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Two-Step Second Derivative Methods with Hermite Polynomial as Basis Function for Solving Stiff Problems

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ABSTRACT
In this paper, a class of two-step second derivative numerical method is developed by incorporating one or more function evaluation at collocation with carefully selected off-grid points. The continuous formulations of the methods are derived through the interpolation and collocation technique with Hermite polynomial as basis function. The two numerical schemes derived are of higher order of accuracy with relatively small error constants. The methods are consistent and zero stable and hence convergent. The stability properties of the methods are carried out via the linear system. Both methods are A-stable as their regions of absolute stability contain the entire left-hand plane of the stability region. Furthermore, the two methods were implemented as block forms in other to simultaneously produce approximate solutions to some standard stiff problems (both linear and nonlinear) found in the literature. Hence our methods are self-starting and do not require separate methods to start the implementation. The errors incurred in our methods on the problems considered, are relatively lower than the methods found in the literature.

Keywords*:* Second Derivatives; Two-Step; Stiff Problems; Hermite Polynomial

INTRODUCTION
Numerical methods for ordinary differential equations (ODEs) are very significant in scientific computation, as they are generally used for solving real life problems. In numerous applications modeled by ordinary differential equations, there exist some problems that exhibit a behavior known as stiffness. Stiff problems are usually difficult to obtain their numerical solutions because certain numerical methods such as explicit methods designed for stiff problems are used with very small step sizes or do not converge at all. The idea of stiffness, occurring in differential equations came as a result of some spearheading works done by the two physicists, Curtiss and Hirschfelder (1952). Shampine and Watt (1969) in their text, explained the characteristics of numerical methods used for solving problems associated with stiffness and examined the diverse realistic objectives when solving stiff problems which includes methods with strong stability properties for solving stiff problems.

Numerical methods for approximating the solution of stiff problems are required to possess good stability properties such as having wide region of absolute stabilities which contain the entire or large enough left half of the complex plan (Akinfenwa *et al.,* 2014; Muka and Obiorah, 2016; Abhulimen and Ukpebor, 2019). Methods of which region of absolute stability contains the entire left half of the complex plane are known as A-stable methods (Butcher, 2008; Ngwane and Jator, 2012; Athe and Muka, 2017). However, A-stable methods are implicit and cannot exceed order p=2 (Lambert, 1991; Hairer, *et al.,* 2002; Butcher, 2008; Athe and Muka, 2017). This restriction is generally known as the second Dahlquist order barrier (Hairer *et al.,* 2002). In a bid to address this barrier, various research attempts have been proposed which include multi-derivative terms in the derivation process (Enright, 1974; Akinfenwa *et al.,* 2017; Bakari *et al.,* 2018; Abhulimen and Ukpebor, 2019), and inclusion of off-grid points (Mehdizadeh *et al.,* 2012; Sahil *et al.,* 2012; Yakubu *et al.,* 2017). Abdelrahim and Omar (2015) adopted the method of interpolation and collocation method in developing a new one step hybrid block method (with two off step points) for solving third order initial value problem of ordinary differential equations. Their method which is of order 4 and zero stable successfully solved third order IVPs. Omar and Abdelrahim (2016) developed an order 4 zero stable one step hybrid block method for solving second order IVPs using collocation and interpolation approach. In deriving their method, the power series

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used as basis function to approximate the solution is interpolated at the off step points while its second derivative is collocated at all points in the selected interval.

In this research, the continuous formulation of a class of two-step implicit linear multistep method incorporating second derivatives in the derivation process and imposing some suitable off-step points is proposed. The continuous scheme derived is expected to generate a number of sufficient schemes in order to solve stiff ODEs as a block method, hence overcoming the problem associated with starting values and predictor-corrector.

Derivation of the Methods

The proposed two-step hybrid block method with second derivative that produces approximations y_{n+k} to the first order ordinary differential equations (ODEs)

$$x' = f\left(t, x\right) \tag{1}$$

is given as follows:

$$\sum_{j=0}^{k}\alpha_j x_{n+j} + \sum_{j=1}^{2}\alpha_{vj} x_{n+vj} = h\left(\sum_{j=0}^{k}\beta_j f_{n+j} + \sum_{j=1}^{2}\beta_{vj} f_{n+vj}\right) + h^2\gamma_k f'_{n+k} \tag{2}$$

α_j, α_{jv} β_j and β_{vj} are constant coefficients.

In order to obtain (2), we approximate the solution by the orthogonal function $X\left(t\right)$ of the form

$$X\left(t\right) = \sum_{j=0}^{r+s-1} a_j \varphi\left(t\right) \tag{3}$$

where

(i) $t \in \left[a, b\right]$

(ii) $\varphi\left(t\right)$ is an orthogonal function defined by the Hermit polynomial

(iii) a_j are unknown coefficients to be determined

(iv) *r* is the number of interpolations for $1 \le r \le k$ and

(v) *s* is the number of distinct collocation points with $s > 0$

The continuous approximation is constructed by imposing the following conditions

$$X\left(t_{n+\mu}\right) = x_{n+\mu}, \quad \left\{j, v_1, v_2\right\}, j = 0, 1, ..., \mathrm{k}-1 \tag{4}$$

$$X'\left(t_{n+\mu}\right) = f_{n+\mu}, \quad \left\{j, v_1, v_2\right\}, j = 0, 1, ..., \mathrm{k} \tag{5}$$

$$X''\left(t_{n+j}\right) = f'_{n+j}, \quad j = k \tag{6}$$

where v_1 and v_2 are not integers. Equations (4) – (6) form a system of nonlinear equations in a_j's which is solved using the matrix inversion technique via Maple software. The values of a_j's obtained are then substituted back into (3) to yield the continuous formulation of our proposed method in the form

$$\sum_{j=0}^{k}\alpha_j\left(t\right) x_{n+j} + \sum_{j=1}^{2}\alpha_{vj}\left(t\right) x_{n+vj} = h\left(\sum_{j=0}^{k}\beta_j\left(t\right) f_{n+j} + \sum_{j=1}^{2}\beta_{vj}\left(t\right) f_{n+vj}\right) + h^2\gamma_k\left(t\right) f'_{n+k}, \tag{7}$$

which upon evaluation at $t = t_{n+k}$, $k = 2$ gives the discrete two-step second derivative hybrid method. However, we intend to implement our methods in a block form, which shall simultaneously generate approximate solutions to (1). In view of this, evaluating the second derivative of (7) at some required points gives a number of discrete schemes necessary to implement the methods in block form. In what follows, two separate block methods for two-step second derivative block hybrid method will be derived following the above procedures. The distinguishing factor in the two proposed block methods is the choice of v_i, $\left(i = 1, 2\right)$

Two-step second derivative hybrid block method with $v_1 = \frac{1}{2}$ and $v_2 = \frac{3}{2}$ (TSDHBM1)

To derive this method, (3) is interpolated at $t = t_n, t_{n+\frac{1}{2}}, t_{n+1}, t_{n+\frac{3}{2}}$ and (4) is collocated at $r + s - 1 = 4 + 6 - 1 = 9$ as follows: $t = t_n, t_{n+\frac{1}{2}}, t_{n+1}, t_{n+\frac{3}{2}}, t_{n+2}$ while (5) is also collocated at $t = t_{n+2}$. These of course are done using the Hermit polynomial of degree

$$\left.\begin{aligned} X(t) &= a_0 + 2ta_1 + \left(8t^2 - 4\right)a_2 + \left(8t^3 - 12t\right)a_3 + \left(16t^4 - 48t^2 + 12\right)a_4 + \left(32t^5 - 160t^3 + 120t\right)a_4 \\ &+ \left(64t^6 - 480t^4 + 720t^2 - 120\right)a_6 + \left(128t^7 - 1344t^5 + 3360t^3 - 1680t\right)a_7 \\ &+ \left(256t^8 - 358t^6 + 13440t^4 - 13440t^2 + 1680\right)a_8 + \left(512t^9 - 9216t^7 + 48384t^5 - 80640t^3 30240t\right)a_9 \end{aligned}\right\} \quad (8)$$

The above procedure leads to a system of nonlinear equations in the form

$$AU = B \quad (9)$$

where

$A = \left[a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9\right]^T$,

$B = \left[x_n, x_{n+\frac{1}{2}}, x_{n+1}, x_{n+\frac{3}{2}}, f_n, f_{n+\frac{1}{2}}, f_{n+1}, f_{n+\frac{3}{2}}, f_{n+2}, f'_{n+2}\right]^T$

and U are the coefficients of a_j's in (8). Using the matrix inversion technique, (9) is solved for the vales of a_j's and then substituted into (8) which gives the continuous scheme in the form

$$\left.\begin{aligned} x(t) &= \alpha_o(t)x_n + \alpha_{\frac{1}{2}}(t)x_{n+\frac{1}{2}} + \alpha_1(t)x_{n+1} + \alpha_{\frac{3}{2}}(t)x_{n+\frac{3}{2}} + \\ & h\left(\beta_o(t)f_n + \beta_{\frac{1}{2}}(t)f_{n+\frac{1}{2}} + \beta_1(t)f_{n+1} + \beta_{\frac{3}{2}}(t)f_{n+\frac{3}{2}} + \beta_2(t)f_{n+2}\right) + h^2\gamma_2(t)f'_{n+2} \end{aligned}\right\} \quad (10)$$

The continuous coefficients $\alpha_j(t)$ $\beta_j(t)$ and $\gamma_2(t)$ are given in the appendix. Evaluating (10), at $t = t_{n+2}$, gives the first discrete scheme as

$$\begin{aligned} x_{n+2} &= \frac{53}{485}x_n + \frac{512}{485}x_{n+\frac{1}{2}} + \frac{432}{485}x_{n+1} - \frac{512}{485}x_{n+\frac{3}{2}} + \frac{6}{485}hf_n + \frac{128}{485}hf_{n+\frac{1}{2}} + \frac{432}{485}hf_{n+1} \\ &+ \frac{384}{485}hf_{n+\frac{3}{2}} + \frac{20}{97}hf_{n+2} - \frac{6}{485}h^2 f'_{n+2} \end{aligned} \quad (11)$$

The discrete Scheme in (11) is however not sufficient to implement the numerical solution for any initial value problem. Therefore, in order to obtain other necessary and sufficient schemes we take the second derivative of the continuous scheme (10) and evaluate $t = t_{n+\frac{1}{2}}$, t_{n+1} and $t_{n+\frac{3}{2}}$ to generate the following schemes are generated.

$$\begin{aligned} x_{n+\frac{1}{2}} &= \frac{1939}{21249}x_n + \frac{503}{787}x_{n+1} + \frac{5729}{21249}x_{n+\frac{3}{2}} + \frac{503}{56664}hf_n - \frac{4706}{21249}hf_{n+\frac{1}{2}} - \frac{449}{1574}hf_{n+1} \\ &- \frac{341}{7083}hf_{n+\frac{3}{2}} + \frac{415}{169992}hf_{n+2} - \frac{485}{14166}h^2 f'_{n+\frac{1}{2}} - \frac{1}{3148}h^2 f'_{n+2} \end{aligned} \quad (12)$$

$$x_{n+1}=\frac{1883}{63963}x_n+\frac{1216}{2369}x_{n+\frac{1}{2}}+\frac{29248}{63963}x_{n+\frac{3}{2}}+\frac{541}{170568}hf_n+\frac{6268}{63963}hf_{n+\frac{1}{2}}+\frac{56}{2369}hf_{n+1}$$
$$-\frac{1492}{21321}hf_{n+\frac{3}{2}}+\frac{1345}{511704}hf_{n+2}-\frac{485}{9476}h^2f'_{n+1}-\frac{7}{21321}h^2f'_{n+2} \qquad (13)$$

$$x_{n+\frac{3}{2}}=\frac{751}{23809}x_n+\frac{8289}{23809}x_{n+\frac{1}{2}}+\frac{14769}{23809}x_{n+1}+\frac{671}{190472}hf_n+\frac{1951}{23809}hf_{n+\frac{1}{2}}+\frac{567}{1642}hf_{n+1}$$
$$+\frac{6338}{23809}hf_{n+\frac{3}{2}}+\frac{1645}{190472}hf_{n+2}-\frac{1455}{47618}h^2f'_{n+\frac{3}{2}}-\frac{93}{95236}h^2f'_{n+2} \qquad (14)$$

Two-step second derivative hybrid block method with $v_1=\dfrac{5}{17}$ and $v_2=\dfrac{29}{17}$ (TSDHBM2)

Similar to (TSDHBM1), (4) is interpolated at $t=t_n,t_{n+\frac{5}{17}},t_{n+1},t_{n+\frac{29}{17}}$ and (5) is collocated at $t=t_n,t_{n+\frac{5}{17}},t_{n+1},t_{n+\frac{29}{17}},t_{n+2}$ while (6) is also collocated at $t=t_{n+2}$ and following similar procedure, the continuous formulation is given as

$$\left.\begin{array}{l} x(t)=\alpha_o(t)x_n+\alpha_{\frac{5}{17}}(t)x_{n+\frac{5}{17}}+\alpha_1(t)x_{n+1}+\alpha_{\frac{29}{17}}(t)x_{n+\frac{29}{17}}+ \\ h\left(\beta_o(t)f_n+\beta_{\frac{5}{17}}(t)f_{n+\frac{5}{17}}+\beta_1(t)f_{n+1}+\beta_{\frac{29}{17}}(t)f_{n+\frac{29}{17}}+\beta_2(t)f_{n+2}\right)+h^2\gamma_2(t)f'_{n+2}\end{array}\right\} \quad (15)$$

The following schemes are generated from (15) and its second derivative:

$$x_{n+\frac{5}{17}}=\frac{51737727016894464}{64986068469543839}x_n+\frac{365653970000}{2664564700051}x_{n+1}+\frac{3049889375}{45769446127}x_{n+\frac{29}{17}}$$
$$+\frac{7321612861440}{1318175583102523}hf_n+\frac{72319104240}{778080584159}hf_{n+\frac{5}{17}}-\frac{230661720000}{2664564700051}hf_{n+1}$$
$$-\frac{686710200}{26830364971}hf_{n+\frac{29}{17}}+\frac{447626970316800}{64986068469543839}hf_{n+2}-\frac{14805079200}{456116204507}h^2f'_{n+\frac{5}{17}} \qquad (16)$$
$$-\frac{1668169728000}{2240898912742891}h^2f'_{n+2}$$

$$x_{n+1}=\frac{46340865220608}{85184079170875}x_n-\frac{24137569}{6985450750}x_{n+\frac{5}{17}}+\frac{626200952567}{1362945266734}x_{n+\frac{29}{17}}$$
$$+\frac{26778428928}{587476408075}hf_n+\frac{301624482081}{1174952816150}hf_{n+\frac{5}{17}}+\frac{779520}{27941803}hf_{n+1}$$
$$-\frac{171860911137}{1174952816150}hf_{n+\frac{29}{17}}+\frac{19581958656}{587476408075}hf_{n+2}-\frac{2387916}{27941803}h^2f'_{n+1} \qquad (17)$$
$$-\frac{13934592}{4051561435}h^2f'_{n+2}$$

$$x_{n+\frac{29}{17}} = \frac{64377523990167552}{111077878514198375} x_n - \frac{13495628761}{78231736375} x_{n+\frac{29}{17}} + \frac{526897739917904}{888623028113587} x_{n+1} + \frac{65702903169024}{1306798570755275} hf_n + \frac{64386092088}{265987903675} hf_{n+\frac{5}{17}} + \frac{410549393010624}{888623028113587} hf_{n+1} + \frac{260006622864}{1329939518375} hf_{n+\frac{29}{17}} + \frac{23752029288185856}{111077878514198375} hf_{n+2} - \frac{498042864288}{4521794362475} h^2 f'_{n+\frac{29}{17}} - \frac{404713710157824}{22215575702839675} h^2 f'_{n+2} \tag{18}$$

$$x_{n+2} = \frac{160805}{2056261} x_n - \frac{120687845}{4737625344} x_{n+\frac{5}{17}} + \frac{442050625}{7994742768} x_{n+1} + \frac{114098288663}{127915884288} x_{n+\frac{29}{17}} + \frac{42050}{6168783} hf_n + \frac{1029396325}{31978971072} hf_{n+\frac{5}{17}} + \frac{442050625}{7994742768} hf_{n+1} + \frac{5970498685}{31978971072} hf_{n+\frac{29}{17}} + \frac{922780}{6168783} hf_{n+2} - \frac{42050}{6168783} h^2 f'_{n+2} \tag{19}$$

Analysis of the Methods

In this section, the local truncation error and order, consistency, zero-stability and convergence analysis of the derived methods are investigated and it is observed that the methods were zero stable, consistence and hence convergence.

Local truncation error and order

Following Nwachukwu and Okor (2018), we can rewrite each of the derived methods as:

$$L\left[y(x);h\right] = \sum_{j=0}^{k}\left(\alpha_j y(x+jh)\right) + h\left(\sum_{j=0}^{k}\beta_j f_{n+j}\right) + h^2\left(\sum_{j=0}^{k}\gamma_j g_{n+j}\right) \tag{20}$$

Expanding (20) in Taylor series, the local truncation error associated with (1) is the linear difference operator

$$L\left[y(x);h\right] = \sum_{j=0}^{k}\left(\alpha_j y(x+jh)\right) - h\left(\sum_{j=0}^{k}\beta_j y'(x+jh)\right) - h^2\left(\sum_{j=0}^{k}\gamma_j y''(x+jh)\right) \tag{21}$$

Assuming that $y(x)$ is sufficiently differentiable, we can expand the terms in (21) as a Taylor series about the point x to obtain the expression

$$L\left[y(x);h\right] = C_0 y(x) + C_1 h y'(x) + \ldots + C_q h^q y^{(q)}(x) + \ldots \tag{22}$$

where the constant $C_q, q = 0,1,\ldots$ are given as follows

$$\left.\begin{aligned} C_0 &= \sum_{j=0}^{k}\alpha_j \\ C_1 &= \sum_{j=1}^{k} j\alpha_j - \sum_{j=0}^{k}\beta_j \\ C_2 &= \frac{1}{2!}\sum_{j=1}^{k}(j)^2\alpha_j - \sum_{j=1}^{k} j\beta_j - \sum_{j=0}^{k}\gamma_j \\ &\vdots \\ C_q &= \frac{1}{q!}\sum_{j=1}^{k}(j)^q\alpha_j - \frac{1}{(q-1)!}\sum_{j=1}^{k} j^{q-1}\beta_j - \frac{1}{(q-2)!}\sum_{j=1}^{k} j^{q-2}\gamma_j \end{aligned}\right\} \tag{23}$$

A linear multistep method is said to be of order of accuracy p if $C_0 = C_1 = ...C_p = 0, C_{p+1} \neq 0.$ C_{p+1} is called the error constant (Akinfenwa *et al.,* 2015)

Table 1: Order and Error Constants for TSDBM1

Equations	Order p	Error constants, C_{p+1}
11	9	$\frac{1}{65184000}$
12	9	$\frac{1}{58060800}$
13	9	$\frac{557}{182775398400}$
14	9	$\frac{199}{91697356800}$

Table 2: Order and Error Constants for TSDBM2

Equation	Order p	Error constants, C_{p+1}
16	9	$\frac{129381561600}{264831058892421889 49}$
17	9	$\frac{5555352}{408402282463525}$
18	9	$\frac{5583159518293248}{2208005846949478269 85325}$
19	9	$\frac{17682025}{4674102375082896}$

RESULTS AND DISCUSSIONS

Numerical Experiments

Problem 1: Consider the linear problem

$$x' = -20x + 20\sin t + \cos t \qquad x(0) = 1, \quad 0 \leq t \leq 2$$

With exact solution

$$x(t) = \sin t + e^{-20t}$$

(Mohamad *et al.*, 2018).

Problem 2: The circuit problem (Source: Musa *et al.*, 2013).

$$x' = -8x + 8t + 1; \quad x(0) = 2, \quad 0 \leq t \leq 10.$$

Exact solution: $x(t) = t - 2e^{-8t}$

Problem 3: We also consider a nonlinear highly stiffed problem. Source: (Mohamad *et al.* 2018).

$$x'(t) = \frac{50}{x(t)} - 50x(t), \quad x(0) = \sqrt{2}, \ 0 \leq t \leq 1$$

Exact solution: $x(t) = \sqrt{1 + e^{-100t}}$

Problem 4. We consider the following linear stiff system of IVP on the range $0 \leq x \leq 1.$ Source: Mohammed (2022).

$$y' = -y + 95z, \qquad y(0) = 1$$

$$z' = -y - 97z, \qquad z(0) = 1$$

Exact solution:

$$y(x)=\frac{95}{47}e^{-2x}-\frac{48}{47}e^{-96x}$$
$$z(x)=\frac{48}{47}e^{-96x}-\frac{1}{47}e^{-2x}$$

Problem 5. We consider the following nonlinear IVP over the range $0 \le x \le 1$. Source: Mohammed *et al.* (2022).

$$y'=-1002y+1000z^2, \qquad y(0)=1$$
$$z'=y-z(1+z), \qquad z(0)=1$$

Exact solution: $y(x)=e^{-2x}$, $z(x)=e^{-x}$

Table 3: Comparing the Maximum errors for Problem 1

h	**Methods**	**Maximum errors**
	TSDBHM1	5.897×10^{-15}
0.01	TSDBHM2	4.108×10^{-15}
0.001	Mohamad *et al.* (2018)	7.35×10^{-04}

Table 3 shows the numerical computations for problem 1 for *h=0.01*. The table displays the exact solutions at the interval of 0.1 and the comparison of the absolute errors for both TSDBHM1 and TSDBHM2 are also shown. Our methods compare favorably with the analytical solution. In table 4, we compare the maximum errors incurred in our numerical methods with that of Mohamad *et al.* (2018). Our methods outperform the method of Mohamad *et al.* (2018) with smaller *h=0.001*.

Table 4: Numerical results for Problem 2 with *h* =0.1

t	**Exact Solution**	**Error in TSDBM 1**	**Error in TSDBM 2**	**Error in Ehiemua and Agbeboh (2019)**
0.1	0.998657928234444	$2.68*10^{-9}$	$1.94*10^{-9}$	$8.30*10^{-7}$
0.2	0.603793035989310	$1.62*10^{-9}$	$1.44*10^{-9}$	$7.46*10^{-7}$
0.3	0.481435906578825	$1.27*10^{-9}$	$1.04*10^{-9}$	$5.03*10^{-7}$
0.4	0.481524407956732	$6.56*10^{-10}$	$5.83*10^{-10}$	$3.01*10^{-7}$
0.5	0.536631277777468	$4.04*10^{-10}$	$3.41*10^{-10}$	$1.69*10^{-7}$
0.6	0.616459494098040	$1.99*10^{-10}$	$1.77*10^{-10}$	$9.12*10^{-8}$
0.7	0.707395727432966	$1.22*10^{-10}$	$9.53*10^{-11}$	$4.78*10^{-8}$
0.8	0.803323114546348	$5.35*10^{-11}$	$4.75*10^{-11}$	$2.45*10^{-8}$
0.9	0.901493171616753	$2.85*10^{-11}$	$2.46*10^{-11}$	$1.24*10^{-8}$
1.0	1.00067092525581	$1.35*10^{-11}$	$1.20*10^{-11}$	$6.20*10^{-8}$

Table 4 above presents the exact solution of problem 2 and absolute error incurred in solving the problem with the newly derived schemes. The two newly derived schemes are of the same order of accuracy but different error constants due the change in the choice of the off-grid points used. The method TSDHBM 2 with lower error constants performs slightly better than its counterpart TSDHBM 1. However, both methods outperform the method of Ehiemua and Agbebor (2019).

Table 5: Comparing the Maximum errors for Problem 3

h	**Methods**	**Maximum errors**
	TSDBHM1	1.212×10^{-03}
0.001	TSDBHM2	7.122×10^{-04}
0.001	Mohamad *et al.* (2018)	3.898×10^{-03}

Table 5 presents the numerical solution of problem 3 which is highly stiffed. The problem was solved using the step size of *h=0.001* and the absolute error are shown. However, our methods still perform better than the method of Mohamed *et al.* (2018) as displayed in table 7 with the same step size.

Table 6: Comparing the Absolute error in the proposed method with existing methods found in literature for problem 4.

h	Ehigie and Okunuga (2013) y_n z_n	Biala *et al* (2015) y_n z_n	Abhulimen and Ukpebor (2018) y_n z_n	Mohammed *et al. (*2022) y_n z_n	TSDBHM1 y_n z_n	TSDBHM2 y_n z_n
0.0625	3.4×10^{-9} 3.6×10^{-9}	4×10^{-10} 8×10^{-10}	5.0×10^{-8} 7.0×10^{-10}	9.25×10^{-11} 9.56×10^{-11}	$1.18*10^{-16}$ $1.24*10^{-18}$	$8.40*10^{-17}$ $8.84*10^{-19}$
0.03125	3.4×10^{-9} 3.5×10^{-9}	7×10^{-12} 7×10^{-14}	6.0×10^{-8} 1.0×10^{-10}	7.8×10^{-13} 1.1×10^{-16}	$5.76*10^{-23}$ $9.08*10^{-24}$	$1.58*10^{-22}$ $4.24*10^{-24}$

The two newly derived methods perform better than existing methods at t=1 as shown in the table 6

Table 7: Comparing the Absolute error in the proposed method with existing methods found in literature for problem 5.

h	Akinfenwa *et al.* (2013) y_n z_n	Mohammed *et al.* (2022) y_n z_n	TSDBHM1 y_n z_n	TSDBHM2 y_n z_n
0.02	9.1102×10^{-13} 1.2527×10^{-12}	2.12×10^{-21} 7.89×10^{-17}	8.55×10^{-24} 9.52×10^{-24}	5.40×10^{-24} 6.68×10^{-24}

The two newly derived methods perform better than existing methods at t=1 as shown in the table 7

CONCLUSION

In this paper, we seek an approximate solution from the Hermit polynomial in order to derive a class of two-step second derivative linear multistep methods for solving stiff problems of first order ordinary differential equations. In the derivation process, some carefully selected off-grid points are incorporated to interpolation and collocation points. The reliability of the derived methods was tested and findings show that they are convergent and A-stable. Hence, we further test the effectiveness of the schemes on some standard stiff problems of first order ordinary differential equations. Numerical results show that the methods are more effective in terms of accuracy than some other methods we compared with in the literature.

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