FORMULATION OF k −STEP ORDER $2k$ FUZZY-STRUCTURED BLOCK HYBRID BACKWARD DIFFERENTIATION FORMULAE ALGORITHMS FOR THE APPROXIMATE SOLUTION OF FUZZY DIFFERENTIAL EQUATIONS

[1]MOHAMMED, U; [2]USMAN, A. M.; [3]SALIHU, N. O.; [4]MA'ALI, A. I.; & [5]AKINTUBOBO, B. G.
[1,3,4] Department of Mathematics, Federal University of Technology, Minna, Nigeria
[2]Department of Mathematics, Federal Polytechnic Bida, Nigeria
[4]Department of Mathematics, Ibrahim Badamasi, Babangida University, Lapai, Nigeria
E-mail: umaru.mohd@futminna.edu.ng **Phone No:** +234-703-215-9609

Abstract
In this paper, a k −step ($k = 2,3,4$), order $2k$ Fuzzy-structured block hybrid backward differentiation formulae were formulated for the approximate solution of fuzzy differential equations (FDE's). k off step points were incorporated at interpolation in the process of formulation. The methods were developed using interpolation and collocation techniques. Convergence of the methods were analyzed and established. The methods were found to be of uniform order $2k$. The methods were implemented as a block method, combining the main method with some additional methods obtained from the same continuous form. Numerical experiments were carried out and the results obtained were found to be effective, efficient and accurate in comparison with the exact solutions and approximations obtained with existing methods.

Introduction
In mathematical modelling, situations arise where variables with certain level of uncertainty or vagueness are involved. Such as materials used in terms of corrosion, thermal expansion or some other measurable materials in electrical engineering, estimating the service life of a given piece of equipment in industrial engineering, to mention but a few. Differential equations associated with such models are called fuzzy differential equations (FDE's). They play important role in many fields of science, engineering, technology and even finance (Dass, 2014).

Several approaches to dealing with FDE's exist in literature. First of which was using H-derivative or its generalization, the Hukuhara differentiability which is introduced by Puri and Ralescu (1983).

However, this approach suffers certain set back that leads to solutions with increasing support since the diameter of the solution is unbounded as time increases (Chalco-Cano *et al.,* 2008). In an attempt to overcome this set back, Bede and Gal (2005) introduced the generalized differentiability by enlarging the class of fuzzy valued function. Bede et al. (2007) also asserted that under certain appropriate conditions, FDEs are equivalent to a system of ordinary differential equations (ODEs) which can be solved by any suitable numerical method.

Several numerical methods for solving FDEs has been presented by various researchers such as Abbasbandi and Viranloo, (2002), Ahmad and Hasan (2011), Balooch-Shahryari and Salahshour (2012), Shokri (2007), Zawawi and Ibrahim (2016), Mehrkanoon *et al* (2009) and Ivaz *et al*. (2013). In this paper, a formulation of k −Step order $2k$ Fuzzy-structures Block Hybrid Backward Differentiation Formulae were derived for the approximate solution of Fuzzy differential equations with the approach of collocation techniques which incorporated k −off-step points

FDE's in a flash

Definition 1 (**Fuzzy set**) Let X be a non-zero set. A fuzzy set A of this set X is defined by the following set of pairs.

$A = \{(x: \mu_A(x))\}: x \in X$ (1)

where $\mu_A: X \to \{0, 1\}$ is a function called the membership function of A and μ_A is the grade of membership of degree of fuzziness of $x \in X$ in A.

Symbolically, $A = \frac{x_1}{\mu_A(x_1)}, \frac{x_2}{\mu_A(x_2)}, \dots, \frac{x_n}{\mu_A(x_n)}$.

Operations that apply to set theory also apply to fuzzy set theory (Dass ,1998).

Definition 2 (**Fuzzy number)** A fuzzy number u is a fuzzy subset of the real line with a normal, convex and semi upper continuous membership function of bounded support. It is completely determined by any pair

$u = \left(\underline{u}(\alpha), \bar{u}(\alpha)\right), 0 \leq \alpha \leq 1$, which satisfy the conditions:

i. $\underline{u}(\alpha)$ is a bounded left continuous monotonic increasing function $\forall\ \alpha \in (0,1]$
ii. $\bar{u}(\alpha)$ is a bounded left continuous monotonic increasing function $\forall\ \alpha \in (0,1]$
iii. $\underline{u}(\alpha) \leq \bar{u}(\alpha)$, $0 \leq \alpha \leq 1$.

Then the α −level set $[\upsilon]^\alpha = \{s | \upsilon(s) \geq \alpha\}, 0 \leq \alpha \leq 1$is a closed, bounded interval denotedby $[\upsilon]^\alpha = [\upsilon_1(\alpha), \upsilon_2(\alpha)]$ (Mehrkanoon *et al*. 2009)

Derivation of the methods

The solution of the differential equation

$\tilde{y}' = f(t, \tilde{y}(t, r))$ (2)

where f is a fuzzy valued function, can be approximated by a polynomial of the form,

$$\hat{y} = \sum_{j=0}^{i+c-1} \alpha_j p_j(t) \quad (3)$$

where i and c are respectively, number of interpolation and collocation points,$\alpha_j's$ are coefficient to be determined and $p_j(x)$can be any orthogonal polynomial.

Incorporating k off-grid points for every k-step method requires that the following conditions must be satisfied:

$$\left.\begin{array}{l} \tilde{y}(t_n, r) = \tilde{y}_n \\ \tilde{y}(t_{n+j}, r) = \tilde{y}_{n+j},\ j = 0, \left(\frac{1}{2}\right), 1, \dots, k - \frac{1}{2}, \\ f(t_{n+k}) = f_{n+k} \end{array}\right\} \quad (4)$$

where f implies the derivative of $\tilde{y}$.

(4) result in $(i + c)$ system of equations which is solved through matrix inversion algorithm to obtain values for $\alpha_j$and β_k. These were then substituted into the continuous form of the method which is expressed as;

$$\hat{y}(t, r) = \sum_{j=0}^{k-\frac{1}{2}} \alpha_j(t) \hat{y}_{n+j} + h\beta_k(t) f_k \quad (5)$$

2-Step Fuzzy-structured Block Hybrid Backward Differentiation formula with 2 off-grid points (2SBHBDF)

To derive a 2-step backward differentiation formula with two off-grid points, the following specifications were considered; $k = 2,\ i = 4,\ c = 1$ and $x \in [t_n, t_{n+2}]$. This results in a system of equations

$$Y_\omega = M\Psi_{\omega-n} \tag{6}$$

where $\tilde{Y}_\omega = (\tilde{y}_n, y_{n+\frac{1}{2}}, y_{n+1}, y_{n+\frac{3}{2}}, f_{n+2})^T$, $\Psi_\omega = (\alpha_0, \alpha_{\frac{1}{2}}, \alpha_1, \alpha_{\frac{3}{2}}, \beta_2)^T$and

$$M = \begin{bmatrix} 1 & t_n & \frac{1}{2}(3t_n{}^2 - 1) & \frac{1}{2}(t_n{}^3 - 3t_n) & \frac{1}{8}(35t_n{}^4 - 30t_n{}^2 + 3) \\ 1 & t_{n+\frac{1}{2}} & \frac{1}{2}(3t_{n+\frac{1}{2}}{}^2 - 1) & \frac{1}{2}(t_{n+\frac{1}{2}}{}^3 - 3t_n) & \frac{1}{8}(35t_{n+\frac{1}{2}}{}^4 - 30t_{n+\frac{1}{2}}{}^2 + 3) \\ 1 & t_{n+1} & \frac{1}{2}(3t_{n+1}{}^2 - 1) & \frac{1}{2}(t_{n+1}{}^3 - 3t_{n+1}) & \frac{1}{8}(35t_{n+1}{}^4 - 30t_{n+1}{}^2 + 3) \\ 1 & t_{n+\frac{3}{2}} & \frac{1}{2}(3t_{n+\frac{3}{2}}{}^2 - 1) & \frac{5}{2}(t_{n+\frac{3}{2}}{}^3 - 3t_{n+\frac{3}{2}}) & \frac{1}{8}(35t_{n+\frac{3}{2}}{}^4 - 30t_{n+\frac{3}{2}}{}^2 + 3) \\ 0 & 1 & 3t_{n+2} & \frac{1}{2}(t_{n+2}{}^2 - 3) & \frac{1}{8}(140t_{n+2}{}^3 - 60t_{n+2}) \end{bmatrix}.$$

Using matrix inversion technique with the aid of maple software, the values of α_0, $\alpha_{\frac{1}{2}}$, α_1, $\alpha_{\frac{3}{2}}$ and β_2 were obtained
substituted into (5)and setting $k = t - t_n$and evaluating at $t = t_n + 2h$ resulted in the main method

$$\tilde{y}_{n+2} = -\frac{3}{25}\tilde{y}_n + \frac{16}{25}\tilde{y}_{n+\frac{1}{2}} - \frac{36}{25}\tilde{y}_{n+1} + \frac{48}{25}\tilde{y}_{n+\frac{3}{2}} + \frac{6}{25}hf_{n+2} \tag{7}$$

To obtain the additional schemes that combine with the main method to form a block, the first derivative of (5)was obtained and evaluated at $t = t_{n+\frac{1}{2}}$, $t = t_{n+1}$ and $t = t_{n+\frac{3}{2}}$ which produced three other discrete schemes given as

$$f_{n+\frac{3}{2}} = \frac{1}{75h}\left[9hf_{n+2} - 17\tilde{y}_n + 99\tilde{y}_{n+\frac{1}{2}} - 279\tilde{y}_{n+1} + 197y_{n+\frac{3}{2}}\right] \tag{8}$$

$$f_{n+1} = -\frac{1}{75h}\left[3hf_{n+2} - 14\tilde{y}_n + 108\tilde{y}_{n+\frac{1}{2}} - 18\tilde{y}_{n+1} - 76\tilde{y}_{n+\frac{3}{2}}\right] \tag{9}$$

$$f_{n+\frac{1}{2}} = \frac{1}{25h}\left[hf_{n+2} - 13\tilde{y}_n - 39\tilde{y}_{n+\frac{1}{2}} + 69\tilde{y}_{n+1} - 17\tilde{y}_{n+\frac{3}{2}}\right] \tag{10}$$

3-Step Fuzzy-structured Block Hybrid Backward Differentiation formula with 3 off-grid points (3SFBHBDF)

In this case, $k = 3$, $i = 6$, $c = 1$ and$x \in [t_n, t_{n+3}]$. Evaluating (5) at $t = t_n + 3h$, the main method below was obtained.

$$y_{n+3} = -\frac{10}{147}y_n + \frac{72}{147}y_{n+\frac{1}{2}} - \frac{225}{147}y_{n+1} + \frac{400}{147}y_{n+\frac{3}{2}} - \frac{450}{147}y_{n+2} + \frac{360}{147}y_{n+\frac{5}{2}} + \frac{30}{147}hf_{n+3} \tag{11}$$

and additional schemes were obtained in order to provide for the available number of unknown as

$$f_{n+\frac{5}{2}} = \frac{1}{4410h}\begin{bmatrix} 300hf_{n+3} - 394\tilde{y}_n + 2925\tilde{y}_{n+\frac{1}{2}} - 9600\tilde{y}_{n+1} + 18700\tilde{y}_{n+\frac{3}{2}} \\ -26550\tilde{y}_{n+2} + 14919\tilde{y}_{n+\frac{5}{2}} \end{bmatrix} \tag{12}$$

$$f_{n+2} = -\frac{1}{4410h}\begin{bmatrix} 60hf_{n+3} - 167\tilde{y}_n + 1320\tilde{y}_{n+\frac{1}{2}} - 4860\tilde{y}_{n+1} + 12560\tilde{y}_{n+\frac{3}{2}} \\ -6045\tilde{y}_{n+2} - 2808\tilde{y}_{n+\frac{5}{2}} \end{bmatrix} \quad (13)$$

$$f_{n+\frac{3}{2}} = \frac{1}{4410h}\begin{bmatrix} 30hf_{n+3} - 157\tilde{y}_n + 1395\tilde{y}_{n+\frac{1}{2}} - 6840\tilde{y}_{n+1} + 400\tilde{y}_{n+\frac{3}{2}} \\ +6165\tilde{y}_{n+2} - 963\tilde{y}_{n+\frac{5}{2}} \end{bmatrix} \quad (14)$$

$$f_{n+1} = -\frac{1}{2205h}\begin{bmatrix} 15hf_{n+3} - 152\tilde{y}_n + 1800\tilde{y}_{n+\frac{1}{2}} + 2460\tilde{y}_{n+1} - 5680\tilde{y}_{n+\frac{3}{2}} \\ +1980\tilde{y}_{n+2} - 408\tilde{y}_{n+\frac{5}{2}} \end{bmatrix} \quad (15)$$

$$f_{n+\frac{1}{2}} = \frac{1}{882}\begin{bmatrix} 12hf_{n+3} - 298\tilde{y}_n - 2235\tilde{y}_{n+\frac{1}{2}} + 4320\tilde{y}_{n+1} - 2780\tilde{y}_{n+\frac{3}{2}} \\ +1290\tilde{y}_{n+2} - 297\tilde{y}_{n+\frac{5}{2}} \end{bmatrix} \quad (16)$$

4-Step Fuzzy-structured Block Hybrid Backward Differentiation formula with 4 off-grid point (4SFBHBDF)

In a similar way as in cases of $k = 2$ and $k = 3$ above, setting $k = 4$, $i = 8$, $c = 1$ and$x \in [x_n, x_{n+4}]$, we obtained the block

$$f_{n+\frac{1}{2}} = \frac{1}{22830h}\begin{bmatrix} 150hf_{n+4} - 5745\tilde{y}_n - 72387\tilde{y}_{n+\frac{1}{2}} + 158410\tilde{y}_{n+1} - 156450\tilde{y}_{n+\frac{3}{2}} - 127925\tilde{y}_{n+2} \\ - 74305\tilde{y}_{n+\frac{5}{2}} + 27762\tilde{y}_{n+3} - 5210\tilde{y}_{n+\frac{7}{2}} \end{bmatrix} \quad (17)$$

$$f_{n+1} = -\frac{1}{479430h}\begin{bmatrix} 1050hf_{n+4} - 17385\tilde{y}_n + 276360\tilde{y}_{n+\frac{1}{2}} + 901117\tilde{y}_{n+1} - 1894200\tilde{y}_{n+\frac{3}{2}} \\ +1161825y_{n+2} - 600040y_{n+\frac{5}{2}} + 210315y_{n+3} - 37992y_{n+\frac{7}{2}} \end{bmatrix} \quad (18)$$

$$f_{n+\frac{3}{2}} = \frac{1}{31962h}\begin{bmatrix} 42hf_{n+4} - 391\tilde{y}_n + 4662\tilde{y}_{n+\frac{1}{2}} - 32354\tilde{y}_{n+1} - 27825\tilde{y}_{n+\frac{3}{2}} + 78435\tilde{y}_{n+2} \\ -30394\tilde{y}_{n+\frac{5}{2}} + 9478\tilde{y}_{n+3} - 1611\tilde{y}_{n+\frac{7}{2}} \end{bmatrix} \quad (19)$$

$$f_{n+2} = -\frac{1}{79905h}\begin{bmatrix} 105hf_{n+4} - 597\tilde{y}_n + 6328\tilde{y}_{n+\frac{1}{2}} - 32942\tilde{y}_{n+1} + 130200\tilde{y}_{n+\frac{3}{2}} - 3675\tilde{y}_{n+2} \\ -123928\tilde{y}_{n+\frac{5}{2}} + 29033\tilde{y}_{n+3} - 4408\tilde{y}_{n+\frac{7}{2}} \end{bmatrix} \quad (20)$$

$$f_{n+\frac{5}{2}} = \frac{1}{479430h}\begin{bmatrix} 1050hf_{n+4} - 368\tilde{y}_n + 36645\tilde{y}_{n+\frac{1}{2}} - 169610\tilde{y}_{n+1} + 502950\tilde{y}_{n+\frac{3}{2}} - 1235325\tilde{y}_{n+2} \\ + 470687\tilde{y}_{n+\frac{5}{2}} + 450030\tilde{y}_{n+3} - 51690\tilde{y}_{n+\frac{7}{2}} \end{bmatrix} \quad (21)$$

$$f_{n+3} = -\frac{1}{159810h}\begin{bmatrix} 1050hf_{n+4} - 2165\tilde{y}_n + 20664\tilde{y}_{n+\frac{1}{2}} - 89705\tilde{y}_{n+1} + 236600\tilde{y}_{n+\frac{3}{2}} - 436275\tilde{y}_{n+2} \\ + 678440\tilde{y}_{n+\frac{5}{2}} - 333039\tilde{y}_{n+3} - 74520\tilde{y}_{n+\frac{7}{2}} \end{bmatrix} \quad (22)$$

$$f_{n+\frac{7}{2}} = \frac{1}{159810h}\begin{bmatrix} 7350hf_{n+4} - 7545\tilde{y}_n + 70070\tilde{y}_{n+\frac{1}{2}} - 292334\tilde{y}_{n+1} + 723975\tilde{y}_{n+\frac{3}{2}} - 1189475\tilde{y}_{n+2} \\ + 1393070\tilde{y}_{n+\frac{5}{2}} - 1324470\tilde{y}_{n+3} + 626709\tilde{y}_{n+\frac{7}{2}} \end{bmatrix} \quad (23)$$

$$\tilde{y}_{n+4} = -\frac{35}{761}\tilde{y}_n + \frac{320}{761}\tilde{y}_{n+\frac{1}{2}} - \frac{3920}{2283}\tilde{y}_{n+1} + \frac{3136}{761}\tilde{y}_{n+\frac{3}{2}} - \frac{4900}{761}\tilde{y}_{n+2} + \frac{15680}{761}\tilde{y}_{n+\frac{5}{2}} - \frac{3920}{761}\tilde{y}_{n+3} + \frac{2240}{761}\tilde{y}_{n+\frac{7}{2}} + \frac{140}{761}hf_{n+3} \qquad (24)$$

Analysis of Methods

Order of accuracy and Error constant

Following S*ü*li (2014), let $y(x_{n+j})$, the solution to $y'(x_{n+j})$be sufficiently differentiable, then $y(x_{n+j})$ and $y'(x_{n+j})$ can be expanded into a Taylor's series about point x_n to obtain

$$T_n = \frac{1}{h\sigma(1)}[C_0 y(x_n) + C_1 h y'(x_n) + C_2 h^2 y''(x_n) + \cdots] \qquad (25)$$

Where

$$C_0 = \sum_{j=0}^{k} \alpha_j$$

$$C_1 = \sum_{j=0}^{k} j\alpha_j - \sum_{j=0}^{k} \beta_j,$$

$$\vdots$$

$$C_q = \frac{1}{q!}\sum_{j=0}^{k} j^q \alpha_j - \frac{1}{(q-1)!}\sum_{j=0}^{k} j^{q-1}\beta_j \qquad (26)$$

Definition 3 A Linear multistep method is said to be of order of accuracy p if $C_0 = C_1 = \ldots C_p = 0$, $C_{p+1} \neq 0$,C_{p+1} is called The error constants.

From our calculations, we have that the block methods of step number k has uniform order $2k$ that is 4 6 and 8 for 2- step, 3- step and 4 -step respectively while the error constants are

$\left(-\frac{29}{320}, -\frac{31}{160}, -\frac{111}{320}, -\frac{3}{40}\right), \left(-\frac{159}{448}, -\frac{81}{224}, -\frac{501}{896}, -\frac{177}{224}, -\frac{1035}{448}, -\frac{15}{224}\right)$

and$\left(-\frac{1335}{1024}, -\frac{12115}{1536}, -\frac{817}{3072}, -\frac{277}{512}, -\frac{12815}{3072}, -\frac{405}{1536}, -\frac{12145}{1024}, -\frac{35}{192}\right)$ for 2- step, 3- step and 4 -step methods.

Consistency

Definition 4 A linear multistep method is said to be consistent if the following conditions are satisfied.

i. the order of accuracy $p > 1$,

ii. $\sum_{j=0}^{k} \alpha_j = 0$,

iii. $\rho'(1) = \sigma(1)$, where $\rho(r)$ and $\sigma(r)$ are respectively, first and second characteristic polynomials of the methods.

Conditions i and ii were taken care of in section 4.1 since the order $p > 1$ and $C_0 = \sum_{j=0}^{k} \alpha_j = 0$ in all cases.

For the third condition, the first and second characteristic polynomials are obtained and evaluated in what follows.

For all the methods, conditions for consistency are satisfied. Hence, they are consistent with uniform order of accuracy, $p = 2k > 0$.

Zero stability

The derived Hybrid Backward Differentiation Formula can be written in a block form as follows.

$$A^{(1)}Y_{\omega+1} = A^{(0)}Y_{\omega-1} + hBF_{\omega+1} \tag{27}$$

whose first characteristics polynomial is given as

$$\rho(R) = \det\left[RA^{(1)} - A^{(0)}\right] \tag{28}$$

*Definition (*Zero stability*)*: The block method (27) is said to be zero stable if no root of the first characteristic polynomial $\rho(R)$ satisfies $|R_j| \leq 1, j = 1,2,3, \ldots$ and for those roots with $|R_j| = 1$, the multiplicity must not exceed 2.

Zero stability of 2-step fuzzy-structured block hybrid backward differentiation formula with 2 off grid points.

Expressing methods (7), (8), (9) and (10) in the form (27), we have

$$A^{(1)} = \begin{pmatrix} 1 & -\frac{23}{13} & \frac{17}{39} & 0 \\ -6 & 1 & \frac{38}{9} & 0 \\ \frac{99}{197} & -\frac{279}{197} & 1 & 0 \\ -\frac{16}{25} & \frac{36}{25} & -\frac{48}{25} & 1 \end{pmatrix}, \quad A^{(0)} = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{3} \\ 0 & 0 & 0 & -\frac{7}{9} \\ 0 & 0 & 0 & \frac{17}{197} \\ 0 & 0 & 0 & -\frac{3}{25} \end{pmatrix} \text{ and } B = \begin{pmatrix} -\frac{25}{39} & 0 & 0 & \frac{1}{39} \\ 0 & \frac{25}{6} & 0 & \frac{1}{6} \\ 0 & 0 & \frac{75}{197} & -\frac{9}{197} \\ 0 & 0 & 0 & \frac{6}{25} \end{pmatrix}$$

$\rho(R) = -\frac{1000}{2561}R^3(R-1) = 0$

$R = \{0, 0, 0, 1\}$.

The method is zero stable since it satisfies $|R_j| \leq 1$.

Zero stability of 3-step Block Hybrid Backward Differentiation Formula with 3 off grid points.

Expressing methods (11), (12), (13), (14), (15) and (16) in the form (27),

$$A^{(1)} = \begin{pmatrix} 1 & -\frac{288}{149} & \frac{556}{447} & -\frac{86}{149} & \frac{99}{745} & 0 \\ \frac{30}{41} & 1 & -\frac{284}{123} & \frac{33}{123} & -\frac{34}{205} & 0 \\ \frac{279}{80} & -\frac{171}{10} & 1 & \frac{1233}{80} & -\frac{963}{400} & 0 \\ -\frac{88}{403} & \frac{324}{403} & -\frac{2512}{1209} & 1 & \frac{72}{155} & 0 \\ \frac{975}{4973} & -\frac{3200}{4973} & \frac{18700}{14919} & -\frac{8850}{4973} & 1 & 0 \\ -\frac{24}{49} & \frac{75}{49} & -\frac{400}{147} & \frac{150}{49} & -\frac{120}{497} & 1 \end{pmatrix}, \quad A^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -\frac{2}{15} \\ 0 & 0 & 0 & 0 & 0 & \frac{38}{615} \\ 0 & 0 & 0 & 0 & 0 & \frac{157}{400} \\ 0 & 0 & 0 & 0 & 0 & -\frac{167}{6045} \\ 0 & 0 & 0 & 0 & 0 & \frac{394}{14919} \\ 0 & 0 & 0 & 0 & 0 & -\frac{10}{147} \end{pmatrix} \text{ and}$$

$$B=\begin{pmatrix} -\frac{294}{745} & 0 & 0 & 0 & 0 & \frac{4}{745} \\ 0 & -\frac{147}{164} & 0 & 0 & 0 & -\frac{1}{164} \\ 0 & 0 & \frac{441}{40} & 0 & 0 & -\frac{3}{40} \\ 0 & 0 & 0 & \frac{294}{403} & 0 & \frac{4}{403} \\ 0 & 0 & 0 & 0 & \frac{1470}{4973} & -\frac{100}{4973} \\ 0 & 0 & 0 & 0 & 0 & \frac{10}{49} \end{pmatrix}$$

$$\rho(R) = -\frac{134481277728}{12243162971}R^5(R-1) = 0$$

$$R = \{0, 0, 0, 0, 0, 1\}$$

The method is zero stable having it satisfied $|R_j| \leq 1$.

Zero stability of 4-step block hybrid backward differentiation formula with 4 off grid points

Expressing methods (17), (18), (19), (20), (21), (22) (23) and (24) in the form of (27),

$$A^{(1)}=\begin{pmatrix} 1 & -\frac{22630}{10341} & \frac{7450}{3447} & -\frac{18275}{10341} & \frac{10615}{10341} & -\frac{1322}{3447} & \frac{5210}{72387} & 0 \\ \frac{39480}{128731} & 1 & -\frac{270600}{128731} & \frac{165975}{128731} & -\frac{85720}{128731} & \frac{30045}{128731} & -\frac{37992}{901117} & 0 \\ -\frac{222}{1325} & \frac{4622}{3975} & 1 & -\frac{747}{265} & \frac{4342}{3975} & -\frac{1354}{3975} & \frac{537}{9275} & 0 \\ -\frac{904}{525} & \frac{3706}{525} & -\frac{248}{7} & 1 & \frac{17704}{525} & -\frac{1382}{175} & \frac{4408}{3675} & 0 \\ \frac{5235}{67241} & -\frac{24230}{67241} & \frac{71850}{67241} & -\frac{176475}{67241} & 1 & \frac{64290}{67241} & -\frac{51690}{470687} & 0 \\ -\frac{984}{15859} & \frac{12815}{47577} & -\frac{33800}{47577} & \frac{20775}{15859} & -\frac{96920}{67241} & 1 & \frac{24840}{111013} & 0 \\ \frac{70070}{626709} & -\frac{292334}{626709} & \frac{241325}{208903} & -\frac{1189475}{626709} & \frac{1393070}{626709} & -\frac{441490}{208903} & 1 & 0 \\ -\frac{320}{761} & \frac{3920}{2283} & -\frac{3136}{761} & \frac{4900}{761} & -\frac{15680}{2283} & \frac{3920}{761} & -\frac{2240}{761} & 1 \end{pmatrix}$$

$$A^{(0)}=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{5}{63} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{17385}{901117} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{391}{27825} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{199}{1225} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3687}{470687} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{2165}{333903} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2515}{208903} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{35}{761} \end{pmatrix}$$

$$B=\begin{pmatrix} -\frac{7610}{24129} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{50}{24129} \\ 0 & -\frac{68490}{128731} & 0 & 0 & 0 & 0 & 0 & -\frac{150}{128731} \\ 0 & 0 & -\frac{1522}{1325} & 0 & 0 & 0 & 0 & \frac{2}{1325} \\ 0 & 0 & 0 & \frac{761}{35} & 0 & 0 & 0 & \frac{1}{35} \\ 0 & 0 & 0 & 0 & \frac{68490}{67241} & 0 & 0 & -\frac{150}{67241} \\ 0 & 0 & 0 & 0 & 0 & \frac{7610}{15859} & 0 & \frac{50}{15859} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{53270}{208903} & -\frac{2450}{208903} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{140}{761} \end{pmatrix}$$

$$\rho(R) = -\frac{143199134699167502254080000}{5821198731115247963453333}R^7(R-1) = 0$$

$R = \{0, 0, 0, 0, 0, 0, 0, 1\}$.

Having satisfied $|R_j| \leq 1$, the method is zero stable.

Convergence

Here, the convergence of the fuzzy-structured hybrid backward differentiation formula developed is considered in agreement with the fundamental theorem of Dahlquist which states that "The necessary and sufficient condition for a LMM to be convergent is for it to be consistent and zero stable" (Henrici, 1962). Following this theorem, the methods developed are convergent having satisfied the necessary and sufficient conditions of consistency and zero stability.

Implementation of the methods and Numerical Experiments

In this section, the self-starting method is implemented efficiently by combining the methods as simultaneous numerical integrator for IVPs for example, the method (7) - (10) are combined to obtain the initial conditions at $t_{n+2}, n(mod 2) \neq 0$ and $0 \leq n \leq N$ using computed values $Y(t_{n+2}, r)$ over sub-interval $[t_0, t_2]$. We consider $Y(t_{n+2}, r) = [\bar{y}(t_{n+2}, r), \underline{y}(t_{n+2}, r)]$.

Problems

The following fuzzy problems were considered.

1. $\widetilde{y'} = \tilde{y}(t)$, $\tilde{y}(0) = (0.75 + 0.25\alpha, 1.125 - 0.125\alpha)$

   Exact solution at $t = 1.0$ is given by

$$Y(1,\alpha) = \begin{bmatrix} (0.75 + 0.25\alpha)e, \\ (1.125 - 0.125\alpha)e \end{bmatrix}, \alpha \in [0,1]$$

   Reduced to the system of ODE

$$\frac{d}{dt}\left(y_1^\alpha(t)\right) = y_1^\alpha(t), y_1^\alpha(0) = 0.75 + 0.25\alpha$$
$$\frac{d}{dt}\left(y_2^\alpha(t)\right) = y_2^\alpha(t), y_2^\alpha(\alpha) = 1.125 - 0.125\alpha$$

2. $\tilde{y}' = -\tilde{y}(t)$, $\tilde{y}(0) = (0.96 + 0.04\alpha, 1.01 - 0.01\alpha)$

   Exact solution:

$$Y(0.1,\alpha) = \begin{bmatrix} (0.985 + 0.015\alpha)e^{-0.1} - (1-\alpha)0.025e^{0.1}, \\ (0.985 + 0.015\alpha)e^{-0.1} + (1-\alpha)0.025e^{0.1} \end{bmatrix}, \alpha \in [0,1]$$

   Reduced to the system of ODE

$$\frac{d}{dt}\left(y_1^{\alpha}(t)\right) = -y_2^{\alpha}(t), y_1^{\alpha}(0) = 0.96 + 0.04\alpha$$
$$\frac{d}{dt}\left(y_2^{\alpha}(t)\right) = -y_1^{\alpha}(t), y_2^{\alpha}(\alpha) = 1.01 - 0.01\alpha$$

3. $\tilde{y}' = -\tilde{y}(t) + t + 1$, $\tilde{y}(0) = (0.96 + 0.04\alpha, 1.01 - 0.01\alpha)$
Exact solution:

$$Y(0.1,\alpha) = \begin{bmatrix} 0.1 + (0.985 + 0.015\alpha)e^{-0.1} - (1-\alpha)0.025e^{0.1}, \\ 0.1 + \ (0.985 + 0.015\alpha)e^{-0.1} + (1-\alpha)0.025e^{0.1} \end{bmatrix}, \alpha \in [0,1]$$

Reduced to the system of ODE

$$\frac{d}{dt}\left(y_1^{\alpha}(t)\right) = -y_2^{\alpha}(t) + t + 1, y_1^{\alpha}(0) = 0.96 + 0.04\alpha$$
$$\frac{d}{dt}\left(y_2^{\alpha}(t)\right) = -y_1^{\alpha}(t) + t + 1, y_2^{\alpha}(\alpha) = 1.01 - 0.01\alpha$$

Results and Discussions

Problems 1, 2 and 3 were taken from Mehrkanoon *et al.*(2009), Ivaz *et al.* (2013). Results obtained with the proposed methods were compared with the exact solution and shown in Figure 1 while the absolute error in the methods is depicted in figure 2 respectively.

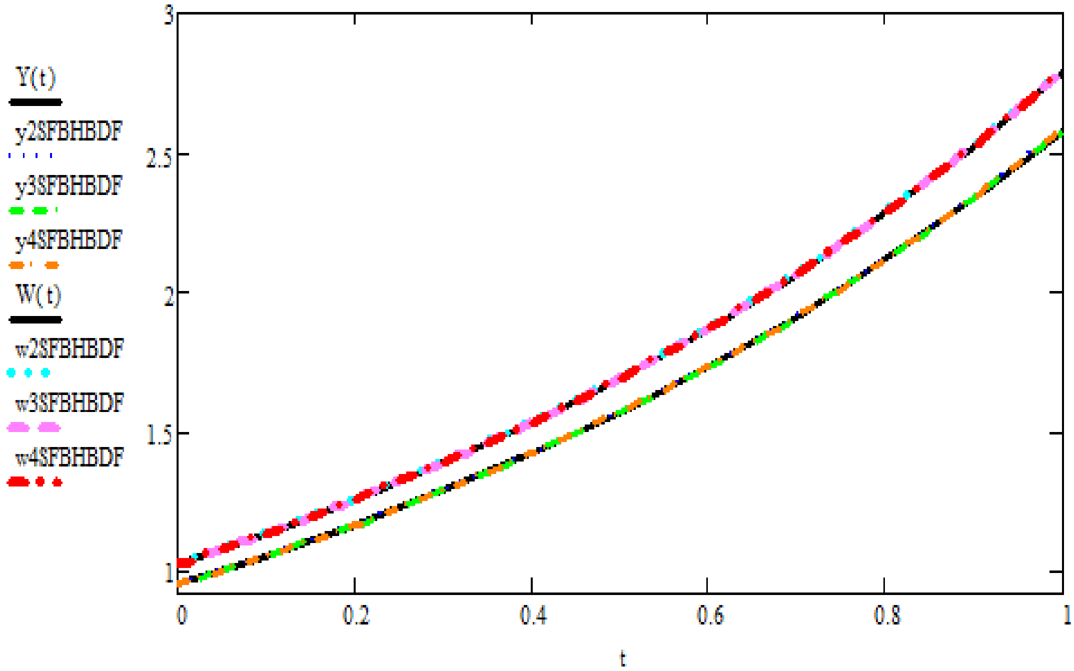

Figure 1: Exact and Numerical approximation of Problem 1

Figure 1 shows the agreement between the exact solution and approximate solution using the formulated methods. Y(t), y2SFBHBDF, y3SFBHBDF and y4SFBHBDF represents exact solution, approximate solution of the respective k-step method developed for the solution of the upper r –cut $\bar{y}(t,r)$ while W(t), w2SFBHBDF, w3SFBHBDF and w4SFBHBDF give the solution of the lower r –cut $\underline{y}(t,r)$ for Problem 1 with$t \in [0,1]$,$r = 0.8$ and $h = 0.1$.

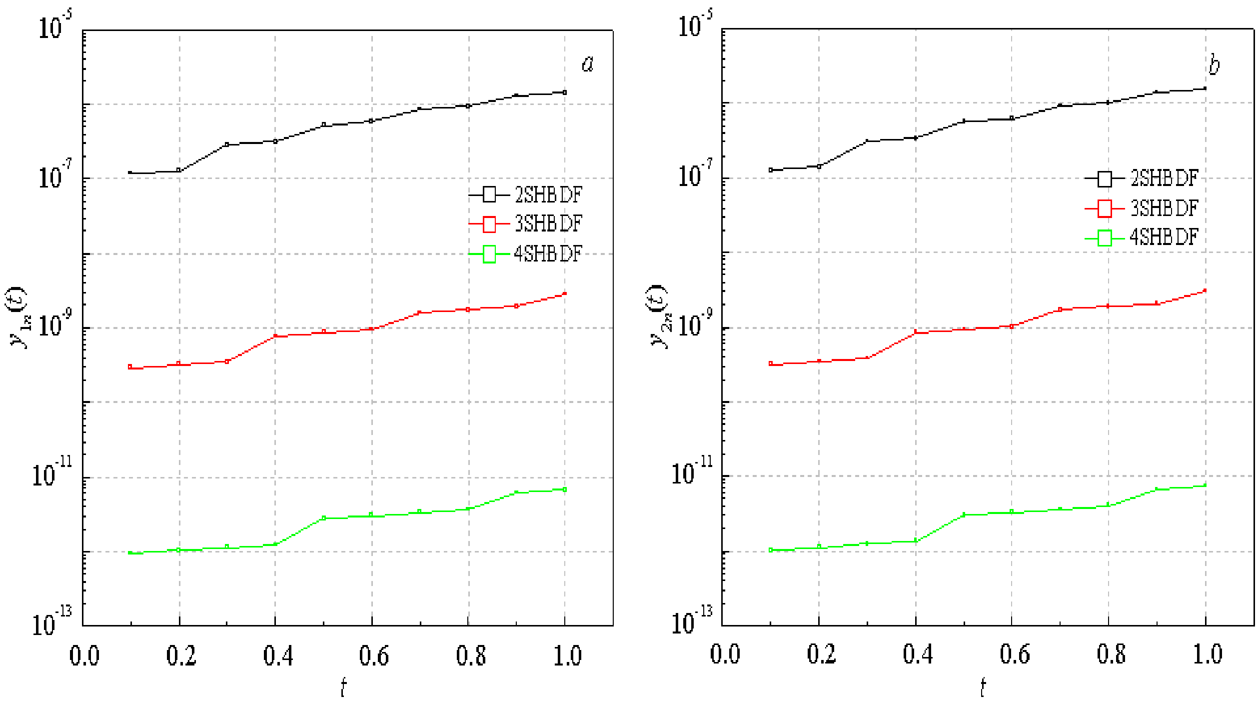

Figure .2: Absolute error in y_{1_n} (left) and y_{2_n} (right) using the proposed methods for problem 1

From Figure 2, we represent the upper r –cut $\bar{y}(t,r)$as y_{1_n} and the lower r –cut $\underline{y}(t,r)$as y_{2_n} for Problem 1.It is observed that as the number of step k increases, the absolute error in the solution obtained with the proposed methods reduces.

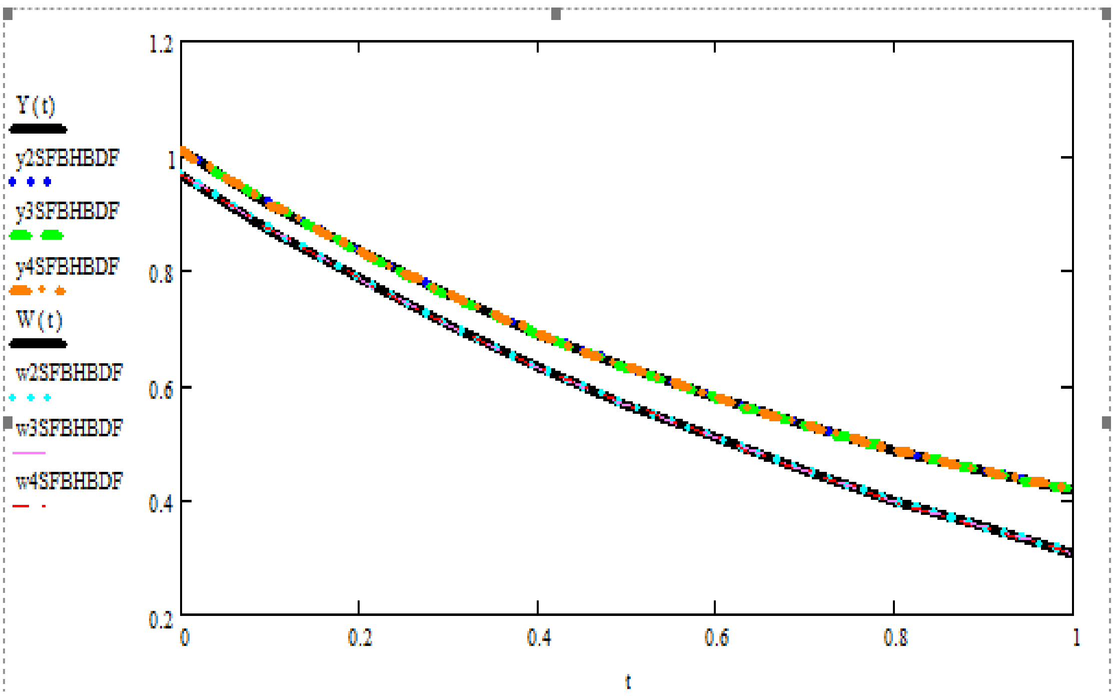

Figure 3: Exact and Numerical approximation of Problem 2

Figure 3 shows the agreement between the exact solution and approximate solution using the formulated methods. Y(t), y2SFBHBDF, y3SFBHBDF and y4SFBHBDF represents exact solution, approximate solution of the respective k-step method developed for the solution of the upper r –cut $\bar{y}(t,r)$ while W(t), w2SFBHBDF, w3SFBHBDF and w4SFBHBDF give the solution of the lower r –cut $\underline{y}(t,r)$ for Problem 2 with $t \in [0,1], h = 0.1$ with $r = 0.2$.

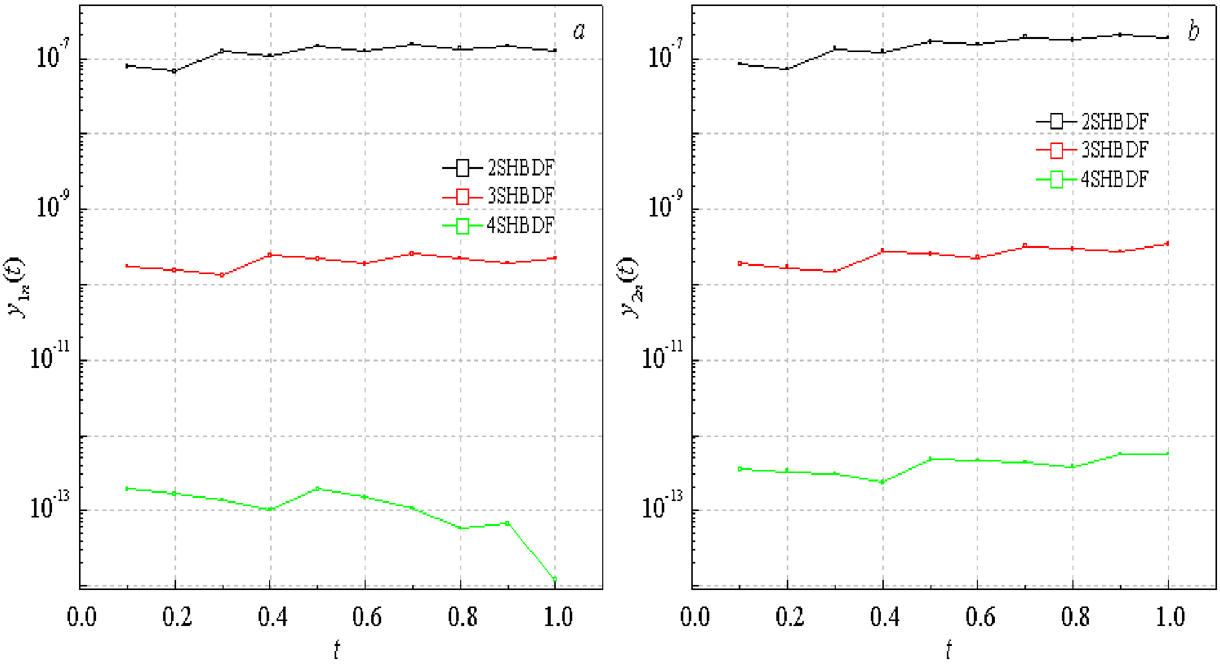

Figure 4: Absolute error in the proposed methods for problem 2

From Figure 4, we represent the upper r –cut $\bar{y}(t,r)$as y_{1_n} and the lower r –cut $\underline{y}(t,r)$as y_{2_n} for Problem 2.It is observed that as the number of step k increases, the absolute error in the solution obtained with the proposed methods reduces.

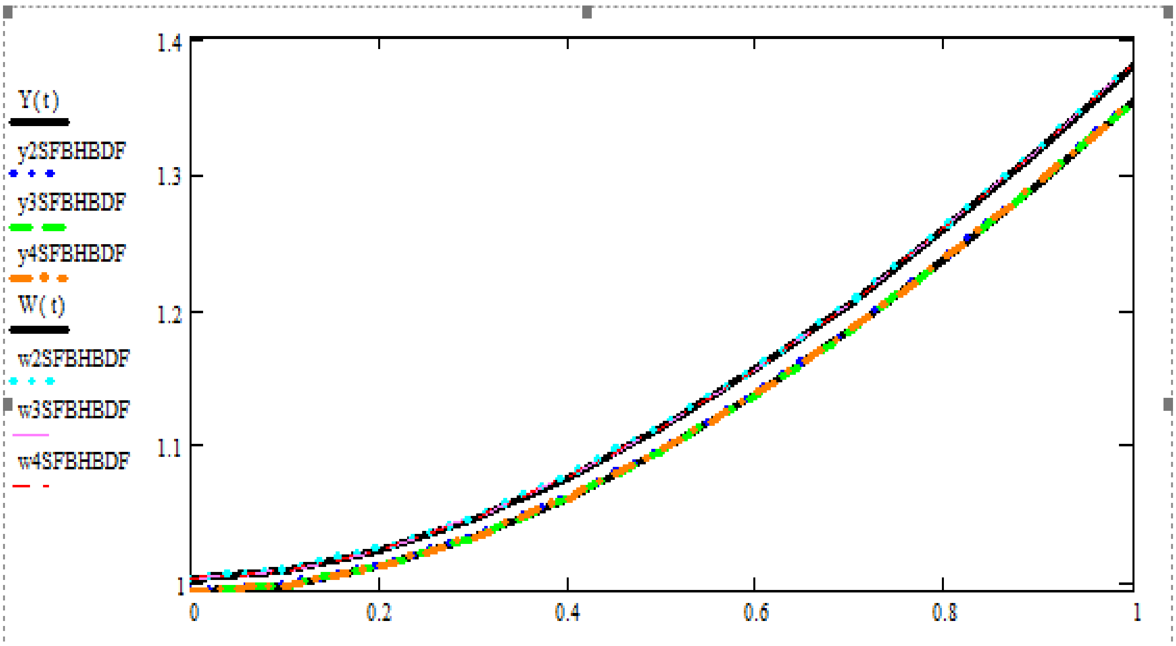

Figure 5: Exact and Numerical approximation of Problem 3

Figure 5 shows the agreement between the exact solution and approximate solution using the formulated methods. Y(t), y2SFBHBDF, y3SFBHBDF and y4SFBHBDF represents exact solution, approximate solution of the respective k-step method developed for the solution of the upper r −cut $\bar{y}(t,r)$ while W(t), w2SFBHBDF, w3SFBHBDF and w4SFBHBDF give the solution of the lower r −cut $\underline{y}(t,r)$ for Problem 3 with $t \in [0,1], h = 0.1$ with $r = 0.8$.

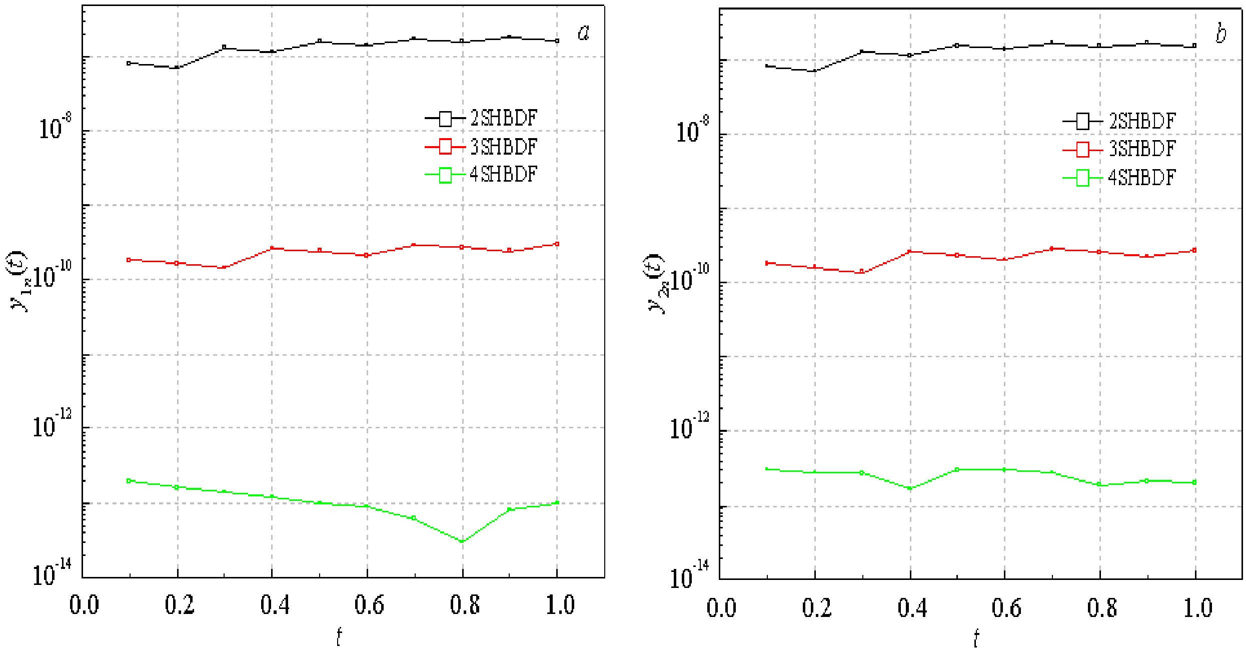

Figure 6: Absolute error in the proposed methods for problem 3

We also solved Problem 3 for different values of r and the results obtained are presented in Figures 7.

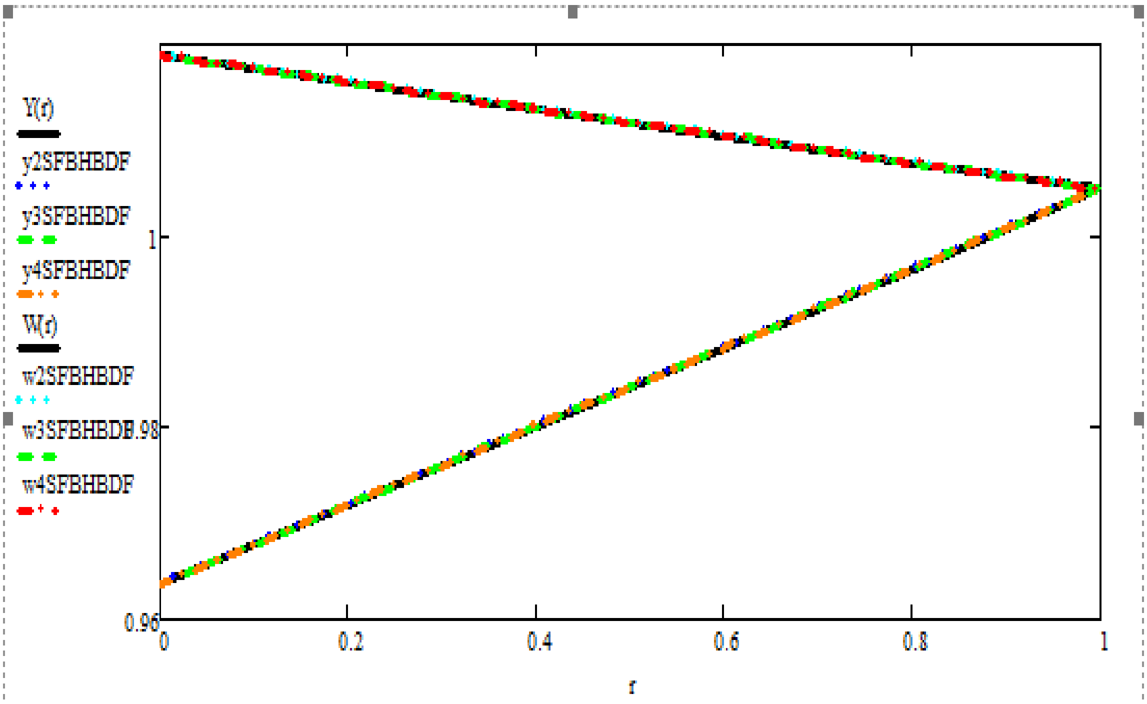

Figure 7: Exact and Numerical approximation of Problem 3 for different values of r

Figure 7 shows the agreement in between the exact solution and approximate solution using the formulated methods. Y(r), y2SFBHBDF, y3SFBHBDF and y4SFBHBDF represents exact solution, approximate solution of the respective k-step method developed for the solution of the upper r −cut $\bar{y}(t,r)$ while W(r), w2SFBHBDF, w3SFBHBDF and w4SFBHBDF give the solution of the lower r −cut $\underline{y}(t,r)$ for Problem 3 at $t = 0.1$ with $r \in [0,1], h = 0.01$.

We also compared the absolute error in the one of the formulated methods (2SFBHBDF) with the methods used in Ivaz *et al*. (2013) for Problem 3 as shown in Table 1. It is observed that the formulated methods perform better than existing methods.

Table 1: Comparing the absolute error in the new method for $k = 2$ with the methods in Ivaz *et a.l* (2013) varying r for problem 6 at $t = 0.1$ and $r \in [0, 1]$.

r	Ivaz et al (2013) $\underline{y}$ (Trapezoidal)	Ivaz et al (2013) $\underline{y}$ (Midpoint)	New method $\underline{y}$ (2SFBHBDF)	Ivaz et al (2013) $\bar{y}$ (Trapezoidal)	Ivaz et al (2013) $\bar{y}$ (Midpoint)	New method $\bar{y}$ (2SFBHBDF)
0.0	8.00E-07	5.06E-03	4.43E-12	7.00E-07	5.06E-03	4.15E-12
0.2	7.00E-07	4.05E-03	4.39E-12	7.00E-07	4.05E-03	4.18E-12
0.4	7.18E-07	3.04E-03	4.38E-12	7.00E-07	3.03E-03	4.25E-12
0.6	7.84E-07	2.02E-03	4.36E-12	8.00E-07	2.02E-03	4.27E-12
0.8	7.51E-07	1.01E-03	4.35E-12	7.00E-07	1.01E-03	4.27E-12
1.0	7.18E-07	1.48E-06	4.34E-12	7.00E-07	1.50E-06	4.32E-12

Conclusion

We now have at our disposal, three different methods having good consistency properties for the approximation of Fuzzy differentia equations, accuracy of which improves as the step number increases. Efficiency and accuracy of the methods have been tested and established from the results.

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