CONTINUOUS FORMULATION OF HYBRID BLOCK MILNE TECHNIQUE FOR SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

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Abstract

In most scientific and engineering problems, ordinary differential equations cannot be solved by analytic methods. Consequently, numerical approaches are frequently required. A block hybrid Milne technique was formulated in this paper in order to develop a suitable algorithm for the numerical solution of ordinary differential equations. Utilizing power series as the basis function, the proposed method is developed. The developed algorithm is used to solve systems of linear and nonlinear differential equations, and it has proven to be an efficient numerical method for avoiding time-consuming computation and simplifying differential equations. The fundamental numerical properties are examined, and the results demonstrate that it is zero-stable and consistent, which ensures convergence. In addition, by comparing the approximate solutions to the exact solutions, we demonstrate that the approximate solutions converge to the exact solutions. The results demonstrate that the developed algorithm for solving systems of ordinary differential equations is straightforward, efficient, and faster than the analytical method.

Keywords: Ordinary differential equations, numerical solution of ODEs, Hybrid Milne method, approximate solutions, algorithm and power series

Introduction

An equation in mathematics that describes the relationship between a function and its derivative is an example of a differential equation. In practical contexts, functions are typically used to represent rates of change. Engineers, physicists, economists, biologists, and others rely heavily on differential equations. Initial value first order ordinary differential equations appear in the process of modeling real-world situations in physical and applied sciences, particularly in algebraic expressions concerning problems related to flow of viscous thin films, disease models, chemical kinetics, quantum mechanics and electromagnetic waves (Aslam *et al.*, 2021; Mazarina and Syahirbanun (2022); Amat *et al.*, 2019; Kwanamu *et al.*, 2021). Understanding the behaviors and propertiesof the investigated physical phenomena requires the resolution of this type of problem (Kashkaria and Syam (2019)). In the majority of instances, available analytical approaches fail to provide an accurate solution to a general first-order initial value problem. To solve such problems that come up in various area of engineering and science, it is important to use numerical approaches that are close to the equations' solutions (Chapra and Canale, 2015). As such, scientific and technological problems involving differential equations are typically solved using numerical methods rather than analytic ones.

In this research, we intend to develop and study a four-step first derivative hybrid block Milne approach for systems of ordinary differential equations taken into account as:

$$\begin{cases} z_1' = f(t, z_1, z_2, \dots \\ z_2' = f(t, z_1, z_2, \dots \\ \vdots \\ z_n' = f(t, z_1, z_2, \dots \end{cases} \quad \text{with initial conditions} \quad \begin{cases} z_1(0) = c_1 \\ z_2(0) = c_2 \\ \vdots \\ z_n(0) = c_n \end{cases} \qquad (1)$$

For arbitrary $z_0 \le z \le z_N$. In this case, the function $f(t,z)$ is assumed to be continuous throughout the integration interval, and a unique solution exists. Numerous research has been carried out to provide numerical solutions to problems modeled as first order ordinary differential equations. These include works of authors such as Ndipmong and Udechukwu (2022), Garba and Mohammed (2020), Gomathi and Rabiyabe (2022), Badmus *et al.* (2015), Ehiemua and Agbeboh (2019), Eziokwu and Okereke (2020). Iyorter, B. V., Luga, T. & Isah, S. S. (2019). Techniques of solution employed by the above researchers include, Euler methods, the Adams Bashforth and Adams Moulton methods, linear multistep methods, Runge-Kutta methods and Milne methods among others.

Few mathematicians have come up with some block Milne techniques regarding solutions to various differential problems. The convergence of some selected properties with respect to block predictor-corrector methods and its applications on differential problems were investigated (Oghonyon *et al.*, 2016a). Again, Oghonyon *et al.,* (2016b) focused on block predictor-corrector method and derived a Milne's scheme. They implemented the scheme on ordinary differential problems and obtained a favourable outcome. Recently, Oghonyon *et al*. (2018a) formulated a suitable exponential fitted block Milne's scheme for ordinary differential equations emerging from oscillating vibrations problems. However, these approaches are limited by their low accuracy rate and low number of steps. The present research was motivated by the need to overcome the shortcomings of existing approaches by expanding the number of steps at both grid and off-step locations. The Milne technique employs the predictor-corrector algorithm and is dahlquist stable and accurate to the second order. For their starting values, the predictor-corrector of the Milne scheme requires single-step methods. In this study, the corrector component is reformulated into a continuous form and implemented as a block method in order to make it self-starting to solve systems of ordinary differential equations. To improve the degree of accuracy of the Milne method, appropriate off-grid points are selected with care.

This paper is structured as follows. In Section two, we describe the construction of the new numerical technique for (1). In section three, we established the order, zero stability, consistency, and convergence of the technique. In Section four, we used the method to solve systems of differential equations of the first order and compared the results of the different problems. Numerical tests with sample problems and their results were resented in section 4 and we concluded the study in section 5.

Construction of the Block Hybrid Milne Technique

To derive the new numerical technique, we apply the notion of a linear multistep collocation procedure using the general format $z(t)=\sum_{n=0}^{s+1} A(t)z(t_{i+n}) = h\sum_{n=0}^{v-1} B(t)f(\overline{t}_n,\overline{z}(\overline{t}_n))$

(2)

where

$$A_i(t)=\sum_{n=0}^{u+v-1} A_{i,n+i}t^n \quad and \quad hB_i(t)=\sum_{n=0}^{w+v-1} B_{i,n+i}t^n \tag{3}$$

Here, we use the basis function of power series to derive a numerical estimate for the ordinary differential equation of the format described in (1).

$$\sum_{n=0}^{w+v-1} d_n t^n \tag{4}$$

where w and v represents the interpolation and collocation points, $t \in [t_0, z_N]$, and d_n's are unknowns. Equation (1) is differentiated to get

$$\sum_{n=0}^{w+v-1} nd_n t^{n-1} \tag{5}$$

Hence, the continuous format of the proposed block technique from (3) with five off grid points at collocation is represented as

$$\begin{aligned} z(t)= A_2(t)z_{i+2}+B_2(t)hf_{i+2}+B_{\frac{9}{4}}(t)hf_{i+\frac{9}{4}}+B_{\frac{5}{2}}(t)hf_{i+\frac{5}{2}}+B_3(t)hf_{i+3}+B_{\frac{13}{4}}(t)hf_{i+\frac{13}{4}} \\ +B_{\frac{7}{2}}(t)hf_{i+\frac{7}{2}}+B_4(t)hf_{i+4} \end{aligned} \tag{6}$$

It generated some non-linear system of equations in the format $Mt = B$ in (7)

$$\begin{pmatrix} 1 & (t_{i+2}) & (t_{i+2})^2 & (t_{i+2})^3 & (t_{i+2})^4 & (t_{i+2})^5 & (t_{i+2})^6 & (t_{i+2})^7 \\ 0 & 1 & 2(t_{i+2}) & 3(t_{i+2})^2 & 4(t_{i+2})^3 & 5(t_{i+2})^4 & 6(t_{i+2})^5 & 7(t_{i+\frac{9}{4}})^6 \\ 0 & 1 & 2(t_{i+\frac{9}{4}}) & 3(t_{i+\frac{9}{4}})^2 & 4(t_{i+\frac{9}{4}})^3 & 5(t_{i+\frac{9}{4}})^4 & 6(t_{i+\frac{9}{4}})^5 & 7(t_{i+\frac{9}{4}})^6 \\ 0 & 1 & 2(t_{i+\frac{5}{2}}) & 3(t_{i+\frac{5}{2}})^2 & 4(t_{i+\frac{5}{2}})^3 & 5(t_{i+\frac{5}{2}})^4 & 6(t_{i+\frac{5}{2}})^5 & 7(t_{i+\frac{5}{2}})^6 \\ 0 & 1 & 2(t_{i+3}) & 3(t_{i+3})^2 & 4(t_{i+3})^3 & 5(t_{i+3})^4 & 6(t_{i+3})^5 & 7(t_{i+3})^6 \\ 0 & 1 & 2(t_{i+\frac{13}{4}}) & 3(t_{i+\frac{13}{4}})^2 & 4(t_{i+\frac{13}{4}})^3 & 5(t_{i+\frac{13}{4}})^4 & 6(t_{i+\frac{13}{4}})^5 & 7(t_{i+\frac{13}{4}})^6 \\ 0 & 1 & 2(t_{i+\frac{7}{2}}) & 3(t_{i+\frac{7}{2}})^2 & 4(t_{i+\frac{7}{2}})^3 & 5(t_{i+\frac{7}{2}})^4 & 6(t_{i+\frac{7}{2}})^5 & 7(t_{i+\frac{7}{2}})^6 \\ 0 & 1 & 2(t_{i+4}) & 3(t_{i+4})^2 & 4(t_{i+4})^3 & 5(t_{i+4})^4 & 6(t_{i+4})^5 & 7(t_{i+4})^6 \end{pmatrix} \begin{pmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \end{pmatrix} = \begin{pmatrix} z_{i+2} \\ f_{i+2} \\ f_{i+\frac{9}{4}} \\ f_{i+\frac{5}{2}} \\ f_{i+3} \\ f_{i+\frac{13}{4}} \\ f_{i+\frac{7}{2}} \\ f_{i+4} \end{pmatrix} \tag{7}$$

Employing Maple 2015 software to compute (7), and evaluation of the desired points results into the following proposed schemes;

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$$z_{i+1}=-\frac{45373}{1260}hf_{i+2}+\frac{48518}{315}hf_{i+3}-\frac{3292832}{945}hf_{i+4}-\frac{134941}{945}hf_{i+\frac{5}{2}}+\frac{3035}{63}hf_{i+\frac{7}{2}}$$
$$+\frac{269296}{2205}hf_{i+\frac{9}{4}}-\frac{27248}{189}hf_{i+\frac{13}{4}}+z_{i+2}$$
(8)

$$z_i=z_{i+2}-\frac{72353}{105}hf_{i+2}+\frac{1994752}{735}hf_{i+\frac{9}{4}}-\frac{3292832}{945}hf_{i+\frac{5}{2}}+\frac{443852}{105}hf_{i+3}-\frac{778240}{189}hf_{i+\frac{13}{4}}$$
$$+\frac{149216}{105}hf_{i+\frac{7}{2}}-\frac{482819}{6615}hf_{i+4}$$
(9)

$$z_{i+\frac{9}{4}}=\frac{2251}{26880}hf_{i+2}+\frac{1661}{26880}hf_{i+3}-\frac{293}{423360}hf_{i+4}-\frac{23021}{241920}hf_{i+\frac{5}{2}}+\frac{85}{5376}hf_{i+\frac{7}{2}}+\frac{5549}{23520}hf_{i+\frac{9}{4}}$$
$$-\frac{311}{6048}hf_{i+\frac{13}{4}}+z_{i+2}$$
(10)

$$z_{i+\frac{5}{2}}=\frac{787}{10080}hf_{i+2}+\frac{73}{2520}hf_{i+3}-\frac{83}{211680}hf_{i+4}+\frac{863}{15120}hf_{i+\frac{5}{2}}+\frac{43}{5040}hf_{i+\frac{7}{2}}+\frac{781}{2205}hf_{i+\frac{9}{4}}$$
$$-\frac{5}{189}hf_{i+\frac{13}{4}}+z_{i+2}$$
(11)

$$z_{i+3}=\frac{37}{420}hf_{i+2}+\frac{44}{105}hf_{i+3}-\frac{47}{26460}hf_{i+4}+\frac{331}{945}hf_{i+\frac{5}{2}}+\frac{1}{21}hf_{i+\frac{7}{2}}+\frac{208}{735}hf_{i+\frac{9}{4}}-\frac{176}{945}hf_{i+\frac{13}{4}}+z_{i+2}$$
(12)

$$z_{i+\frac{13}{4}}=\frac{1405}{16128}hf_{i+2}+\frac{8875}{16128}hf_{i+3}-\frac{125}{84672}hf_{i+4}+\frac{16375}{48384}hf_{i+\frac{5}{2}}+\frac{575}{16128}hf_{i+\frac{7}{2}}+\frac{4075}{14112}hf_{i+\frac{9}{4}}$$
$$-\frac{295}{6048}hf_{i+\frac{13}{4}}+z_{i+2}$$
(13)

$$z_{i+\frac{7}{2}}=\frac{99}{1120}hf_{i+2}+\frac{141}{280}hf_{i+3}-\frac{17}{7840}hf_{i+4}+\frac{197}{560}hf_{i+\frac{5}{2}}+\frac{15}{112}hf_{i+\frac{7}{2}}+\frac{69}{245}hf_{i+\frac{9}{4}}+\frac{1}{7}hf_{i+\frac{13}{4}}+z_{i+2}$$
(14)

$$z_{i+4}=\frac{17}{315}hf_{i+2}+\frac{404}{315}hf_{i+3}-\frac{869}{6615}hf_{i+4}+\frac{32}{945}hf_{i+\frac{5}{2}}+\frac{352}{315}hf_{i+\frac{7}{2}}+\frac{1024}{2205}hf_{i+\frac{9}{4}}-\frac{1024}{945}hf_{i+\frac{13}{4}}+z_{i+2}$$
(15)

Analysis of the Proposed Technique

This section is concerned on analyses with respect to zero stability and consistency of the novel technique.

Consistency

The proposed technique described in section 2 is frequently written as;

$$\sum_{i=0}^{4} \mathrm{A}_i z_{n+i} - \sum_{i=0}^{4} h\mathrm{B} f_{n+i} = 0 \quad (16)$$

Following Oghonyon *et al.* (2018b) and Mohammed *et al.* (2021), the local truncation error is a linear difference operator as;

$$L[z(t);h] = z_i - h\left(\mathrm{B}_2(t)z'_n + \mathrm{B}_3(t)z'_n + \mathrm{B}_{\frac{5}{2}}(t)z'_n + \mathrm{B}_{\frac{7}{2}}(t)z'_n + \mathrm{B}_{\frac{9}{4}}(t)z'_n + \mathrm{B}_{\frac{13}{4}}(t)z'_n + \mathrm{B}_4(t)z'_n\right) \quad (17)$$

Assuming that $z(t)$ is sufficiently differentiable, then the Taylor's expansion of (17) about the point t, can be represented as;

$$L[z(t);h] = E_0 z(t) + E_1 h z'(t) + E_2 h^2 z''(t) + \cdots + E_p h^p z^p(t) + E_{p+1} h^{p+1} z^{p+1}(t) \quad (18)$$

The discrete scheme in (9) is said to be consistent if $p \geq 1$ for $E_0 = E_1 = E_2 = \cdots = E_p = 0, E_{p+1} \neq 0$, where E_{p+1} denotes the error constant, and p denotes the order of the hybrid technique (Tiamiyu *et al.,* 2021). The summary of the order and error constant of the block schemes is given in Table 1.

Table 1 –Error Constants and Order of the Proposed Technique

Equation	Order	Error constant
(8)	7	$\frac{1643}{15680}$
(9)	7	$\frac{70099}{27095040}$
(10)	7	$\frac{1051}{2055208960}$
(11)	7	$\frac{9925}{11098128384}$
(12)	7	$\frac{1}{1003520}$
(13)	7	$\frac{139}{433520640}$
(14)	7	$\frac{17}{16056320}$
(15)	7	$-\frac{1}{211680}$

Zero Stability

To determine the zero stability of the new derived schemes, the first characteristic polynomial $R(\lambda)$ of (8) to (15) denoted as $\det(\lambda\times A(1)-A(0))$ is normalized as follows;

$R(\lambda)=\det(\lambda\times A(1)-A(0))$ such that we obtain

$$R(\lambda)=\lambda\times\begin{pmatrix}1&1&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&1&1&0&0&0&0&0\\0&1&0&1&0&0&0&0\\0&1&0&0&1&0&0&0\\0&1&0&0&0&1&0&0\\0&1&0&0&0&0&1&0\\0&1&0&0&0&0&0&1\end{pmatrix}-\begin{pmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&-1\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}=(\lambda^2-\lambda)\lambda^6$$

for $|\lambda|\le 1$ and the roots $|\lambda|=1$, the multiplicity must not exceed one. Hence, we arrive at the deduction $R(\lambda)=\det(\lambda\times A(1)-A(0))=(\lambda^2-\lambda)\lambda^6=0$ and $\lambda=(0\quad 0\quad 0\quad 0\quad 0\quad 0\quad 0\quad 1)$. Therefore, the developed hybrid block Milne technique is said to be zero stable.

Convergence

According to Ma'ali *et al.* (2020), Dahlquist's fundamental theorem asserts that "the necessary and sufficient requirements for a linear multi-step procedure to be convergent are consistency and zero-stability. By Kashkaria and Syam (2019) and Oghonyon *et al*. (2018b), since the hybrid block approach provided is consistent and zero stable, the convergence requirement is met.

Numerical Tests

Problem 1: We consider a set of linear differential equations in the form;

$$\begin{aligned}&z_1'=-21z_1+19z_2-20z_3, && z_1(0)=1\\&z_2'=19z_1-21z_2+20z_3, && z_2(0)=0\\&z_3'=40z_1-40z_2-40z_3, && z_3(0)=-1\\&0\le t\le 3, \quad h=0.2\end{aligned}$$

The exact solution is provided as

$$z_1(t)=\frac{1}{2}e^{-2t}+\frac{1}{2}e^{-40t}\sin(40t)+\frac{1}{2}e^{-40t}\cos(40t)$$
$$z_2(t)=\frac{1}{2}e^{-2}-\frac{1}{2}e^{-40t}\sin(40t)-\frac{1}{2}e^{-40t}\cos(40t)$$
$$z_3(t)=e^{-40t}\sin(40t)-e^{-40t}\cos(40t)$$

Problem 2: Considering the systems of initial value problem of first order differential equation of the form;

$$z_1' = -z_1 + 95z_2, \qquad z_1(0)=1$$
$$z_2' = -z_1 - 97z_2, \qquad z_2(0)=1$$
$$h = 0.0625$$

The real solution is provided as;

$$z_1(t)=\frac{95}{47}e^{-2t}-\frac{48}{47}e^{-96t}$$
$$z_2(t)=\frac{48}{47}e^{-96t}-\frac{1}{47}e^{-2t}$$

Problem 3: We consider the systems of initial value problem of first order differential equation of the form;

$$z_1' = -\left(2+10^4\right)z_1 + 10^4 z_2, \qquad z_1(0)=1$$
$$z_2' = z_1 - z_2 - z_2^2, \qquad z_2(0)=1$$

With $h = 0.1$ and the exact solution given as

$$z_1(t)=e^{-2t}$$
$$z_2(t)=e^{-t}$$

Problem 4. Solving the non-linear system of initial value problem of first order differential equation of the form;

$$z_1' = -1002z_1 + 100{z_2}^2, \qquad z_1(0)=1$$
$$z_2' = z_1 - z_2(1+z_2), \qquad z_2(0)=1$$

$0 \le t \le 1$ and the exact solution can be obtained from the following relations

$z_1(t)=e^{-2t},\ z_2(t)=e^{-t}$

Test Results

This section presents the test results for problems 1 to 4 considered in previous section. Comparison of the computations are displayed in some Figures and Tables. The exact solutions are represented by $z(t)$ and the new hybrid Block Milne solutions are denoted as $z_s(t), s = 1, 2$.

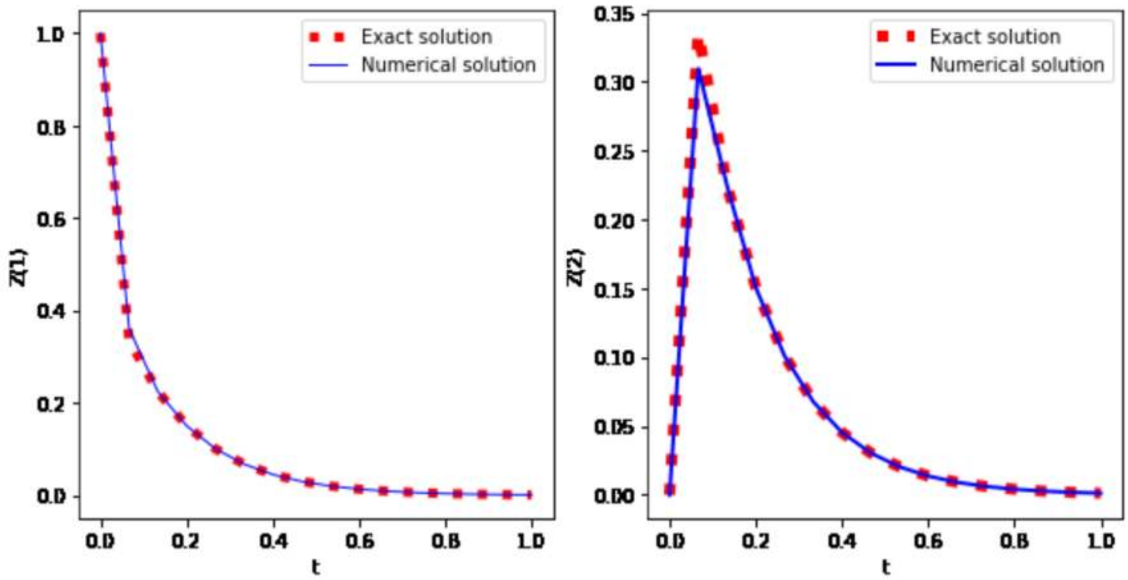

Figure 1: Profile solution for Problem 1

Table 2: Comparison Result of z_1 for Problem 1

t	$z(t)$	$z_1(t)$	$\lvert z(t) - z_1(t)\rvert$
0.20	0.33530156446464362999	0.067672002734714790247	2.54284×10^{-2}
0.40	0.22466441197379730427	0.22469925984189926044	3.48478×10^{-5}
0.60	0.15059710594701431165	0.15059886462150065052	1.75860×10^{-6}
0.80	0.10094825899733647829	0.10097264863920704481	2.43896×10^{-5}
1.00	0.06766764161830634611	0.067672002734714790247	4.36116×10^{-6}
1.20	0.04535897664470625168	0.045361183829914490593	2.20718×10^{-6}
1.40	0.03040503131260898249	0.030406509816399355942	1.47850×10^{-7}
1.60	0.02038110198918310758	0.020382094187318619971	9.92198×10^{-7}
1.80	0.01366186122364628040	0.013662846293445486353	9.85069×10^{-7}
2.00	0.00915781944436709014	0.009158487495676968640	6.68051×10^{-7}
2.20	0.00613866995153422058	0.0061391177562605422358	4.47804×10^{-7}
2.40	0.00411487352451001442	0.0041151737191594553502	3.00194×10^{-7}
2.60	0.00275828221038038621	0.0027585481612003116988	2.65950×10^{-7}
2.80	0.00184893185824146541	0.0018491116996924581333	1.79841×10^{-7}
3.00	0.00123937608833317921	0.0012394966389739651914	1.20550×10^{-7}

Table 3: Comparison Result of z_2 for Problem 1

t	$z(t)$	$z_2(t)$	$\lvert z(t) - z_2(t)\rvert$
0.20	0.33535037428834497180	0.30960572256641213744	2.57446×10^{-2}
0.40	0.22466451974417424401	0.22464062855752519152	2.38911×10^{-5}
0.60	0.15059710593100098342	0.15060266988864076734	5.56395×10^{-6}
0.80	0.10094825899732591356	0.10092877892490726787	1.94800×10^{-6}

1.00	0.06766764161830634894	0.06766974703465251893	2.10541×10^{-6}
1.20	0.04535897664470625168	0.04536118109942907980	2.20445×10^{-6}
1.40	0.03040503131260898249	0.03040650998570001563	1.47867×10^{-7}
1.60	0.02038110198918310758	0.02038209226510866320	9.90275×10^{-7}
1.80	0.01366186122364628040	0.01366284619401490224	9.84970×10^{-7}
2.00	0.00915781944436709014	0.00915848749555003001	6.68051×10^{-7}
2.20	0.00613866995153422058	0.00613911775626806871	4.47804×10^{-7}
2.40	0.00411487352451001442	0.00411517371907533865	3.00194×10^{-7}
2.60	0.00275828221038038620	0.00275854816119593290	2.65950×10^{-7}
2.80	0.00184893185824146541	0.00184911169969245197	1.79841×10^{-7}
3.00	0.00123937608833317921	0.00123949663897396534	1.20550×10^{-7}

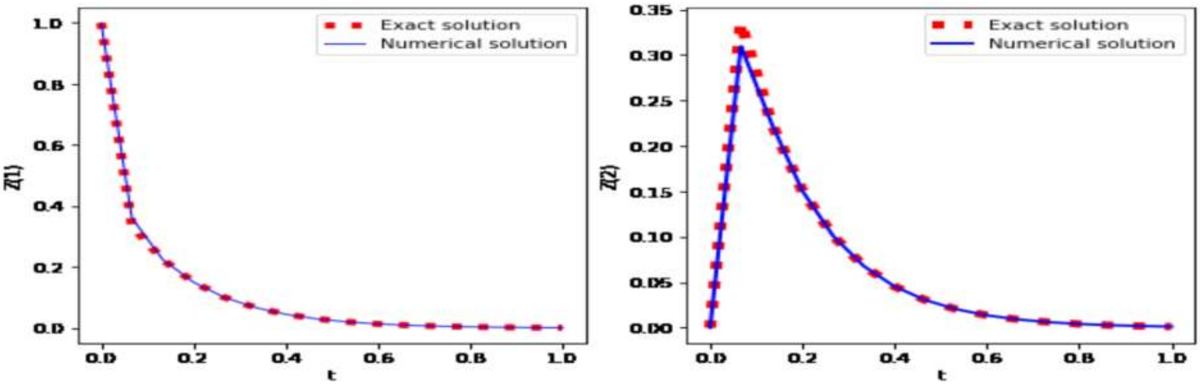

Figure 2: Profile solution for Problem 2

Table 4: Comparison Result of z_1 for Problem 2

t	$z(t)$	$z_1(t)$	$\lvert z(t)-z_1(t)\rvert$
0.0625	1.78123884342673570357 14	1.72247412014931656559 75	5.87647×10^{-2}
0.1250	1.57416552062958516445 15	1.57357845743733081884 85	5.87063×10^{-7}
0.1875	1.38920171817241130033 16	1.38920240376031707740 55	6.85587×10^{-7}
0.2500	1.22596622704017256725 75	1.22593889940576352663 42	2.73276×10^{-7}
0.3125	1.08191139807020385801 81	1.08190976716386954721 52	1.63090×10^{-9}
0.3750	0.95478345766800821371 07	0.95478345038751260245 16	7.28049×10^{-9}
0.4375	0.84259344403102762165 96	0.84259345163790892362 09	7.60688×10^{-9}
0.5000	0.74358610449546852237 24	0.74358611046158447387 20	5.96611×10^{-9}
0.5625	0.65621243402219626235 57	0.65621244276269182035 25	8.74049×10^{-9}
0.6250	0.57910544046208637299 71	0.57910544828568151689 21	7.82359×10^{-9}
0.6875	0.51105875747767905089 45	0.51105876438232431325 80	6.90464×10^{-9}
0.7500	0.45100777051278369678 00	0.45100777660626361088 81	6.09347×10^{-9}
0.8125	0.39801296051911563310 26	0.39801296763962172528 92	7.12050×10^{9}
0.8750	0.35124520484664440499 29	0.35124521117402657367 55	6.32738×10^{-9}
0.9375	0.30997280532485540453 35	0.30997281090873313204 70	5.58387×10^{-9}
1.0000	0.27355004058464267510 48	0.27355004551250170880 59	4.92785×10^{-9}

Table 5: Comparison Result of z_2 for Problem 2

t	$z(t)$	$z_2(t)$	$\lvert z(t) - z_2(t)\rvert$
0.0625	-0.0162450382575448978 41	0.0425196927492879193219	5.87647×10^{-4}
0.1250	-0.0165639544867754279 61	-0.0159768842801852529 55	5.70702×10^{-6}
0.1875	-0.0146231605904669032 41	-0.0146238399883204099 86	9.75732×10^{-7}
0.2500	-0.0129049076149057200 49	-0.0128775745173324912 21	2.73330×10^{-7}
0.3125	-0.0113885410322233741 04	-0.0113869006165371897 55	1.64041×10^{-8}
0.3750	-0.0100503521859787994 34	-0.0100503363963568831 39	1.57896×10^{-11}
0.4375	-0.0088694046740108164 89	-0.0088694047716614395 35	9.76506×10^{-10}
0.5000	-0.0078272221525838791 81	-0.0078272214915466573 76	6.61037×10^{-11}
0.5625	-0.0069074993054968027 61	-0.0069074993540578018 68	4.85609×10^{-11}
0.6250	-0.0060958467417061723 47	-0.0060958468236392669 86	8.19330×10^{-11}
0.6875	-0.0053795658681860952 72	-0.0053795659408670421 76	7.26809×10^{-11}
0.7500	-0.0047474502159240389 13	-0.0047474502800465601 13	6.41225×10^{-11}
0.8125	-0.0041896101107275329 80	-0.0041896101856790659 57	7.49515×10^{-11}
0.8750	-0.0036973179457541516 31	-0.0036973180123581632 04	6.66040×10^{-11}
0.9375	-0.0032628716349984779 42	-0.0032628716937761382 44	5.87776×10^{-11}
1.0000	-0.0028794741114172913 16	-0.0028794741632894911 53	5.18721×10^{-11}

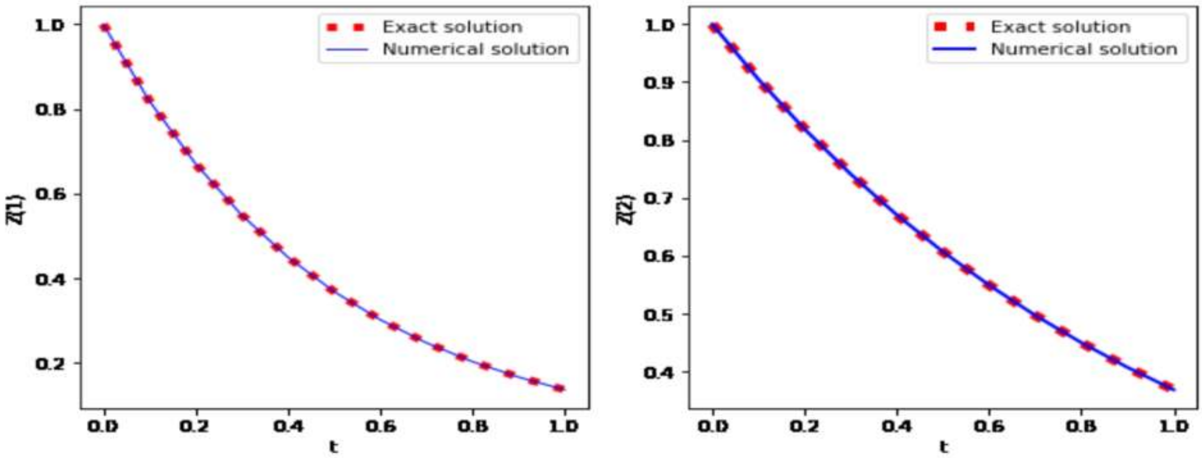

Figure 3: Profile solution for Problem 3

Table 6: Comparison Result of z_1 for Problem 3

t	$Z(t)$	$z_1(t)$	$\lvert Z(t) - z_1(t)\rvert$
0.100	0.1353352832366126918	0.1353352839005536003 80	6.63940×10^{-10}
0.200	0.0183156388887341802	0.0183156390379789136 19	1.49244×10^{-10}
0.300	0.0024787521766663584	0.0024787522088799435 11	3.22135×10^{-11}
0.400	0.0003354626279025118	0.0003354626333453063 06	5.44279×10^{-12}
0.500	0.0000453999297624848	0.0000453999307188030 95	9.56318×10^{-13}
0.600	$6.144212353328209 \times 10^{-6}$	$6.14421250257915187 \times 10^{-6}$	1.49250×10^{-13}
0.700	$8.315287191035678 \times 10^{-7}$	$8.31528743325846039 \times 10^{-7}$	2.42222×10^{-14}
0.800	$1.125351747192591 \times 10^{-7}$	$1.12535178360510108 \times 10^{-7}$	3.64125×10^{-15}
0.900	$1.522997974471262 \times 10^{-8}$	$1.52299803111905810 \times 10^{-8}$	5.66477×10^{-16}
1.000	$2.061153622438557 \times 10^{-9}$	$2.06115370575375547 \times 10^{-9}$	8.33151×10^{-17}

Table 7: Comparison Result of z_2 for Problem 3

t	$Z(t)$	$z_2(t)$	$\lvert Z(t) - z_2(t)\rvert$
0.100	0.36787944117144232160	0.36787944207382158129	9.02379×10^{-10}
0.200	0.13533528323661269189	0.13533528378795869403	5.51346×10^{-10}
0.300	0.049787068367863942979	0.049787068691376165830	3.23512×10^{-10}
0.400	0.018315638888734180294	0.018315639037311634828	1.48577×10^{-11}
0.500	0.006737946999085467096	0.006737947070050389561	7.09649×10^{-11}
0.600	0.002478752176666358423	0.002478752206771637136	3.01052×10^{-11}
0.700	0.000911881965554516208	0.000911881978835968039	1.32814×10^{-12}
0.800	0.000335462627902511838	0.000335462633329613693	5.42710×10^{-12}
0.900	0.000123409804086679549	0.000123409806381785372	2.29510×10^{-13}
1.000	0.000045399929762484851	0.000045399930680040142	9.17555×10^{-13}

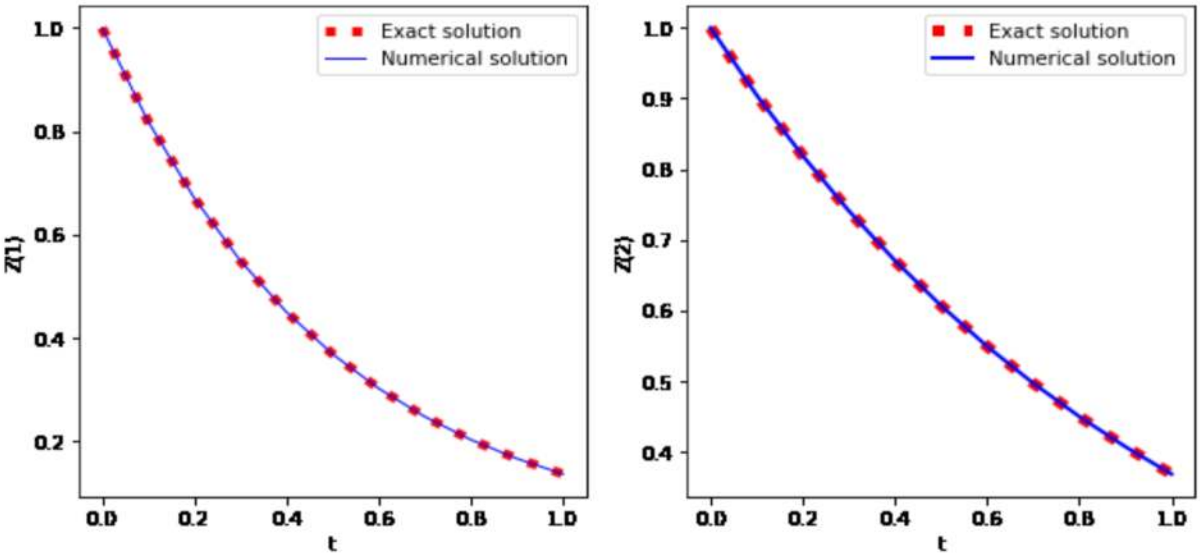

Figure 4: Profile solution for Problem 4

Table 8: Comparison Result of z_1 for Problem 4

t	$Z(t)$	$z_1(t)$	$\lvert Z(t) - z_1(t)\rvert$
0.100	0.81873075307798185867	0.81873075717909515030	4.10111×10^{-9}
0.200	0.67032004603563930074	0.67032004732826214578	1.29262×10^{-9}
0.300	0.54881163609402643263	0.54881163715192314629	1.05789×10^{-9}
0.400	0.44932896411722159143	0.44932896498600046686	8.68778×10^{-9}
0.500	0.36787944117144232160	0.36787944368730752857	2.51586×10^{-9}
0.600	0.30119421191220209664	0.30119421304245564822	1.13025×10^{-10}
0.700	0.24659696394160647694	0.24659696486679413398	9.25187×10^{-10}
0.800	0.20189651799465540849	0.20189651875332366913	7.58668×10^{-10}
0.900	0.16529888822158653830	0.16529888964265752646	1.42107×10^{-9}
1.000	0.13533528323661269189	0.13533528398193333375	7.45320×10^{-10}

Table 9: Comparison Result of z_2 for Problem 4

t	$Z(t)$	$z_2(t)$	$\lvert Z(t) - z_2(t)\rvert$
0.100	0.90483741803595957316	0.90483741888208201073	8.46122×10^{-10}
0.200	0.81873075307798185867	0.81873075386712958296	7.89147×10^{-10}
0.300	0.74081822068171786607	0.74081822139576027818	7.14042×10^{-10}
0.400	0.67032004603563930074	0.67032004668177876646	6.46139×10^{-10}
0.500	0.60653065971263342360	0.60653066083442824593	1.12179×10^{-9}
0.600	0.54881163609402643263	0.54881163712358090997	1.02955×10^{-9}
0.700	0.49658530379140951470	0.49658530472298378291	9.31574×10^{-10}
0.800	0.44932896411722159143	0.44932896496017530462	8.42953×10^{-10}
0.900	0.40656965974059911188	0.40656966084988405816	1.10928×10^{-9}
1.000	0.36787944117144232160	0.36787944218432163944	6.63940×10^{-9}

Discussion of Results

The newly derived block Milne technique is applied to stiff initial value problems in ordinary differential equations of the first order. The present technique associates numerical results with their exact solutions and summarizes the results in graphs and tables. The graphs of the exact solutions versus the numerical solutions for problems 1 to 4 are presented in Figures 1 to 4, which demonstrate that the numerical results are in good agreement with the exact solutions. In addition, the absolute errors associated with the numerical results and the analytic solutions are compared in Tables 2–9. The relatively small difference between the exact answer and the computed results proves the validity of the derived technique.

Conclusion

Using the collocation methodology, we established a self-starting hybrid block Milne technique with a greater degree of accuracy in this study. The novel technique aims to increase the efficacy and precision of Linear Multistep techniques by increasing the number of steps at both grid and off-grid locations. In the creation of the new approach, four off-step points and four step points were selected. The convergence of the suggested technique's fundamental attributes was analyzed. Systems of stiff problems were solved numerically to illustrate the accuracy of the suggested technique. The numerical outcomes of the problems demonstrate the effectiveness of the proposed technique, as the computed outcomes corresponded well with the exact solutions. Based on the graphs and tabulated data, we can infer that the proposed technique is an appropriate alternative for dealing with stiff problems that exist in all disciplines of science and engineering. For all computations, Maple 2015 was utilized.

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