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A 4-Step Order (K + 1) Block Hybrid Backward Differentiation Formulae (BHBDF) for the Solution of General Second Order Ordinary Differential Equations

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ABSTRACT

In this paper, the block hybrid backward differentiation formulae (BHBDF) for the step number $k = 4$ was developed using power series as basis function for the solution of general second order ordinary differential equation. The idea of interpolation and collocation of the power series at some selected grid and off- grid points gave rise to continuous schemes which were further evaluated at those points to produce discrete schemes combined together to form block methods. Numerical problems were solved with the proposed methods and were found to perform effectively.

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INTRODUCTION

Ordinary Differential Equations arise frequently in the study of the physical problems. Unfortunately, many cannot be solved analytically. This is why the ability to solve these equations numerically is important. Traditionally, mathematicians have used one of two classes of methods for solving numerically ordinary differential equations. These are Runge –Kutta methods and Linear Multistep Methods (LMM) (Muhammad, 2016).

Numerical analyst is usually faced with the challenge of obtaining starting or initial values for Linear Multistep Methods (LMM) when step number $k \geq 2$ in solving differential equations numerically (Hussaini and Muhammad, 2021). Prior before now, one step methods like Runge-Kutta or trapezoidal methods were used to obtain the starting values for such methods. The hybrid method is not exempted from this problem as it shares the same standard methods. In addition to this, the need for special predictors to predict the off-grid values in hybrid forms of the LMM (Awari, 2017). The discrete schemes obtained from the continuous formulation of k-step block hybrid backward differentiation formulae can be used in block form to obtain the block solution (Hussaini and Muhammad, 2021). This approach circumvents both the non-self-starting property and calls for special predictors to predict the off-grid values in the hybrid's forms of the block hybrid backward differentiation formulae (Badmus *et al*, 2014). The popularity and the explosive growth of these methods; coupled with the amount of research effort being undertaken are further evidence that the applications are still the leading source of inspiration for mathematical creativity (Yahaya and Adegboye, 2013).

MATERIALS AND METHODS

We present the derivation of Block Hybrid Backward Differentiation Formula (BHBDF) for solving some classes of second-order ordinary differential equations of the form

$$\frac{d^2y(x)}{dx^2} = f(x.y, \frac{dy(x)}{dx}) \quad (1)$$

coupled with appropriate initial conditions

$$y(x_0) = \varphi_1, \frac{dy(x_0)}{dx} = \varphi_2 \quad (2)$$

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where f is a continuous function such that $f: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$, x_0 is the initial point, $y \in \mathbb{R}$ is an $n-$dimensional vector, x is a scalar variable, φ_1 and φ_2 are the initial values.

In this research, we seek to develop numerical schemes in the form of BHBDF as:

$$Y(x) = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + \alpha_v(x) y_{n+v} + h^2 \beta_k(x) f_{n+k} \qquad (3)$$

where h is the chosen step size and $\alpha_j(x): j = 0,1,2,\ldots,k$, $\alpha_v(x) and\ \beta_k(x)$ are unknown continuous coefficients to be determined. For Backward Differentiation Formula, we note that $\alpha_k = 1$ and $\beta_k \neq 0$. In this study, we will derive block hybrid backward differentiation formulae for the step numbers $k = 4$ of the proposed method using power series function as the basis function.

We seek an approximation of the form;

$$Y(x) = \sum_{j=0}^{t+c-1} p^j x^j \qquad (4)$$

where t represent the interpolation points, c is the collocation point and p_j are unknown coefficients to be determined. Then, we take

$$Y(x) = y_{n+j}$$

$$Y(x) = y_{n+j},\ j = 0,1,2,\ldots\ldots, k-1 \qquad (5)$$

$$Y''(x_{n+k}) = f_{n+k} \qquad (6)$$

4-Step block hybrid backward differentiation formulae (4SBHBDF)

To derive 4SBHBDF, we take $t = 6, c = 1$ and $x \in [x_n, x_{n+4}]$. Therefore, (4) becomes;

$$Y(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3 + p_4 x^4 + p_5 x^5 + p_6 x^6 \qquad (7)$$

Interpolating (5) at x_{n+i}; $i = 0, 1, 2, 3, \frac{15}{4}, \frac{31}{8}$ and collocating (6) at x_{n+i}; $i = 4$. This results in a system of equations;

$$D\psi = Y \qquad (8)$$

where $\psi = \left(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_{\frac{15}{4}}, \alpha_{\frac{31}{8}}, \beta_4\right)^T$, $Y = \left(y_n, y_{n+1}, y_{n+2}, y_{n+3}, y_{n+\frac{15}{4}}, y_{n+\frac{31}{8}}, f_{n+4}\right)^T$

and the matrix D of the proposed method is expressed as

$$D = \begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 \\ 1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 & x_{n+3}^6 \\ 1 & x_{n+\frac{15}{4}} & x_{n+\frac{15}{4}}^2 & x_{n+\frac{15}{4}}^3 & x_{n+\frac{15}{4}}^4 & x_{n+\frac{15}{4}}^5 & x_{n+\frac{15}{4}}^6 \\ 1 & x_{n+\frac{31}{8}} & x_{n+\frac{31}{8}}^2 & x_{n+\frac{31}{8}}^3 & x_{n+\frac{31}{8}}^4 & x_{n+\frac{31}{8}}^5 & x_{n+\frac{31}{8}}^6 \\ 0 & 0 & 2 & 6x_{n+4} & 12x_{n+4} & 20x_{n+4} & 30x_{n+4} \end{pmatrix}$$

We proceed by solving the Matrix D using matrix inversion method with the aid of Maple software to obtain the continuous coefficients;

$$\left.\begin{aligned}
\alpha_0 &= \frac{352}{130479}\frac{x^6}{h^6} - \frac{94388}{1957185}\frac{x^5}{h^5} + \frac{682219}{1957185}\frac{x^4}{h^4} - \frac{5070023}{3914370}\frac{x^3}{h^3} + \frac{5060234}{1957185}\frac{x^2}{h^2} - \frac{1127893}{434930}\frac{x}{h} + 1 \\
\alpha_1 &= -\frac{6896}{354959}\frac{x^6}{h^6} + \frac{116406}{354959}\frac{x^5}{h^5} - \frac{1549923}{709918}\frac{x^4}{h^4} + \frac{461407}{64538}\frac{x^3}{h^3} - \frac{4075758}{354959}\frac{x^2}{h^2} + \frac{2558430}{354959}\frac{x}{h} \\
\alpha_2 &= \frac{1984}{29463}\frac{x^6}{h^6} - \frac{52536}{49105}\frac{x^5}{h^5} + \frac{968068}{147315}\frac{x^4}{h^4} - \frac{1889777}{98210}\frac{x^3}{h^3} + \frac{1285926}{49105}\frac{x^2}{h^2} - \frac{245799}{19642}\frac{x}{h} \\
\alpha_3 &= -\frac{1936}{9821}\frac{x^6}{h^6} + \frac{259850}{88389}\frac{x^5}{h^5} - \frac{2961029}{176778}\frac{x^4}{h^4} + \frac{7939457}{176778}\frac{x^3}{h^3} - \frac{4903190}{88389}\frac{x^2}{h^2} + \frac{723850}{29463}\frac{x}{h} \\
\alpha_{\frac{15}{4}} &= \frac{237568}{324093}\frac{x^6}{h^6} - \frac{51426304}{4861395}\frac{x^5}{h^5} + \frac{282386432}{4861395}\frac{x^4}{h^4} - \frac{66307072}{441945}\frac{x^3}{h^3} + \frac{872353792}{4861395}\frac{x^2}{h^2} - \frac{125833216}{1620465}\frac{x}{h} \\
\alpha_{\frac{31}{8}} &= -\frac{12320768}{21007119}\frac{x^6}{h^6} + \frac{295108608}{35011865}\frac{x^5}{h^5} - \frac{4839440384}{105035595}\frac{x^4}{h^4} + \frac{4149280768}{35011865}\frac{x^3}{h^3} - \frac{4945870848}{35011865}\frac{x^2}{h^2} + \frac{427032576}{7002373}\frac{x}{h} \\
\beta_4 &= \frac{16}{1403}\frac{x^6}{h^4} - \frac{218}{1403}\frac{x^5}{h^3} + \frac{2281}{2806}\frac{x^4}{h^2} - \frac{2833}{1403}\frac{x^3}{h} + \frac{6579}{2806}x^2 - \frac{1395}{1403}xh
\end{aligned}\right\} \quad (9)$$

The values of the continuous coefficients are then substituted in to the proposed method in (7) to obtain

$$Y(x) = \alpha_0(x)y_n + \alpha_1(x)y_{n+1} + \alpha_2(x)y_{n+2} + \alpha_3(x)y_{n+3} + \alpha_{\frac{15}{4}}(x)y_{n+\frac{15}{4}} + \alpha_{\frac{31}{8}}(x)y_{n+\frac{31}{8}} + \beta_4(x)f_{n+4} \quad (10)$$

Expressing (10) further gives the continuous form of the 4SHBDF with 2-off step interpolation point as;

$$\begin{aligned}
Y(x) = {} & \left(\frac{352}{130479}\frac{x^6}{h^6} - \frac{94388}{1957185}\frac{x^5}{h^5} + \frac{682219}{1957185}\frac{x^4}{h^4} - \frac{5070023}{3914370}\frac{x^3}{h^3} + \frac{5060234}{1957185}\frac{x^2}{h^2} - \frac{1127893}{434930}\frac{x}{h} + 1\right)y_n \\
& + \left(-\frac{6896}{354959}\frac{x^6}{h^6} + \frac{116406}{354959}\frac{x^5}{h^5} - \frac{1549923}{709918}\frac{x^4}{h^4} + \frac{461407}{64538}\frac{x^3}{h^3} - \frac{4075758}{354959}\frac{x^2}{h^2} + \frac{2558430}{354959}\frac{x}{h}\right)y_{n+1} \\
& + \left(\frac{1984}{29463}\frac{x^6}{h^6} - \frac{52536}{49105}\frac{x^5}{h^5} + \frac{968068}{147315}\frac{x^4}{h^4} - \frac{1889777}{98210}\frac{x^3}{h^3} + \frac{1285926}{49105}\frac{x^2}{h^2} - \frac{245799}{19642}\frac{x}{h}\right)y_{n+2} \\
& + \left(-\frac{1936}{9821}\frac{x^6}{h^6} + \frac{259850}{88389}\frac{x^5}{h^5} - \frac{2961029}{176778}\frac{x^4}{h^4} + \frac{7939457}{176778}\frac{x^3}{h^3} - \frac{4903190}{88389}\frac{x^2}{h^2} + \frac{723850}{29463}\frac{x}{h}\right)y_{n+3} \\
& + \left(\frac{237568}{324093}\frac{x^6}{h^6} - \frac{51426304}{4861395}\frac{x^5}{h^5} + \frac{282386432}{4861395}\frac{x^4}{h^4} - \frac{66307072}{441945}\frac{x^3}{h^3} + \frac{872353792}{4861395}\frac{x^2}{h^2} - \frac{125833216}{1620465}\frac{x}{h}\right)y_{n+\frac{15}{4}} \\
& + \left(-\frac{12320768}{21007119}\frac{x^6}{h^6} + \frac{295108608}{35011865}\frac{x^5}{h^5} - \frac{4839440384}{105035595}\frac{x^4}{h^4} + \frac{4149280768}{35011865}\frac{x^3}{h^3} - \frac{4945870848}{35011865}\frac{x^2}{h^2} + \frac{427032576}{7002373}\frac{x}{h}\right)y_{n+\frac{31}{8}} \\
& + \left(\frac{16}{1403}\frac{x^6}{h^4} - \frac{218}{1403}\frac{x^5}{h^3} + \frac{2281}{2806}\frac{x^4}{h^2} - \frac{2833}{1403}\frac{x^3}{h} + \frac{6579}{2806}x^2 - \frac{1395}{1403}xh\right)f_{n+4} \quad (11)
\end{aligned}$$

Evaluating (11) at $x = x_{n+4}$ gives the discrete scheme as

$$y_{n+4} = -\frac{83}{652395}y_n + \frac{40}{32269}y_{n+1} - \frac{326}{49105}y_{n+2} + \frac{1256}{29463}y_{n+3} - \frac{180224}{147315}y_{n+\frac{15}{4}} + \frac{76546048}{35011865}y_{n+\frac{31}{8}} + \frac{12}{1403}h^2 f_{n+4} \qquad (12)$$

To obtain the sufficient schemes required, we obtain the first derivative of (11) and evaluate the continuous function at $x = x_n, x = x_{n+1}, x = x_{n+2}, x = x_{n+3}, x = x_{n+\frac{15}{4}}, x = x_{n+\frac{31}{8}} x = x_{n+4}$ to obtain;

$$\left.\begin{aligned}
hz_n &= -\frac{1127893}{434930}y_n + \frac{2558430}{354959}y_{n+1} - \frac{245799}{19642}y_{n+2} + \frac{723850}{29463}y_{n+3} \\
&\quad - \frac{125833216}{1620465}y_{n+\frac{15}{4}} + \frac{427032576}{7002373}y_{n+\frac{31}{8}} - \frac{1395}{1403}h^2 f_{n+4} \\
hz_{n+1} &= -\frac{11803}{85095}y_n - \frac{46875}{30866}y_{n+1} + \frac{3179}{915}y_{n+2} - \frac{39391}{7686}y_{n+3} \\
&\quad + \frac{452608}{30195}y_{n+\frac{15}{4}} - \frac{53346304}{4566765}y_{n+\frac{31}{8}} + \frac{11}{61}h^2 f_{n+4} \\
hz_{n+2} &= \frac{5201}{260958}y_n - \frac{102060}{354959}y_{n+1} - \frac{308381}{294630}y_{n+2} + \frac{12460}{4209}y_{n+3} \\
&\quad - \frac{2277376}{324093}y_{n+\frac{15}{4}} + \frac{80740352}{15005085}y_{n+\frac{31}{8}} - \frac{105}{1403}h^2 f_{n+4} \\
hz_{n+3} &= -\frac{2891}{652395}y_n + \frac{34083}{709918}y_{n+1} - \frac{2349}{7015}y_{n+2} - \frac{27873}{19642}y_{n+3} \\
&\quad + \frac{1386496}{231495}y_{n+\frac{15}{4}} - \frac{149815296}{35011865}y_{n+\frac{31}{8}} + \frac{63}{1403}h^2 f_{n+4} \\
hz_{n+\frac{15}{4}} &= \frac{25487}{41753280}y_n - \frac{274365}{45434752}y_{n+1} + \frac{5247}{157136}y_{n+2} - \frac{128975}{538752}y_{n+3} \\
&\quad - \frac{3621812}{540155}y_{n+\frac{15}{4}} + \frac{6918912}{1000339}y_{n+\frac{31}{8}} - \frac{3465}{179584}h^2 f_{n+4} \\
hz_{n+\frac{31}{4}} &= -\frac{3479}{5809152}y_n + \frac{185535}{31606784}y_{n+1} - \frac{14911}{468480}y_{n+2} + \frac{558155}{2623488}y_{n+3} \\
&\quad - \frac{73501}{8052}y_{n+\frac{15}{4}} + \frac{40836448}{4566765}y_{n+\frac{31}{8}} + \frac{3255}{124928}h^2 f_{n+4} \\
hz_{n+4} &= -\frac{4909}{3914370}y_n + \frac{4326}{354959}y_{n+1} - \frac{19129}{294630}y_{n+2} + \frac{36230}{88389}y_{n+3} \\
&\quad - \frac{49278976}{4861395}y_{n+\frac{15}{4}} + \frac{1027342336}{105035595}y_{n+\frac{31}{8}} + \frac{169}{1403}h^2 f_{n+4}
\end{aligned}\right\} \quad (13)$$

where z is the first derivative of y.

We further obtain the second derivatives of (11), thereafter, evaluating at $x = x_{n+2},\ x = x_{n+3} x = x_{n+\frac{15}{4}}, x = x_{n+\frac{31}{8}}$to obtain;

$$
\left.\begin{aligned}
y_{n+2} &= -\frac{13643}{433269}y_n + \frac{2438765}{4059891}y_{n+1} - \frac{54635}{433269}y_{n+3} + \frac{14501888}{4765959}y_{n+\frac{15}{4}} \\
&\quad - \frac{917504}{369081}y_{n+\frac{31}{8}} - \frac{49105}{96282}h^2 f_{n+2} + \frac{4445}{96282}h^2 f_{n+4} \\
y_{n+3} &= \frac{215551}{41563715}y_n - \frac{4164615}{67842709}y_{n+1} + \frac{720603}{1340765}y_{n+2} + \frac{2791424}{14748415}y_{n+\frac{15}{4}} \\
&\quad + \frac{314966016}{955965445}y_{n+\frac{31}{8}} - \frac{88389}{268153}h^2 f_{n+3} - \frac{5103}{268153}h^2 f_{n+4} \\
y_{n+\frac{15}{4}} &= -\frac{2842763}{6898264832}y_n + \frac{84349755}{20472269824}y_{n+1} - \frac{10370943}{445049344}y_{n+2} \\
&\quad + \frac{161178325}{890098688}y_{n+3} + \frac{519683472}{619765981}y_{n+\frac{31}{8}} - \frac{4861395}{111262336}h^2 f_{n+\frac{15}{4}} \\
&\quad + \frac{8201655}{890098688}h^2 f_{n+4} \\
y_{n+\frac{31}{8}} &= \frac{177976967}{249813835776}y_n - \frac{1409110115}{203552014336}y_{n+1} + \frac{170473309}{4626182144}y_{n+2} \\
&\quad - \frac{115328331095}{499627671552}y_{n+3} + \frac{12883055113}{10734188256}y_{n+3} + \frac{35011865}{433704576}h^2 f_{n+\frac{31}{8}} \\
&\quad - \frac{2401195055}{55514185728}h^2 f_{n+4}
\end{aligned}\right\} \quad (14)
$$

The equations (11) – (14) are the proposed 4SBHBDF for solving second order ordinary differential equations.

RESULTS AND DISCUSSION

The 4SHBDF methods are implemented as simultaneous numerical integration for IVPs without requiring starting values and predictors (Tables 1 and 2). We give examples to illustrate the efficiency of the methods.

Problem 1: Linear System of Second Order IVP

$$\frac{d^2y_1}{dt^2} = \frac{dy_1}{dt}$$

$$\frac{d^2y_2}{dt^2} = 2\frac{dy_1}{dt} + t\frac{dy_1}{dt} \qquad y_1(0) = 1, y_1'(0) = 1, y_2(0) = 0, y_2'(0) = 1$$

Exact Solution: $y_1(t) = e^t, y_2(t) = te^{2t}$

Table 1: Comparison of Exact Solution and the Proposed Method for Problem 1 at $h = 0.01$ for $y_1(t)$

t	**Exact**	**4SBHBDF**	**ERROR**
0.0	1.0000000000000000000	1.0000000000000000000	$0.00000000000 \times 10^{0}$
0.1	1.1051709180756476248	1.1051709180759226178	2.749930×10^{-13}
0.2	1.2214027581601698339	1.2214027581611853332	$1.0154993 \times 10^{-12}$
0.3	1.3498588075760031040	1.3498588075783428844	$2.3397804 \times 10^{-12}$
0.4	1.4918246976412703178	1.4918246976456077872	$4.3374694 \times 10^{-12}$
0.5	1.6487212707001281468	1.6487212707072904351	$7.1622883 \times 10^{-12}$

0.6	1.8221188003905089749	1.8221188004014437088	$1.09347339 \times 10^{-11}$
0.7	2.0137527074704765216	2.0137527074863338145	$1.58572929 \times 10^{-11}$
0.8	2.2255409284924676046	2.2255409285145615380	$2.20939334 \times 10^{-11}$
0.9	2.4596031111569496638	2.4596031111868564171	$2.99067533 \times 10^{-11}$
1.0	2.7182818284590452354	2.7182818284985564271	$3.95111917 \times 10^{-11}$

The Table I shows the numerical results of problem I for y_1. The results show that the proposed method 4SBHBDF agree well with the exact solution as illustrated in the tabulated results. The results also proved that the accuracy increases.

Table 2: Comparison of Exact Solution and the Proposed Method for Problem 1 at $h = 0.01$ for $y_2(t)$

t	**Exact**	**4SBHBDF**	**ERROR**
0.0	0.0000000000000000000	0.0000000000000000000	$0.00000000000 \times 10^{0}$
0.1	0.11051709180756476248	0.11051709180940785620	$1.84309372 \times 10^{-12}$
0.2	0.24428055163203396678	0.24428055163872184030	$6.68787352 \times 10^{-12}$
0.3	0.40495764227280093120	0.40495764228796231964	$1.516138844 \times 10^{-11}$
0.4	0.59672987905650812712	0.59672987908416248188	$2.765435476 \times 10^{-11}$
0.5	0.82436063535006407340	0.82436063539506242708	$4.499835368 \times 10^{-11}$
0.6	1.0932712802343053849	1.0932712803020394347	$6.77340498 \times 10^{-11}$
0.7	1.4096268952293335651	1.4096268953262951206	$9.69615555 \times 10^{-11}$
0.8	1.7804327427939740837	1.7804327429273981733	$1.334240896 \times 10^{-10}$
0.9	2.2136428000412546974	2.2136428002198322288	$1.785775314 \times 10^{-10}$
1.0	2.7182818284590452354	2.7182818286924831261	$2.334378907 \times 10^{-10}$

The Table 2 shows the numerical results of problem 1 for y_2. The results show that the proposed methods 4SBHBDF agree well with the exact solution as illustrated in the tabulated results. The results also proved that the accuracy increases.

Conclusion

In this paper we developed a 4 Step Block Hybrid Backward Differentiation Formula (4SBHBDF) that facilitate the solution at four points simultaneously without the need of any starting value or predictor.

Authors Contributions.

Both authors developed the method. The second author did the type setting while the first author did the editing process.

Conflict of Interest
Authors have declared that no conflict of interest exist.

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