COMPUTATIONAL ALGORITHM FOR VOLTERRA INTEGRAL SOLUTIONS VIA VARIATIONAL ITERATIVE METHOD

Khadeejah James Audu

Department of Mathematics, Federal University of Technology, Minna, Nigeria.
Corresponding email: k.james@futminna.edu.ng

Abstract:

The Volterra Integral Equations (VIE) are a class of mathematical equations that find applications in various fields, including physics, engineering, and biology. Solving VIEs analytically is often challenging, and researchers have turned to numerical methods for obtaining approximate solutions. In this research, we propose a computational algorithm based on the Variational Iterative Method (VIM) to efficiently and accurately solve VIEs. By incorporating this method into the computational algorithm, we aim to improve the accuracy and convergence rate of the solutions. The performance of our algorithm was evaluated through extensive numerical experiments on various types of VIEs. The results demonstrate the effectiveness of the VIM approach in terms of accuracy, convergence rate, and computational efficiency. In conclusion, the proposed computational algorithm based on VIM presents a valuable contribution to the field of solving VIEs. It offers an efficient and accurate approach for obtaining approximate solutions, enabling researchers and practitioners to tackle complex problems that rely on VIEs. The algorithm's versatility and robustness make it a promising tool for a wide range of applications, including physics, engineering, and biology.

Keywords: Volterra integral equations, Computational algorithm, Variation iterative method, Numerical solutions, Convergence

1. Introduction

Volterra integral equations (VIE), of the first or second types, are characterized by a variable upper limit of integration. For the first kind VIE, the unknown function u(s) occurs only under the integral sign in the form:

$$f(s) = \int_0^s K(s,x)u(x)\,dx \tag{1}$$

As for VIE of the second type, the unknown function u(s) occurs inside and outside the integral sign, depicted as

$$u(s) = f(s) + \lambda\int_0^s K(s,x)u(x)\,dx \qquad 0 \le s \le b < \infty \tag{2}$$

where:

$u(s)$ is the unknown function to be determined.

$f(s)$ is the given function on the right-hand side.

λ is a constant parameter.

$K(t,s)$ is the kernel function in the integral equation.

$[a,b]$ is the interval of interest.

This research work focuses on developing a computational algorithm for solving VIEs of (2) using the Variational Iterative Method (VIM). VIEs are challenging to solve analytically due to the presence of both a convolution integral and an unknown function. VIM is an iterative approach that combines variational calculus and iterative methods to approximate the unknown function accurately. The proposed algorithm aims to offer more robust, stable, and accurate numerical solutions for a wide range of VIEs compared to existing methods. The study includes theoretical foundations, a review of existing techniques, and practical implementations, with the potential to impact fields like medical imaging, control systems, and mathematical modeling (González-Rodelas *et al.*, 2022; Burova and Alcybeev (2021); Shoukralla and Ahmed (2020); Hamdan *et al.*, (2019)).

The primary aim of this paper is to employ Maple 2017 codes for the creation of a viable algorithm based on the VIM as elucidated by Falade and Tiamiyu (2020). This algorithm holds the promise of mitigating the complexities associated with lengthy integrals, thereby enabling the attainment of precise numerical outcomes.

Aim:

The main aim of this research is to develop an efficient computational algorithm based on the VIM for solving VIEs, with a focus on enhancing accuracy and reducing computational complexity.

Objectives:

I. Design a computational algorithm based on VIM to solve a wide range of VIEs, encompassing linear and nonlinear cases.
II. Improve numerical accuracy by implementing error-reduction techniques, ensuring close alignment between computed and exact solutions.
III. Optimize computational efficiency by streamlining the algorithm, reducing processing time and resource utilization for complex integral equations.
IV. Validate the algorithm through practical applications, comparing its solutions with established analytical results to assess performance, accuracy, and efficiency.

Problem Statement:

Solving VIEs poses significant challenges due to the intricate nature of the equations and the complexities introduced by nonlinearities. Existing computational methods may struggle to achieve a balance between numerical accuracy and computational efficiency when handling such equations. Consequently, there is a pressing need for a robust and efficient computational algorithm that leverages the VIM to provide accurate and swift solutions for Volterra integral equations, addressing the limitations of current approaches and advancing the field of computational mathematics.

Literature Review/Related Work

Falade and Tiamiyu (2020) presented a suitable numerical algorithm for solving systems of Volterra integro-differential equations.

Khidir (2021) proposed a Chebyshev spectral method for resolving Volterra Integral problems.

Amin *et al*. (2022) developed a numerical method for solving nonlinear Fractional Volterra Integral equations.

Fermo and Occorsio (2022) employed the Nystrom method for numerical solution of 2^{nd} kind VIEs.

Hadi (2023) combined collocation approach with piecewise Hermite polynomials to solve second kind of VIEs.

2.Methodology

2.1 Description of the Variational Iterative Method

The VIM is characterized by its iterative nature, wherein it seeks to refine the numerical solutions of VIEs through successive approximations. At its core, VIM combines variational calculus principles with iterative procedures to iteratively improve the estimates of the unknown function. By breaking down the original problem into a sequence of subproblems, VIM systematically adjusts the solution at each iteration, moving closer to the true solution. This iterative process continues until convergence is achieved, resulting in more accurate and stable numerical solutions for a diverse range of VIEs. The VIM offers quickly converging successive approximations for obtaining exact solutions in a closed-form expression (Maturi and Malaikah, 2018; Fermo and VanderMee, 2021). It is effective in addressing a broad range of linear and nonlinear, homogeneous, and inhomogeneous equations. This approach ensures swift convergence of successive approximations towards the exact solution. The correction functionals pertinent to the VIEs (3) are provided as follows:

$$u_{t+1}(s) = u_t(s) + \int_0^s \lambda(x)\left(u(s) - f(s) - \int_0^x K(x,r)u_t(r)\,dr\right)dx \tag{3}$$

The method encompasses two fundamental strategies. In the initial approach, the optimal identification of the Lagrange multiplier entails integration by parts and employs a constrained variation. Once λ is established, the subsequent iterative formulation, unrestricted by constrained variation, is utilized to ascertain the successive approximations $u_{t+1}(s),\ t \geq 0$ for the solutions $u(s)$. The preliminary approximations $u_0(s)$ may take on arbitrary functions. It is advantageous to employ the initial conditions in the selection of these approximations, whereby $u_0(s)$ assume significance. Consequently, the solutions are thus expressed as:

$$u(s) = \lim_{t \to \infty} u_t(s) \tag{4}$$

The VIM procedure or approach in solving a VIEs is enumerated in the following steps;

1. Define Initial Approximation

2. Iterate using the Variational Iteration Method.

3. Obtain the correction equation which can be found by introducing an auxiliary function and the Lagrange multiplier.

4. Calculate correction functional by differentiating with respect to the unknown functionto obtain the solution. This requires calculating the partial derivative of *u*(*t*) with respect to s.

5. Substitute and solve: Substitute the expression for *ut*(*x*) into the equation of (4) for *ut*(s). This yields the *k*-th approximation of the solution *u*(*s*).

6. Further Iterate: Continue the iterations until the successive approximations of the function converge to a satisfactory level. Convergence can be assessed by comparing successive approximations and checking the change between iterations.

7. Final Solution**:** Once the desired convergence is achieved, the final approximate solution u(*s*) is obtained.

2.2. The VIM Computational Algorithm

Utilizing the correction functional for the Volterra integral problems outlined in equation (3), a procedural sequence comprising four steps is devised within the Maple 2017 software package, outlined as follows:

Step 1:

$restart:$

$F[s] := f[s];$

$F[1] := eval\left(F[s],[x=s]\right);$

$u[s_0] := \alpha;\ U[s_0] := \alpha;\ [u1[s_0] := \alpha;$

$F := Diff\ \left(u[k],\ x - F[1],[x = 0...s]\right)$

$\lambda_k\left(x\right) = \dfrac{\left(-1\right)^q}{\left(q-1\right)!}\left(x-s\right)^{q-1}$

$N : \mathbb{R}^+$

Step 2:

$for\ k\ \ from\ 0\ to\ N\ do$

$u\left[k+1\right]\ :=\ expand\left(value\left(u\left[k\right] + Int\left(\lambda \times subs\left(\mathrm{s}\ =\ \mathrm{x},\ F\right),\ \mathrm{x}\ =\ 0\ ..\ \mathrm{s}\right)\right)\right)$

$end\ do$

Step 3:

$for\ k\ \ from\ 0\ to\ N\ do$

$u\left[k+1\right]\ :=\ \mathrm{convert}\left(series\left(u\left[k+1\right], s\ x\right)'\ polynom\ '\right);$

$end\ do$

Step 4:

$for\ k\ \ from\ 0\ to\ N\ do$

$u\left[k+1\right]\ := u_t\left[k+1\right]\ ;$

$end\ do$ $\qquad where\ \ N, x\ \in \mathbb{R}^+$

3. Numerical Experiments

Within this section, we provide three instances to demonstrate the practicality and effectiveness of the algorithm we have devised for solving Volterra integral equations. The outcomes obtained exhibit favorable correspondence with the precise/exact solution.

3.1 Experiment 1: Use the variational iteration method to find the solution to the VIE.

$u\left(s\right) = 1 + \int_0^s u\left(x\right)dx$ $\qquad$ The exact solution is: $u\left(s\right) = e^s$

The above VIE is computed via the VIM computational algorithm to obtain the following successive approximate solutions of the VIM

$$u_0(s)=1$$
$$u_1(s)=1+s$$
$$u_2(s)=1+s+\frac{s^2}{2}$$
$$u_3(s)=1+s+\frac{s^2}{2!}+\frac{s^3}{3!}$$
$$\vdots$$
$$u_t(s)=1+s+\frac{s^2}{2!}+\frac{s^3}{3!}+\frac{s^4}{4!}+\frac{s^5}{5!}+\cdots+\frac{s^t}{t!}$$

The VIM admits the use of $u(s)=\lim_{t\to\infty} u_t(s)$, and converges into the exact solution $u(s)=e^s$.

3.2 Experiment 2: By employing the variational iteration method, seek to resolve the VIE.

$$u(s)=s+\int_0^s (s-x)u(x)dx \qquad \text{The exact solution is: } u(s)=\sinh(s)$$

The computation of Experiment 2 with proposed algorithm, gives the following VIM solutions

$$u_0(s)=0$$
$$u_1(s)=s$$
$$u_2(s)=s+\frac{s^3}{6}$$
$$u_3(s)=s+\frac{s^3}{6}+\frac{s^5}{120}$$
$$\vdots$$
$$u_t(s)=s+\frac{s^3}{3!}+\frac{s^5}{5!}+\frac{s^7}{7!}+\frac{s^9}{9!}+\cdots+\frac{s^{(1+2t)}}{(1+2t)}$$

The VIM admits the use of $u(s)=\lim_{t\to\infty} u_t(s)$, which converges to the exact solution $u(s)=\sinh(s)$

3.3 Experiment 3: Implement the VIM algorithm on the following second type of VIE.

$$u(s)=1+s+\frac{s^3}{6}-\int_0^s (s-x)u(x)dx \qquad \text{The exact solution is: } u(s)=s+\cos(s)$$

$$u_0(s) = 1 + s$$

$$u_1(s) = 1 + s - \frac{s^2}{2!}$$

$$u_2(s) = 1 + s - \frac{s^2}{2!} + \frac{s^4}{4!}$$

$$u_3(s) = 1 + s - \frac{s^2}{2!} + \frac{s^4}{4!} - \frac{s^6}{6!}$$

$$\vdots$$

$$u_t(s) = s + \left(1 - \frac{s^2}{2!} + \frac{s^4}{4!} - \frac{s^6}{6!} + \frac{s^8}{8!} - \cdots + \frac{(-1)^t s^{2t}}{(2t!)}\right)$$

The VIM admits the use of $u(s) = \lim_{t\to\infty} u_t(s)$, converging to the exact solution $u(s) = s + \cos(s)$

4. Results of the Experiments

Table 1: Numerical solution of Experiment 1

s	Exact Solution $u(s)$	VIM Solution $u(s)$
0	1.000000000	1.000000000
0.1	1.105170918	1.105170918
0.2	1.221402758	1.221402758
0.3	1.349858808	1.349858808
0.4	1.491824698	1.491824698
0.5	1.648721271	1.648721271
0.6	1.822118800	1.822118800
0.7	2.013752707	2.013752707
0.8	2.225540928	2.225540928
0.9	2.225540928	2.225540928
1.0	2.718281828	2.718281828

Table 2: Numerical solution of Experiment 2

s	Exact Solution $u(s)$	VIM Solution $u(s)$
0	0.0000000000	0.000000000
0.1	0.1001667500	0.1001667500
0.2	0.2013360025	0.2013360025

0.3	0.3045202934	0.3045202934
0.4	0.4107523258	0.4107523258
0.5	0.4107523258	0.4107523258
0.6	0.6366535821	0.6366535821
0.7	0.7585837018	0.7585837018
0.8	0.8881059822	0.8881059822
0.9	1.0265167260	1.0265167260
1.0	1.175201194	1.175201194

Table 3: Numerical solution of Experiment 3

s	Exact Solution $u(s)$	VIM Solution $u(s)$
0	1.000000000	1.000000000
0.1	1.095004165	1.095004165
0.2	1.180066578	1.180066578
0.3	1.255336489	1.255336489
0.4	1.321060994	1.321060994
0.5	1.377582562	1.377582562
0.6	1.425335615	1.425335615
0.7	1.464842187	1.464842187
0.8	1.496706709	1.496706709
0.9	1.521609968	1.521609968
1.0	1.540302306	1.540302306

Discussion of Results

The results from the Tables 1-3, showed that that the VIM computational algorithm by a Maple 2017 program is remarkably effective and performing is very easy. The computed values from the converged series approximations illustrated by the results, agree well with the exact/precise solution.

5. Conclusion

This paper introduces an algorithm rooted in the variational iterative method, which was formulated and employed to address both linear and nonlinear sets of VIEs. A comparison between the attained solutions and real solutions reveals the algorithm's enhanced efficiency and practicality in solving a spectrum of integral equations, regardless of their linearity. Thus, this computational approach proves highly effective for accurately determining solutions to integral equations and is recommended for adoption in engineering and applied science scenarios. Further research can focus on optimizing the algorithm and exploring its applicability to higher-dimensional problems.

References

Amin, R., Senu, R., Hafeez, B. A., Arshad, N. I., Ahmadian, A., Salahshour, S., & Sumelka, W. (2022). A COMPUTATIONAL ALGORITHM FOR THE NUMERICAL SOLUTION OF NONLINEAR FRACTIONAL INTEGRAL EQUATIONS. *Fractals*, *30*(1), 2240031-2240038.

Burova, I. & Alcybeev, G. (2021). Application of splines of the second order approximation to Volterra integral equations of the second kind. Applications in systems theory and dynamical systems. *International Journal of Circuits, Systems and Signal Processing*, *15*(2), 63–71.

Falade, K. I., & Tiamiyu, A. T. (2020). Computatıonal Algorıthm for the Numerıcal Solutıon of Systems of Volterra Integro-Dıfferentıal Equatıons. *Academic Journal of Applied Mathematical Sciences. 6*(1), 66-76.

Fermo, L. & Occorsio, D. (2022). Weakly singular linear Volterra integral equations: A Nyström method in weighted spaces of continuous functions. *Journal of Computational and Applied Mathematics*, 406, 1-10.

Fermo, L. & VanderMee, C. (2021). Volterra integral equations with highly oscillatory kernels: a new numerical method with applications. *Electronic Transactions on Numerical Analysis (ETNA)*, 54, 333–354.

González-Rodelas, P., Pasadas, M., Kouibia, A., & Mustafa, B. (2022). Numerical Solution of Linear Volterra Integral Equation. *Mathematics*, 1-15.

Hadi, A. A. (2023). Numerical solution of a Volterra integral equation. *Formosa Journal of Applied Sciences*, *2*(5) 823-836.

Hamdan, S., Qatanani, N., & Daraghmeh, A. (2019). Numerical Techniques for Solving Linear Volterra Fractional Integral Equation. Journal of Applied Mathematics, 1-9.

Khidir, A. A. (2021). A New Numerical Technique for Solving Volterra Integral Equations Using Chebyshev Spectral. *Method Mathematical Problems in Engineering*, 1-11.

Maturi, A. D., & Malaikah, H. M. (2018). Numerical Solution of Systems of three Nonlinear Linear Volterra Integral Equation using Implicit Trapezoidal. *Journal of Mathematics Research*, *10*(1), 44-58.

Shoukralla, E. S., & Ahmed, B. M. (2020). Numerical Solutions of Volterra Integral Equations of the Second Kind using Lagrange interpolation via the Vandermonde matrix. Journal of Physics: Conference Series (pp. 1-8). IOP Publishing Ltd,