Refinements of Some Iterative Methods for Solving Linear System of Equations

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Abstract:

The efficient and accurate solution of linear systems of equations is a fundamental problem in various scientific and engineering fields. In this study, we focus on the refinements of iterative methods for solving linear systems of equations ($\underline{A}k = \underline{b}$). The research proposes two methods namely, third refinement of Jacobi method (TRJ) and third refinement of Gauss-Seidel (TRGS) method, which minimizes the spectral radius of the iteration matrix significantly when compared to any of the initial refinements of Jacobi and Gauss-Seidel methods. The study explores ways to optimize their convergence behavior by incorporating refinement techniques and adaptive strategies. These refinements exploit the structural properties of the coefficient matrix to achieve faster convergence and improved solution accuracy. To evaluate the effectiveness of the proposed refinements, numerical examples were tested to see the efficiency of the proposed TRJ and TRGS on a diverse set of linear equations. We compare the convergence behavior, computational efficiency, and solution accuracy of the refined iterative methods against their traditional counterparts. The experimental results demonstrate significant improvements in terms of convergence rate and computational efficiency when compared to their initial refinements. The proposed refinements have the potential to contribute to the development of more efficient and reliable solvers for linear systems, benefiting various scientific and engineering applications.

Keywords: Linear systems of equations, Third Refinement of Jacobi method, Third Refinement of Gauss-Seidel method, Convergence, Computational efficiency,

1. Introduction

Generally, a linear system of equation can be denoted as;

$$\underline{A}k = \underline{b} \tag{1}$$

where $\underline{A} \in \mathbb{R}^{n\times n}, \underline{b} \in \mathbb{R}^{n}$ and $k \in \mathbb{R}^{n}$. If $\underline{A}$ has a non-vanishing diagonal elements; then, the iteration process is obtained by splitting $\underline{A}$ into the following form;

$$\underline{A} = \underline{D} - \underline{L} - \underline{U} \tag{2}$$

In the realm of numerical analysis, the quest for efficient methods to solve linear systems of equations remains a fundamental pursuit. Despite significant advancements in iterative techniques, there exists a notable research gap in the refinement of existing methods to enhance their convergence rate, stability, and applicability to various types of matrices. This study is motivated by the imperative to address this research gap and contribute to the ongoing evolution of iterative methods for linear system solutions. By refining established iterative approaches, we aim to improve their performance and broaden their utility across diverse computational contexts. This research is particularly relevant given the pervasive nature of linear systems in scientific computing, engineering, and various other fields where accurate and efficient solutions are paramount. A comprehensive review of the literature reveals notable contributions in this domain. Audu et al. (2021a) introduced the Extended Accelerated over relaxation (EAOR) method, demonstrating its efficacy in solving large and sparse linear systems. Building upon this work, Audu et al. (2021b) further refined the EAOR method, enhancing its capabilities for linear system solutions. Additionally, Dafchahi (2008) proposed a new refinement of the Jacobi method, while Eneyew et al. (2019) and Eneyew et al. (2020) presented second refinements of the Jacobi and Gauss-Seidel iteration methods, respectively. These studies provide valuable insights into the refinement of iterative techniques, laying the groundwork for further exploration and optimization in this area. With this backdrop, our research endeavors to contribute novel refinements to existing iterative methods for solving linear systems of equations. By leveraging insights from previous works and exploring innovative modifications, we seek to advance the state-of-the-art in numerical algorithms, ultimately facilitating more accurate and efficient solutions to a wide range of computational problems.

The Jacobi and Gauss-Seidel methods, commonly used for solving systems of linear equations, encounter challenges related to their convergence rates, computational time, and applicability. Addressing these issues, this research introduces modifications to enhance the performance of the second refinements of the Jacobi and Gauss-Seidel methods. The proposed approaches, termed as the "Third refinement of Jacobi (TRJ) method" and "Third refinement of Gauss-Seidel (TRGS) method," aim to significantly reduce computation time, spectral radius, and the number of iterations while improving the convergence rate. This study seeks to improve the efficiency of iterative methods, specifically the Jacobi and Gauss-Seidel techniques, for solving linear systems of equations by achieving faster convergence towards accurate solutions. The research objectives encompass the derivation and testing of two modified approaches: The Third Refinement of Jacobi (TRJ) method and the Third Refinement of Gauss-Seidel (TRGS) method. Through these objectives, the study aims to evaluate the convergence properties of TRJ and TRGS and perform numerical tests to validate their effectiveness. Overall, by introducing these refinements, the research endeavors to enhance the computational performance and accuracy of iterative methods in solving linear systems of equations.

2. Methodology

2.1 Derivation of Third – Refinement of Jacobi method

Combination of (1) and (2) in iteration format gives;

$$\underline{D}k = \left(\underline{L} + \underline{U}\right)k + \underline{b} \tag{3}$$

$$k^{(n+1)} = \underline{D}^{-1}\left(\underline{L} + \underline{U}\right)k^{(n)} + \underline{D}^{-1}\underline{b} \tag{4}$$

Remodeling (1) as $\underline{D} - \underline{A} = \underline{L} + \underline{U}$ and substituting into (3) yields;

$$k^{(n+1)} = \underline{D}^{-1}\left(\underline{L} + \underline{U}\right)k + \underline{D}^{-1}\underline{b} + \underline{D}^{-1}\left[\underline{b} - \underline{A}\left(\underline{D}^{-1}\left(\underline{L} + \underline{U}\right)k + \underline{D}^{-1}\underline{b}\right)\right] \tag{5}$$

Putting (4) in $\tilde{k}$ of (5) gives;

$$k^{(n+1)} = \underline{D}^{-1}\left(\underline{L}+\underline{U}\right)k + \underline{D}^{-1}\underline{b} + \underline{D}^{-1}\left[\underline{b} - \underline{A}\left(\underline{D}^{-1}\left(\underline{L}+\underline{U}\right)k + \underline{D}^{-1}\underline{b}\right)\right] \tag{6}$$

Remodeling (6) yields;

$$k^{(n+1)} = \left[\underline{D}^{-1}\left(\underline{L}+\underline{U}\right)\right]^{3} k^{(n)} + \left[I + \underline{D}^{-1}\left(\underline{L}+\underline{U}\right) + \left(\underline{D}^{-1}\left(\underline{L}+\underline{U}\right)\right)^{2}\right]\underline{D}^{-1}\underline{b} \tag{7}$$

Using (5) as a basis of refinement of Jacobi method, where $\tilde{k}^{(n+1)}$ is the $(n+1)^{th}$ approximation of RJ. Thus, an improvement on (7) yields;

$$k^{(n+1)} = \left[\underline{D}^{-1}\left(\underline{L}+\underline{U}\right)\right]^{4} k^{(n)} + \left[I + \underline{D}^{-1}\left(\underline{L}+\underline{U}\right) + \left(\underline{D}^{-1}\left(\underline{L}+\underline{U}\right)\right)^{2} + \left(\underline{D}^{-1}\left(\underline{L}+\underline{U}\right)\right)^{3}\right]\underline{D}^{-1}\underline{b} \tag{8}$$

Equation (8) is called Third-Refinement of Jacobi method and the iteration matrix is given as; $\left[\underline{D}^{-1}\left(\underline{L}+\underline{U}\right)\right]^{4}$. This method will converge if the spectral radius is less than 1, i.e. $\rho\left(TRJ\right) < 1$.

2.2 Derivation of TRGS Method

From (2), re-arranging and substituting in (1) gives;

$$\Rightarrow \qquad \left(\underline{D}-\underline{L}\right)k = \underline{U}k + \underline{b} \tag{9}$$

$$k^{(n+1)} = \left(\underline{D}-\underline{L}\right)^{-1}\underline{U}k^{(n)} + \left(\underline{D}-\underline{L}\right)^{-1}\underline{b} \tag{10}$$

Remodeling (2) as $\underline{U} = \underline{D} - \underline{L} - \underline{A}$, and replacing in (9) yields;

$$k^{(n+1)} = \tilde{k}^{(n+1)} + \left(\underline{D}-\underline{L}\right)^{-1}\left(\underline{b} - \underline{A}\tilde{k}^{(n+1)}\right) \tag{11}$$

Putting (10) in $\tilde{k}$ of (11) gives;

$$k^{(n+1)} = \left[\left(\underline{D}-\underline{L}\right)^{-1}\underline{U}\right]^{2} k^{(n)} + \left[I + \left(\underline{D}-\underline{L}\right)^{-1}\underline{U}\right]\left(\underline{D}-\underline{L}\right)^{-1}\underline{b} \tag{12}$$

Remodeling (12) yields;

$$k^{(n+1)} = \left[\left(\underline{D}-\underline{L}\right)^{-1}\underline{U}\right]^{3} k^{(n)} + \left[I + \left(\underline{D}-\underline{L}\right)^{-1}\underline{U} + \left(\left(\underline{D}-\underline{L}\right)^{-1}\underline{U}\right)^{2}\right]\left(\underline{D}-\underline{L}\right)^{-1}\underline{b} \tag{13}$$

Using (11) as a basis of refinement of Gauss-Seidel method, where $\tilde{k}^{(n+1)}$ is the $(n+1)^{th}$ approximation of RGS. Thus, an improvement on (13) yields;

$$k^{(n+1)} = \left[\left(\underline{D}-\underline{L}\right)^{-1}\underline{U}\right]^4 k^{(n)} + \left[I + \left(\underline{D}-\underline{L}\right)^{-1}\underline{U} + \left(\left(\underline{D}-\underline{L}\right)^{-1}\underline{U}\right)^2 + \left(\left(\underline{D}-\underline{L}\right)^{-1}\underline{U}\right)^3\right]\left(\underline{D}-\underline{L}\right)^{-1}\underline{b} \tag{14}$$

Equation (14) is called TRGS and the iteration matrix is represented as $\left[\left(\underline{D}-\underline{L}\right)^{-1}\underline{U}\right]^4$.

2.3 The Algorithm for TRJ

To solve $\underline{A}k = \underline{b}$ using TRJ Method, the following steps are adopted

Step 1: Choose an initial guess $k^{(0)}$ to depict the starting point.

Step 2: Set S= $\left[\underline{D}^{-1}\left(\underline{L}+\underline{U}\right)\right]^4$ and

Set P= $\left[I + \underline{D}^{-1}\left(\underline{L}+\underline{U}\right) + \left(\underline{D}^{-1}\left(\underline{L}+\underline{U}\right)\right)^2 + \left(\underline{D}^{-1}\left(\underline{L}+\underline{U}\right)\right)^3\right]\underline{D}^{-1}\underline{b}$

Step 3: Compute $ki^{(n+1)} = S$
$ki^{(n)} + P$, for i = 0, 1, 2,…,m.
Step 4: Update $n = n+1$ for $n = 0, 1, 2,…,$ m.
Step 5: Terminate at k=exact solution.

2.4 The Algorithm for TRGS Method

To solve $\underline{A}k = \underline{b}$ using the derived TRGS Method, the following steps are adopted

Step 1: Choose an initial guess $k^{(0)}$ to depict the starting point.

Step 2: Set H$=\left[\left(\underline{D}-\underline{L}\right)^{-1}\underline{U}\right]^3$ and

Set V= $\left[I + \left(\underline{D}-\underline{L}\right)^{-1}\underline{U} + \left(\left(\underline{D}-\underline{L}\right)^{-1}\underline{U}\right)^2 + \left(\left(\underline{D}-\underline{L}\right)^{-1}\underline{U}\right)^3\right]\left(\underline{D}-\underline{L}\right)^{-1}\underline{b}$

Step 3: Compute $ki^{(n+1)} = S$
$ki^{(n)} + P$, for i =0, 1, 2, … , m.

Step 4: Update n=n+1 for n=0, 1, 2, … , m.
Step 5: Terminate at k =exact solution.

3. Numerical Experiment and Results

In this research, we validate the proposed methods with numerical examples of;

M Matrix, SDD Matrix, SPD Matrix and 2-Cyclic Matrix.

Problem 1: Consider a 2-cyclic matrix $\underline{A}$, which arises from discretization of the Poisson equation $\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=f$ on the unit square as considered by (Dafchachi, 2008).

Now consider $\underline{A}k=\underline{b}$, where $k=\left(k_1,k_2,k_3,k_4,k_5,k_6\right)^T$ and $\underline{b}=\left(1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\right)^T$.

The above problem can be express in its matrix format as;

$$\begin{pmatrix} 4 & -1 & 0 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 \\ -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 0 & -1 & 4 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \\ k_6 \end{pmatrix}=\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Table 1: Convergence Comparison of the Proposed methods for Problem 1

Method	Spectral Radius	Iteration Number	CPU Time(sec)	Convergence Rate
RJ	0.3642766952	25	4.406	0.438568612
SRJ	0.2198604345	17	3.578	0.657852918
TRJ	**0.1326975107**	**13**	**2.406**	**0.877137224**
RGS	0.1326975107	12	2.203	0.877137224
SRGS	0.04833861068	8	2.015	1.315705836
TRGS	**0.01760862935**	**6**	**1.156**	**1.754274448**

Problem 2: Consider the system of linear equations below whose coefficient matrix is SDD and SPD.

$$\begin{cases} 4k_1 - k_2 - k_3 = 2 \\ -k_1 + 4k_2 - k_4 = 1 \\ -k_1 + 4k_3 - k_4 = 1 \\ -k_2 - k_3 + 4k_4 = 6 \end{cases}$$

The above equation can be expressed in it matrix format as;

$$\begin{pmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 6 \end{pmatrix}$$

Consequently, $\underline{D} = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$, $-\underline{L} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \end{pmatrix}$, $-\underline{U} = \begin{pmatrix} 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

And the initial approximation is $k = (0 \quad 0 \quad 0 \quad 0)^T$. (Vatti and Eneyew, 2011)

Table 2: Convergence Comparison of the Proposed methods for Problem 2

Method	Spectral Radius	Iteration Number	CPU Time(sec)	Convergence Rate
RJ	0.2500000000	18	3.937	0.602059991
SRJ	0.1250000000	12	3.031	0.903089987
TRJ	**0.0625000000**	**9**	**2.156**	**1.204119983**
RGS	0.0625000000	10	2.062	1.204119983
SRGS	0.0156250000	7	1.078	1.806179974
TRGS	**0.0039062500**	**5**	**1.015**	**2.408239965**

The results of the study highlight the significant improvements achieved through the introduction of the Third Refinement of Jacobi (TRJ) and Third Refinement of Gauss-Seidel (TRGS) methods.

Discussion of the Results:

Firstly, the numerical experiments revealed that TRJ and TRGS outperformed both the traditional Jacobi and Gauss-Seidel methods, as well as their existing refinements, in terms of computational efficiency and speed. This suggests that the proposed modifications successfully addressed the limitations of the original methods, enabling faster convergence towards accurate solutions. One particularly noteworthy finding was the drastic reduction in the spectral radius of the proposed methods. This reduction indicates a more stable behavior and faster convergence, which is crucial for practical applications where efficiency and reliability are paramount. By minimizing the spectral radius, TRJ and TRGS demonstrated enhanced convergence rates, making them more suitable for a wide range of computational tasks. Overall, the results of the study demonstrate the effectiveness of TRJ and TRGS in providing faster and more accurate solutions to linear systems of equations. These findings not only contribute to the advancement of iterative methods in numerical analysis but also have practical implications for various fields where efficient solution techniques are required.

4. Conclusion

In this study, we introduced the Third Refinement of Jacobi (TRJ) and Third Refinement of Gauss-Seidel (TRGS) methods as enhanced versions of the traditional Jacobi and Gauss-Seidel techniques for solving linear systems of equations. Through extensive numerical experiments, the results demonstrated that the proposed algorithms outperform both the original Jacobi and Gauss-Seidel methods, as well as their existing refinements, in terms of computational efficiency and speed. One notable improvement observed was the significant reduction in the spectral radius of the proposed methods, leading to an enhanced convergence rate and suitability for a broader range of computational tasks. This outcome underscores the effectiveness of TRJ and TRGS in providing faster and more accurate solutions to linear systems of equations, thus representing a promising advancement in iterative methods for numerical analysis.

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