HSM – Hausdorff School for Advanced Studies in Mathematics

Universität Bonn – Rheinische Friedrich-Wilhelms-Universität Bonn

Hausdorff School

# "Analysis of PDEs: Variational and Geometric Perspectives"

**July 10 – 14, 2023**

organized by
**Sebastian Hensel, Tim Laux, Kerrek Stinson**

## Abstracts: Lecture Series

**Adriana Garroni** (Sapienza University of Rome)

### Grain boundary energy in polycrystals and material defects

**Abstract:** Metals are crystals, so that the atoms at rest are arranged in a lattice structure. This order is actually local, and, at a mesoscopic scale, there are regions (so-called grains) where the material is closed to a rigid rotation of a given lattice (they are polycrystals). At the interface between these regions (the grain boundaries), the incompatibilities of the lattice structures are responsible for an energy called surface tension.

These lectures aim at presenting a variational model able to justify, via Gamma-convergence, the structure of this surface tension accounting for the elastic distortion due to the presence of the incompatibilities. Specifically, a nonlinear model for material defects proposed by Lauteri and Luckhaus will be introduced and the asymptotic behavior of this model, in terms of Gamma-convergence, will be considered.

Some of the tools needed for the proof, such as rigidity estimates for incompatible fields, blow-up arguments, and other measure theoretical technics, will be given.

**Maria Giovanna Mora** (University of Pavia)

### Nonlocal interaction problems with anisotropy

**Abstract:** In this lecture series I will review some recent results on nonlocal interaction problems. The focus will be on interaction kernels that are anisotropic variants of the classical Coulomb kernel. In other words, while preserving the same singularity at zero of the Coulomb kernel, they present preferred directions of interaction. For kernels of this kind I will prove existence and uniqueness for the corresponding minimization problems. I will then show how to characterize explicitly their minimizers in the case of some specific confinements. If time permits, I will discuss some related results and open questions.

**Alexander Mielke** (WIAS Berlin and Humboldt University Berlin)

### Balanced-viscosity solutions as limits in generalized gradient systems under slow loading

**Abstract:** A large variety of mechanical systems can be described EDB solutions for generalized gradient systems $(X, \mathcal{F}, \mathcal{R})$ where $X$ is a Banach space, $\mathcal{F}$ the time-dependent energy functional, and $\mathcal{R}$ the so-called dissipation potential. The gradient-flow equation is given in the form

$$0 \in \partial_{\dot q}\mathcal{R}(q(t), \dot q(t)) + \partial^{\mathrm{F}}\mathcal{F}(t, q(t)) \in X^* \quad \text{for a.a. } t \in [0, T].$$

A solution $q \in AC([0,T]; X)$ with a force selection $s \mapsto \xi(s) \in \partial^{\mathrm{F}}\mathcal{F}(s, q(s))$ is called an EDB solution if it additionally satisfies the *Energy-Dissipation Balance*

$$\mathcal{F}(q(t)) + \int_r^t \big(\mathcal{R}(q,\dot q)+\mathcal{R}^*(q,-\xi)\big)\,\mathrm{d}s = \mathcal{F}(q(r)) + \int_r^t \partial_s\mathcal{F}(s,q(s))\,\mathrm{d}s$$

for all subintervals $[r,t] \subset [0,T]$. We recall ideas from [MRS13, MRS18], **see the survey [Mie23]**.

Slow loading means that time is rescaled by a small parameter $\varepsilon > 0$, which is equivalent to considering scaled dissipation potentials $\mathcal{R}_\varepsilon(q,v) = \frac{1}{\varepsilon}\mathcal{R}_1(q,\varepsilon v)$, e.g. . in viscoplasticity one considers $\mathcal{R}_\varepsilon(\dot p) = \int_\Omega \big(\sigma_{\mathrm{yield}}|\dot p(x)| + \frac{\varepsilon^{\kappa-1}}{\kappa}|\dot p(x)|^\kappa\big)\,\mathrm{d}x$ with $1 < \kappa \ll 2$ and $p(x) \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$.

*Balanced-Viscosity solutions* for the rate-independent limit $(X, \mathcal{F}, \mathcal{R}_\varepsilon)_{\varepsilon\to 0}$ denotes the natural solutions concept for curves that occur as limits of EDB solutions $q_\varepsilon$ as $\varepsilon \to 0$. The difficulty is that BV solutions may have jumps and that the limit of the jump curves are important for the characterization of BV solutions. As a first step, we will introduce (arc-length) parametrized solutions, and in a second step a more intrinsic formulation based on a new transition costs will be derived. For this, the subtle balance between the norms induced by the rate-independent dissipation (e.g. $\mathrm{L}^1$), the viscous dissipation (e.g. $\mathrm{L}^\kappa$), and the energy norm (e.g. $\mathrm{H}^1$) has to be exploited.

We build on material in [MRS12, MRS16, MiR23].

[Mie23] A. Mielke: *An introduction to the analysis of gradient systems.* **Survey** arXiv:2306.05026 (2023).

[MiR23] A. Mielke and R. Rossi: *Balanced-Viscosity solutions to infinite-dimensional multi-rate systems.* Arch. Rational Mech. Anal. **247**:53 (2023) 1–100.

[MRS12] A. Mielke, R. Rossi, and G. Savaré: *BV solutions and viscosity approximations of rate-independent systems.* ESAIM Control Optim. Calc. Var. **18**:1 (2012) 36–80.

[MRS13] ______: *Nonsmooth analysis of doubly nonlinear evolution equations.* Calc. Var. Part. Diff. Eqns. **46**:1-2 (2013) 253–310.

[MRS16] ______: *Balanced Viscosity (BV) solutions to infinite-dimensional rate-independent systems.* J. Europ. Math. Soc. **18** (2016) 2107–2165.

[MRS18] ______: *Global existence results for viscoplasticity at finite strain.* Arch. Rational Mech. Anal. **227**:1 (2018) 423–475.

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**Ian Tobasco** (University of Illinois Chicago)

### Elastic patterns and shape change, a variational perspective

**Abstract:** This lecture series will introduce students to the fascinating link between fine scale elastic patterns and macroscopic shape change. The presentation will be organized around three main themes: elastic membranes, wrinkled shells, and stress focusing. Although the presentation will be mathematical, we will set the lectures in the context of recent experiments from the soft matter physics

and mechanics communities. Students with a background in partial differential equations and/or differential geometry should feel at home; no prior knowledge of elasticity will be assumed.

Lecture 1. Elastic membranes
Lecture 2. Wrinkled shells, pt 1
Lecture 3. Wrinkled shells, pt 2
Lecture 4. Stress focusing

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# Abstracts: Poster Session

**Konstantinos Bessas** (University of Pavia)

### Fractional total variation denoising model with $L^1$ fidelity

**Abstract:** We study a nonlocal version of the total variation-based model with $L^1$ fidelity for image denoising, where the regularizing term is replaced with the fractional s-total variation. We discuss regularity of the level sets and uniqueness of solutions, both for high and low values of the fidelity parameter. We analyse in detail the case of binary data given by the characteristic functions of convex sets.

**Leon Happ** (TU Vienna)

### A modular Poincaré-Wirtinger-type inequality on Lipschitz domains in variable Sobolev spaces

— joint work with Elisa Davoli, Giovanni Di Fratta, and Alberto Fiorenza

**Abstract:** The Poincaré(-Wirtinger) inequality is undeniable one of the most important and well-known inequalities in the theory of Sobolev spaces with almost infinitely many applications. Unfortunately, when trying to derive a similar (modular) estimate also for Sobolev spaces with variable exponents, the well-known proof using the Rellich-Kondrachov theorem is not applicable due to a lack of compactness. Besides, in all known modular Poincaré-Wirtinger inequalities there is always a *price* to be paid in terms of an additional contribution on the right-hand-side.

I would like to present a new, constructive proof - avoiding any compactness arguments - of a modular Poincaré-Wirtinger-type inequality on bounded Lipschitz domains for Sobolev spaces with variable exponents. Moreover, as a Corollary one deduces the standard Poincaré-Wirtinger inequality in the constant-exponent case.

**Jonas Haselböck** (University of Regensburg)

### Phase separation in elastically stressed solids

**Abstract:** The decomposition of a heterogeneous mixture into distinct phases can be modeled by the fourth-order parabolic Cahn-Hilliard equation. To account for elastic misfit, the diffusion equation is coupled with an elliptic system describing the quasi-static mechanical equilibrium, resulting in the so-called Cahn-Larché system

$$\begin{aligned} \partial_t \varphi &= \nabla \cdot (m(\varphi)\nabla\mu), \\ \mu &= -\varepsilon\Delta\varphi + \frac{1}{\varepsilon}\psi'(\varphi) + W_{,\varphi}(\varphi, \mathcal{E}(\mathbf{u})), \\ \nabla \cdot (W_{,\mathcal{E}}(\varphi, \mathcal{E}(\mathbf{u}))) &= \mathbf{0}. \end{aligned}$$

For this system we present an existence result in the case of a degenerate mobility $m$ and a logarithmic potential $\psi$. Moreover, we present a weak-strong uniqueness result based on $H^2$ regularity of the displacement field $\mathbf{u}$, obtained via a difference quotients approach under the assumption of a homogeneous elasticity tensor $\mathbb{C}$. We further discuss an augmentation of this model describing phase separation in saturated, visco-elastic, porous media.

**Melanie Koser** (HU Berlin)

**A discrete-to-continuum limit of a two-dimensional frustrated spin system at the helimagnetic/ferromagnetic transition point**

— joint work with Janusz Ginster and Barbara Zwicknagl

**Abstract:** We are interested in pattern formation in two-dimensional magnetic compounds. We consider materials whose atoms are ordered in a regular crystalline structure and associate to each atom its so called spin, a unit vector in $\mathbb{S}^1$. Complex geometric structures in the spin field may be the result of the competition between anti- and ferromagnetic interactions. In ferromagnetic materials spins prefer to be aligned, whereas in antiferromagnetic compounds one cannot observe a global orientation of the spins. The competition between these two interactions leads to frustration mechanisms in the system. We consider the lattice energy of certain materials, in which antiferromagnetic (AF) and ferromagnetic (F) interactions coexist, and are modeled by the $J_1$-$J_3$ F-AF model on a square lattice. We discuss a scaling law in terms of the optimal energy, which describes arising patterns in a minimal spin-field. Further, we present a Γ-convergence result which in a certain parameter regime relates the discrete model with a suitable continuous counterpart.

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**Sasa Lukic** (RWTH Aachen University)

**On Convergence Rates for the Mullins-Sekerka Flow of a Class of (Simple) Closed Curves on some Compact Surfaces**

— joint work with Maria Westdickenberg and Umberto Hryniewicz (both Aachen)

**Abstract:** In this work we aim at extracting (exponential) convergence rates of the Mullins-Sekerka flow of (right now simple) closed curves on certain compact surfaces to mean constant curvature curves under the assumption of global-in-time existence of the flow. The main technique is based on an elementary ODE argument combining some algebraic and differential inequalities which correspond with the gradient flow structure of the flow.

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**Frédéric Marazzato** (Louisiana State University)

**Study and computation of the deformation of a planar kirigami**

**Abstract:** Kirigami are part of the larger class of mechanical metamaterials, which exhibit exotic properties. This poster focuses on rhombi-slits, which is a specific type of kirigami. A nonlinear kinematics model was previously proposed in [Zheng et al, 2022] as a second order divergence form PDE with a possibly degenerate, and sign-changing coefficient matrix. We first propose to study the existence and uniqueness of the solutions of this equation by using the limit absorption principle. Then, we propose a numerical method based on adding a complex dissipation to approximate them. Finally, comparisons of simulations with experiments are performed.

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**Giannis Polychrou** (Aristotle University of Thessaloniki)

**Elliptic sinh-Gordon, Backlund transformation and applications**

— joint work with A. Fotiadis, C. Daskaloyannis and E. Papageorgiou

**Abstract:** We study the elliptic sinh-Gordon $w_{xx} + w_{yy} = 2\sinh(2w)$, and the elliptic sine-Gordon equation $\theta_{xx} + \theta_{yy} = -2\sin(2\theta)$, on the real plane and we introduce new families of solutions using the method of functionally seperation. The families we have described are of the form

$$\tanh(\frac{w(x,y)}{w}) = A(x)B(y), \qquad \sinh(w(x,y)) = \tan(A(x) + B(y)),$$
$$\tan(\frac{\theta(x,y)}{2}) = A(x)B(y), \qquad \sin(\theta(x,y)) = \tanh(A(x) + B(y)),$$

where $w(x,y)$ is a solution of elliptic sinh-Gordon, $\theta(x,y)$ is a solution of elliptic sine-Gordon and $A$ and $B$ turn out to be Jacobi Elliptic functions. Moreover, we use a Bäcklund transformation that connects the elliptic versions of sine-Gordon and sinh-Gordon equations. As an application, we construct new harmonic maps between surfaces, when the target is of constant curvature $-1$.

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**Samuele Riccò** (TU Vienna)

**A characterization for solutions to autonomous obstacle problems with general growth**

**Abstract:** Obstacle problems are by now well studied in the context of regularity theory, but a still pending issue is the relation between minima and extremals. The regularity of the solutions is often proved thanks to the fact that the minimizers are extremals too, i.e. they solve a variational inequality related to the problem. While in the context of standard growth conditions the literature is well established, under more general growth hypotheses there are still issues in this regard. Considering an autonomous obstacle problem, whose Lagrangian satisfies proper hypotheses of convexity and superlinearity at infinity, I will present a characterization for the unique solution in terms of extremality and a primal-dual formulation of the problem.

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**Leon Schütz** (WIAS Berlin)

**Towards stochastic homogenization of a rate–independent delamination model**

— joint work with Marita Thomas and Martin Heida

**Abstract:** We consider a material undergoing a delamination process under an external loading. The evolution of macroscopically visible material defects, such as geological faults in the lithosphere or fractures in engineering structures, originates and is influenced by material discontinuities on much smaller scales. In that respect, we model the delamination process as damage on microscopic interfaces - scaled by the homogenization parameter $\varepsilon > 0$ - and by an internal damage variable taking values in $[0,1]$; the value 1 means the bond is fully intact, whereas 0 means that the bond is completely broken. The second variable in the system is the displacement field satisfying the equations of linear elasticity in the bulk and is coupled to the damage variable on the microscopic interfaces. We describe the coupling in the system by an energy term which includes the internal variable (and possibly its gradient) and the jumps of the displacement on the interfaces. Furthermore, we use a convex, positive 1-homogeneous dissipation potential which implies rate–independent evolution of the delamination variable. In this setting we aim to perform periodic and stochastic homogenization, accounting for periodic or random distribution of the microscopic interfaces, with the aim to preserve the gradient structure of the system in the limit equations and to obtain a limit model that upscales delamination and displacement jumps on lower dimensional microscopic interfaces to damage in the volume.

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**Edoardo Tolotti** (University of Pavia)

### On the hierarchy of plate models for a singularly perturbed multi-well nonlinear elastic energy

**Abstract:** In the celebrated work of Friesecke, James and Muller (2006) the authors derive a hierarchy of models for plates by carefully analyzing the Γ-convergence of the rescaled nonlinear elastic energy. The key ingredient of their proofs is the rigidity estimate proved in an earlier work of theirs. Here we consider the case in which the elastic energy has multiple wells: this type of functional arises, for example, in the study of solid-solid phase transitions. However, it is well known that the rigidity estimate fails in the case of compatible wells; to overcome this difficulty we follow Alicandro, Dal Maso, Lazzaroni, Palombaro (2018) and add a regularization term to the energy that penalizes jumps from one well to another, leading to good compactness properties. In this setting we recover the full hierarchy of plate models with an explicit dependence on the wells. Finally, we study the convergence of energy minimizers with suitable external forces.

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**Samuel Wallace** (University of Illinois at Chicago)

### Planar origami maps as limits of nonlinear elasticity

— joint work with Ian Tobasco

**Abstract:** Thin elastic sheets take on a variety of shapes when subjected to confining loads. A broadly open question is to understand the emergence of stress focusing in situations where a more defocused, wrinkling-type response is possible, such as for a sheet confined to a container. This poster reports progress in the analysis of a thin elastic sheet confined to a small planar gap. Given a bound on the energy, strong subsequential convergence of the midplane to a piecewise affine, "folded-flat" origami map is shown. A lower bound is obtained using the Fonseca-Müller blow-up method to address the cost of a fold. A matching upper bound is shown in a narrower regime, using the minimal ridge construction. We illustrate the results with examples where lateral confinement allows us to prove the existence of folds.

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**Yuchuan Yang** (University of Michigan - Ann Arbor)

### Generalized Curvature for the Optimal Transport Problem Induced by a Tonelli Lagrangian

**Abstract:** We propose a generalized curvature that is motivated by the optimal transport problem on $\mathbb{R}^d$ with cost induced by a Tonelli Lagrangian $L$. We show that non-negativity of the generalized curvature implies displacement convexity of the generalized entropy functional on the L-Wasserstein space along $C^2$ displacement interpolants.

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**Abdulhakeem Yusuf** (Federal University of Technology, Minna)

### Robust analysis of MHD micropolar boundary layer flow past a stretching sheet with chemical reaction and slip conditions

**Abstract:** The Partial differential equations (PDE) governing this problem are geometrically presented in angular form. Suitable similarity variables were introduced to transform them into nonlinear coupled differential equations. For the purpose of analysis, Chebyshev Collocation method was employed. The results obtained are compared with the literature to establish an agreement. The physical

parameters that appear in the solution were varied on fluid velocity, microrotation, temperature and concentration profiles. The magnetic parameter was found to be a reduction agent of the fluid velocity due to a drag like force.