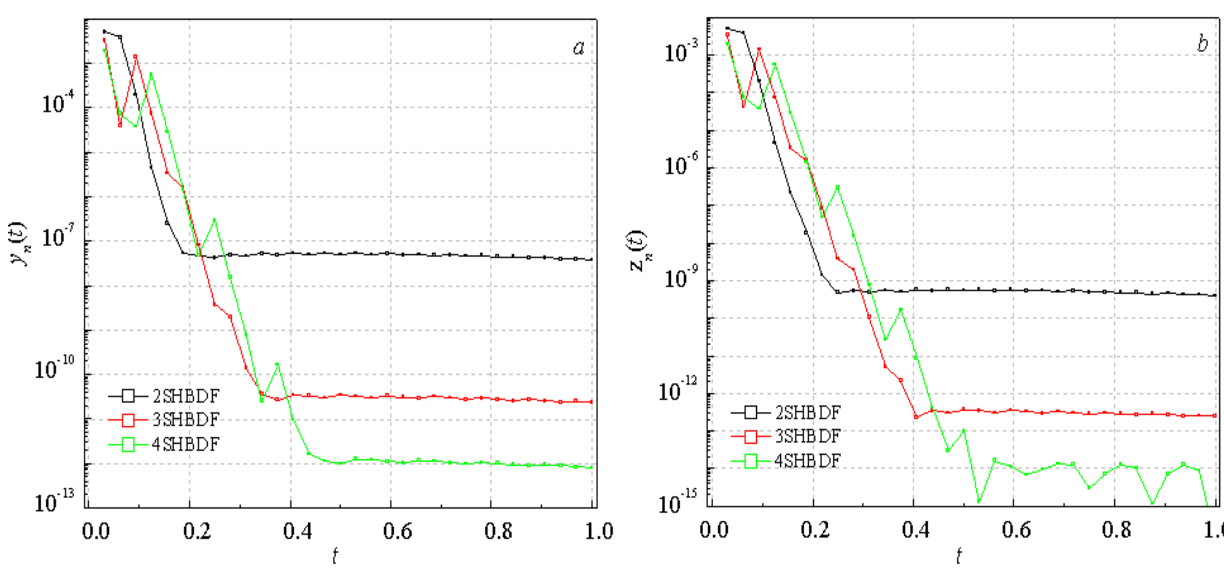

Fig. 5. Absolute Error in the Proposed Methods For Problem4.11 with $\boldsymbol{h} = 0.03125$

Table7. Comparing the Absolute error in the proposed method with existing methods found in literature for problem 1.

H	Biala *et al* (2015) y_n z_n	Abhulimen and Omeike (2011) y_n z_n	Abhulimen and Ukpebor (2018) y_n z_n	Ehigie et al. (2013) y_n z_n	Sahi et al (2012) y_n z_n	New Method y_n z_n
0.0625	4×10^{-10} 8×10^{-10}	3.2×10^{-10} 2.4×10^{-10}	5.0×10^{-8} 7.0×10^{-10}	3.4×10^{-9} 3.6×10^{-9}	9×10^{-11} 1×10^{-8}	9.25×10^{-11} 9.56×10^{-11}
0.03125	7×10^{-12} 7×10^{-14}	1.2×10^{-10}. 8.1×10^{-10}	6.0×10^{-8} 1.0×10^{-10}	3.4×10^{-9} 3.5×10^{-9}	4×10^{-12} 4×10^{-12}	7.8×10^{-13} 1.1×10^{-16}

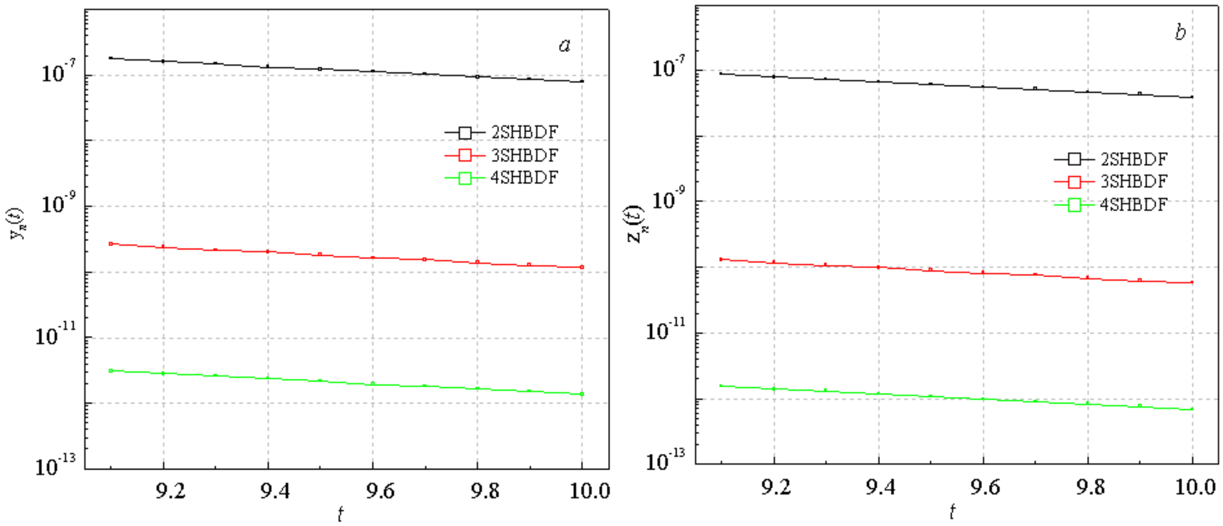

Fig.6. Absolute Error in the Proposed Methods For Problem 2

Table 8. Comparing the Absolute error in the proposed method with existing methods found in literature for problem 2.

Akinfenwa, et al (2011) y_n z_n	Ehigie et al (2013) y_n z_n	New Method y_n z_n
4.183×10^{-13} 2.092×10^{-13}	4.18×10^{-13} 8.92×10^{-18}	1.36×10^{-14} 6.82×10^{-15}

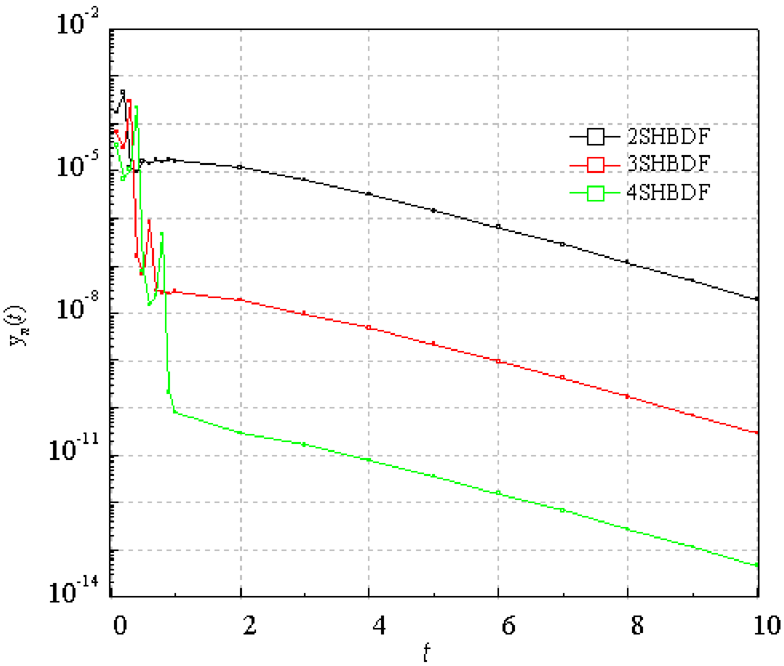

Fig. 7. Absolute Error in the Proposed Methods for Problem3

H	Abhulimen and Omeike (2011) y_n	Akinfenwa et al (2014) y_n	New Method y_n
0.1	1.4×10^{-8}	1.56×10^{-14}	4.65×10^{-16}

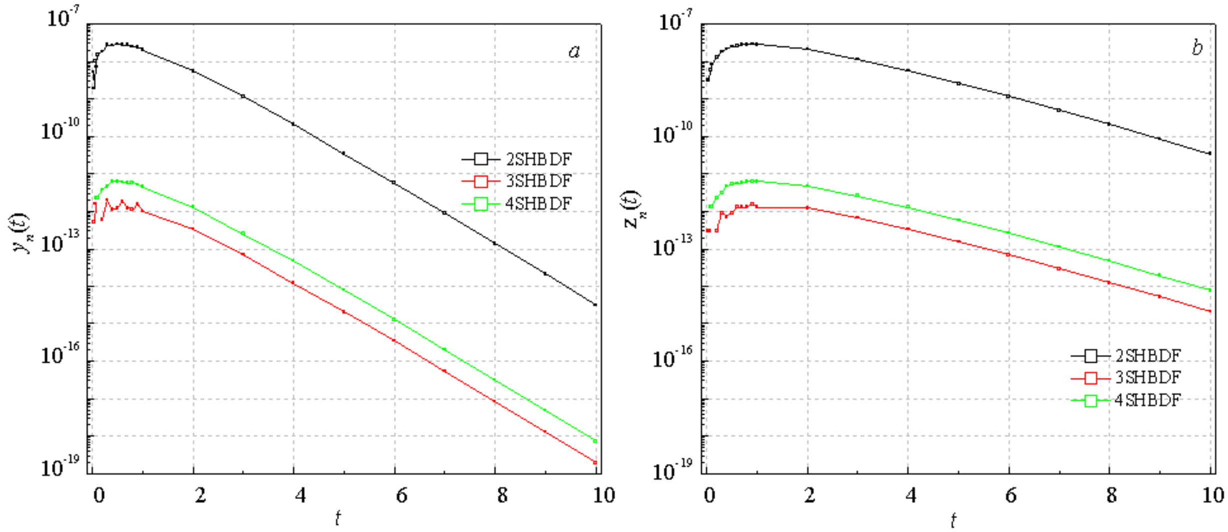

Fig. 9. Absolute Error in the Proposed Method for Problem4

Table 10. Comparing the Absolute error in the proposed method with existing methods found in literature for problem 4.

h	Akinfenwa et al (2013) y_n z_n	New Method y_n z_n
0.02	9.1102×10^{-13} 1.2527×10^{-12}	2.12×10^{-21} 7.89×10^{-17}

5. Conclusion

In this paper, a continuous (k -step) Block Hybrid Backward Differentiation Formula of order $2k$ have been developed by the interpolation and collocation techniques with the incorporation of k off-step points at interpolation for the approximation of the solutions of stiff system and system of fuzzy of ordinary differential equations. The Legendre polynomial of first kind was employed as basis function, which of course, produces exactly the same continuous form as the popularly adopted power series on inspection.

Analysis of basic properties of numerical methods was carried out and findings show that the methods are of maximum order $2k$ in general and the 2-step block hybrid backward differentiation formula is of optimal order. They are consistent, zero-stable and convergent. The stability region was plotted using the idea of General Linear Method (GLM). The methods were reformulated and stability polynomials were obtained and found to have a moderate region of absolute stability.

The schemes were implemented as block method and therefore have the capacity to generate k simultaneous solutions at different points in a single application of the methods.

Four test problems have been considered and compared with existing methods to test the efficiency and accuracy of the new methods.

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