*Science World Journal Vol. 16(No 3) 2021*
*www.scienceworldjournal.org*
*ISSN: 1597-6343 (Online), ISSN: 2756-391X (Print)*
*Published by Faculty of Science, Kaduna State University*

Full Length Research Article

# REFINEMENT OF PRECONDITIONED OVERRELAXATION ALGORITHM FOR SOLUTION OF THE LINEAR ALGEBRAIC SYSTEM $\boldsymbol{Ax = b}$

[1]Ramatu Abdullahi, [2]Raihanatu Muhammad

[1]Department of Mathematics and Statistics, Niger State Polytechnic, Zungeru, Nigeria
[2]Department of Mathematics, Federal University of Technology, Minna, Nigeria

Authors' Email Addresses:arahmatu@yahoo.com /raihanatu.muhammad@futminna.edu.ng

## ABSTRACT

In this paper, a refinement of preconditioned successive overrelaxation method for solving the linear system $Bx = c$ is considered. The coefficient matrix $B \in R^{n,n}$ is a nonsingular real matrix, $c \in R^n$ and $x$ is the vector of unknowns. Based on the usual splitting of the coefficient matrix $B$ as $B = D - L_B - U_B$, the linear system is expressed as $Ax = b$ or $(I - L - U)x = b$; where $L = D^{-1}L_B$, $U = D^{-1}U_B$ and $b = D^{-1}c$. This system is further preconditioned with a preconditioner of the type $P = I + S$ as $\bar{A}x = \bar{b}$ or $(\bar{D} - \bar{L} - \bar{U})x = \bar{b}$. A refinement of the resulting preconditioned successive overrelaxation (SOR) method is performed. Convergence of the resulting refinement of preconditioned SOR iteration is established and numerical experiments undertaken to demonstrate the effectiveness and efficiency of the method. Results comparison revealed that the refinement of SOR method converges faster than the preconditioned as well as the classical SOR method.

**Keywords:** SOR method, Preconditioned SOR, Convergence, Refinement, Nonsingular Matrix, *L*-Matrix A

## INTRODUCTION

The rate of convergence of an iterative technique depends on the spectral radius of the coefficient matrix associated with the method. One way to select a procedure to accelerate convergence is to choose a method whose associated matrix has minimal spectral radius. Thus, there is the need to introduce a new means of measuring the amount by which an approximation to the solution to a linear system differs from the true solution to the system (Burden *et al.*, 2014). Suppose $\bar{x} \in R^n$ is an approximation to the solution of the linear system defined by $Ax = b$. The residual vector for $\bar{x}$ with respect to this system is $r = b - A\bar{x}$. In procedures such as the Jacobi, Gauss-Seidel or SOR methods, a residual vector is associated with each calculation of an approximate component to the solution vector. The true objective is to generate a sequence of approximations that will cause the residual vectors to converge rapidly to zero. Consider the linear system

$$Ay = r \qquad (1)$$

The approximate solution $\bar{y}$ of the above system satisfies

$$\bar{y} \approx A^{-1}r = A^{-1}(b - A\bar{x}) = A^{-1}b - A^{-1}A\bar{x} = x - \bar{x}$$

and

$$x \approx \bar{x} + \bar{y}$$

So $\bar{y}$ is an estimate of the error produced when $\bar{x}$ approximates the solution $x$ to the original system. In general, $\bar{x} + \bar{y}$ is a more accurate approximation to the solution of the linear system $Ax = b$ than the original approximation $\bar{x}$. The method using this assumption is called *iterative refinement*, or *iterative improvement*, and consists of performing iterations on the system whose right-hand side is the residual vector for successive approximations until satisfactory accuracy results.

Several studies have been made in this regard. For example, Yuan (1998) proposed some iterative refinement methods for linear system $Ax = b$, which proved to be better than the usual iterative refinement methods in respect of complexity and storage requirements. These methods are based on the obtained results for the nonsingular matrix $A$, which show there exists a convergent splitting $A = M - N$ with $M = DQ$ or $M = QD$ where $Q$ and $D$ are unitary and diagonal matrices respectively. The study further revealed that, if $LU$ decomposition of $A$ is obtainable, there exists a convergent splitting $A = M - N$ with triangular $M$. An example of the desired triangular matrix $M$ is constructed for $p$ −cyclic matrices. Dafchahi (2008) presented a refinement of the Jacobi method whose rate of convergence measures up with that of the SOR, and by consequence more than that of the Gauss-Seidel method. It was recommended that this method be used in place of the SOR method in view of the associated difficulty in finding the optimal relaxation parameter $\omega_{opt}$ for the SOR technique. Further application of this method to the Poisson partial differential equation yielded favourable results. Vatti and Gonfa (2011) proposed refinement of generalized Jacobi method (RGJ) for solving system of linear equations. Convergence conditions and theorems of RGJ were advanced and established. The efficiency of the RGJ over the generalized Jacobi method was demonstrated through some numerical experiments. Vatti and Eneyew (2011) discussed a refinement of the Gauss-Seidel method for solution of the linear system $Ax = b$. Comparison results between refinement of the Gauss-Seidel method and the Gauss-Seidel method went further to validate the convergence theorems presented. Kyurkchiev and Iliev (2013) focused on refinement of some successive overrelaxation (SOR) methods. Based on the reverse Gauss–Seidel method, a splitting of coefficient matrix of the linear system $Ax = b$ is obtained as $A = T_m - E_m - F_m$, where $T_m$ is a banded matrix of bandwidth $2m + 1$. Theoretical convergence of the methods were studied and established and numerical examples with results are provided with the aid of Mathematica software package. Laskar and Behera (2014) surveyed three iterative refinement methods for the solution of system of linear equations. By comparing the number of iterations required to converge, storage requirements and level of accuracy, it was conclusive that the refinement of generalized Jacobi method is much more efficient than the refinement of Jacobi iterative method and is as fast as the refinement of Gauss-Seidel method.

Gonfa (2016) studied the refinement of generalized Gauss-Seidel (RGGS) method for solving systems of linear equations. Sufficient convergence conditions are established and appropriate numerical experiments conducted. The results of RGGS are compared with those of refinement of generalized Jacobi and Successive-Over Relaxation methods. The results analysis indicates that, for all the parameters considered, that is, CPU time, iteration count and computer storage, the RGGS proved to be more efficient than the other methods when the coefficient matrix is either an M-matrix or a diagonally dominant matrix. Kebede (2017) presented a second degree refinement of Jacobi method. A comparison of spectral radii of second degree refinement of Jacobi method (SDRJ) with other methods of first degree Jacobi (FDJ), first degree refinement of Jacobi (FDRJ) and second degree Jacobi (SDJ) method was done in order to demonstrate the effectiveness of the method. Numerical results of spectral radius show that, SDRJ methods converge with a small number of iteration steps for solving systems of linear equations. Gebregiorgis and Gofe (2018) is concerned about extension of refined generalized numerical algorithms for solving systems of linear equations whose coefficient matrices are $M-$matrices, to solving fuzzy linear systems like Refined Generalized Jacobi (RGJ), and Refined Generalized Gauss-Seidel (RGGS) iteration methods. These methods are developed via the embedding approach and splitting strategy of the $M-$matrix together with the refinement process. The results of numerical experiments demonstrated that the proposed methods perform better when compared with similar work. Muleta and Gofe (2018) presented refinement of generalized accelerated over relaxation (RGAOR) iterative method based on the Nekrassov-Mehmke 1 (NM1) method procedure for solving the linear system $Ax = b$. A splitting of the coefficient matrix $A$ is obtained in the form $A = T_m - E_m - F_m$, where $T_m$ is a banded matrix of band width $2m + 1$ and $-E_m$ and $-F_m$ are the strictly lower and strictly upper triangular parts of the matrix $A - T_m$respectively. The results show that the iteration matrix of RGAOR is the square of the iteration matrix of generalized accelerated successive over relaxation method. When compared to generalized accelerated successive over relaxation (SOR2GNM1, SOR1GNM1), the results reveal that the new method (RSOR1GNM1, RSOR2GNM1) converges faster and its errors at any predefined error of tolerance is lesser than the other methods. Vatti *et al.* (2018) considered a refinement of accelerated over relaxation (AOR) method for solving the linear system $Ax = b$, where convergence conditions were also identified and established. It was capped by a numerical experiment in order to validate the superiority of the proposed method over existing methods. Eneyew *et al.* (2019) proposed a new method christened second refinement of Jacobi (SRJ) method for solving linear system of equations. This method is applicable to the solution of ODE and PDE problems reduced to linear system of equations with coefficient matrices being strictly diagonally dominant (SDD) or symmetric positive definite (SPD) or $M-$matrices. Few numerical examples established the increased rate of convergence of the SRJ over Jacobi and refinement of Jacobi (RJ) methods as noticed in the minimized number of iterations as well as the spectral radius. Eneyew *et al.* (2020) proposed the second-refinement of Gauss-Seidel method for solving linear system of equations. In this study, the convergence rate of the Gauss-Seidel method was enhanced by minimization of the spectral radius, and by implication, a further reduction in the number of iterations. Another distinct feature of the method is that it can be applied towards the solution of linear systems resulting from finite difference discretisation of systems of differential equations. These systems are usually composed of coefficient matrices that are strictly diagonally dominant, symmetric positive definite, or M-matrices. In Theorem 1 of this research, it was established that, if the coefficient matrix $A$ is strictly diagonally dominant, then the new modified method converges to the exact solution. Similarly, in Theorems 2 and 3 it was proved that, if the coefficient matrix is symmetric positive definite or $M$ $-$matrix, then the modified method converges. More so, Theorem 4 showed that, second-refinement of Gauss-Seidel method exhibits faster convergence when compared to the Gauss-Seidel and refinement of Gauss-Seidel methods. The numerical experiments went further to demonstrate the efficiency of second-refinement of Gauss-Seidel method over the Gauss-Seidel and refinement of Gauss-Seidel methods. Assefa and Teklehaymanot (2021) proposed a second refinement of accelerated over relaxation method; it is a two-parameter generalization of the refinement of accelerated over relaxation methods and the optimal value of each parameter is derived. In general the *kth* refinement of accelerated methods are also derived. The spectral radius of the iteration matrix and convergence criteria of the second refinement of accelerated over relaxation (SRAOR) are discussed.

This research work is aimed to formulate refinement of preconditioned successive overrelaxation iterative method for the solution of linear algebraic system $Ax = b$ resulting from finite difference discretization of elliptic partial differential equations.

## MATERIALS AND METHODS

### Formulation of Refinement of SOR Method

Following Vatti *et al.* (2011), Vatti and Gonfa (2011), Vatti *et al.* (2018) the refinement of SOR is obtained as

From the linear system

$$Ax = b \tag{2}$$

which becomes

$$Dx + \omega Ax = Dx + \omega b$$

A splitting of the matrix $A$ is considered as

$$Dx + \omega(D - L - U)x = Dx + \omega b$$
$$(D - \omega L)x = (D - \omega D + \omega U)x + \omega b$$
$$(D - \omega L)x = Dx - \omega(D - U)x + \omega b$$
$$(D - \omega L)x = Dx - \omega(L + A)x + \omega b$$
$$(D - \omega L)x = (D - \omega L)x + \omega(b - Ax)$$
$$x = x + \omega(D - \omega L)^{-1}(b - Ax)$$

From whence the refinement of SOR (RSOR) formula is obtained as

$$x^{(n+1)} = x^{(n+1)} + \omega(D - \omega L)^{-1}\left(b - Ax^{(n+1)}\right) \tag{3}$$

Or,

$$x^{(n+1)} = x^{(n+1)} + \omega\left(I - \omega\tilde{L}\right)^{-1}\left(\tilde{b} - \tilde{A}x^{(n+1)}\right) \tag{4}$$

where $x^{(n+1)}$ appearing in the RHS of (3.2) is the $(n + 1)\text{th}$ approximation of SOR method given by

$$x^{(n+1)} = (D - \omega L)^{-1}\{(1 - \omega)D + \omega U\}x^{(n)} + (D - \omega L)^{-1}\omega b \tag{5}$$

Or

$$x^{(n+1)} = \left(I - \omega\tilde{L}\right)^{-1}\{(1 - \omega)I + \omega\tilde{U}\}x^{(n)} + \left(I - \omega\tilde{L}\right)^{-1}\omega\tilde{b} \tag{6}$$

where, $\tilde{L} = D^{-1}L,\ \tilde{U} = D^{-1}U,\ \tilde{b} = D^{-1}b$ and $\tilde{A} = D^{-1}A$

Substituting (5) in (3),

$$\begin{aligned} x^{(n+1)} &= (D-\omega L)^{-1}\{(1-\omega)D+\omega U\}x^{(n)} \\ &\quad + (D-\omega L)^{-1}\omega b + \omega(D-\omega L)^{-1}[b \\ &\quad - (D-L-U)(D-\omega L)^{-1} \\ &\quad \times [\{(1-\omega)D+\omega U\}x^{(n)}+\omega b]] \end{aligned}$$

$$\begin{aligned} x^{(n+1)} &= (D-\omega L)^{-1}\{(1-\omega)D+\omega U\}x^{(n)} \\ &\quad + 2\omega(D-\omega L)^{-1}b - (D-\omega L)^{-1}[\omega D \\ &\quad - \omega L - \omega U + D \\ &\quad - D](D-\omega L)^{-1}[\{(1-\omega)D \\ &\quad + \omega U\}x^{(n)}+\omega b] \end{aligned}$$

$$\begin{aligned} x^{(n+1)} &= (D-\omega L)^{-1}\{(1-\omega)D+\omega U\}x^{(n)} \\ &\quad + 2\omega(D-\omega L)^{-1}b \\ &\quad - (D-\omega L)^{-1}[(D-\omega L) \\ &\quad - \{(1-\omega)D+\omega U\}](D-\omega L)^{-1}[\{(1 \\ &\quad - \omega)D+\omega U\}x^{(n)}+\omega b] \end{aligned}$$

$$\begin{aligned} x^{(n+1)} &= (D-\omega L)^{-1}\{(1-\omega)D+\omega U\}x^{(n)} \\ &\quad + 2\omega(D-\omega L)^{-1}b - [I \\ &\quad - (D-\omega L)^{-1}\{(1-\omega)D \\ &\quad + \omega U\}](D-\omega L)^{-1}[\{(1-\omega)D \\ &\quad + \omega U\}x^{(n)}+\omega b] \end{aligned}$$

$$\begin{aligned} x^{(n+1)} &= (D-\omega L)^{-1}\{(1-\omega)D+\omega U\}x^{(n)} \\ &\quad + 2\omega(D-\omega L)^{-1}b \\ &\quad - (D-\omega L)^{-1}\{(1-\omega)D+\omega U\}x^{(n)} \\ &\quad + [(D-\omega L)^{-1}\{(1-\omega)D \\ &\quad + \omega U\}]^2x^{(n)} - (D-\omega L)^{-1}\omega b \\ &\quad + \omega[(D-\omega L)^{-1}\{(1-\omega)D \\ &\quad + \omega U\}](D-\omega L)^{-1}b \end{aligned}$$

$$\begin{aligned} x^{(n+1)} &= [(D-\omega L)^{-1}\{(1-\omega)D+\omega U\}]^2x^{(n)} \\ &\quad + \omega(D-\omega L)^{-1}b \\ &\quad + \omega(D-\omega L)^{-1}\{(1-\omega)D \\ &\quad + \omega U\}(D-\omega L)^{-1}b \end{aligned}$$

$$x^{(n+1)} = {G_\omega}^2x^{(n)} + \omega(D-\omega L)^{-1}b + \omega G_\omega(D-\omega L)^{-1}b$$

$$x^{(n+1)} = {G_\omega}^2x^{(n)} + \omega[I+G_\omega](D-\omega L)^{-1}b \qquad (7)$$

That is,

$$\begin{aligned} x^{(n+1)} &= [(D-\omega L)^{-1}\{(1-\omega)D+\omega U\}]^2x^{(n)} \\ &\quad + \omega[I \\ &\quad + (D-\omega L)^{-1}\{(1-\omega)D+\omega U\}](D \\ &\quad - \omega L)^{-1}b \qquad (8) \end{aligned}$$

Or,

$$\begin{aligned} x^{(n+1)} &= \left[\left(I-\omega\tilde{L}\right)^{-1}\{(1-\omega)I+\omega\tilde{U}\}\right]^2 x^{(n)} \\ &\quad + \omega\left[I\right. \\ &\quad \left. + \left(I-\omega\tilde{L}\right)^{-1}\{(1-\omega)I+\omega\tilde{U}\}\right]\left(I\right. \\ &\quad \left. - \omega\tilde{L}\right)^{-1}\tilde{b} \qquad (9) \end{aligned}$$

where, $\tilde{L} = D^{-1}L, \quad \tilde{U} = D^{-1}U \quad \text{and} \quad \tilde{b} = D^{-1}b$

And the refinement of $\text{SOR (i.e., RSOR)}$ method is defined by

$$x^{(n+1)} = R_{G_\omega}x^{(n)} + \tilde{d}\,, \qquad n = 0,1,2,\cdots \qquad (10)$$

where,

$$R_{G_\omega} = \left[\left(I-\omega\tilde{L}\right)^{-1}\{(1-\omega)I+\omega\tilde{U}\}\right]^2$$

is the $\text{RSOR}$ iteration matrix, and

$$\tilde{d} = \omega\left[I + \left(I-\omega\tilde{L}\right)^{-1}\{(1-\omega)I+\omega\tilde{U}\}\right]\left(I-\omega\tilde{L}\right)^{-1}\tilde{b}$$

Here, $G_\omega$ is the SOR iteration matrix.

Now, by letting

$E = G_\omega,\ \bar{E} = R_{G_\omega} = E^2,$

$f = \left(I-\omega\tilde{L}\right)^{-1}\tilde{b},\ \bar{f} = \left(I-\omega\tilde{L}\right)^{-1}\tilde{b}.$

**Convergence of Refinement of Preconditioned SOR Method**

**Lemma 3.1** (Hadjidimos (1978)): If A is an irreducible matrix with weak diagonal dominance, then the methods of Jacobi, of Gauss-Seidel, of Simultaneous Overrelaxation, of Successive Overrelaxation and of Accelerated Overrelaxation converge.

**Theorem 3.1** Let $A$ be irreducible matrix with weak diagonal dominance. Then the refinement of SOR method converges for any arbitrary choice of the initial approximation.

Proof: Suppose $x_E$ is the exact solution of linear system (2). Because matrix $A$ is an irreducible matrix with weak diagonal dominance, the SOR method is convergent by Lemma 3.1. Let $\hat{x}^{(n+1)}$ be the $(n+1)\text{th}$ approximation to the solution of (2) by the SOR method (6). Then,

$$\begin{aligned} \left\|\hat{x}^{(n+1)} - x_E\right\| &= \left\|x^{(n+1)} + \omega\left(I-\omega\tilde{L}\right)^{-1}\left(\tilde{b} - \tilde{A}x^{(n+1)}\right)\right. \\ &\quad \left. - x_E\right\| \\ &\leq \left\|x^{(n+1)} - x_E\right\|\left\|\left(\tilde{b} - \tilde{A}x^{(n+1)}\right)\right\|\left\|\omega\left(I-\omega\tilde{L}\right)^{-1}\right\| \end{aligned}$$

Since $\left\|x^{(n+1)} - x_E\right\| \to 0$ and $\left\|\left(\tilde{b} - \tilde{A}x^{(n+1)}\right)\right\| \to 0,$ we have $\left\|\hat{x}^{(n+1)} - x_E\right\| \to 0.$

Hence, the refinement of SOR method converges to the solution of the linear system (2).

**Theorem 3.2** Let $A$ be irreducible matrix with weak diagonal dominance, then $\left\|R_{G_\omega}\right\|_\infty = \|G_\omega\|_\infty^2 < 1.$

Proof: Consider $\left\|R_{G_\omega}\right\|_\infty$; then we have,

$$\begin{aligned} \left\|R_{G_\omega}\right\|_\infty &= \left\|\left[\left(I-\omega\tilde{L}\right)^{-1}\{(1-\omega)I+\omega\tilde{U}\}\right]^2\right\|_\infty \\ &= \left\|\left[\left(I-\omega\tilde{L}\right)^{-1}\{(1-\omega)I+\omega\tilde{U}\}\right]\right\|_\infty^2 \\ &= \|G_\omega\|_\infty^2 < 1 \quad \text{by Theorem 3.1} \end{aligned}$$

**Theorem 3.3** If $A$ is irreducible matrix with weak diagonal dominance, then $\left\|R_{G_\omega}\right\|_\infty < \|G_\omega\|_\infty$.

Proof: By Theorem 3.2, we have $\left\|R_{G_\omega}\right\|_\infty = \|G_\omega\|_\infty^2 < \|G_\omega\|_\infty$.

**Theorem 3.4** The refinement of SOR method converges faster than the SOR method when SOR method is convergent.

Proof: Let $\bar{\bar{x}}$ be the solution of linear system (2) obtained by the refinement of SOR method (10) and $\hat{x}$ be the solution of (2) obtained by the SOR method (6).

From (10), we get

$$\begin{aligned} \bar{\bar{x}} &= R_{G_\omega}\hat{x} + \tilde{d} \\ \bar{\bar{x}} &= {G_\omega}^2\hat{x} + \tilde{d} \\ \bar{\bar{x}}^{(n+1)} - \bar{\bar{x}} &= {G_\omega}^2x^{(n)} + \tilde{d} - \bar{\bar{x}} \\ &= {G_\omega}^2\left(x^{(n)} - \hat{x}\right) + \tilde{d} - \bar{\bar{x}} + {G_\omega}^2\hat{x} \\ &= {G_\omega}^2\left(x^{(n)} - \hat{x}\right) - \bar{\bar{x}} + ({G_\omega}^2\hat{x} + \tilde{d}) \\ &= {G_\omega}^2\left(x^{(n)} - \hat{x}\right) - \bar{\bar{x}} + \bar{\bar{x}} \\ &= {G_\omega}^2\left(x^{(n)} - \hat{x}\right) \end{aligned}$$

Now,

$$\begin{aligned} \left\|\bar{\bar{x}}^{(n+1)} - \bar{\bar{x}}\right\|_\infty &= \left\|{G_\omega}^2\left(x^{(n)} - \hat{x}\right)\right\|_\infty \\ &\leq \left\|{G_\omega}^2\right\|_\infty\left\|\left(x^{(n)} - \hat{x}\right)\right\|_\infty \\ &\leq \|G_\omega\|_\infty^2\left\|\left(x^{(n)} - \hat{x}\right)\right\|_\infty \end{aligned}$$

Therefore, by Theorems 3.1 and 3.2 the refinement of SOR method converges faster than the SOR method.

### Numerical Experiments

**Problem** 1: The solution of the following elliptic linear second-order partial differential equation

$$(x+1)\frac{\partial^2 u}{\partial x^2}+(y^2+1)\frac{\partial^2 u}{\partial y^2}+u=-1$$

with the boundary values $u(0,y)=y, u(1,y)=y^2, u(x,0)=0, u(x,1)=1$, is sought in the region $0\le x\le 1,\quad 0\le y\le 1$, with $h=\frac{1}{3}$.
Source: (Ndanusa and Adeboye, 2012).

**Problem 2:** Consider the Poisson equation

$$\nabla^2 u=-1$$

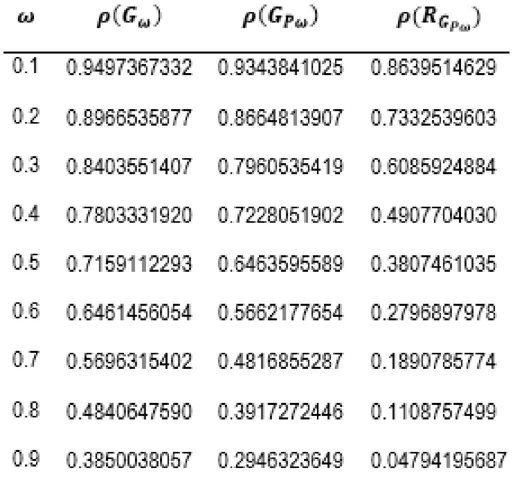
in the region $|x|\le 1,\quad |y|\le 1$, with boundary values $u=0, |x|=1, |y|=1$.

**Problem 3:** The solution of Problem 1 is sought for $h=1/4$.

### RESULTS AND DISCUSSION

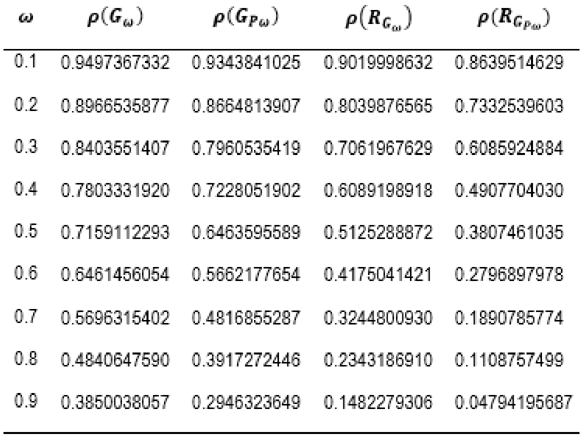
The spectral radii of iteration matrices of Successive Overrelaxation (SOR) method, Preconditioned Successive Overrelaxation (PSOR), Refinement of Successive Overrelaxation (RSOR) and Refinement of Preconditioned Successive Overrelaxation (RPSOR) schemes are computed with the aid of Maple 2019 mathematical software package. In what follows, the following notations are employed: $G_\omega=$ Iteration matrix of the SOR iteration method; $G_{P\omega}=$ Iteration matrix of the preconditioned SOR iteration method of Ndanusa and Adeboye (2012); $R_{G_\omega}=$ Iteration matrix of the Refinement of SOR iteration method; $R_{G_{P\omega}}=$ Iteration matrix of the Refinement of Preconditioned SOR iteration method; $\rho(G_\omega)=$ Spectral radius of $G_\omega$; $\rho(G_{P\omega})=$ Spectral radius of $G_{P\omega}$; $\rho\left(R_{G_\omega}\right)=$ Spectral radius of $R_{G_\omega}$; $\rho\left(R_{G_{P\omega}}\right)=$ Spectral radius of $R_{G_{P\omega}}$; $R(G)=$Rate of convergence of linear iteration $x^{(n+1)}=Gx^{(n)}+k$, where $\rho(G)$ is the spectral radius for that iterative method.

**Table 1:** Comparison of spectral radii of $G_\omega, G_{P\omega}$and $R_{G_{P\omega}}$ for Problem 1



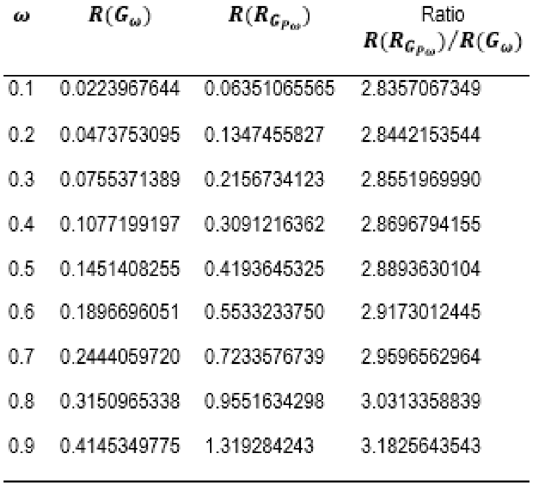
| $\omega$ | $\rho(G_\omega)$ | $\rho(G_{P\omega})$ | $\rho(R_{G_{P\omega}})$ |
|---|---|---|---|
| 0.1 | 0.9497367332 | 0.9343841025 | 0.8639514629 |
| 0.2 | 0.8966535877 | 0.8664813907 | 0.7332539603 |
| 0.3 | 0.8403551407 | 0.7960535419 | 0.6085924884 |
| 0.4 | 0.7803331920 | 0.7228051902 | 0.4907704030 |
| 0.5 | 0.7159112293 | 0.6463595589 | 0.3807461035 |
| 0.6 | 0.6461456054 | 0.5662177654 | 0.2796897978 |
| 0.7 | 0.5696315402 | 0.4816855287 | 0.1890785774 |
| 0.8 | 0.4840647590 | 0.3917272446 | 0.1108757499 |
| 0.9 | 0.3850038057 | 0.2946323649 | 0.04794195687 |

**Table 2:** Comparison of spectral radii of $R_{G_\omega}$ and $R_{G_{P\omega}}$ for Problem 1



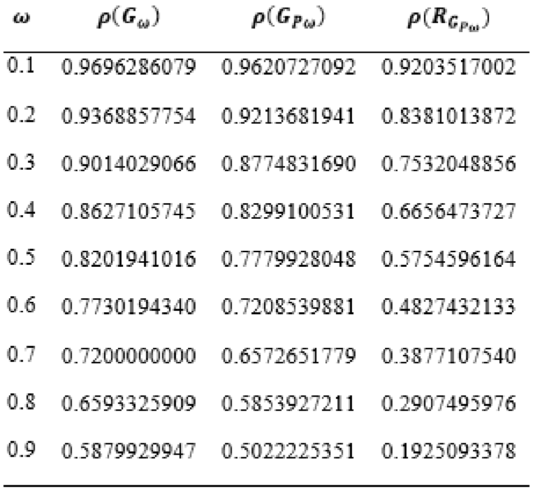
| $\omega$ | $\rho(G_\omega)$ | $\rho(G_{P\omega})$ | $\rho\left(R_{G_\omega}\right)$ | $\rho\left(R_{G_{P\omega}}\right)$ |
|---|---|---|---|---|
| 0.1 | 0.9497367332 | 0.9343841025 | 0.9019998632 | 0.8639514629 |
| 0.2 | 0.8966535877 | 0.8664813907 | 0.8039876565 | 0.7332539603 |
| 0.3 | 0.8403551407 | 0.7960535419 | 0.7061967629 | 0.6085924884 |
| 0.4 | 0.7803331920 | 0.7228051902 | 0.6089198918 | 0.4907704030 |
| 0.5 | 0.7159112293 | 0.6463595589 | 0.5125288872 | 0.3807461035 |
| 0.6 | 0.6461456054 | 0.5662177654 | 0.4175041421 | 0.2796897978 |
| 0.7 | 0.5696315402 | 0.4816855287 | 0.3244800930 | 0.1890785774 |
| 0.8 | 0.4840647590 | 0.3917272446 | 0.2343186910 | 0.1108757499 |
| 0.9 | 0.3850038057 | 0.2946323649 | 0.1482279306 | 0.04794195687 |

**Table 3:** Comparison of $R(G_\omega)$ and $R(R_{G_{P\omega}})$ for Problem 1



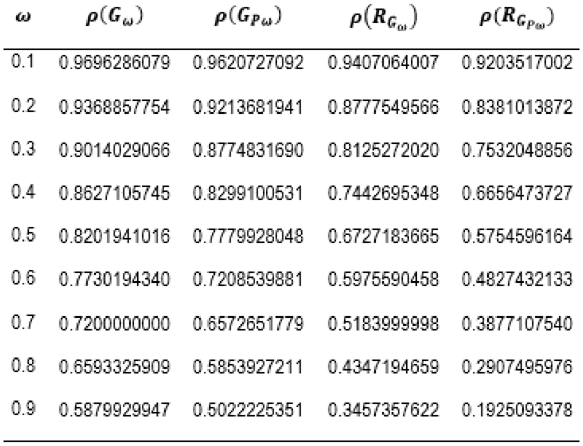
| $\omega$ | $R(G_\omega)$ | $R(R_{G_{P\omega}})$ | Ratio<br>$R(R_{G_{P\omega}})/R(G_\omega)$ |
|---|---|---|---|
| 0.1 | 0.0223967644 | 0.06351065565 | 2.8357067349 |
| 0.2 | 0.0473753095 | 0.1347455827 | 2.8442153544 |
| 0.3 | 0.0755371389 | 0.2156734123 | 2.8551969990 |
| 0.4 | 0.1077199197 | 0.3091216362 | 2.8696794155 |
| 0.5 | 0.1451408255 | 0.4193645325 | 2.8893630104 |
| 0.6 | 0.1896696051 | 0.5533233750 | 2.9173012445 |
| 0.7 | 0.2444059720 | 0.7233576739 | 2.9596562964 |
| 0.8 | 0.3150965338 | 0.9551634298 | 3.0313358839 |
| 0.9 | 0.4145349775 | 1.319284243 | 3.1825643543 |

**Table 4:** Comparison of spectral radii of $G_\omega, G_{P\omega}$and $R_{G_{P\omega}}$ for Problem 2



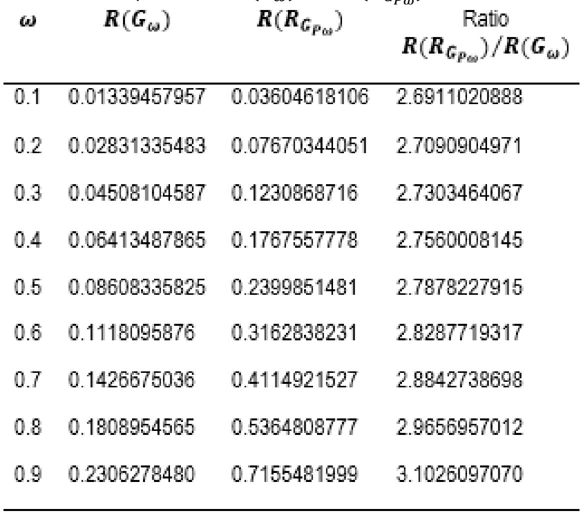
| $\omega$ | $\rho(G_\omega)$ | $\rho(G_{P\omega})$ | $\rho(R_{G_{P\omega}})$ |
|---|---|---|---|
| 0.1 | 0.9696286079 | 0.9620727092 | 0.9203517002 |
| 0.2 | 0.9368857754 | 0.9213681941 | 0.8381013872 |
| 0.3 | 0.9014029066 | 0.8774831690 | 0.7532048856 |
| 0.4 | 0.8627105745 | 0.8299100531 | 0.6656473727 |
| 0.5 | 0.8201941016 | 0.7779928048 | 0.5754596164 |
| 0.6 | 0.7730194340 | 0.7208539881 | 0.4827432133 |
| 0.7 | 0.7200000000 | 0.6572651779 | 0.3877107540 |
| 0.8 | 0.6593325909 | 0.5853927211 | 0.2907495976 |
| 0.9 | 0.5879929947 | 0.5022225351 | 0.1925093378 |

**Table 5:** Comparison of spectral radii of $R_{G_\omega}$ and $R_{G_{P\omega}}$ for Problem 2



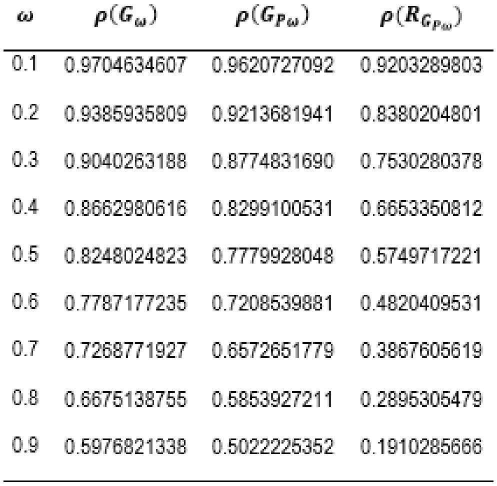
| $\omega$ | $\rho(G_\omega)$ | $\rho(G_{P\omega})$ | $\rho(R_{G_\omega})$ | $\rho(R_{G_{P\omega}})$ |
|---|---|---|---|---|
| 0.1 | 0.9696286079 | 0.9620727092 | 0.9407064007 | 0.9203517002 |
| 0.2 | 0.9368857754 | 0.9213681941 | 0.8777549566 | 0.8381013872 |
| 0.3 | 0.9014029066 | 0.8774831690 | 0.8125272020 | 0.7532048856 |
| 0.4 | 0.8627105745 | 0.8299100531 | 0.7442695348 | 0.6656473727 |
| 0.5 | 0.8201941016 | 0.7779928048 | 0.6727183665 | 0.5754596164 |
| 0.6 | 0.7730194340 | 0.7208539881 | 0.5975590458 | 0.4827432133 |
| 0.7 | 0.7200000000 | 0.6572651779 | 0.5183999998 | 0.3877107540 |
| 0.8 | 0.6593325909 | 0.5853927211 | 0.4347194659 | 0.2907495976 |
| 0.9 | 0.5879929947 | 0.5022225351 | 0.3457357622 | 0.1925093378 |

**Table 6:** Comparison of $R(G_\omega)$ and $R(R_{G_{P\omega}})$ for Problem 2



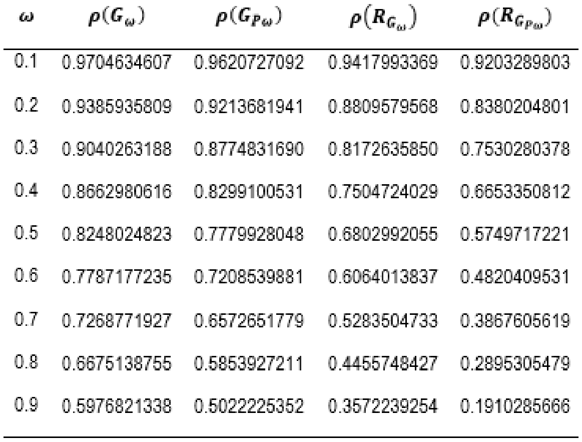
| $\omega$ | $R(G_\omega)$ | $R(R_{G_{P\omega}})$ | Ratio<br>$R(R_{G_{P\omega}})/R(G_\omega)$ |
|---|---|---|---|
| 0.1 | 0.01339457957 | 0.03604618106 | 2.6911020888 |
| 0.2 | 0.02831335483 | 0.07670344051 | 2.7090904971 |
| 0.3 | 0.04508104587 | 0.1230868716 | 2.7303464067 |
| 0.4 | 0.06413487865 | 0.1767557778 | 2.7560008145 |
| 0.5 | 0.08608335825 | 0.2399851481 | 2.7878227915 |
| 0.6 | 0.1118095876 | 0.3162838231 | 2.8287719317 |
| 0.7 | 0.1426675036 | 0.4114921527 | 2.8842738698 |
| 0.8 | 0.1808954565 | 0.5364808777 | 2.9656957012 |
| 0.9 | 0.2306278480 | 0.7155481999 | 3.1026097070 |

**Table 7:** Comparison of spectral radii of $G_\omega, G_{P\omega}$and $R_{G_{P\omega}}$ for Problem 3



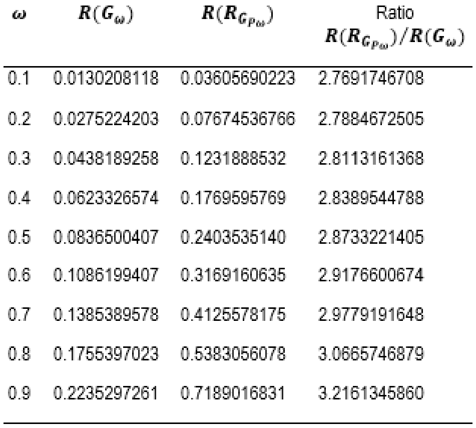
| $\omega$ | $\rho(G_\omega)$ | $\rho(G_{P\omega})$ | $\rho(R_{G_{P\omega}})$ |
|---|---|---|---|
| 0.1 | 0.9704634607 | 0.9620727092 | 0.9203289803 |
| 0.2 | 0.9385935809 | 0.9213681941 | 0.8380204801 |
| 0.3 | 0.9040263188 | 0.8774831690 | 0.7530280378 |
| 0.4 | 0.8662980616 | 0.8299100531 | 0.6653350812 |
| 0.5 | 0.8248024823 | 0.7779928048 | 0.5749717221 |
| 0.6 | 0.7787177235 | 0.7208539881 | 0.4820409531 |
| 0.7 | 0.7268771927 | 0.6572651779 | 0.3867605619 |
| 0.8 | 0.6675138755 | 0.5853927211 | 0.2895305479 |
| 0.9 | 0.5976821338 | 0.5022225352 | 0.1910285666 |

**Table 8:** Comparison of spectral radii of $R_{G_\omega}$ and $R_{G_{P\omega}}$ for Problem 3



| $\omega$ | $\rho(G_\omega)$ | $\rho(G_{P\omega})$ | $\rho(R_{G_\omega})$ | $\rho(R_{G_{P\omega}})$ |
|---|---|---|---|---|
| 0.1 | 0.9704634607 | 0.9620727092 | 0.9417993369 | 0.9203289803 |
| 0.2 | 0.9385935809 | 0.9213681941 | 0.8809579568 | 0.8380204801 |
| 0.3 | 0.9040263188 | 0.8774831690 | 0.8172635850 | 0.7530280378 |
| 0.4 | 0.8662980616 | 0.8299100531 | 0.7504724029 | 0.6653350812 |
| 0.5 | 0.8248024823 | 0.7779928048 | 0.6802992055 | 0.5749717221 |
| 0.6 | 0.7787177235 | 0.7208539881 | 0.6064013837 | 0.4820409531 |
| 0.7 | 0.7268771927 | 0.6572651779 | 0.5283504733 | 0.3867605619 |
| 0.8 | 0.6675138755 | 0.5853927211 | 0.4455748427 | 0.2895305479 |
| 0.9 | 0.5976821338 | 0.5022225352 | 0.3572239254 | 0.1910285666 |

**Table 9:** Comparison of $R(G_\omega)$ and $R(R_{G_{P\omega}})$ for Problem 3



| $\omega$ | $R(G_\omega)$ | $R(R_{G_{P\omega}})$ | Ratio<br>$R(R_{G_{P\omega}})/R(G_\omega)$ |
|---|---|---|---|
| 0.1 | 0.0130208118 | 0.03605690223 | 2.7691746708 |
| 0.2 | 0.0275224203 | 0.07674536766 | 2.7884672505 |
| 0.3 | 0.0438189258 | 0.1231888532 | 2.8113161368 |
| 0.4 | 0.0623326574 | 0.1769595769 | 2.8389544788 |
| 0.5 | 0.0836500407 | 0.2403535140 | 2.8733221405 |
| 0.6 | 0.1086199407 | 0.3169160635 | 2.9176600674 |
| 0.7 | 0.1385389578 | 0.4125578175 | 2.9779191648 |
| 0.8 | 0.1755397023 | 0.5383056078 | 3.0665746879 |
| 0.9 | 0.2235297261 | 0.7189016831 | 3.2161345860 |

Table 1 shows the spectral radii for iteration matrices of $G_\omega, G_{P\omega}$and $R_{G_{P\omega}}$ respectively, for various values of the relaxation parameter $\omega$, between 0 and 1, for Problem 1. The refinement of preconditioned SOR, RPSOR, exhibits faster convergence than the PSOR and SOR methods, i.e., $\rho(G_\omega) < \rho(G_{P\omega}) < \rho\left(R_{G_{P\omega}}\right) < 1$, for all values of relaxation parameter $\omega$. It further reveals that optimum convergence is attained at $\omega = 0.9$ for all methods. Table 2 specifically compares the spectral radius of refinement of SOR (RSOR) method, $\boldsymbol{\rho}\left(R_{G_\omega}\right)$, with that of refinement of preconditioned SOR (RPSOR) method, $\boldsymbol{\rho}(R_{G_{P\omega}})$, for various values of $\omega$, in order to demonstrate the effect of preconditioning on the refinement technique. As is shown in the table, the RPSOR converges faster than RSOR; while RSOR converges faster than PSOR. In Table 3, the rate of convergence of refinement of preconditioned SOR method, RPSOR is compared with the rate of convergence of SOR method. It revealed that the RPSOR converges almost thrice as fast as the SOR method.

In Table 4, the spectral radii for iteration matrices of $G_\omega, G_P$and $R_{G_\omega}$ are compared, for Problem 2, for various values of the relaxation parameter $\omega$, between 0 and 1. The refinement of preconditioned SOR, RSOR, exhibits faster convergence than the PSOR and SOR methods, i.e., $\rho(G_\omega) < \rho(G_P) < \rho\left(R_{G_\omega}\right) < 1$, for all values of relaxation parameter $\omega$. It also revealed that optimum convergence is attained at $\omega = 0.9$ for all methods. Table 5 specifically compares the spectral radius of refinement of SOR (RSOR) method, $\boldsymbol{\rho}\left(R_{G_\omega}\right)$, with that of refinement of preconditioned SOR (RPSOR) method, $\boldsymbol{\rho}(R_{G_{P_\omega}})$, for various values of $\omega$, in order to demonstrate the effect of preconditioning on the refinement technique. As is shown in the table, the RPSOR converges faster than RSOR. In Table 6, the rate of convergence of refinement of preconditioned SOR method, RPSOR is compared with the rate of convergence of SOR method. It revealed that the RPSOR converges thrice as fast as the SOR method

In Table 7 the spectral radii for iteration matrices of $G_\omega, G_P$and $R_{G_\omega}$ are compared, for Problem 3, for various values of the relaxation parameter $\omega$, between 0 and 1. The refinement of preconditioned SOR, RSOR, exhibits faster convergence than the PSOR and SOR methods, i.e., $\rho(G_\omega) < \rho(G_P) < \rho\left(R_{G_\omega}\right) < 1$, for all values of relaxation parameter $\omega$. It also reveals that optimum convergence is attained at $\omega = 0.9$ for all methods. Table 8 specifically compares the spectral radius of refinement of SOR (RSOR) method, $\boldsymbol{\rho}\left(R_{G_\omega}\right)$, with that of refinement of preconditioned SOR (RPSOR) method, $\boldsymbol{\rho}(R_{G_{P_\omega}})$, for various values of $\omega$, in order to demonstrate the effect of preconditioning on the refinement technique. As is shown in the table, the RPSOR converges faster than RSOR. In Table 9, the rate of convergence of refinement of preconditioned SOR method, RPSOR is compared with the rate of convergence of SOR method. It revealed that the RPSOR converges thrice as fast as the SOR method.

## Conclusion

A refinement iteration technique for the Successive Overrelaxation (SOR) iterative method is proposed; this refinement iteration is applied to a preconditioned linear system of equations arising from discretization of elliptic partial differential equations. The main thrust of refinement of preconditioned successive overrelaxation (RPSOR) is to reduce to the barest minimum the spectral radius of the iteration matrix so as to increase the rate of convergence of the method in comparison to the successive overrelaxation (SOR) method. Convergence theorems were advanced and the refinement of preconditioned SOR method is shown to be convergent; for irreducible matrix with weak diagonal dominance then the RPSOR converges for any initial approximation. Also, the RPSOR is shown to converge almost thrice as fast as the SOR method. Sample numerical experiments conducted further corroborate the theoretical analysis.

## REFERENCES

Assefa, W. L. and Teklehaymanot, A. W. (2021). Second refinement of accelerated over relaxation method for the Solution of Linear System. *Pure and Applied Mathematics Journal*, 10(1): 32-37. doi: 10.11648/j.pamj.20211001.13

Burden, R.L, Faires, J.D, and Burden, A.M. (2014). *Numerical Analysis*, (10th ed.) Boston,U.S.A: Brooks/Cole, Cengage Learning.

Dafchahi, F. N. (2008). A new refinement of Jacobi method for solution of linear system equations $\mathrm{Ax} = \mathrm{b}$. *International Journal of Contemporary Mathematical Sciences*, 3(17): 819-827.

Eneyew, T. K., Awgichew, G., Haile, E. and Abie, G. D. (2019). Second refinement of Jacobi iterative method for solving linear system of equations. *International Journal of Computing Science and Applied Mathematics*, 5(2): 41-47.

Eneyew, T. K., Awgichew, G., Haile, E. and Abie, G. D. (2020). Second refinement of Gauss-Seidel iterative method for solving linear system of equations. *Ethiopian Journal of Science and Technology,* 13(1): 1-15.

Gebregiorgis, S. and Gofe, G. (2018). Refined generalized iterative methods in solving fuzzy linear systems. *Ethiopian Journal of Education and Science,* 13(2): 73-81.

Gonfa, G. G. (2016). Refined iterative method for solving system of linear equations. *American Journal of Computational and Applied Mathematics*, 6(3): 144-147.

Hadjidimos, A. (1978). Accelerated overrelaxation method. *Mathematics of Computation*, (32) 141: 149-157.

Kebede, T. (2017). Second degree refinement Jacobi iteration method for solving system of linear equation. *International Journal of Computing Science and Applied Mathematics*, 3(1): 5-10.

Kyurkchiev, N. and Iliev, A. (2013). A refinement of some overrelaxation algorithms for solving a system of linear equations. *Serdica Journal of Computing,* 7(3): 245–256.

Laskar, A. H. and Behera, S. (2014). Refinement of iterative methods for the solution of system of linear equations $\mathrm{Ax} = \mathrm{b}$. *IOSR Journal of Mathematics (IOSR-JM)*, 10(3): 70-73.

Muleta, H. and Gofe, G. (2018). Refinement of generalized accelerated over relaxation method for solving system of linear equations based on the Nekrassov-Mehmke1-method. *Ethiopian Journal of Education and Science,* 13(2): 1-18.

Ndanusa, A. and Adeboye, K. R. (2012). Preconditioned SOR iterative methods for $\mathrm{L}$-matrices. *American Journal of Computational and Applied Mathematics,* 2(6): 300-305.

Vatti, V. B. K. and Eneyew, T. K. (2011). A refinement of Gauss-Seidel method for solving of linear system of equations. *International Journal of ContemporaryMathematical Sciences*, 6(3): 117-121.

Vatti, V. B. K. and Gonfa, G. G. (2011). Refinement of Generalized Jacobi (RGJ) method
for solving system of linear equations. *International Journal of Contemporary Mathematical Sciences*, 6(3): 109-116.

Vatti, V. B. K. Sri, R. and Mylapalli, M. S. K. (2018). A refinement of accelerated over relaxation method for the solution of linear systems. *International Journal of Pure and Applied Mathematics,* 118(18): 1571-1577.

Yuan, J. Y. (1998). Iterative refinement using splitting methods. *Linear Algebra and its Applications,* 273:199-214.