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A Comparative Study Of Two Iterative Techniques For Systems Of Linear Algebraic Equations

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Abstract

This study compares numerically two iterative methods for solving systems of linear algebraic equations: the Symmetric Accelerated Overrelaxation technique and the Symmetric Successive Overrelaxation method. Four numerical problems are applied to analyze and compare the convergence speeds of the two approaches. On the basis of performance metrics including spectral radius, convergence time, accuracy, and number of iterations required to converge, the numerical results demonstrate that the Symmetric Accelerated Overrelaxation approach needed less computing time, a smaller spectral radius, and fewer iterations than the Symmetric Successive Overrelaxation approach. This demonstrates that the Symmetric Accelerated Overrelaxation is superior to the Symmetric Successive Overrelaxation. Researchers and numerical analysts can benefit from the findings of this study; it will help them comprehend iteration techniques and adopt an appropriate or more efficient iterative strategy for solving systems of linear algebraic equations.

Keywords: Iteration technique, Symmetric Accelerated Overrelaxation method, linear equations, rapid convergence, Symmetric Successive Overrelaxation method

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1.Introduction

Linear algebraic equations are notoriously difficult to solve using only direct methods. Controlling the accumulation of rounding errors is a problem with the Gauss elimination approach. For this reason, many researchers have turned to indirect methods to investigate the solutions of linear equations, such as Young (2014), Audu (2022), Assefa and Teklehaymanot (2021), Tesfaye *et al.* (2020), Mohammed and Rivaie (2017), Vatti *et al.* (2018), Audu *et al.* (2021a), Vatti *et al.* (2020), Zhang *et al.* (2020) and Audu *et al.* (2021b). Models of linear algebraic equations can be found in a wide variety of contexts, including physical situations and numerical solutions to other mathematical models. There are numerous examples of these applications in all fields of physical, biological, and social science. Nonlinear equations, partial differential equations, and optimization problems all use linear systems. Solving square linear systems is the most common challenge. Finding solutions of the system of linear algebraic equations designated as:

$$\mathrm{A}f = b \qquad (1)$$

where A is a square nonsingular matrix possessing a solution which is unique in the form

$$f = \mathrm{A}^{-1}b \qquad (2)$$

Suppose A possess diagonals that are non-vanishing, then one can split A into

$$\mathrm{A} = D - S - T \qquad (3)$$

where D is the diagonal, S is the strict lower and T is strict upper section of A. Decomposing the matrix A in the form

$$\mathrm{A} = G - H \qquad (4)$$

And inserting $A = G - H$ into $Af = b$, gives the following expressions:

$$\left.\begin{aligned} (G-H)f &= b \\ Gf &= Hf + b \\ Gf^{(r+1)} &= Hf^{(r)} + b \\ f^{(r+1)} &= Zf^{(r)} + q \end{aligned}\right\} \qquad (5)$$

$$r = 0, 1, 2\ldots, N$$

where 1ZGH-= is the matrix of iteration and 1qGb-= is column vector of the iteration scheme. By applying 1DLU-A=111,T,LDSUDqDb---=== and 1DD-I=.

In order to find a solution to the system of linear algebraic equations that has the form (1), the symmetric successive overrelaxation method (SSOM) and symmetric accelerated overrelaxation method (SAOM) are utilized for a nonsingular matrix with n rows and n columns.

Many scientists and engineers have explored the SSOM and SAOM techniques to finding numerical solutions in relation to systems of linear equations in different application problems. Darvishi and Hessari (2011) established a modification of the SSOM in solving augmented linear systems after examining the convergence criteria of the method. The scheme utilizes two parameters that enable rapid convergence. Salkuyeh *et al*. (2012) introduced an improved SSOM and studied its convergence properties and noticed that the method converges faster than the established modified symmetric successive overrelaxation method (MSSOM).

Darvishi *et al.* (2011) proposed the symmetric modified AOM technique for solution of system of linear equations, through the numerical tests carried out, it was shown that the method has a faster rate of convergence when compared with AOM and modified AOM schemes. Noor *et al.* (2013) proposed and analyzed the symmetric acceleration super-relaxation method for solving absolute complementarity problems, when A in L matrix, the method's convergence is proven.

Turek (2019) investigated the SSOM preconditioner for rapid computational performance in providing solutions to models of fluid dynamics. The investigation indicates that the SSOM can serve as an appropriate fallback preconditioner for the computationally sensitive but rapidly incomplete lower–upper factorization. Other authors like

Huang *et al.* (2019) and Tan (2017) employed the SSOM preconditioner to solve non-Hermitian matrices and symmetric complex linear systems. Huang (2017) proposed a four parameters symmetric Successive Overrelaxation technique for problems related to augmented systems. Deng (2020) developed a massive detection technique based on the SSOM approach for finding solutions to applied problems. Saudi and Dahalan (2022) combined the modified and skewed accelerated method to proffer solutions to two dimensional Poisson equations

When solving linear algebraic equations, it is possible to obtain the root system's values to the specified accuracy as an upper bound on the sequence of some vectors. An iteration is the process of putting together a sequence like this one. There are two iterative approaches studied in this research that differs from the direct approach, which attempts to calculate an exact solution in a finite number of operations, in that it starts with an initial estimate and produces successive enhanced estimations in an unbounded sequence whose limit is the exact answer. Since rounding errors are more likely to occur in a direct solution, this has the advantage in practice. The following are detailed descriptions of the various methods' procedures:

2.0 Methodology

2.1 Analysis of Symmetric Successive Overrelaxation Method (SSOM)

The Symmetric Successive Overrelaxation method was introduced by Young (2014) as two half iterations. The initial half is the usual or forward SOM scheme, while the second half is the SOM scheme with the equations used in backward order. The SSOM iteration scheme is represented by the following equations
to equation (3), we obtain

$$f^{r+1} = X_{\omega} f^{r} + \left(\mathrm{I} - \omega L\right)^{-1} \omega q \tag{6}$$

where $X_\omega = (\mathrm{I}-\omega L)^{-1}\left[(1-\omega)\mathrm{I}+\omega U\right]$ and explicitly as:

$$\begin{aligned} X_\omega &= (\mathrm{I}-\omega L)^{-1}\left[\mathrm{I}-\omega\mathrm{I}+\omega U\right] \\ &= (\mathrm{I}-\omega L)^{-1}\left[(\mathrm{I}-\omega L)-\omega(\mathrm{I}-L-U)\right] \qquad (7) \\ &= \mathrm{I}-\omega(\mathrm{I}-\omega L)^{-1} D^{-1}\mathrm{A} \end{aligned}$$

The symmetric SOM is constructed by firstly obtaining f^{r+1} of the forward SOM as;

$$f^{r+\frac{1}{2}} = X_\omega f^r + (\mathrm{I}-\omega L)^{-1}\omega q \qquad (8)$$

And computing the second part f^{r+1} from $f^{r+\frac{1}{2}}$ with respect to backward SOM as

$$f^{r+1} = Y_\omega f^{r+\frac{1}{2}} + (\mathrm{I}-\omega U)^{-1}\omega q \qquad (9)$$

Which is further simplified to obtain the following equations in (10)

$$\begin{aligned} Y_\omega &= (\mathrm{I}-\omega U)^{-1}\left[\mathrm{I}-\omega\mathrm{I}+\omega L\right] \\ &= (\mathrm{I}-\omega U)^{-1}\left[(\mathrm{I}-\omega U)-\omega(\mathrm{I}-L-U)\right] \qquad (10) \\ &= \mathrm{I}-\omega(\mathrm{I}-\omega U)^{-1} D^{-1}\mathrm{A} \end{aligned}$$

By inserting $f^{r+\frac{1}{2}}$ from (8) into (9) gives the symmetric SOM scheme in the format

$$f^{r+1} = \mathrm{K}_\omega f^r + s_\omega \qquad (11)$$

where $\mathrm{K}_\omega = X_\omega Y_\omega$ and. $s_\omega = \omega(2-\omega)\mathrm{I}(\mathrm{I}-\omega U)^{-1}(\mathrm{I}-\omega L)^{-1} q$ For the fact that the forward and backward SOM iteration schemes are simplified to get

$$\begin{aligned} X_\omega &= \mathrm{I}-\omega(\mathrm{I}-\omega L)^{-1} D^{-1}\mathrm{A} \\ Y_\omega &= \mathrm{I}-\omega(\mathrm{I}-\omega U)^{-1} D^{-1}\mathrm{A} \end{aligned} \qquad (12)$$

Then after some algebraic manipulation on the product of $X_\omega Y_\omega$, we get

$$\begin{aligned} X_\omega Y_\omega &= \left(\mathrm{I}-\omega(\mathrm{I}-\omega L)^{-1} D^{-1}\mathrm{A}\right)\left(\mathrm{I}-\omega(\mathrm{I}-\omega U)^{-1} D^{-1}\mathrm{A}\right) \\ &= \mathrm{I}-D^{-1}\mathrm{A}(\mathrm{I}-\omega U)^{-1}(\mathrm{I}-\omega L)^{-1}\left[\omega(\mathrm{I}-\omega U)+\omega(\mathrm{I}-\omega L)-\omega^2 D^{-1}\mathrm{A}\right] \\ &= \mathrm{I}-\omega(2-\omega)\left[(\mathrm{I}-\omega U)^{-1}(\mathrm{I}-\omega L)^{-1} D^{-1}\mathrm{A}\right] \qquad (13) \\ &= \mathrm{I}-\omega(2-\omega)\left[(\mathrm{I}-\omega U)^{-1}(\mathrm{I}-\omega L)^{-1}\left[\mathrm{I}-L-U\right]\right] \end{aligned}$$

As such, the symmetric SOM iteration matrix obtained from (13) is represented as $M_{SSOM} = X_{\omega}Y_{\omega} = \mathrm{I} - \omega(2-\omega)\left[(\mathrm{I}-\omega U)^{-1}(\mathrm{I}-\omega L)^{-1}[\mathrm{I}-L-U]\right]$. It is possible to obtain a preconditioner from the SAOM iteration matrix, which takes the format. This preconditioner can be utilized to accelerate some stationary and non-stationary iterative schemes $P_{SSOM} = \left[(\mathrm{I}-\omega U)^{-1}(\mathrm{I}-\omega L)^{-1}\right]$. The vector component corresponding to the SSOM iteration matrix is given as $\omega(2-\omega)\mathrm{I}(\mathrm{I}-\omega U)^{-1}(\mathrm{I}-\omega L)^{-1}q$. The scheme is said to be convergent if the spectral radius of its iteration matrix is less than 1, that is $\rho(M_{SSOM}) < 1$. In addition, the SSOM iteration converges for positive definite, M, symmetric and irreducible diagonally dominant linear systems.

2.2. Analysis of Symmetric Accelerated Overrelaxation Method (SAOM)

The Symmetric Accelerated Overrelaxation method (SAOM) was proposed by Hadjidimos and Yeyious (1982) as two half iterations. The initial half is the usual or forward AOM technique, while the second half is the AOM technique with the equations used in backward order. The SAOM iteration scheme is represented by the following equations. The classical forward AOM scheme for solution of (1) is: $f^{r+1} = X_{\omega,\beta} f^r + (\mathrm{I}-\beta L)^{-1}\omega q$

$$(14)$$

where $X_{\omega,\beta} = (\mathrm{I}-\beta L)^{-1}\left[(1-\omega)\mathrm{I}+(\omega-\beta)L+\omega U\right]$ and explicitly as:

$$\begin{aligned} X_{\omega,\beta} &= (\mathrm{I}-\beta L)^{-1}\left[(1-\omega)\mathrm{I}+(\omega-\beta)L+\omega U\right] \\ &= (\mathrm{I}-\beta L)^{-1}\left[(\mathrm{I}-\beta L)-\omega(\mathrm{I}-L-U)\right] \\ &= \mathrm{I}-(\mathrm{I}-\beta L)^{-1}\omega D^{-1}\mathrm{A} \end{aligned} \tag{15}$$

Similarly, the classical backward AOM scheme for computing the linear system (1) is depicted as:

$$f^{r+1} = Y_{\omega,\beta} f^r + (\mathrm{I}-\beta U)^{-1}\omega q \tag{16}$$

where $Y_{\omega,\beta} = (\mathrm{I}-\beta U)^{-1}\left[(1-\omega)\mathrm{I}+(\omega-\beta)U+\omega L\right]$ and explicitly as:

$$Y_{\omega,\beta} = (\mathrm{I}-\beta U)^{-1}\left[(1-\omega)\mathrm{I}+(\omega-\beta)U+\omega L\right] \quad (17)$$
$$= (\mathrm{I}-\beta U)^{-1}\left[(\mathrm{I}-\beta U)-\omega(\mathrm{I}-U-L)\right]$$
$$= \mathrm{I}-(\mathrm{I}-\beta U)^{-1}\omega D^{-1}\mathrm{A}$$

The symmetric AOM is constructed by firstly obtaining $f^{r+\frac{1}{2}}$ of the forward AOM as:

$$f^{r+\frac{1}{2}} = X_{\omega,\beta} f^{r} + (\mathrm{I}-\beta L)^{-1}\omega q \quad (18)$$

And computing the second part f^{r+1} with respect to backward AOM as thus

$$f^{r+1} = Y_{\omega,\beta} f^{r+\frac{1}{2}} + (\mathrm{I}-\beta U)^{-1}\omega q \quad (19)$$

By inserting (18) into (19) gives the symmetric AOM scheme as thus

$$f^{r+1} = Z_{\omega,\beta} f^{r} + p_{\omega,\beta} \quad (20)$$

where $Z_{\omega,\beta} = X_{\omega,\beta}Y_{\omega,\beta}$ and $p_{\omega,\beta} =.Y_{\omega,\beta}(\mathrm{I}-\beta L)^{-1}\omega q+(\mathrm{I}-\beta U)^{-1}\omega q$ Next, the product of the backward and forward AOM iteration schemes in (15) and (17) is evaluated to obtain

$$Z_{\omega,\beta} = \left[\mathrm{I}-(\mathrm{I}-\beta U)^{-1}\omega D^{-1}\mathrm{A}\right]\left[\mathrm{I}-(\mathrm{I}-\beta L)^{-1}\omega D^{-1}\mathrm{A}\right]$$
$$= \mathrm{I}-(\mathrm{I}-\beta U)^{-1}\left[\mathrm{I}-\beta L+1-\beta U-\omega+\omega L\right]\left[\mathrm{I}-(\mathrm{I}-\beta L)^{-1}\omega D^{-1}\mathrm{A}\right] \quad (21$$
$$= (\mathrm{I}-\beta U)^{-1}\, 2\mathrm{I}-\omega+(\omega-\beta)L+U(\mathrm{I}-\beta L)^{-1}\omega D^{-1}\mathrm{A}$$
$$= \mathrm{I}-\omega(\mathrm{I}-\beta U)^{-1}\left[(2-\omega)\mathrm{I}+(\omega-\beta)L+U\right](\mathrm{I}-\beta L)^{-1}(\mathrm{I}-L-U)$$

Thus, the SAOM iteration matrix is represented in the format as

$$Z_{\omega,\beta} = \mathrm{I}-\omega(\mathrm{I}-\beta U)^{-1}\left[(2-\omega)\mathrm{I}+(\omega-\beta)L+U\right](\mathrm{I}-\beta L)^{-1}(\mathrm{I}-L-U) \quad (22)$$

A preconditioner can equally be obtained from the SAOM iteration in the form of $K = \frac{1}{\omega}\left[(2-\omega)\mathrm{I}+(\omega-\beta)L+U\right]$, which can be used to accelerate slow convergent iterative schemes. Similarly, the vector component of the SAOM is simplified to get equation (23).

$$p_{\omega,\beta} = \omega(\mathrm{I}-\beta U)^{-1}\left[(2-\omega)\mathrm{I}+(\omega-\beta)(L+U)\right](\mathrm{I}-\beta L)^{-1} q \quad (23)$$

The convergence of the SAOM scheme is associated with the spectral radius of its iteration matrix. It converges whenever the spectral radius is smaller than 1, represented as $\rho(M_{SAOM})<1$. Furthermore, it is convergent for linear systems whose coefficient matrices are, positive definite, M, symmetric and irreducible diagonally dominant.

2.3 Algorithms for Numerical Computations

2.3.1. Algorithm for SSOM

To solve

$$Af=b \quad or \quad (I-L-U)f=b$$

Step 0: Insert elements of matrix $A(a_{i,j})$ *and* b_i select an initial guess of f^0, desired iteration number of tolerance (ξ) and $\omega\in(0,2)$

Step 1: Get the diagonal and triangular matrices U,D and L from matrix $A, q=D^{-1}b$ and $D^{-1}A$

Step 2: Create and obtain the inverse of the matrices $(I-\omega L)$ and $(I-\omega U)$

Step 3: Get the matrix $X_\omega = I-\omega(I-\omega L)^{-1}D^{-1}A$

Step 4: Create the matrix $Y_\omega = I-\omega(I-\omega U)^{-1}D^{-1}A$

Step 5: Establish $M = X_\omega Y_\omega = I-\omega(2-\omega)\left[(I-\omega U)^{-1}(I-\omega L)^{-1}[I-L-U]\right]$

Step 6: Establish $s=\omega(2-\omega)I(I-\omega U)^{-1}(I-\omega L)^{-1}q$

Step 7: Calculate $f^{r+1}=Mf^r+s$

Step 8: For $r=0,1,2,3,\ldots,N$ terminate if $\|f^{r+1}-f^r\|<\xi$, then, STOP.

2.3.2. Algorithm for SAOM

To solve

$$Af=b \quad or \quad (I-L-U)f=b$$

Step 0: Insert elements of matrix $A(a_{i,j})$ *and* b_i select an initial guess f^0, desired iteration number of tolerance (ξ) and $\omega\in(0,2)$.

Step 1: Get the diagonal and triangular matrices ,UD and L from matrix A, $q=D^{-1}b$ and. $D^{-1}A$

Step 2: Create and obtain the inverse of the matrices $(I-\omega L)$ and $(I-\omega U)$

Step 3: Get the matrix $X_{\omega,\beta} = \mathrm{I} - \omega\left(\mathrm{I} - \beta L\right)^{-1} D^{-1}\mathrm{A}$

Step 4: Create the matrix $Y_{\omega,\beta} = \mathrm{I} - \omega\left(\mathrm{I} - \beta U\right)^{-1} D^{-1}\mathrm{A}$

Step 5: Establish. $Z = \mathrm{I} - \omega\left(\mathrm{I} - \beta U\right)^{-1}\left[(2-\omega)\mathrm{I} + (\omega - \beta)L + U\right]\left(\mathrm{I} - \beta L\right)^{-1}\left(\mathrm{I} - L - U\right)$

Step 6: Establish. $p = \omega\left(\mathrm{I} - \beta U\right)^{-1}\left[(2-\omega)\mathrm{I} + (\omega - \beta)(L + U)\right]\left(\mathrm{I} - \beta L\right)^{-1} q$

Step 7: Calculate $f^{r+1} = Zf^{r} + p$

Step 8: For $r = 0,1,2,3,\ldots,N$ terminate if $\left\|f^{r+1} - f^{r}\right\| < \xi$, then, STOP.

3. **Numerical Computations**

This section is concerned with carrying out numerical experiments to investigate the performances of the two iterative approaches discussed in previous section. The Maple 2017 software was employed for the computations and the results are presented in Tables 1 to 5 and Figures 1 to 4.

Problem 1: We consider the system of linear equations in the form $\mathrm{A}f = b$

$$\begin{pmatrix} 7 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 \\ -1 & 7 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & -1 & 7 & -1 & 0 & -1 & 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 7 & -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 & 7 & -1 & 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 & -1 & 7 & -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 & 7 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 & -1 & 0 & -1 & 7 & -1 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 7 & -7 \\ 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 7 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \\ f_9 \\ f_{10} \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}$$

Problem 2: We consider the linear system from Heat transfer analysis represented as $\mathrm{A}f = b$

$$\begin{pmatrix} 4.0 & -1.0 & 0 & -1.0 & 0 & 0 & 0 & 0 & 0 \\ -1.0 & 4.0 & -1.0 & 0 & -1.0 & 0 & 0 & 0 & 0 \\ 0 & -1.0 & 4.0 & 0 & 0 & -1.0 & 0 & 0 & 0 \\ -1.0 & 0 & 0 & 4.0 & -1.0 & 0 & -1.0 & 0 & 0 \\ 0 & -1.0 & 0 & -1.0 & 4.0 & -1.0 & 0 & -1.0 & 0 \\ 0 & 0 & -1.0 & 0 & -1.0 & 4.0 & 0 & 0 & -1.0 \\ 0 & 0 & 0 & -1.0 & 0 & 0 & 4.0 & -1.0 & 0 \\ 0 & 0 & 0 & 0 & -1.0 & 0 & -1.0 & 4.0 & -1.0 \\ 0 & 0 & 0 & 0 & 0 & -1.0 & 0 & -1.0 & 4.0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \\ f_9 \end{pmatrix} = \begin{pmatrix} 1500.0 \\ 1000.0 \\ 1000.0 \\ 2000.0 \\ 0 \\ 0 \\ 1500.0 \\ 1000.0 \\ 1000.0 \end{pmatrix}$$

Problem 3: Considering the linear system of the form $Af = b$

$$\begin{pmatrix} 1.0 & -\frac{12}{43} & -\frac{10}{43} & 0.0 \\ -\frac{15}{49} & 1.0 & 0.0 & -\frac{10}{49} \\ -\frac{13}{49} & 0.0 & 1.0 & \frac{12}{49} \\ 0.0 & -\frac{13}{55} & -\frac{3}{11} & 1.0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} \frac{5}{43} \\ \frac{8}{147} \\ \frac{22}{49} \\ \frac{62}{165} \end{pmatrix}$$

Problem 4: Considering the linear system of the format $Af = b$

$$\begin{pmatrix} 4.0 & -1.0 & -1.0 & 0.0 \\ -1.0 & 4.0 & 0.0 & -1.0 \\ -1.0 & 0.0 & 4.0 & -1.0 \\ 0.0 & -1.0 & -1.0 & 4.0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} 1000.00 \\ 1000.00 \\ 0.00 \\ 0.00 \end{pmatrix}$$

Table 1: Comparison Result of Spectral Radii for Problem 1

??	??	$\rho(M_{SSOM})$	Abs Error of $\rho(M_{SSOM})$	$\rho(M_{SAOM})$	Abs Error of $\rho(M_{SAOM})$
0.150	0.100	0.907322817701074	0.092677	0.875606868802594	0.124393
0.250	0.200	0.842795580595991	0.157204	0.813479825838499	0.186520
0.350	0.300	0.776403077657459	0.223596	0.749645003657818	0.250355
0.450	0.400	0.708612344518829	0.291388	0.684539306028665	0.315461
0.550	0.500	0.640132390060952	0.359867	0.618816478363021	0.381184
0.650	0.600	0.572024255267111	0.427976	0.553442316732589	0.446558
0.750	0.700	0.505875307724584	0.494124	0.489845401571549	0.510155
0.850	0.800	0.444082853367090	0.555917	0.430167286097516	0.569833
0.950	0.900	0.390304764638412	0.609695	0.377688667030804	0.622311
1.050	1.000	0.350048270122173	0.649951	0.337522335637017	0.662478

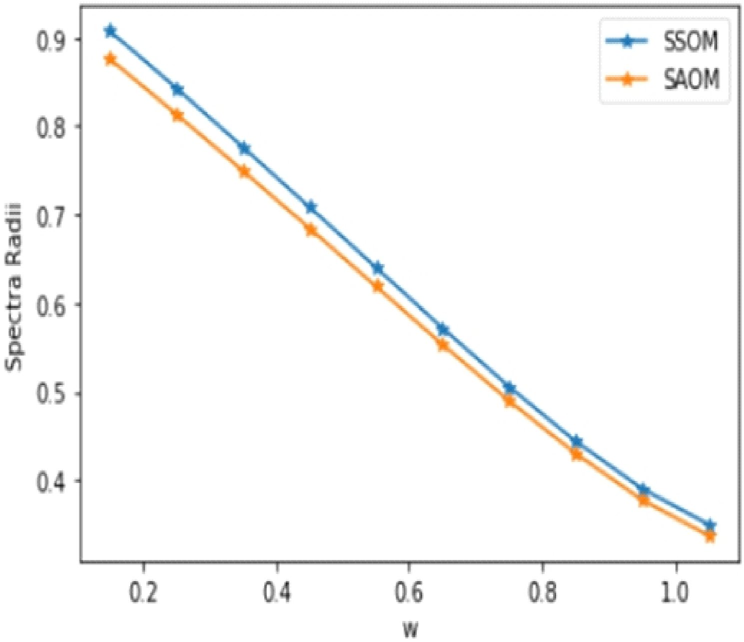

Figure 1: Plot of Spectral Radii for Problem 1

Table 2: Comparison Result of Spectral Radii for Problem 2

??	??	$\rho(M_{SSOM})$	Abs Error of $\rho(M_{SSOM})$	$\rho(M_{SAOM})$	Abs Error of $\rho(M_{SAOM})$
0.150	0.100	0.909434700765402	0.0905653	0.877329102547252	0.122671
0.250	0.200	0.846168821672753	0.153831	0.816476311839150	0.183524
0.350	0.300	0.780809651870280	0.219190	0.753726513552910	0.246273
0.450	0.400	0.713691056449360	0.286309	0.689400604748524	0.310599
0.550	0.500	0.645339768131172	0.354660	0.623993740906722	0.376006
0.650	0.600	0.576562916944380	0.423437	0.558250426921560	0.441749
0.750	0.700	0.508585825259910	0.491414	0.493280660295151	0.506719
0.850	0.800	0.443280342285813	0.556719	0.430749869902163	0.569250
0.950	0.900	0.383562497576713	0.622311	0.373210885969380	0.626789
1.050	1.000	0.334102448266658	0.665898	0.324728249498409	0.675271

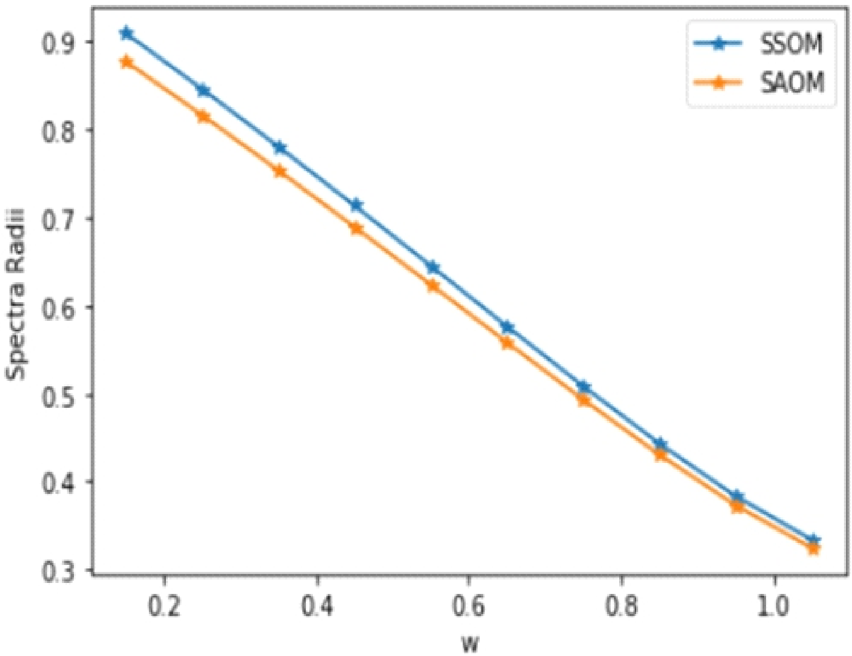

Figure 2: Plot of Spectral Radii for Problem 2

Table 3: Comparison Result of Spectral Radii for Problem 3

⁇	⁇	$\rho(M_{SSOM})$	Abs Error of $\rho(M_{SSOM})$	$\rho(M_{SAOM})$	Abs Error of $\rho(M_{SAOM})$
0.150	0.100	0.853212608447515	0.146787	0.831447530967531	0.168552
0.250	0.200	0.756175086423092	0.243823	0.736955755441031	0.263044
0.350	0.300	0.660741132813221	0.339259	0.644062881570968	0.355937
0.450	0.400	0.567937017265531	0.432062	0.553745926536980	0.446254
0.550	0.500	0.479043069637498	0.520956	0.467214958582709	0.532785
0.650	0.600	0.395691885293556	0.604308	0.385998841962154	0.614001
0.750	0.700	0.320050325924644	0.679949	0.312105520917686	0.687894
0.850	0.800	0.255189330192984	0.744811	0.248363031066074	0.751636
0.950	0.900	0.205833577477588	0794166	0.199209797172208	0.800790
1.050	1.000	0.179481146832074	0.82052	0.1663923048192620	0.833607

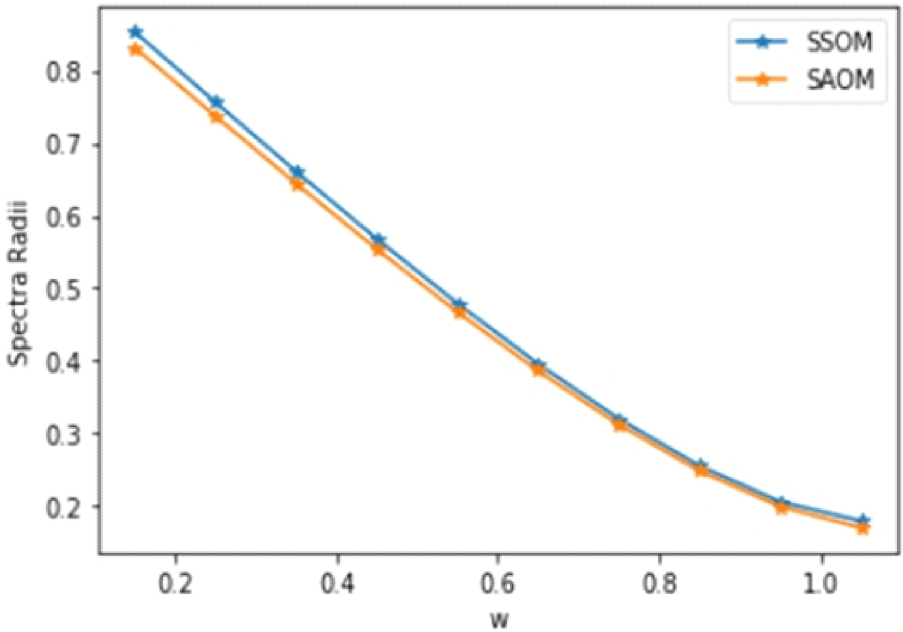

Figure 3: Plot of Spectral Radii for Problem 3

Table 4: Comparison Result of Spectral Radii for Problem 4

??	??	$\rho(M_{SSOM})$	Abs Error of $\rho(M_{SSOM})$	$\rho(M_{SAOM})$	Abs Error of $\rho(M_{SAOM})$
0.150	0.100	0.850341516381201	0.168552	0.829091419238791	0.170909
0.250	0.200	0.751666513451280	0.248333	0.732954653234820	0.267045
0.350	0.300	0.654827908831615	0.345172	0.638646321722955	0.361354
0.450	0.400	0.560855179339077	0.439145	0.547148358736471	0.452852
0.550	0.500	0.471021462693750	0.528979	0.459666195979610	0.540332
0.650	0.600	0.386934086479990	0.613066	0.377706653902068	0.622293
0.750	0.700	0.310701150650525	0.689299	0.303220669276965	0.696779
0.850	0.800	0.245266982896251	0.754733	0.238897864506877	0.761102
0.950	0.900	0.195070423818982	0.804929	0.183528016546302	0.816472
1.050	1.000	0.166774904542064	0.833225	0.156449499192320	0.843551

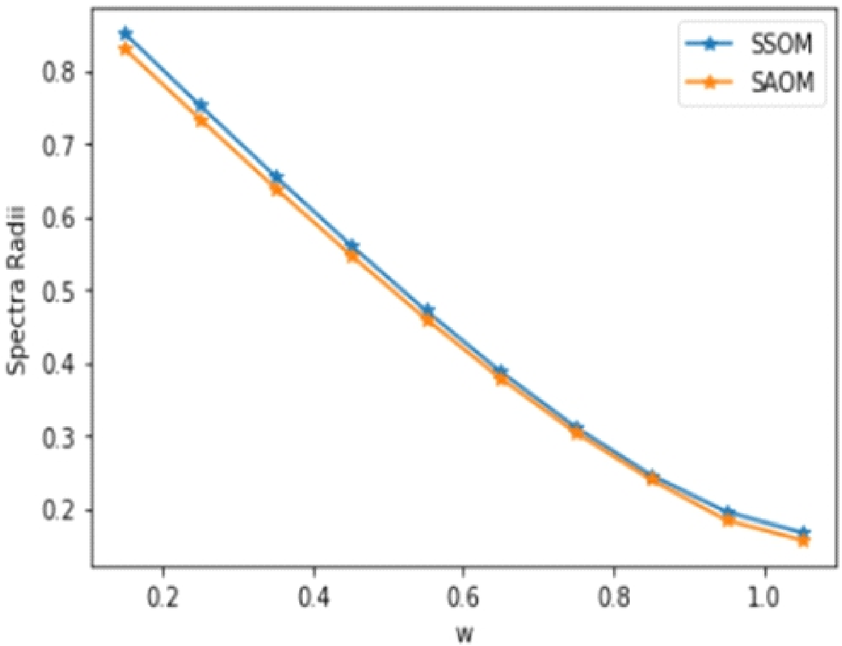

Figure 4: Plot of Spectral Radii for Problem 4

Table 5: Convergence results for the Numerical computations

Numerical computations	Techniques employed	Number of iterations	Computational time
Problem 1	SAOM	17	5.81 secs
	SSOM	23	7.53 secs
Problem 2	SAOM	27	5.11 secs
	SSOM	35	6.18 secs
Problem 3	SAOM	14	2.28 secs
	SSOM	20	3.12 secs
Problem 4	SAOM	12	2.45 sec
	SSOM	17	3.09 secs

4. Discussion of Results

In Tables 1- 4, the spectral radius of the SSOM iteration matrix is represented as $\rho(M_{SSOM})$ while the spectral radius of the SAOM iteration matrix is denoted as $\rho(M_{SAOM})$. Tables 1, 2, 3, and 4 display the findings of Problems 1, 2, 3, and 4 for various spectral radii of SAOM and SSOM iteration matrices. All the spectral radii of the two iterative techniques, for a wide range of ω and β values, are less than one, that is to say, $\rho(M_{SAOM}) < 1$ and $\rho(M_{SSOM}) < 1$, as shown by Tables 1 through 4. As their spectral radii are less than one, this means that the two approaches are convergent. Nevertheless, the SAOM spectral radii are smaller than the SSOR spectral radii due to the fact that $\rho(M_{SAOM}) < \rho(M_{SSOM}) < 1$. Clearly, the SAOM iteration will converge to the exact solution more quickly than the SSOM iteration due to the smaller spectral radii of its iteration matrices.

For the sake of clarity, Figures 1 through 4 demonstrate the performance of the contrasted spectral radii for the two iterative techniques. Because the spectral radii of its iteration matrices are smaller, the SAOM method yields superior results to the SSOM method. The figures demonstrate the applicability of SAOM to linear algebraic equations and reveal that the spectral radii of the SAOM technique produced superior results compared to those of SSOM.

The convergence findings that were achieved from the two different iteration approaches that were used in the study to solve the numerical problems are illustrated in Table 5. The SAOM technique arrived at the exact solution in a shorter amount of time and with fewer iterations than the SSOM approach did, while still achieving success in solving Problems 1 through 4. Therefore, the SAOM technique is superior to the SSOM technique in terms of performance.

5. Conclusion

This work explored and analyzed two iterative strategies for solving linear systems: the Symmetric Successive Overrelaxation method and the Symmetric Accelerated Overrelaxation method. According to the numerical computations, the Symmetric Accelerated Overrelaxation approach converges more quickly than the Symmetric Successive Overrelaxation method. Because the spectral radii of its iteration matrices are smaller, the SAOM method yields superior results to the SSOM method. In addition, the obtained computational time indicates that SSOM iterations required more time than SAOM iterations to converge to the exact answer. The Symmetric Accelerated Overrelaxation uses less iterations than the Symmetric Successive Overrelaxation to get its final solutions. Therefore, the SAOM approach outperforms the SSOM approach. As a result, the Symmetric Accelerated Overrelaxation technique is more effective and precise when locating solutions to linear systems. Consequently, the two iteration approaches may effectively solve linear algebraic problems. Nevertheless, the Symmetric Accelerated Overrelaxation approach is regarded as the most effective.

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