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AN OPTIMIZED SINGLE-STEP BLOCK HYBRID NYSTRÖM-TYPE METHOD FOR SOLVING SECOND ORDER INITIAL VALUE PROBLEMS OF BRATU-TYPE

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ABSTRACT: *In this paper, a global single-step implicit block hybrid Nyström-type method (BHNTM) for solving nonlinear second-order initial-boundary value problems of Bratu-type is developed. The mathematical derivation of the proposed BHNTM is based on the interpolation and multistep collocation techniques with power series polynomials as the trial function. Unlike previous approaches, BHNTM is applied without linearization or restrictive assumptions. The basic properties of the proposed method, such as zero stability, consistency and convergence are analysed. The numerical results from three test problems demonstrate its superiority over existing methods which emphasize the effectiveness and reliability in numerical simulations. Furthermore, as the step size decreases as seen in the test problems, the error drastically reduces, indicating BHNTM's precision. These findings underscore BHNTM's significance in numerical methods for solving differential equations, offering a more precise and dependable approach for addressing complex problems.*

KEYWORDS: Initial value problems, Bratu-type equation, Block Hybrid Nyström-type Method, Second-Order differential equation.

INTRODUCTION

Nonlinear phenomena are of fundamental importance in various fields of science and engineering. The nonlinear models of real-life problems are still difficult to solve either numerically or analytically. There has been much attention devoted to the search for better and more efficient solution methods for determining a solution, approximate or exact, analytic or numerical, to nonlinear models, according to Jeffrey and Mohammad [9]. Bratu-type equation is a nonlinear differential equation with many applications in Mathematics, Physics, Engineering and other sciences, according to Safaa and Thair [15]. Moreover, the Bratu-type equation is formulated in the form of a nonlinear problem with either initial or boundary conditions.

This research paper is aimed at providing numerical solution to the second order initial value problem of Bratu-type with initial conditions given as follows:

$$\left.\begin{array}{l} y''(x) \ + \ \lambda e^{y(x)} = 0, \ 0 \le x \le 1, \\ y(0) = y'(0) = 0 \end{array}\right\} \tag{1.1}$$

which is used to model a combustion problem in a numerical slab. This problem derived its importance from the first thermal combustion theory, which was created by the simplification of the solid fuel ignition model. Moreover, it appeared in the Chandrasekhar model of the expansion of the universe. It stimulates a thermal reaction process in a rigid material where the process depends on a balance between chemically generated heat and heat transfer by conduction, according to Wazwaz [17].

The block methods were first proposed by Milne for solving initial value problems in differential equations (Milne [11]). Several block methods had been developed since then by researchers. Notable among them are: S. N. Jator, S. O. Fatunla, Z. A. Majid. Several numerical techniques such as the variational iteration method by Batiha [2], modified homotopy perturbation method by Feng *et al.* [5]; Adomian decomposition method by Wazwaz [16]; the finite difference method, finite element approximation, and weighted residual method by Aregbesola [1]; successive differentiation by Wazwaz [17]; an algorithm using Runge-Kutta methods of orders four and five developed by Debela *et al.* [3]; Runge-Kutta seven stages method by Fenta and Derese [5]; and Block Nyström-type integrator by Jator and Manathunga [8] have been implemented independently to handle the Bratu's initial value problems numerically. The choice of these methods depends on the specific problem characteristics, from cumbersomeness to enormous computational burden, time wastage and desired accuracy. Despite the extensive research carried out by numerous researchers to address the challenges attributed to solving second-order initial value problems of Bratu-type described by (1.1), the accuracies of many existing methods still require improvements. In order to address these issues, this research study introduces a novel approach called an optimized single-step block hybrid Nyström-type method for solving second order initial value problems of Bratu-type. This method is particularly developed to enhance the accuracy of the numerical solutions by directly solving second order IVPs of Bratu-type without transforming them into a system of first-order initial value problems of ODEs.

The research of IVPs is one of the important aspects of applied and computational mathematics because it plays a tremendous role in modelling real-life problems in dynamics, heat transfer,

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chemical and physical phenomena such as radiation reactions, chemical reactor theory and numerous other real-world differential problems, which can be modelled by Equation (1.1). Practically, the standard Nyström method is used to solve only first order initial value problems. Motivated by the different applications of the second-order IVPs in real-world modelling problems in applied sciences and engineering with the aims of improving the accuracy of some existing methods by using the global block hybrid Nyström-type method, more reliable and accurate results are provided when compared with existing methods, contributing to the advancement of solution to the difficulties associated with problems of the form represented in (1.1). The present manuscript is organized as follows. In Section 2, the BHNTM for solving second-order IVP is presented. The characteristics of the derived methods are analysed in Section 3. In Section 4, the numerical experiments were described. Finally, the presentation of numerical results of some physical modelled problems to demonstrate the efficiency and reliability of the method proposed, followed by the conclusion of the paper were discussed in Section 5.

Development of the BHNTM

Consider the problem

$$y''(x) = f(x, y, y'), \qquad y(a) = 0, \qquad y(b) = 0, \qquad x \in [a,\ b] \tag{2.1}$$

Subject to the following conditions on $f(x, y, y')$:

i. $f(x, y, y')$ is continuous,

ii. derivative of $f(x, y, y')$ exist and continuous.

considering also the problem on the interval [a, b], such that the partitioning is done using the Bhaskara cosine approximation formula to generate the hybrid points which in turn is used to develop the algorithm by approximating the cosine functions $\left\{\cos\frac{\pi i}{M}\right\}_{i=0}^{M} \approx \frac{M^2 - 4i^2}{M^2 + i^2}$, where $M = w+1, \quad w \in N : w$ (number of off-grids) $w \geq 2$, the number $h = \frac{(b-a)}{N}$ is referred to as the constant step size. Supposing the exact solution is approximated by the power series polynomial of the form

$$y(x) = \sum_{0}^{9} a_j x^j \tag{2.2}$$

where a_j are coefficients obtained distinctly and $x \in [x_n,\ x_{n+1}]$, for some natural number n. In order to determine the coefficients a_j, the following conditions hold:

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$$\left.\begin{aligned} &y(x_n) = y_n, \\ &y'(x_n) = y'_n, \\ &y''(x_{n+v}) = f_{n+v}, v = 0, \frac{5}{74}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{69}{74}, 1. \end{aligned}\right\} \tag{2.3}$$

Differentiating (2.2) once and twice gives

$$\left.\begin{aligned} &y'(x) = \sum_{j=1}^{9} j a_j x^{j-1}, \\ &y''(x) = \sum_{j=2}^{9} j(j-1) a_j x^{j-2} = f(x, y, y'). \end{aligned}\right\} \tag{2.4}$$

Using Equations (2.2) and (2.4) on the criteria given in (2.3), a system of nonlinear equations of the form are obtained.

$$\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & \dots & x_n^8 \\ 0 & 1 & 2x_n & 3x_n^2 & \dots & 8x_n^7 \\ 0 & 0 & 2 & 6x_n & \dots & 56x_n^6 \\ \dots & \dots & 2 & 6x_{n+\frac{5}{74}} & \dots & 56x_{n+\frac{5}{74}}^6 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 2 & 6x_{n+1} & \dots & 56x_{n+1}^6 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ . \\ . \\ . \\ a_8 \end{pmatrix} = \begin{pmatrix} y_n \\ y'_n \\ f_n \\ f_{n+\frac{5}{74}} \\ f_{n+\frac{1}{4}} \\ . \\ . \\ f_{n+1} \end{pmatrix} \tag{2.5}$$

The above system of nonlinear equations is solved using the matrix inversion algorithm. The unknown coefficients a_j are then substituted into (2.2) to derive the continuous Hybrid Block Nystrom-type Method with five off-step points ($\mathrm{BHNTM}_{1,5}$):

$$\begin{aligned} y(x) &= \alpha_0(x) y_n + \alpha_1(x) h y'_n \\ &+ h^2 \left(\beta_n f_n + \beta_{n+\frac{5}{74}} f_{n+\frac{5}{74}} + \beta_{n+\frac{1}{4}} f_{n+\frac{1}{4}} + \beta_{n+\frac{1}{2}} f_{n+\frac{1}{2}} + \beta_{n+\frac{3}{4}} f_{n+\frac{3}{4}} + \beta_{n+\frac{69}{74}} f_{n+\frac{69}{74}} + \beta_{n+1} f_{n+1} \right), \end{aligned} \tag{2.6}$$

and

$$y'(x) = y'_n + h \left(\beta_n f_n + \beta_{n+\frac{5}{74}} f_{n+\frac{5}{74}} + \beta_{n+\frac{1}{4}} f_{n+\frac{1}{4}} + \beta_{n+\frac{1}{2}} f_{n+\frac{1}{2}} + \beta_{n+\frac{3}{4}} f_{n+\frac{3}{4}} + \beta_{n+\frac{69}{74}} f_{n+\frac{69}{74}} + \beta_{n+1} f_{n+1} \right) \tag{2.7}$$

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where $\alpha_0(x),\ \alpha_1(x)$ and $\beta_j(x)$ are continuous coefficients that are distinctly determined. Assuming that y_n is the numerical approximation to the analytical solution $y(x_n),\ y'_n$ is the numerical approximation to $y'(x_n)$ and f_{n+v} is the numerical approximation to $f\left(x_{n+hv}, y_{n+hv}, y'_{n+hv}\right)$.

The main methods are obtained by evaluating (2.6) at points $x = x_n, x_{n+\frac{5}{74}}, x_{n+\frac{1}{4}}, x_{n+\frac{1}{2}}, x_{n+\frac{3}{4}}, x_{n+\frac{69}{74}}, x_{n+1}$ to give the discrete schemes which form the continuous block hybrid Nystrom-type method with five off-grid points:

$$
\left.
\begin{aligned}
y_{n+\frac{5}{74}} &= y_n + \frac{5}{74}hy'_n + h^2\left(\begin{aligned}&\frac{58052434225}{44612850799692}f_n + \frac{15063275}{13592395776}f_{n+\frac{5}{74}} - \frac{2933478125}{16325717140467}f_{n+\frac{1}{4}} + \frac{24930298125}{294258030395392}f_{n+\frac{1}{2}}\\ &-\frac{24485719375}{440794362792609}f_{n+\frac{3}{4}} + \frac{7823810875}{177258433314816}f_{n+\frac{69}{74}} - \frac{6595375}{323281527534}f_{n+1}\end{aligned}\right)\\
y_{n+\frac{1}{4}} &= y_n + \frac{1}{4}hy'_n + h^2\left(\begin{aligned}&\frac{348899}{89026560}f_n + \frac{68719861387}{3069160980480}f_{n+\frac{5}{74}} + \frac{449}{77568}f_{n+\frac{1}{4}} - \frac{92961}{73400320}f_{n+\frac{1}{2}} + \frac{155707}{219905280}f_{n+\frac{3}{4}}\\ &-\frac{8806682539}{16573469294592}f_{n+\frac{69}{74}} + \frac{7177}{29675520}f_{n+1}\end{aligned}\right)\\
y_{n+\frac{1}{2}} &= y_n + \frac{1}{2}hy'_n + h^2\left(\begin{aligned}&\frac{211}{28980}f_n + \frac{7142427571}{129480228864}f_{n+\frac{5}{74}} + \frac{48253}{859005}f_{n+\frac{1}{4}} + \frac{115}{16384}f_{n+\frac{1}{2}} - \frac{241}{286335}f_{n+\frac{3}{4}} + \frac{69343957}{215800381440}f_{n+\frac{69}{74}}\\ &-\frac{1}{8694}f_{n+1}\end{aligned}\right)\\
y_{n+\frac{3}{4}} &= y_n + \frac{3}{4}hy'_n + h^2\left(\begin{aligned}&\frac{37377}{3297280}f_n + \frac{29332493811}{341017886720}f_{n+\frac{5}{74}} + \frac{103833}{904960}f_{n+\frac{1}{4}} + \frac{4690143}{73400320}f_{n+\frac{1}{2}} + \frac{449}{77568}f_{n+\frac{3}{4}}\\ &-\frac{762783527}{1023053660160}f_{n+\frac{69}{74}} + \frac{729}{3297280}f_{n+1}\end{aligned}\right)\\
y_{n+\frac{69}{74}} &= y_n + \frac{69}{74}hy'_n + h^2\left(\begin{aligned}&\frac{4946912289}{359201697260}f_n + \frac{17400136203}{158577950720}f_{n+\frac{5}{74}} + \frac{1411081172199}{9069842855815}f_{n+\frac{1}{4}} + \frac{165964330541889}{147129015976960}f_{n+\frac{1}{2}}\\ &+\frac{1127174082469}{27209528567445}f_{n+\frac{3}{4}} + \frac{15063275}{13592395776}f_{n+\frac{69}{74}} + \frac{10840797}{35920169726}f_{n+1}\end{aligned}\right)\\
y_{n+1} &= y_n + hy'_n + h^2\left(\frac{643}{43470}f_n + \frac{69343957}{586414080}f_{n+\frac{5}{74}} + \frac{48976}{286335}f_{n+\frac{1}{4}} + \frac{4671}{35840}f_{n+\frac{1}{2}} + \frac{48976}{859005}f_{n+\frac{3}{4}} + \frac{69343957}{8092514304}f_{n+\frac{69}{74}}\right)
\end{aligned}
\right\}
\tag{2.8}
$$

The additional methods are obtained by evaluating (2.7) at $x = \left(\frac{5}{74}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{69}{74}, 1\right)$ to give the following discrete schemes:

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$$\left.\begin{aligned}
y'_{n+\frac{5}{74}} &= y'_n + h\left(\begin{aligned}&\frac{132171928615}{4823010897264} f_n + \frac{13113969205}{299423029248} f_{n+\frac{5}{74}} - \frac{61021705750}{11913361156557} f_{n+\frac{1}{4}} + \frac{1181523375}{497057483776} f_{n+\frac{1}{2}} - \frac{18479790250}{11913361156557} f_{n+\frac{3}{4}}\\ &+ \frac{368391325}{299423029248} f_{n+\frac{69}{74}} - \frac{2741432975}{4823010897264} f_{n+1}\end{aligned}\right)\\
y'_{n+\frac{1}{4}} &= y'_n + h\left(\begin{aligned}&\frac{23579}{5564160} f_n + \frac{199779940117}{1294802288640} f_{n+\frac{5}{74}} + \frac{1422961}{13744080} f_{n+\frac{1}{4}} - \frac{20709}{1146880} f_{n+\frac{1}{2}} + \frac{139651}{13744080} f_{n+\frac{3}{4}} - \frac{9916185851}{1294802288640} f_{n+\frac{69}{74}}\\ &+ \frac{19421}{5564160} f_{n+1}\end{aligned}\right)\\
y'_{n+\frac{1}{2}} &= y'_n + h\left(\frac{277}{12880} f_n + \frac{901471441}{8092514304} f_{n+\frac{5}{74}} + \frac{217162}{859005} f_{n+\frac{1}{4}} + \frac{4671}{35840} f_{n+\frac{1}{2}} - \frac{2362}{95445} f_{n+\frac{3}{4}} + \frac{69343957}{4495841280} f_{n+\frac{69}{74}} - \frac{467}{69552} f_{n+1}\right)\\
y'_{n+\frac{3}{4}} &= y'_n + h\left(\frac{2329}{206080} f_n + \frac{2149662667}{15985213440} f_{n+\frac{5}{74}} + \frac{36973}{169680} f_{n+\frac{1}{4}} + \frac{319653}{1146880} f_{n+\frac{1}{2}} + \frac{63389}{509040} f_{n+\frac{3}{4}} - \frac{1317535183}{47955640320} f_{n+\frac{69}{74}} + \frac{435}{41216} f_{n+1}\right)\\
y'_{n+\frac{69}{74}} &= y'_n + h\left(\begin{aligned}&\frac{596477423}{38832615920} f_n + \frac{10024207}{803604480} f_{n+\frac{5}{74}} + \frac{168853796338}{735392663985} f_{n+\frac{1}{4}} + \frac{641903629419}{2485287418880} f_{n+\frac{1}{2}} + \frac{514439521514}{2206177991955} f_{n+\frac{3}{4}}\\ &+ \frac{200151221}{2410813440} f_{n+\frac{69}{74}} - \frac{489781527}{38832615920} f_{n+1}\end{aligned}\right)\\
y'_{n+1} &= y'_n + h\left(\frac{643}{43470} f_n + \frac{2565726409}{20231285760} f_{n+\frac{5}{74}} + \frac{195904}{859005} f_{n+\frac{1}{4}} + \frac{4671}{17920} f_{n+\frac{1}{2}} + \frac{195904}{859005} f_{n+\frac{3}{4}} + \frac{2565726409}{20231285760} f_{n+\frac{69}{74}} + \frac{643}{43470} f_{n+1}\right)
\end{aligned}\right\}$$

(2.9)

Analysis of the BHNTM

In this section, mathematical analysis on some basic properties of the derived schemes are discussed extensively. Existing definitions and theorems well support these properties.

Order and Error Constant

The proposed BHNTM of (2.8) and (2.9) is classified as a member of the linear multistep method (LMM) which can be represented generally as:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j f_{n+j} \tag{3.1}$$

Definition 1 (Rufai & Ramos, 2020)

Associated with a numerical method is a linear difference operator η which is supposed to have $y(x_n)$ with higher derivatives. The term $y(x_n + v_i h)$ and its second derivative $y''(x_n + v_i h)$ can be expanded as Taylor's series about the point x_n. The local truncation error associated with a second-order ordinary differential equation is defined by the difference operator:

$$\eta\left[y(x_n);h\right] = \sum_{i=0}^{k}\left[\alpha_i y(x_n + v_i h) - h^2 \beta_i f(x_n + v_i h)\right] \tag{3.2}$$

where $y(x_n)$ is an arbitrary function continuously differentiable on the interval $[x_n, x_{n+1}]$. Expanding the expression (3.2) in Taylor series approximation about the point x_n gives

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$$\eta\left[y(x_n);h\right]=\hat{C}_0 y(x_n)+\hat{C}_1 h y'(x_n)+\hat{C}_2 h^2 y''(x_n)+\ldots+\hat{C}_{p+1}h^{p+1}y^{p+1}(x_n)+\ldots+\hat{C}_{p+2}h^{p+2}y^{p+2}(x_n)+\ldots \tag{3.3}$$

where the vectors

$$\begin{cases}\hat{C}_0=\sum_{v_i=0}^{k}\alpha_{v_i}, \quad \hat{C}_1=\sum_{i=0}^{k}v_i\alpha_{v_i}, \quad \hat{C}_2=\frac{1}{2!}\sum_{v_i=0}^{k}v_i\alpha_{v_i}-\beta_{v_i},\ldots,\\ \hat{C}_p=\frac{1}{p!}\sum_{v_i=0}^{k}v_i{}^{p}\alpha_{v_i}-p(p-1)(p-2)v_i{}^{p-2}\beta_{v_i}.\end{cases} \tag{3.4}$$

Following Lambert (1991), the associated methods are said to be of order p if, in (3.3)

$\hat{C}_0=\hat{C}_1=C_2=\ldots=\hat{C}_p=\hat{C}_{p+1}=0$ and $\hat{C}_{p+2}\neq 0$.

Therefore, $\hat{C}_{p+2}$ is the error constant and $\hat{C}_{p+2}h^{p+2}y^{p+2}(x_n)$ is the principal local truncation error at the point x_n. The local truncation error of the proposed BHNTM obtained are given as:

$$\eta\left[y(x_n;h)\right]=\begin{cases}\frac{28132172125}{1207309213103041069056}h^{(9)}y^{(9)}(x)+O\left(h^{10}\right)\\ -\frac{147697}{651143046758400}h^{(9)}y^{(9)}(x)+O\left(h^{10}\right)\\ -\frac{43}{317940940800}h^{(9)}y^{(9)}(x)+O\left(h^{10}\right)\\ -\frac{351}{8038803046400}h^{(9)}y^{(9)}(x)+O\left(h^{10}\right)\\ -\frac{109474638723}{372626300340444774400}h^{(9)}y^{(9)}(x)+O\left(h^{10}\right)\\ -\frac{43}{158970470400}h^{(9)}y^{(9)}(x)+O\left(h^{10}\right)\end{cases} \tag{3.5}$$

which shows that the proposed method has order $p=7$, where the error constant $\hat{C}_{p+2}$ is a 1×6 vector given by

$$\hat{C}_9=\begin{pmatrix}\frac{28132172125}{1207309213103041069056},-\frac{147697}{651143046758400},-\frac{43}{317940940800},-\frac{351}{8038803046400},\\ -\frac{109474638723}{372626300340444774400},-\frac{43}{158970470400}\end{pmatrix}. \tag{3.6}$$

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Zero Stability of HBNTM

Zero stability is a property concerning the proposed method when limiting *h* to zero. Therefore, as *h* tends to zero in the main method (2.8), the following system of equations are formed:

$$\left.\begin{aligned} y_{n+\frac{5}{74}} &= y_n \\ y_{n+\frac{1}{4}} &= y_n \\ y_{n+\frac{1}{2}} &= y_n \\ y_{n+\frac{3}{4}} &= y_n \\ y_{n+\frac{69}{74}} &= y_n \\ y_{n+1} &= y_n \end{aligned}\right\}$$

(3.7)

which can be written in matrix form as

$$A^0 Y_i - A^i Y_{i-1} = 0, \tag{3.8}$$

where

$$A^0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad A^i = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad Y_i = \begin{pmatrix} y_{n+\frac{5}{74}} \\ y_{n+\frac{1}{4}} \\ y_{n+\frac{1}{2}} \\ y_{n+\frac{3}{4}} \\ y_{n+\frac{69}{74}} \\ y_{n+1} \end{pmatrix}, \quad Y_{i-1} = \begin{pmatrix} y_n \\ y_n \\ y_n \\ y_n \\ y_n \\ y_n \end{pmatrix}$$

Following Lambert (1991), a method is said to be zero stable if the root r_i of the first characteristic polynomial $\rho(r) = \det\left|A^0 r - A^i\right|$ does not exceed one ($|r_i| \le 1$).

The first characteristic polynomial of the BHNTM is given by

$$r^5\left[r-1\right] = 0. \tag{3.9}$$

The roots of (3.9) are $r = 0,0,0,0,0,1$ in which none of them is greater than one. Therefore, the BHNTM is zero-stable.

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Consistency of the BHNTM

Definition 2 (Jator & Li, 2009)

The linear multistep method (3.1) is said to be consistent if it is of order $p \geq 1$ and its first and second characteristic polynomials defined as $\rho(z) = \sum_{j=0}^{k} \alpha_j z^j$ and $\sigma(z) = \sum_{j=0}^{k} \beta_j z^j$

where z satisfied (i) $\sum_{j}^{k} \alpha_j = 0$ (ii) $\rho(1) = \rho'(1) = 0$ and (iii) $\rho''(1) = 2!\sigma(1)$.

The discrete schemes derived are all of order greater than one and satisfies the conditions therein.

Convergence of the BHNTM

This section focuses on the discussion of the convergence analysis of the proposed BHNTM. By defining convergence, the proposed method is shown to be convergent by writing the formulas in (2.8) and (2.9) in an appropriate matrix notation form.

Definition 3 (Rufai & Ramos, 2021)

Let $y(x)$ be the exact solution of the given second order boundary value problems and $\{y_i\}_{i=0}^{N}$ the approximate solutions obtained with the derived numerical technique. The proposed method is said to be convergent of order *p* if, for sufficiently small *h*, there exists a constant *C* independent of *h*, such that:

$$\max_{0 \leq i \leq N} \left| y(x_i) - y_i \right| \leq Ch^p.$$

Note that in this circumstance, $\max_{0 \leq i \leq N} \left| y(x_i) - y_i \right| \to 0$ as $h \to 0$.

Theorem 1 (***Convergence Theorem***)

Let $y(x)$ denote the true solution of the second order BVP in (1.1) with boundary condition in (1.2), and $\{y_i\}_{i=0}^{N}$ the discrete solution provided by the proposed method. Then, the proposed method is convergent to order seven.

Proof

Following Jator and Manathunga (2018), suppose A be matrix of dimension $12N \times 12N$ defined by:

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$$A = \begin{bmatrix} A_{1,1} & \cdots & \cdots & \cdots & A_{1,2N} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{2n,1} & \cdots & \cdots & \cdots & A_{2N,2N} \end{bmatrix},$$

where the elements of $A_{i,j}$ are 6×6 matrices as follows:

$$A_{i,i} = I = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}; \qquad A_{i,i-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \quad i = N+2,\ldots,2N;$$

$$A_{i,N+i-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -\frac{5}{74} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & -\frac{3}{4} \\ 0 & 0 & 0 & 0 & 0 & -\frac{69}{74} \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \quad \text{where } 1 < i < N+1,$$

$A_{i,j} = 0$ otherwise, where **0** is a zero matrix.

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Suppose B be a $12N \times 12N$ matrix defined by

$$B = \begin{bmatrix} \beta_{1,1} & \beta_{1,2} & \beta_{1,3} & \beta_{1,4} & \cdots & \beta_{1,2N} \\ \vdots & & & & \ddots & \vdots \\ \beta_{2N,1} & \beta_{2N,2} & \beta_{2N,3} & \beta_{2N,4} & \cdots & \beta_{2N,2N} \end{bmatrix},$$

where the elements of $B_{i,j}$ are 6×6 matrices given as follows:

$$\mathrm{B}_{i,j} = \begin{bmatrix} \frac{15063275}{13592395776} & -\frac{2933478125}{16325717140467} & \frac{24930298125}{294258030395392} & -\frac{24485719375}{440794362792609} & \frac{7823810875}{177258433314816} & -\frac{6595375}{323281527534} \\ \frac{68719861387}{3069160980480} & \frac{449}{77568} & -\frac{92961}{73400320} & \frac{155707}{219905280} & -\frac{8806682539}{16573469294592} & \frac{7177}{29675520} \\ \frac{7142427571}{129480228864} & \frac{48253}{859005} & \frac{115}{16384} & -\frac{241}{286335} & \frac{69343957}{215800381440} & -\frac{1}{8694} \\ \frac{29332493811}{341017886720} & \frac{103833}{904960} & \frac{4690143}{73400320} & \frac{449}{77568} & -\frac{762783527}{1023053660160} & \frac{729}{3297280} \\ \frac{17400136203}{158577950720} & \frac{1411081172199}{9069842855815} & \frac{165964330541889}{147129015976960} & \frac{1127174082469}{27209528567445} & \frac{15063275}{13592395776} & \frac{10840797}{35920169726} \\ \frac{69343957}{586414080} & \frac{48976}{286335} & \frac{4671}{35840} & \frac{48976}{859005} & \frac{69343957}{8092514304} & 0 \end{bmatrix}, \text{ where } 1 \le i \le N$$

$$\mathrm{B}_{i,i-N} = \begin{bmatrix} \frac{13113969205}{299423029248} & -\frac{61021705750}{11913361156557} & \frac{1181523375}{497057483776} & -\frac{18479790250}{11913361156557} & \frac{368391325}{299423029248} & -\frac{2741432975}{4823010897264} \\ \frac{199779940117}{1294802288640} & \frac{1422961}{13744080} & -\frac{20709}{1146880} & \frac{139651}{13744080} & -\frac{9916185851}{1294802288640} & \frac{19421}{5564160} \\ \frac{901471441}{8092514304} & \frac{217162}{859005} & \frac{4671}{35840} & -\frac{2362}{95445} & \frac{69343957}{4495841280} & -\frac{467}{69552} \\ \frac{2149662667}{15985213440} & \frac{36973}{169680} & \frac{319653}{1146880} & \frac{63389}{509040} & -\frac{1317535183}{47955640320} & \frac{435}{41216} \\ \frac{10024207}{803604480} & \frac{168853796338}{735392663985} & \frac{641903629419}{2485287418880} & \frac{514439521514}{2206177991955} & \frac{200151221}{2410813440} & -\frac{489781527}{38832615920} \\ \frac{2565726409}{20231285760} & \frac{195904}{859005} & \frac{4671}{17920} & \frac{195904}{859005} & \frac{2565726409}{20231285760} & \frac{643}{43470} \end{bmatrix}, \text{ where } N+1 \le i \le 2N$$

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$$B_{i,i-1}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{58052434225}{44612850799692} \\ 0 & 0 & 0 & 0 & 0 & \frac{348899}{89026560} \\ 0 & 0 & 0 & 0 & 0 & \frac{211}{28980} \\ 0 & 0 & 0 & 0 & 0 & \frac{37377}{3297280} \\ 0 & 0 & 0 & 0 & 0 & \frac{4946912289}{359201697260} \\ 0 & 0 & 0 & 0 & 0 & \frac{643}{43470} \end{bmatrix}, \text{ where } 1<i\le N$$

$$B_{i,i-(N+1)}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{132171928615}{4823010897264} \\ 0 & 0 & 0 & 0 & 0 & \frac{23579}{5564160} \\ 0 & 0 & 0 & 0 & 0 & \frac{277}{12880} \\ 0 & 0 & 0 & 0 & 0 & \frac{2329}{206080} \\ 0 & 0 & 0 & 0 & 0 & \frac{596477423}{38832615920} \\ 0 & 0 & 0 & 0 & 0 & \frac{643}{43470} \end{bmatrix}, \text{ where } N+1<i\le 2N,$$

$B_{i,j}=0,$ otherwise, where **0** is a zero matrix.

Hence,

$$A=\left[\begin{array}{c|c} \hat{A}_{1,1} & \hat{A}_{1,2} \\ \hline 0 & \hat{A}_{2,2} \end{array}\right],$$

$$B=\left[\begin{array}{c|c} \hat{B}_{1,1} & 0 \\ \hline \hat{B}_{2,1} & 0 \end{array}\right].$$

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Furthermore, the following vectors are defined as:

$$\left.\begin{aligned}
Y &= \begin{pmatrix} y\left(x_{\frac{5}{74}}\right), y\left(x_{\frac{1}{4}}\right), \ldots, y(x_1), \ldots, y\left(x_{N-1+\frac{5}{74}}\right), y\left(x_{N-1+\frac{1}{4}}\right), \ldots, y(x_N), hy'(x_{\frac{5}{74}}), \\ hy'(x_{\frac{1}{4}}), \ldots, hy'(x_1), \ldots, hy'\left(x_{N-1+\frac{5}{74}}\right), hy'\left(x_{N-1+\frac{1}{4}}\right), \ldots, hy'(x_N) \end{pmatrix}^W, \\
C &= \begin{pmatrix} -y_0 - \frac{5h}{74} y_0' - h^2\beta_0(1) f_0, -y_0 - \frac{h}{4} y_0' - h^2\beta_0(2) f_0, \ldots, -y_0 - hy_0' - h^2\beta_0(6) f_0, \\ 0,0,0,\ldots,0, -h^2\beta_0'(1), -h^2\beta_0'(2), \ldots, -h^2\beta_0'(6), 0,0,0,\ldots,0 \end{pmatrix}^W \\
T(h) &= \left(\xi_{\frac{5}{74}}, \xi_{\frac{1}{4}}, \ldots, \xi_1, \xi_{1+\frac{5}{74}}, \ldots, \xi_N, \xi_{\frac{5}{74}}, \eta'_{\frac{1}{4}}, \ldots, \xi'_1, \xi'_{1+\frac{5}{74}}, \ldots, \xi'_N \right)^W,
\end{aligned}\right\}$$

where $T(h)$ is the local truncation error. The exact form of the system is given by

$$AY - h^2 BF(Y) + C + T(h) = 0, \tag{3.10}$$

and the approximate form of the system is given by

$$A\overline{Y} - h^2 BF(\overline{Y}) + C = 0, \tag{3.11}$$

where $\overline{Y} = \left(y_{\frac{5}{74}}, y_{\frac{1}{4}}, \ldots, y_1, y_{1+\frac{5}{74}}, \ldots, y_N, hy'_{\frac{5}{74}}, hy'_{\frac{1}{4}}, \ldots, hy'_1, hy'_{1+\frac{5}{74}}, \ldots, hy'_N \right).$

Subtracting (3.10) from (3.11) gives

$$AE - h^2 BF(\overline{Y}) + h^2 BF(Y) + C = T(h), \tag{3.12}$$

where $E = \overline{Y} - Y = \left(e_{\frac{5}{75}}, e_{\frac{1}{4}}, e_{\frac{1}{2}}, \ldots e_1, e'_{\frac{5}{75}}, e'_{\frac{1}{4}}, e'_{\frac{1}{2}}, \ldots e'_1 \right)$. Applying the mean-value theorem (Jator & Manathunga, 2018), which can be written as

$$F(\overline{Y}) = F(Y) + J_F(Y)(\overline{Y} - Y) + o\left(\|\overline{Y} - Y\|\right),$$

where J_F is a Jacobian matrix. From this equation, $\dfrac{F(\overline{Y}) - F(Y)}{\overline{Y} - Y} = \dfrac{F(\overline{Y}) - F(Y)}{E} = J_F(Y)$.

Hence, we obtain $\left(A - h^2 BJ_F(Y)E\right) = T(h)$, where $J_F = \begin{bmatrix} J_{1,1} & J_{1,2} \\ J_{2,1} & J_{2,2} \end{bmatrix}$, and $J_{i,j}$ are $6N \times 6N$ matrices.

Considering the matrix $A - h^2 BJ_F(Y)$, asserting that for sufficiently small h, $A - h^2 BJ_F(Y)$ is invertible. Observe that $\det A = \det \hat{A}_{1,1} \det \hat{A}_{2,2}$. Since $\hat{A}_{1,1} = \hat{A}_{2,2}$, it is enough to prove only the

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invertibility of $\hat{A}_{1,1}$. The diagonal elements of $\hat{A}_{1,1}$ being non-zero certainly implies that its determinant is non-zero; hence, it is invertible. In fact, it is a lower triangular matrix. Therefore, A is invertible. Let $Z = -h^2 BJ_F$, we have $Q = A + Z$ then $\det(Q) = \det(A)\det(I + ZA^{-1})$.

Let $C = ZA^{-1}$, then $\det(Q) = \det(A)\det(I - C)$. Note that $\det(I - C)$ is the characteristic polynomial of C. Therefore,

$\det(I - C) = (\varphi - \varphi_1)(\varphi - \varphi_2)\ldots(\varphi - \varphi_{12N})$, where φ_j are eigenvalues of C. For $\varphi = 1$, gives

$\det(I - C) = (1 - \varphi_1)(1 - \varphi_2)\ldots(1 - \varphi_{12N})$.

If each $\varphi_j \neq 1$, then $\det(I - C) \neq 0$. Suppose that φ_j is an eigenvalue of $BJ_F A^{-1}$, then $h^2 \hat{\varphi}_j$. If $h^2 \hat{\varphi}_j \neq 1$ is proved successfully, then we are done. Thus, choosing $h^2 \notin \left\{ \frac{1}{\hat{\varphi}_j} : \hat{\varphi}_j \text{ is a non-zero eigenvalue of } BJ_F A^{-1} \right\}$ Therefore, there exists an h such that $\det(I - C) \neq 0$. Thus, $\det Q \neq 0$. This means that Q is invertible and $\|Q\|_\infty = O(h^2)$. Now, $(A + Z)E = T(h)$. This implies that $QE = T(h)$. Since Q is invertible, $E = Q^{-1}T(h)$. Taking the maximum norm gives

$\|E\|_\infty = \|Q^{-1}T(h)\|_\infty \leq \|Q^{-1}\|_\infty \|T(h)\|_\infty \leq O(h^{-2})O(h^9) \leq O(h^7)$. Thus, the BHNTM is convergent, providing seventh-order approximations.

NUMERICAL EXPERIMENT

In this section, the accuracy of the seventh order single-step implicit block method is experimented on three test problems with the cases of their step sizes $h = 0.1$ and $h = 0.01$ respectively. In each case, the computed result is compared with results obtained from reported methods found in literature. The absolute errors are given at some selected points of evaluation also.

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Implementation of the Proposed BHNTM

Step 1: Input the number of iterations N, step size $h=\frac{(x_N - x_0)}{N}$, initial conditions on the interval $[x_0, x_N]$.

Step 2: Using main method and additional method, for $n=0$, generate a system of equations $\left(y_{\frac{5}{74}}, y_{\frac{1}{4}}, y_{\frac{1}{2}}, y_{\frac{3}{4}}, y_{\frac{69}{74}}, y_1\right)$ and $\left(y'_{\frac{5}{74}}, y'_{\frac{1}{4}}, y'_{\frac{1}{2}}, y'_{\frac{3}{4}}, y'_{\frac{69}{74}}, y'_1\right)$ on the sub-interval $[x_0, x_1]$ and do not solve yet.

Step 3: For $n = 1$, generate a system of equations $\left(y_{\frac{5}{74}}, y_{\frac{1}{4}}, y_{\frac{1}{2}}, y_{\frac{3}{4}}, y_{\frac{69}{74}}, y_1\right)$ and $\left(y'_{\frac{5}{74}}, y'_{\frac{1}{4}}, y'_{\frac{1}{2}}, y'_{\frac{3}{4}}, y'_{\frac{69}{74}}, y'_1\right)$ on the subinterval $[x_1, x_2]$, and do not solve yet.

Step 4: The process is continued for $n = 2, ..., N-1$ until all the variables on the sub-interval $[x_0, x_1], [x_2, x_3], ..., [x_{N-1}, x_N]$ are obtained.

Step 5: Create a single block matrix equation to simultaneously obtain all solutions for (1.1) on the entire interval $[x_0, x_N]$ obtained in Step 3 and Step 4.

The following notations are used:

BHNTM$_{1,5}$: Single-step Block Hybrid Nystrom-Type Method with five off-grid points.

BNM: Block Nyström Method

Problem 1: Source: [3]. Consider the IVP of Bratu-Type. Take $h = 0.1$ and $h = 0.01$

$$y''(x) - 2e^{y(x)} = 0,\ 0 \le x \le 1,$$
$$y(0) = y'(0) = 0$$
$$\text{Exact solution: } y(x) = -2\ln(\cos x)$$

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Table 1: Comparison of Absolute Error of Problem 1 for $h = 0.1$

x	**Debela *et al.* [3]**	**Raja *et al.* [12]**	**Fenta & Derese [5]**	**Jator & Manathunga [8]**	$BHNTM_{1,5}$
0.0	0.00	0.00	0.00	0.00	0.00
0.1	4.9027×10^{-9}	6.88×10^{-8}	4.7354×10^{-11}	3.50×10^{-15}	1.9241×10^{-16}
0.2	1.0071×10^{-8}	5.71×10^{-8}	4.4009×10^{-10}	1.63×10^{-14}	8.6484×10^{-16}
0.3	1.5973×10^{-8}	8.43×10^{-8}	1.2764×10^{-9}	4.46×10^{-14}	2.3693×10^{-15}
0.4	$2.3233.\times 10^{-8}$	7.18×10^{-8}	2.7775×10^{-9}	1.05×10^{-13}	5.5794×10^{-15}
0.5	3.2789×10^{-8}	4.33×10^{-8}	5.3998×10^{-9}	2.37×10^{-13}	1.2641×10^{-14}
0.6	4.6204×10^{-8}	1.30×10^{-7}	1.0104×10^{-8}	5.47×10^{-13}	2.9148×10^{-14}
0.7	6.6335×10^{-8}	2.59×10^{-7}	1.9048×10^{-8}	1.33×10^{-12}	7.1039×10^{-14}
0.8	9.8959×10^{-8}	5.89×10^{-8}	3.7529×10^{-8}	3.54×10^{-12}	1.8912×10^{-13}
0.9	1.5718×10^{-7}	3.13×10^{-7}	8.0833×10^{-8}	1.06×10^{-11}	5.6978×10^{-13}
1.0	2.7544×10^{-7}	1.51×10^{-6}	1.9336×10^{-7}	3.78×10^{-11}	2.0332×10^{-12}

Table 1 presents a comparison of the absolute error for Problem 1 with a step size of $h = 0.1$. It becomes evident that the newly introduced single-step block hybrid Nyström-type method (BHNTM) of order seven outperforms the one-step five hybrids Block Nyström method (BNM) in Jator and Manathunga [8] as well as the sixth-order seven-stage Runge-Kutta method proposed by Fenta and Derese [5]. The superior accuracy of the BHNTM, as showcased in Table 1, highlights its sophistication and reliability in numerical simulations for Problem 1. This improvement in accuracy demonstrates the significance of the BHNTM as a valuable addition to the field of numerical methods for solving differential equations, offering a more precise and dependable approach for addressing complex problems.

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Table 2: Comparison of Absolute Error of Problem 1 for $h = 0.01$

x	**Debela *et al.* [3]**	**Fenta & Derese [5]**	$BHNTM_{1,5}$
0.01	4.8904×10^{-15}	2.9300×10^{-17}	1.8987×10^{-26}
0.02	9.7697×10^{-15}	4.8247×10^{-17}	7.5237×10^{-25}
0.03	1.4542×10^{-15}	4.2718×10^{-17}	1.6961×10^{-25}
0.04	1.9526×10^{-15}	7.0256×10^{-17}	3.0235×10^{-25}
0.05	2.4300×10^{-15}	3.8164×10^{-17}	4.7402×10^{-25}
0.06	2.9238×10^{-15}	2.6021×10^{-18}	6.8560×10^{-25}
0.07	3.4202×10^{-15}	5.0307×10^{-17}	9.3791×10^{-25}
0.08	3.9077×10^{-15}	1.3878×10^{-17}	1.2322×10^{-24}
0.09	4.4138×10^{-15}	7.9797×10^{-17}	1.5698×10^{-24}
0.10	4.9098×10^{-15}	3.9899×10^{-17}	1.9525×10^{-24}

It is observed here that as the step size reduces, there is a drastic reduction in the error posed.

Problem 2: Source: [3]. Consider the IVP of Bratu-Type. Take $h = 0.1$ and $h = 0.01$

$$y''(x) + \pi^2 e^{-y(x)} = 0, \quad 0 \le x \le 1,$$
$$y(0) = 0, y'(0) = \pi$$

with Exact solution: $y(x) = \ln(1 + \sin(\pi x))$

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Table 3: Comparison of Absolute Error of Problem 2 for $h = 0.1$

x	**Debela *et al.* [3]**	**Fenta & Derese [5]**	$BHNTM_{1,5}$
0.1	8.3507×10^{-8}	3.7804×10^{-7}	5.5626×10^{-12}
0.2	4.4458×10^{-8}	5.4006×10^{-7}	8.2280×10^{-12}
0.3	2.6380×10^{-7}	6.5775×10^{-7}	1.0724×10^{-11}
0.4	5.3872×10^{-7}	7.7979×10^{-7}	1.3631×10^{-11}
0.5	8.6014×10^{-7}	9.2484×10^{-7}	1.7173×10^{-11}
0.6	1.2299×10^{-6}	1.1035×10^{-6}	2.1532×10^{-11}
0.7	1.6559×10^{-6}	1.3242×10^{-6}	2.6932×10^{-11}
0.8	2.1494×10^{-6}	1.5934×10^{-6}	3.3645×10^{-11}
0.9	2.7194×10^{-6}	1.9045×10^{-6}	4.1805×10^{-11}
0.1	3.3454×10^{-6}	2.1817×10^{-6}	4.9883×10^{-11}

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Table 4: Comparison of Absolute Error of Problem 2 for $h = 0.01$

x	**Debela *et al.* [3]**	**Fenta & Derese [5]**	$BHNTM_{1,5}$
0.01	2.2668×10^{-13}	5.9442×10^{-14}	1.0332×10^{-20}
0.02	4.1313×10^{-13}	1.1242×10^{-13}	1.9228×10^{-20}
0.03	5.6410×10^{-13}	1.5964×10^{-13}	2.6951×10^{-20}
0.04	6.8325×10^{-13}	2.0214×10^{-13}	3.3713×10^{-20}
0.05	7.7396×10^{-13}	2.4061×10^{-13}	3.9691×10^{-20}
0.06	8.3944×10^{-13}	2.7536×10^{-13}	4.5021×10^{-20}
0.07	8.8199×10^{-13}	3.0720×10^{-13}	4.9822×10^{-20}
0.08	9.0389×10^{-13}	3.3654×10^{-13}	5.4188×10^{-20}
0.09	9.0730×10^{-13}	3.6360×10^{-13}	5.8197×10^{-20}
0.10	8.9412×10^{-13}	3.8858×10^{-13}	6.1915×10^{-20}

In Table 3, For Problem 2, the methods of Debela *et al.* [3] and Derese and Fenta [5] show a divergence at $x = 0.5$ but the BHNTM solutions approach the bounded exact solutions with an accuracy of 11 to 12 decimal points.

Problem 3: Source: [3]. Consider the IVP of Bratu-Type. Take $h = 0.1$ and $h = 0.01$

$$y''(x) - e^{2y(x)} = 0, \; 0 \le x \le 1,$$
$$y(0) = y'(0) = 0$$

with Exact solution: $y(x) = \ln(\sec x)$

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Table 5: Comparison of Absolute Error of Problem 3 for $h = 0.1$

x	**Debela *et al.* [3]**	**Fenta & Derese [5]**	$BHNTM_{1,5}$
0.1	2.4514×10^{-9}	2.3677×10^{-11}	9.6207×10^{-17}
0.2	5.0353×10^{-9}	2.2004×10^{-10}	4.3242×10^{-16}
0.3	7.9863×10^{-9}	6.3819×10^{-10}	1.1846×10^{-15}
0.4	1.1616×10^{-8}	1.3888×10^{-9}	2.7891×10^{-15}
0.5	1.6395×10^{-8}	2.6999×10^{-9}	6.3204×10^{-15}
0.6	2.3102×10^{-8}	5.0521×10^{-9}	1.4574×10^{-14}
0.7	3.3168×10^{-8}	9.5241×10^{-9}	3.5519×10^{-14}
0.8	4.9480×10^{-8}	1.8764×10^{-8}	9.4558×10^{-14}
0.9	7.8591×10^{-8}	4.0066×10^{-8}	2.8489×10^{-13}
1.0	1.3772×10^{-7}	9.6681×10^{-8}	1.0166×10^{-12}

Table 6: Comparison of Absolute Error of Problem 3 for $h = 0.01$

x	**Debela *et al.* [3]**	**Fenta & Derese [5]**	$BHNTM_{1,5}$
0.01	2.3599×10^{-15}	7.0582×10^{-17}	9.3937×10^{-27}
0.02	4.8279×10^{-15}	3.2878×10^{-17}	3.7619×10^{-26}
0.03	7.3091×10^{-15}	1.6697×10^{-17}	8.4806×10^{-26}
0.04	9.6700×10^{-15}	5.7896×10^{-17}	1.5118×10^{-25}
0.05	1.2101×10^{-14}	6.8522×10^{-17}	2.3704×10^{-25}
0.06	1.4654×10^{-14}	3.6429×10^{-17}	3.4280×10^{-25}
0.07	1.7124×10^{-14}	4.8138×10^{-17}	4.6895×10^{-25}
0.08	1.9546×10^{-14}	8.6736×10^{-19}	6.1609×10^{-25}
0.09	2.2018×10^{-14}	1.1276×10^{-17}	7.8492×10^{-25}
0.10	2.4441×10^{-14}	8.7604×10^{-17}	9.7623×10^{-25}

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CONCLUSION

The main goal of this research paper is the development of a single-step implicit Block hybrid Nyström-type method (BHNTM) to solve nonlinear second order initial-boundary value problems of Bratu-type. The method was applied directly without the use of linearization, or any restrictive assumptions. The convergence analysis has been discussed and the order of the method is consistent and zero stable. To further justify the applicability of the proposed method, tables of pointwise absolute errors for three test problems at different step size h are displayed. Table 1, 2, 3, 4, 5 and 6 demonstrate that the seventh-order BHNTM improves the findings of Fenta and Derese [5], Jator and Manathunga [8], and Debela *et al.* [3]. Moreover, it is evidently seen that all the absolute errors decrease rapidly as the step size h decreases, which in turn shows that the smaller the step size h, the better approximate values.

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