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AN $M^X/E_k/c$ QUEUING SYSTEM WITH BERNOULLI SCHEDULE SERVER VACATIONS AND RANDOM BREAKDOWNS

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Abstract

In this work, the batch arrival multiple server queuing system with Bernoulli schedule server vacations and random breakdowns was analyzed. In this queuing system, it has been assumed that the customers arrive to the system in batches of variable size, but are served individually by a multiple server in a first come, first served (FCFS) basis. It has been assumed that the service time's distribution is an Erlang-k service time. After any service completion, the server may take a single vacation of random length. On the other hand, it has also been assumed that the system is subject to random breakdowns. Whenever the system breaks down, the customer whose service is interrupted comes back to the head of the queue and the system undergoes a repair process of variable length. Introducing the elapsed service time as a supplementary variable enabled us to obtain a set of time-dependent differential equations. It has been shown how to solve these equations to obtain the queue length at an arbitrary point of time.

Keywords: Breakdowns, Customer, Distribution, Queuing systems, Server, Vacation, Waiting Time.

Introduction

Everyone has experienced the annoyance of having to wait in queues (Adan et al., 2001). Unfortunately, this phenomenon continues to be common in congested, urbanized societies (Adan and Resing, 2001). For example, in the United States, it has been estimated that Americans spend $37{,}000{,}000$ hours per year waiting in queues. If this time could be spent productively instead, it would amount to nearly 20 million person-years of useful work (Adan & Wessels, 1996; Ahn & Jeon, 2002).

The customers do not generally like to wait in queues, and the managers of the establishment at which we wait also do not like us to wait, since it may cost their business, but this is the situation whenever the current demand for a service exceeds the current capacity to provide that service. Decision regarding the amount of capacity to provide must be made frequently in industry and elsewhere. Alfa (2003) however submitted that, these are difficult decisions since it is often impossible to accurately predict when customer will arrive requiring a service and/or how much time will be needed to provide the required service. Adding more servers in the system is costly; on the other hand, fewer servers would cause the waiting queue to become excessively long. Therefore, the goal is to achieve an economical balance between the cost and the waiting line length. According to Gross and Harris (1998), queuing theory itself does not directly find this balance. However, it does contribute vital information required for such decision by predicting various characteristics of the waiting queue such as the average waiting time, the average number of customers in the queue, for instance, Gupta and Sikdar (2006) and Hillier and Boling (1967) described a queuing system as a service center and a population of customers that may enter the service center at various points of time in order to get service. In many cases, the service center can only serve a limited number of customers at a time. If a new customer arrives and there is no free server, the customers enters a waiting line and wait until the service facility becomes available. Figure 1 shows the elements of a simple queuing Model.

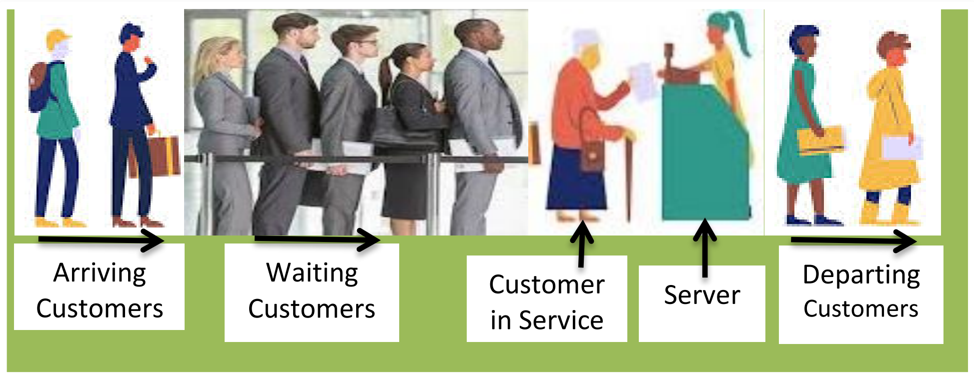

Figure 1: Elementary Queuing System

The term customer is used in general sense and doesn't imply necessarily a human customer. For example, a customer could be a ball bearing waiting to be polished, an airplane waiting in line to take off or land, or a computer command waiting to be performed (Hillier & Lieberman, 2005).

It is not necessary that there actually be a physical waiting line in front of the service provider. In other words, the members of the queue may be scattered throughout an area waiting for the server to come to them, e.g. machines waiting to be repaired (Maraghi *et al.,* 2009).

There are many valuable applications of the theory, most of which have been documented in the literature of probability, operations research, management science, and industrial engineering (Maraghi *et al.,* 2009).

Materials and Methods

Probabilistic Waiting line Models

A probabilistic waiting line system will result; however, if either the arrival and/or the service time is a random variable. The arrival time and the service time can follow any distribution, empirical or analytical. It depends on the case under consideration for study. The service rate is an expected value from the poison distribution. The models are based on the assumptions of an infinite population, given that the size of the population is large relative to the arrival rate.

In this case, individuals leaving the population do not significantly affect the arrival potential of the remaining systems.

Assume that both the arrival rate and the service rate are expected values from independent poison distributions. This assumption holds when the rates are independents of time, queue length, or any other property of waiting line system. The expected number of arrivals per period may be express as $\frac{1}{\lambda}$. The expected number of service completion per period may be express as $\frac{1}{c\mu}$. Where λ and μ are the mean time between arrivals and the mean service time in the system for the assumed distributions, respectively. If the number of arrival per period or the number of services per period have a poison distribution, then the time between arrivals or the service duration will have an exponential distribution. (It is assumed that μ is greater than λ, and that the arrival population is infinite).

The Model Assumptions

The mathematical model of this study is characterized by the following assumptions:

a. Customers arrive at the system in batches of variable sizes in a compound poison process.

Let $\lambda c_i dt(i=1,2,3,4,...)$ be the first order probability that a batch of i customers arrives at the system during a short interval of time $(t, t+dt)$, where

$0 \le c_i \le 1, \sum_{i=1}^{\infty} c_i = 1$, and $\lambda > 0$ is the mean arrival rate of batches.

b. There is multiple server which provide one by one service to arriving customers on a "First Come First Served" basis and the service time follows a general (arbitrary) distribution with distribution function $G(s)$ and density function $g(s)$. Let $\mu(x)$ be the conditional probability density of service completion during the interval $(x, x+dx)$ given that the elapse service time is x so that

$$\mu(x) = \frac{g(x)}{1-G(x)} \tag{1}$$

And

$$g(s) = \mu(x) e^{-\int_0^s \mu(x)dx} \tag{2}$$

This is the gap this study seeks to fill.

c. The system breaks down at random and break downs are assumed to occur according to a poison stream with mean break down rate $\alpha > 0$. Further we assume that once the system breaks down, the customer whose service was interrupted comes to the head of the queue.

d. Once the system breaks down, it enters a repair process immediately. The repair time are exponentially distributed with mean repair rate $\beta > 0$.

e. Initially there are no customers in the system before arrival.

f. As soon as a service is completed, the server may take a vacation with probability p or may stay in the system providing service with probability $1-p$, $0 \le p \le 1$.

g. The duration of vacation follows an exponential distribution with rate $\gamma > 0$ and hence mean vacation time $\frac{1}{\gamma}$

h. Various stochastic processes involved in the system are independent

Definition of Variable

λ_n = Mean arrival rate of customers when there are n customers present

μ_n = Mean service rate of the whole system when there are n customers present

ρ = Server utilization factor ($\rho = \lambda / c\mu$ when the arrival rate is $\lambda_n = \lambda$ for all n, and the service rate of each server is μ

n Number of customers in the queue.

λ The average number of customers entering the queue system.

μ The average rate of service customers per server.

c The number of server available for service.

ρ A measure of traffic congestion for multiple server system which is defined as $\rho \equiv \frac{\lambda}{c\mu}$

$p_n(t)$ Transient state probability of having exactly n customers in the system at time t.

p_n Steady state probability of having exactly n customers in the system.

$N(t)$ Random variable representing the total number of customers in the system at time t.

$N_q(t)$ Random variable representing the total number of customers in the queue at time t.

$N_s(t)$ Random variable representing the total number of customers in service at time t.

T Random variable representing the time a customer spends in the s

T_q Random variable representing the time a customer spends waiting in the queue prior to entering service.

S Random variable representing the service time with mean service time, $E(S) = 1/\mu$.

L The mean number of customers in the system, $L = E[N]$

L_q The mean number of customers in the queue, $L_q = E[N_q]$

W The mean waiting time in the system, $W = E[T]$

W_q The mean waiting time in the queue, $W_q = E[W_q]$

$p_n(x,t)$ Probability that at time t, the servers are providing a service and there are

are n $(n \geq c)$ customers in the queue excluding the one being served and the elapsed service time for this customer is x.

$p_n(t)$ $p_n(t) = \int_0^\infty p_n(x,t)dx$ Denotes the probability that at time t, the server is providing service and there are n $(n \geq c)$ customers in the queue excluding the one being served irrespective of the value of x.

$V_n(x,t)$ Probability that at time t the server is on vacation with elapsed vacation time x and there are n $(n \geq c)$ customers waiting in the queue for service.

$V_n(t)$ $V_n(t) = \int_0^\infty V_n(x,t)dx$ Denotes the probability that at time t there are n customers in the queue and the server is on vacation irrespective of the value of x.

$R_n(t)$ Probability that at time t, the system is inactive due to system breakdown and the system is under repair with elapsed repair time x.

$R_n(t)$ $R_n(t) = \int_0^\infty R_n(x,t)dx$

Denotes the probability that at time t, the system is in-active due to system breakdown and the system is under repair irrespective of the value of x, while there n $(n \geq c)$ in the queue waiting for service.

$Q(t)$ Probability that at time t, there are no customers in the system and the server is idle but available for service in the system.
$p_0(t)$ Probability that there are no customers in the system and the system is inactive.

Equations Governing the System
According to the assumptions mentioned above, the system has the following set of differential-difference equations

$$\frac{\partial}{\partial x}p_n(x,t)+\frac{\partial}{\partial t}p_n(x,t)+(\lambda+c\mu(x)+\alpha)p_n(x,t)=\lambda C_i\sum_{i=c}^{n-c}(n-c)p_{n-c}(x,t), n\geq c \quad (3)$$

$$\frac{\partial}{\partial x}p_0(x,t)+\frac{\partial}{\partial t}p_0(x,t)+(\lambda+c\mu(x)+\alpha)p_0(x,t)=0 \quad (4)$$

p

$$\lambda C_i\sum_{i=c}^{n-s}(n-c)V_{n-c}(t)+p\int_0^\infty p_n(x,t)c\mu(x)dx, n\geq c \quad (5)$$

$$\frac{d}{dt}V_0(t)+(\lambda+\gamma)V_0(t)=p\int_0^\infty p_0(x,t)c\mu(x)dx, n\geq 0 \quad (6)$$

$$\frac{d}{dt}R_n(t)+(\lambda+\beta)R_n(t)=\lambda C_i\sum_{i=c}^{n-c}(n-c)R_{n-c}(t)+\alpha\int_0^\infty p_{n-c}(x,t)dx, n\geq c \quad (7)$$

$$\frac{d}{dt}R_0(t)+(\lambda+\beta)R_0(t)=0 \quad (8)$$

$$\frac{d}{dt}Q(t)+\lambda Q(t)=\gamma V_0(t)+\beta R_0(t)+(1-p)\int_0^\infty p_0(x,t)c\mu(x)dx \quad (9)$$

The above equations are to be solved subject to the following boundary conditions.

$$p_n(0,t)=(1-p)\int_0^\infty p_{n+1}(x,t)c\mu(x)+\gamma V_{n+1}(t)+\beta R_{n+1}(t)+\lambda C_{n+1}Q(t), n\geq 0 \quad (10)$$

We assume that initially there is no customers in the system and the server is idle. So that the initial conditions are:

$$p_n(0)=0, V_n(0)=0, R_0(0)=0, Q(0)=1, n\geq 0 \quad (11)$$

To find the solution, first we define the Probability Generating Functions (Z- Transform)

$$\left.\begin{aligned} p_q(x,z,t) &= \sum_{n=s}^{\infty} Z^n p_n(x,t) \\ V_q(x,z,t) &= \sum_{n=s}^{\infty} Z^n p_n(x,t) \\ R_q(x,z,t) &= \sum_{n=s}^{\infty} Z^n p_n(x,t) \\ C(z) &= \sum_{i=s}^{\infty} Z^i C_i \end{aligned}\right\}$$

(12)

We take the Laplace transform of equations $3-10$ using the initial conditions given in (11)

$$\frac{\partial}{\partial x}\tilde{p}_n(x,s)+(s+\lambda+c\mu(x)+\alpha)\tilde{p}_n(x,s)=\lambda C_i\sum_{i=c}^{n-c}(n-c)\tilde{p}_{n-c}(x,s), n\geq c$$

(13)

$$\frac{\partial}{\partial x}\tilde{p}_0(x,s)+(s+\lambda+c\mu(x)+\alpha)\tilde{p}_0(x,s)=0, n=0$$

(14)

$$(s+\lambda+\gamma)\tilde{V}_n(s)=\lambda C_i\sum_{i=c}^{n-c}(n-c)\tilde{V}_{n-c}(s)+p\int_0^{\infty}\tilde{p}_{n-c}(x,s)c\mu(x)dx, n\geq c$$

(15)

$$(s+\lambda+\gamma)\tilde{V}_0(s)=p\int_0^{\infty}\tilde{p}_0(x,s)c\mu(x)dx, n\geq 0$$

(16)

$$(s+\lambda+\beta)\tilde{R}_n(s)=\lambda C_i\sum_{i=c}^{n-c}(n-c)\tilde{R}_{n-c}(s)+\alpha\int_0^{\infty}\tilde{p}_{n-c}(x,s)dx, n\geq c$$

(17)

$$(s+\lambda+\beta)\tilde{R}_0(s)=0$$

(18)

$$(s+\lambda)\tilde{Q}(s)=1+(1-p)\int_0^{\infty}p_0(x.s)c\mu(x)dx+\gamma\tilde{V}_0(s)+\beta\tilde{R}_0(s)$$

(19)

$$\tilde{p}_n(0,s)=(1-p)\int_0^{\infty}\tilde{p}_{n+1}(x,s)c\mu(x)+\gamma\tilde{V}_{n+1}(s)+\beta\tilde{R}_{n+1}(s)+\lambda C_{n+1}\tilde{Q}(s), n\geq 0$$

(20)

Next, to obtain the z –transform (probability generating functions) of (13) we multiply (13) by z^n and take summation over n from 1 to ∞ we get.

$$\frac{\partial}{\partial x}\sum_{n=c}^{\infty}z^n\tilde{p}_n(x,s)+(s+\lambda+c\mu(x)+\alpha)\sum_{n=c}^{\infty}z^n\tilde{p}_n(x,s)=\lambda C_i\sum_{n=c}^{\infty}\sum_{i=1}^{n-c}z^n\tilde{p}_{n-i}(x,s)$$

Adding the result to (14) yields

$$\frac{\partial}{\partial x}\sum_{n=c}^{\infty} z^{n}\widetilde{p}_{n}(x,s)+(s+\lambda+c\mu(x)+\alpha)\sum_{n=s}^{\infty} z^{n}\widetilde{p}_{n}(x,s)=\lambda C_{i}\sum_{i=1}^{n-c} z^{n}\sum_{n=c}^{\infty} z^{n}\widetilde{p}_{n-c}(x,s)$$

Using the z transform define in equation (12), adjusting limit on the RHS we get.

$$\frac{\partial}{\partial x}\widetilde{p}_{q}(x,z,s)+(s+\lambda-\lambda c+c\mu(x)+\alpha)\widetilde{p}_{q}(x,z,s)=0$$

(21)

Similarly multiply equation (15) by z^{n} and taking the summation over n from 1 to ∞ adding it to (16) and invoking the generating function defined in (12) we get.

$$(s+\lambda-\lambda c(z)+\gamma)\widetilde{V}_{q}(z,s)=p\int_{0}^{\infty}\widetilde{p}_{q}(x,z,s)c\mu(x)dx$$

(22)

Again, multiply equation 17 by z^{n}, take the summation over n from 1 to ∞ adding the result to equation 18 invoking the z- transform defined in equation 12 we get.

$$(s+\lambda-\lambda c(z)+\beta)\widetilde{R}_{q}(z,s)=\alpha z\int_{0}^{\infty}\widetilde{p}_{q}(x,z,s)dx$$

(23)

For the boundary conditions, we multiply equation 15 by z^{n+1} take summation over n from 0 to ∞ and using the generating functions defined in equation 12 and adjusting the limit we obtain

$$z\widetilde{p}_{q}(0,z,s)=(1-p)\int_{0}^{\infty}\widetilde{p}_{q}(x,z,s)c\mu(x)+\gamma\widetilde{p}_{q}(z,s)+\beta\widetilde{R}_{q}(z,s)+\lambda C(z)\widetilde{Q}(s)-\left[(1-p)\int_{0}^{\infty}\widetilde{p}_{0}(x,s)c\mu(x)dx+\gamma\widetilde{V}_{0}(s)+\beta\widetilde{R}_{0}(s)\right]$$

(24)

Equation (19) can be rewritten in the form:

$$1-(s+\lambda)\widetilde{Q}(s)=-\left[(1-p)\int_{0}^{\infty}\widetilde{p}_{0}(x,s)c\mu(x)dx+\gamma\widetilde{V}_{0}(s)+\beta\widetilde{R}_{0}(s)\right]$$

(25)

Utilizing (25) in (24), we get

$$z\widetilde{p}_{q}(0,z,s)=(1-p)\int_{0}^{\infty}\widetilde{p}_{q}(x,z,s)c\mu(x)+\gamma\widetilde{p}_{q}(z,s)+\beta\widetilde{R}_{q}(z,s)+\lambda C(z)\widetilde{Q}(s)+-1(s+\lambda)\widetilde{Q}(s)$$

Simplifying we have

$$z\widetilde{p}_{q}(0,z,s)=1+(1-p)\int_{0}^{\infty}\widetilde{p}_{q}(x,z,s)c\mu(x)+\gamma\widetilde{p}_{q}(z,s)+\beta\widetilde{R}_{q}(z,s)+[\lambda(C(z)-1)-s]\widetilde{Q}(s)$$

(26)

Now equation (26) is a first order linear differential equation with constant coefficient. Using the method of integrating factor, we integrate between 0 and x and the boundary condition we have.

$$\tilde{p}_q(x,z,s) = \tilde{p}_q(0,z,s)\exp\left[-(s+\lambda C(z)+\alpha)x - \left\{\int_0^z c\mu(t)dt + c\left(\int_0^z \mu(t)dt\right)\right\} - c\int_0^z \mu(t)dt\right]$$

(27)

Where $\tilde{p}_q(0,z,s)$ is given by

(26)

Again, integrating (27) by part with respect to x between 0 and ∞ using (20), we obtained

$$\tilde{p}_q(z,s) = \tilde{p}_q(0,z,s)\left[\exp\frac{\left[-(s+\lambda C(z)+\alpha)x - \left\{\int_0^z c\mu(t)dt + c\left(\int_0^z \mu(t)dt\right)\right\} - c\int_0^z \mu(t)dt - 1\right]}{-s-\lambda+\lambda C(z)-\alpha}\right]$$

Rearranging we have

$$\tilde{p}_q(z,s) = \tilde{p}_q(0,z,s)\left[\exp\frac{\left[1-\tilde{G}(s+\lambda-\lambda C(z)+\alpha)\right]}{s+\lambda-\lambda C(z)+\alpha}\right] \tag{28}$$

Where $\tilde{G}(s+\lambda-\lambda C(z)+\alpha) = \int_0^\infty e^{-(s+\lambda-\lambda C(z)+\alpha)x - \left\{\int_0^z c\mu(t)dt + c\left(\int_0^z \mu(t)dt\right)\right\}}.dG(x)$

is the Laplace Stieltjes transforms of the service time. From the result obtained in (28), Equation (23) given by

$(s+\lambda-\lambda c(z)+\beta)\tilde{R}_q(z,s) = \alpha z\int_0^\infty \tilde{p}_q(x,z,s)dx$ becomes

$$(s+\lambda-\lambda c(z)+\beta)\tilde{R}_q(z,s) = \alpha z\,\tilde{p}_q(0,z,s)\left[\frac{1-\tilde{G}(s+\lambda-\lambda C(z)+\alpha)}{s+\lambda-\lambda C(z)+\alpha}\right] \tag{29}$$

Multiply (27) through by $c\mu(x)$ and integrate over x we get.

$$\int_0^\infty \tilde{p}_q(x,z,s)c\mu(x)dx = \tilde{p}_q(0,z,s)\tilde{G}((s+\lambda-\lambda C(z))+\alpha) \tag{30}$$

Utilizing equation (30), Equation (20) can be written as

$$(s+\lambda-\lambda c(z)+\gamma)\tilde{V}_q(z,s) = p\,\tilde{p}_q(0,z,s)\tilde{G}((s+\lambda-\lambda C(z))+\alpha) \tag{31}$$

Similarly we utilize (30) in (26) and simplifying we get

$$\left[z-(1-p)\overline{G}\{s+\lambda-\lambda C(z)+\alpha\}\right]\tilde{P}_q(0,z,s) = 1+\gamma\tilde{V}_q(z,s)+\beta\tilde{R}(z,s)+[\lambda\{C(z)-1\}-s]\tilde{Q}(s) \tag{32}$$

We now substitute the expression for $\tilde{V}_q(z,s)$ and $\tilde{R}_q(z,s)$ from equation (29) and (31) in equation (32) and solve for $\tilde{p}_q(0,z,s)$ we get

$$\tilde{p}_q(0,z,s)= \frac{f_1(z,s)\, f_2(z,s)\, f_3(z,s)\{1+[\lambda(C(z)-1)-s]\tilde{Q}(s)\}}{f_1(z,s)\, f_2(z,s)\, f_3(z,s)\, f_4(z,s)-\alpha\beta z\, f_3(z,s)-\tilde{G}[f_1(z,s)]\{\gamma p\, f_1(z,s)\, f_2(z,s)-\alpha\beta z\, f_3(z,s)\}} \tag{33}$$

Where

$$f_1(z,s)=s+\lambda-\lambda C(z)+\alpha$$
$$f_2(z,s)=s+\lambda-\lambda C(z)+\beta$$
$$f_3(z,s)=s+\lambda-\lambda C(z)+\gamma$$
$$f_4(z,s)=z-(1-p)\tilde{G}[s+\lambda-\lambda C(z)+\alpha]$$

Substituting (33) in (28), (29) and (31) we obtain:

$$\tilde{p}_q(0,z,s)= \frac{f_2(z,s)\, f_3(z,s)\{1-\tilde{G}[f_1(z,s)]\}\{1+[\lambda(C(z)-1)-s]\tilde{Q}(s)\}}{f_1(z,s)\, f_2(z,s)\, f_3(z,s)\, f_4(z,s)-\alpha\beta z\, f_3(z,s)-\tilde{G}[f_1(z,s)]\{\gamma p\, f_1(z,s)\, f_2(z,s)-\alpha\beta z\, f_3(z,s)\}} \tag{34}$$

$$\tilde{V}_q(0,z,s)= \frac{p f_1(z,s)\, f_2(z,s)\tilde{G}[f_1(z,s)]\{1+[\lambda(C(z)-1)-s]\tilde{Q}(s)\}}{f_1(z,s)\, f_2(z,s)\, f_3(z,s)\, f_4(z,s)-\alpha\beta z\, f_3(z,s)-\tilde{G}[f_1(z,s)]\{\gamma p\, f_1(z,s)\, f_2(z,s)-\alpha\beta z\, f_3(z,s)\}} \tag{35}$$

$$\tilde{R}_q(0,z,s)= \frac{\alpha z\, f_3(z,s)\{1-\tilde{G}[f_1(z,s)]\}\{1+[\lambda(C(z)-1)-s]\tilde{Q}(s)\}}{f_1(z,s)\, f_2(z,s)\, f_3(z,s)\, f_4(z,s)-\alpha\beta z\, f_3(z,s)-\tilde{G}[f_1(z,s)]\{\gamma p\, f_1(z,s)\, f_2(z,s)-\alpha\beta z\, f_3(z,s)\}} \tag{36}$$

Let $W_q(z,s)$ denote the probability generating function of the queue size irrespective of the rate of the system. Then adding (34), (35) and (36) we obtain:

$$\tilde{W}_q(z,s)=\tilde{p}_q(z,s)+\tilde{V}_q(z,s)+\tilde{R}_q(z,s)$$

$$\tilde{W}_q(z,s)= \frac{\{f_2(z,s) f_3(z,s)[1-\tilde{G}[f_1(z,s)]]+p f_1(z,s)\, f_2(z,s)\tilde{G}[f_1(z,s)]\}\{1+[\lambda(C(z)-1)-s]\tilde{Q}(s)\}}{f_1(z,s)\, f_2(z,s)\, f_3(z,s)\, f_4(z,s)-\alpha\beta z\, f_3(z,s)-\tilde{G}[f_1(z,s)]\{\gamma p\, f_1(z,s)\, f_2(z,s)-\alpha\beta z\, f_3(z,s)\}}$$
$$+\frac{\alpha z\, f_3(z,s)\{1-\tilde{G}[f_1(z,s)]\}\{1+[\lambda(C(z)-1)-s]\tilde{Q}(s)\}}{f_1(z,s)\, f_2(z,s)\, f_3(z,s)\, f_4(z,s)-\alpha\beta z\, f_3(z,s)-\tilde{G}[f_1(z,s)]\{\gamma p\, f_1(z,s)\, f_2(z,s)-\alpha\beta z\, f_3(z,s)\}} \tag{37}$$

It can be shown that the denominator of the right hand side of (37) has one zero inside the unit circle $|z|=1$

Which is sufficient to determine the only unknown $\tilde{Q}(s)$ appearing in the numerator. Therefore,

$$\tilde{p}_q(z,s), \tilde{V}_q(z,s), \tilde{R}_q(z,s)$$

The Steady State Results

To define the steady state probability and the corresponding probability generating function we drop the argument t and for that matter, the argument s, wherever it appears in the time-dependent analysis up to this point. Then the corresponding steady state results can be obtained by the well-known Tauberian property

$$\lim_{s\to 0} s\tilde{f}(s)=\lim_{t\to\infty} f(t).$$

Thus multiplying both sides of (37) by s, taking limit as s approaches zero, applying the Tauberian property, and after some simplification we get

$$p_q(z)=\lim_{s\to 0} s\,\tilde{p}_q(z,s)=$$

$$\frac{\lim\limits_{s\to 0} s\left\{f_2(z,s)\, f_3(z,s)\left[1-\tilde{G}\left[f_1(z,s)\right]\right]\left\{1+\left[\lambda(C(z)-1)-s\right]\tilde{Q}(s)\right\}\right\}}{\lim\limits_{s\to 0}\left\{f_1(z,s)\, f_2(z,s)\, f_3(z,s)\, f_4(z,s)-\alpha\beta z\, f_3(z,s)-\tilde{G}\left[f_1(z,s)\right]\left\{\gamma p\, f_1(z,s)\, f_2(z,s)-\alpha\beta z\, f_3(z,s)\right\}\right\}} \tag{38}$$

$$=\frac{\lim\limits_{s\to 0}\left\{f_2(z,s)\, f_3(z,s)\left[1-\tilde{G}\left[f_1(z,s)\right]\right]\left\{s+s\left[\lambda(C(z)-1)-s\right]\tilde{Q}(s)\right\}\right\}}{\lim\limits_{s\to 0}\left\{f_1(z,s)\, f_2(z,s)\, f_3(z,s)\, f_4(z,s)-\alpha\beta z\, f_3(z,s)-\tilde{G}\left[f_1(z,s)\right]\left\{\gamma p\, f_1(z,s)\, f_2(z,s)-\alpha\beta z\, f_3(z,s)\right\}\right\}}$$

$$p_q(z)=\frac{\left\{f_2(z)\, f_3(z)\left[1-\tilde{G}\left[f_1(z)\right]\right]\left\{\left[\lambda(C(z)-1)\right]Q\right\}\right\}}{\left\{f_1(z)\, f_2(z)\, f_3(z)\, f_4(z)-\alpha\beta z\, f_3(z)-\tilde{G}\left[f_1(z)\right]\left\{\gamma p\, f_1(z)\, f_2(z)-\alpha\beta z\, f_3(z)\right\}\right\}}$$

Where

$$f_1(z)=\lambda-\lambda C(z)+\alpha$$

$$f_2(z)=\lambda-\lambda C(z)+\beta$$

$$f_3(z)=\lambda-\lambda C(z)+\gamma$$

$$f_4(z)=z-(1-p)\tilde{G}[\lambda-\lambda C(z)+\alpha]$$

Performing similar operation to (35) and (36) leads to

$$V_q(z)=\lim_{s\to 0} s\,\tilde{V}_q(z,s)=$$

$$\frac{\lim\limits_{s\to 0} s\left\{p f_1(z,s)\, f_2(z,s)\,\tilde{G}\left[f_1(z,s)\right]\left\{1+\left[\lambda(C(z)-1)-s\right]\tilde{Q}(s)\right\}\right\}}{\lim\limits_{s\to 0}\left\{f_1(z,s)\, f_2(z,s)\, f_3(z,s)\, f_4(z,s)-\alpha\beta z\, f_3(z,s)-\tilde{G}\left[f_1(z,s)\right]\left\{\gamma p\, f_1(z,s)\, f_2(z,s)-\alpha\beta z\, f_3(z,s)\right\}\right\}} \tag{39}$$

$$V_q(z)=\frac{\left\{p f_1(z)\, f_2(z)\,\tilde{G}\left[f_1(z)\right]\left\{\left[\lambda(C(z)-1)\right]Q\right\}\right\}}{\left\{f_1(z)\, f_2(z)\, f_3(z)\, f_4(z)-\alpha\beta z\, f_3(z)-\tilde{G}\left[f_1(z)\right]\left\{\gamma p\, f_1(z)\, f_2(z)-\alpha\beta z\, f_3(z)\right\}\right\}}$$

$$R_q(z)=\lim_{s\to 0} s\,\tilde{R}_q(z,s)=$$

$$\frac{\lim\limits_{s\to 0} s\left\{\alpha z\, f_3(z,s)\left(1-\tilde{G}\left[f_1(z,s)\right]\right)\left\{1+\left[\lambda(C(z)-1)-s\right]\tilde{Q}(s)\right\}\right\}}{\lim\limits_{s\to 0}\left\{f_1(z,s)\, f_2(z,s)\, f_3(z,s)\, f_4(z,s)-\alpha\beta z\, f_3(z,s)-\tilde{G}\left[f_1(z,s)\right]\left\{\gamma p\, f_1(z,s)\, f_2(z,s)-\alpha\beta z\, f_3(z,s)\right\}\right\}} \tag{40}$$

$$R_q(z)=\frac{\left\{\alpha z\, f_3(z)\left(1-\tilde{G}\left[f_1(z)\right]\right)\left\{\left[\lambda(C(z)-1)\right]Q\right\}\right\}}{\left\{f_1(z)\, f_2(z)\, f_3(z)\, f_4(z)-\alpha\beta z\, f_3(z)-\tilde{G}\left[f_1(z)\right]\left\{\gamma p\, f_1(z)\, f_2(z)-\alpha\beta z\, f_3(z)\right\}\right\}}$$

Now for the steady state by adding (38), (39), and (40) we get

$$W_q(z)=p_q(z)+V_q(z)+R_q(z)$$

$$W_q(z)=$$

$$\frac{\left\{f_3(z)\, f_2(z)+\alpha z\right\}+\tilde{G}\left[f_1(z)\right]\left\{p\, f_1(z)\, f_2(z)-f_2(z)\, f_3(z)-\alpha z\, f_3(z)\right\}\left\{\lambda\left[C(z)-1\right]\right\}\lambda Q}{f_1(z)\, f_2(z)\, f_3(z)\, f_4(z)-\alpha\beta z\, f_3(z)-\tilde{G}\left[f_1(z)\right]\left\{\gamma p\, f_1(z)\, f_2(z)-\alpha\beta z\, f_3(z)\right\}} \tag{41}$$

According to the normalization condition we have $Q+W_q(1)=1$

We see that for $z=1, W_q(z)$ is indeterminate of $\frac{0}{0}$ form. Therefore, we apply L'Hopital's rule on equation (41), where we differentiate both the numerator and denominator with respect to z accordingly, we get

$$W_q(1)=\frac{\lambda QE(I)\{\alpha\gamma+\beta\gamma+\tilde{G}[\alpha]\{p\alpha\beta-\alpha\gamma-\beta\gamma\}\}}{\tilde{G}(\alpha)\{\lambda\beta\gamma E(I)+\lambda\alpha\gamma E(I)-\lambda\alpha\beta pE(I)+\alpha\beta\gamma\}-\lambda\gamma E(I)\{\alpha+\beta\}}$$
(42)

Where, $C(1)=1, C'(1)=E(I)$ is the mean batch size of the arriving customers adding Q to equation (42), equating it to 1 and solving for Q we get

$$Q=\frac{\tilde{G}(\alpha)\{\alpha\beta\gamma+\lambda\beta\gamma E(I)+\lambda\alpha\gamma E(I)-\lambda\alpha\beta pE(I)\}-\lambda\gamma E(I)\{\alpha+\beta\}}{\alpha\beta\gamma\tilde{G}[\alpha]} \tag{43}$$

After simplification, equation (43) can be written as

$$Q=1-\lambda E(I)\left\{\frac{1}{\beta\tilde{G}[\alpha]}+\frac{1}{\alpha\tilde{G}[\alpha]}-\frac{1}{\alpha}-\frac{1}{\beta}+\frac{p}{\gamma}\right\} \tag{44}$$

Where

$$\rho=\lambda E(I)\left\{\frac{1}{\beta\tilde{G}[\alpha]}+\frac{1}{\alpha\tilde{G}[\alpha]}-\frac{1}{\alpha}-\frac{1}{\beta}+\frac{p}{\gamma}\right\}\prec 1$$

Emerges out of the stability condition under which the steady state exist. Hence

$$\rho=\lambda E(I)\left\{\frac{1}{\beta\tilde{G}[\alpha]}+\frac{1}{\alpha\tilde{G}[\alpha]}-\frac{1}{\alpha}-\frac{1}{\beta}+\frac{p}{\gamma}\right\} \tag{45}$$

$$g(x)=\frac{(c\mu)^k x^{k-c} e^{(c\mu)x}}{(k-c)!}, c\mu\succ 0, k=1,2,3,...$$

Equation (44) gives the probability that the server is idle, substituting for Q from (44) in (41), we have completely and explicitly determine $W_q(z)$, the probability generating function of the queue size at a random epoch.

3.10 The Mean Queue Size L_q and the Mean Waiting Time in the Queue W_q

To find the mean number of customers in the queuing system, recall that we write $W_q(z)$ obtained in (41) as $\frac{N(z)}{D(z)}$ where $N(z)$ and $D(z)$ are the numerator and denominator of the right hand side of (44) and (45) respectively. Then we use

$$L_q=\frac{D'(1)N''(1)-N'(1)D''(1)}{2\{D'(1)\}^2} \tag{46}$$

Note that primes and double primes in (46) denote first and second derivatives at $z=1$, respectively. Then carrying out the first and second derivative for the expression for $W_q(z)$ obtained in equation (41) and letting $z=1$ we get the following:

$$N'(1)=\lambda QE(I)\{\alpha\gamma+\beta\gamma+\tilde{G}[\alpha](p\alpha\beta-\alpha\gamma-\beta\gamma)\} \tag{47}$$

$$N''(1)=2Q(\lambda E(I))^2\left\{\left(-\gamma-\beta-\alpha+\frac{\alpha\gamma}{\lambda E(I)}\right)\right\}+\tilde{G}(\alpha)\left\{\gamma+\beta+\alpha-\frac{\alpha\gamma}{\lambda E(I)}-p\beta-p\alpha\right\}+ \tag{48}$$

$$\tilde{G}'(\alpha)[\alpha\gamma+\beta\gamma-p\alpha\beta]+\lambda QE[I(I-1)]\{\alpha\gamma+\beta\gamma+\tilde{G}(\alpha)[p\alpha\beta-\alpha\gamma-\beta\gamma]\}$$

$$D'(1)=-\lambda\gamma E(I)[\alpha+\beta]+\tilde{G}(\alpha)[\lambda\beta\gamma E(I)+\lambda\alpha\gamma E(I)-\lambda\alpha\beta pE(I)+\alpha\beta\gamma] \tag{49}$$

$$D''(1)=2\{E(I)\}^2\left\{\left(\alpha+\beta+\gamma-\frac{\gamma\alpha}{\lambda E(I)}-\frac{\gamma\beta}{\lambda E(I)}\right)\right\}+\tilde{G}(\alpha)\left[p\alpha+p\beta-\alpha-\beta-\gamma-\frac{\alpha\beta}{\lambda E(I)}\right]+$$
$$\tilde{G}'(\alpha)\left[\alpha\beta p-\alpha\gamma-\beta\gamma-\frac{\alpha\beta\gamma}{\lambda E(I)}\right]-\lambda E[I(I-1)]\{\alpha\gamma+\beta\gamma+\tilde{G}(\alpha)[p\alpha\beta-\alpha\gamma-\beta\gamma]\} \tag{50}$$

Where $E[I(I-1)]$, is the factorial moment of the batch is size of the arriving customers, and Q has been found in (44). Then if we substitute for $N'(1), N''(1), D'(1), D''(1)$ from (47)-(50) in equation (46) we obtain L_q in a close form. Further, the mean waiting time of a customer can be found using Little's laws discussed in chapter one. Specifically the equation $L_q=\lambda W_q$ can be used to find the mean waiting time, knowing the mean queue size.

K –Erlang Service Time

When customers arrive at the system one by one, then $c_1=1$ and $c_i=0$ for $i\neq 1$ consequently, $C(z)=z, E(I)=1, E\big(I(I-1)\big)=0$

In the case of K –Erlang service time, the service time has a k-Erlang distribution, and hence

$$\tilde{G}[\lambda-\lambda C(z)+\alpha]=\frac{(c\mu)^k}{\big(\lambda-\lambda C(z)+\alpha+c\mu\big)^k},\tilde{G}[\alpha]=\frac{(c\mu)^k}{(\alpha+c\mu)^k}$$

Substituting for $\tilde{G}[\lambda-\lambda C(z)+\alpha]$ in equation 41we have

$$W_q(z)=\frac{\big(\lambda-\lambda c(z)+\alpha+c\mu\big)^k f_3(z)\big(f_2(z)+\alpha z\big)\lambda(C(z)-1)Q}{\big(\lambda-\lambda c(z)+\alpha+c\mu\big)^k f_3(z)\begin{Bmatrix}\{f_1(z)f_2(z)f_3(z)f_4(z)-\alpha\beta z\, f_3(z)\}\\-(c\mu)^k\{\gamma p\, f_1(z)f_2(z)-\alpha\beta z\, f_3(z)\}\end{Bmatrix}}$$
$$+\frac{\{(c\mu)^k\big(p\, f_1(z)f_2(z)-f_2(z)f_3(z)-\alpha z\, f_3(z)\big)\}\lambda(C(z)-1)Q}{\big(\lambda-\lambda c(z)+\alpha+c\mu\big)^k f_3(z)\begin{Bmatrix}\{f_1(z)f_2(z)f_3(z)f_4(z)-\alpha\beta z\, f_3(z)\}\\-(c\mu)^k\{\gamma p\, f_1(z)f_2(z)-\alpha\beta z\, f_3(z)\}\end{Bmatrix}}$$

(51)

Where

$$f_1(z)=\lambda-\lambda C(z)+\alpha$$
$$f_2(z)=\lambda-\lambda C(z)+\beta$$
$$f_3(z)=\lambda-\lambda C(z)+\gamma$$
$$f_4(z)=z-\frac{(1-p)(c\mu)^k}{\big(\lambda-\lambda c(z)+\alpha+c\mu\big)^k}$$

And Q and p are given by

$$Q=1-\lambda E(I)\left(\frac{(\alpha+c\mu)^k}{\beta(c\mu)^k}+\frac{(\alpha+c\mu)^k}{\alpha(c\mu)^k}-\frac{1}{\alpha}-\frac{1}{\beta}+\frac{p}{\gamma}\right) \tag{52}$$

$$\rho = \lambda E(I)\left(\left(\frac{(\alpha + c\mu)^k}{\beta(c\mu)^k} + \frac{(\alpha + c\mu)^k}{\alpha(c\mu)^k} - \frac{1}{\alpha} - \frac{1}{\beta} + \frac{p}{\gamma}\right)\right) \tag{53}$$

Further L_q can be found from (3.89) and then W_q is obtained using little's formula. Thus substituting for $\tilde{G}[\alpha] = \frac{(c\mu)^k}{(\alpha + (c\mu))^k}$ and $\tilde{G}[\alpha] = \frac{-k(c\mu)^k}{(\alpha + (c\mu))^{k+1}}$ in equation (47) – (50), We get

$$N'(1) = \lambda QE(I)\left\{\alpha\gamma + \beta\gamma + \frac{(c\mu)^k}{(\alpha+(c\mu))^k}(p\alpha\beta - \alpha\gamma - \beta\gamma)\right\} \tag{54}$$

$$\begin{aligned} N''(1) = {} & 2Q(\lambda E(I))^2\left\{\left(-\gamma - \beta - \alpha + \frac{\alpha\gamma}{\lambda E(I)}\right)\right\} \\ & + \frac{(c\mu)^k}{(\alpha+(c\mu))^k}\left(\gamma + \beta + \alpha - \frac{\alpha\gamma}{\lambda E(I)} - p\beta - p\alpha\right) - \frac{k(c\mu)^k}{(\alpha+(c\mu))^{k+1}}(\alpha\gamma + \beta\gamma - p\alpha\beta) \\ & + \lambda QE((I)(I-1))\left\{\alpha\gamma + \beta\gamma + \frac{(c\mu)^k}{(\alpha+(c\mu))^k}(p\alpha\beta - \alpha\gamma - \beta\gamma)\right\} \end{aligned} \tag{55}$$

$$D'(1) = -\lambda\gamma E(I)(\alpha + \beta) + \frac{(c\mu)^k}{(\alpha+(c\mu))^k}(\lambda\beta\gamma E(I) + \lambda\alpha\gamma E(I) - \lambda\alpha\beta p E(I) + \alpha\beta\gamma) \tag{56}$$

$$\begin{aligned} D''(1) = {} & 2Q(\lambda E(1))^2\left\{\left(\alpha + \beta + \gamma - \frac{\gamma\alpha}{\lambda E(I)} - \frac{\lambda\beta}{\lambda E(1)}\right)\right\} + \frac{(c\mu)^k}{(\alpha + (c\mu))^k}\left(p\alpha + p\beta - \alpha - \beta - \gamma - \frac{\alpha\beta}{\lambda E(I)}\right) \\ & \frac{k(c\mu)}{(\alpha + (c\mu))^{k+1}}\left(\alpha\beta p - \alpha\gamma - \beta\gamma - \frac{\alpha\gamma}{\lambda E(I)}\right) - \lambda QE((I(I-1)\left\{\alpha\gamma + \beta\gamma + \frac{(c\mu)^k}{(\alpha + (c\mu)^k)}\left(p\alpha + p\beta - \alpha - \beta - \gamma - \frac{\alpha\beta}{\beta\lambda E(I)}\right)\right\} \end{aligned} \tag{57}$$

Result and Discussion of Numerical Illustration

The systems of differential-difference equations describing the state of the servers was solved using the Laplace and the (LST) with the complementary variable technique. For the purpose of a numerical illustration, we obtain some tables for the first two particular cases with primary data collected from Guarantee Trust Bank, Wuse II, F.C.T Abuja in early July, 2023. The following values were arrived at:

$\lambda = 10$ Customers/ quarter hr
$\mu = 5$
$E(I) = 1,\ E(I(I-1)) = 0,$
$c = 4$
$C(1) = 1, C_i \neq 1 = 0$

All the tables give the computed values of various states of the servers, the proportion of idle time (Q), the traffic intensity (ρ), the mean queue length (L_q), the mean waiting time in the queue (w_q), the probability that the server is working mindless the number of customers present in the queue $P_q(1)$, the probability that the server is on vacation irrespective of the number of customers in the queue ($V_q(1)$, the probability that the system is in repair time due to breakdown irrespective of the number of customers in the queue $R_q(1)$, and the probability that the server is not idle $w_q(1)$.

Table 1: Queuing characteristics of $k - Erlang$ Service Time Distribution $k = 5, \beta = 3,\ \gamma = 4,\ \lambda = 10,\ \mu = 5$ this table shows the impact of vacation rate and vacation probabilities on various queue characteristics.

α	P	Q	ρ	L_q	W_q	$P_q(1)$	$V_q(1)$	$R_q(1)$	$W_q(1)$
1	0.500	0.7954	0.2046	1.623	0.9990	0.8172	0.0681	0.0391	0.6245
1	0.750	0.7907	0.2093	1.8096	1.3040	0.4955	0.0761	0.0390	0.6047
1	0.600	0.7461	0.2539	2.8940	1.9570	0.3433	0.1523	0.0392	0.5347
1	0.7500	0.7091	0.2909	2.9890	2.0560	0.3555	0.1492	0.0391	0.5439
2	0.250	0.6330	0.3670	3.5850	2.5670	0.3034	0.1488	0.0390	0.4933
2	0.750	0.5430	0.4570	3.9050	2.7660	0.306	0.1479	0.0391	0.4931
2	0.450	0.4184	0.5816	5.6250	3.1590	0.2675	0.1480	0.0392	0.4547
3	0.650	0.3980	0.6020	5.725	3.6140	0.2322	0.1469	0.0391	0.4183
3	0.550	0.3091	0.6909	6.4980	3.9480	0.1293	0.1420	0.0391	0.3104
3	0.580	0.2881	0.7119	6.9980	4.3760	0.1169	0.1481	0.0392	0.3041
3	0.750	0.2081	0.7919	9.5350	5.4100	0.0293	0.1499	0.0391	0.2094

Table **1** shows the impact of server vacation and breakdown on the proportion of idle time, mean waiting time, mean length of the queue and the utilization factor. Every customer is interested in the duration of time he/she will have to wait before joining any queue. After jockeying through the queues, if the customers waiting time increases, he/she reneges from the queue. It is observed that, the impact of mean breakdown rate fluctuating through from 2 to 7 and the mean vacation of the servers on the customers waiting time, length of the queue, traffic intensity, proportion of idle time in the system and the probability that the servers are not idle when the service time distribution is Erlang-k.

When the breakdown rate increases from 2 to 4 with a corresponding increase in the scheduled vacation with probability ranging from 0.5 to 0.75, there is an increase in the customers waiting time in the queue more so now that the number of server is just 4, the length of the queue (a factor that every arriving customer will want to avoid). It can also be seen that the remaining servers increase their intensity (every service provider is interested in this parameter). Again, the probability that the servers are not idle decrease with increase in the rate of break down. It can also be observed that as the traffic intensity increases with an increase in the mean breakdown (the more servers are being utilized the more their chances of breaking down) rate and server vacation, there is an increase in the idle time of the servers, a factor that every service provider will strive to avoid. In general, as the mean breakdown rate and the server vacation increase, it is seen a corresponding increase in the length of the queue, waiting time in the queue and the traffic intensity. In contrast it is seen a decrease in the idle time of the server. This is so because the remaining available severs are being over worked.

Table 2: Queuing characteristics of $k - Erlang$ Service Time Distribution $k = 4,\ \alpha = 3,\ P = 0.7,\ \lambda = 10,\ \mu = 5$ This table shows the impact of break down rate and repair rate on various queue characteristics

β	γ	Q	ρ	L_q	W_q	$P_q(1)$	$V_q(1)$	$R_q(1)$	$W_q(1)$
1	1	0.4781	0.5219	8.5410	5.4551	0.3581	0.0163	0.275	0.7959
1	2	0.4923	0.5077	7.8410	4.112	0.3580	0.0135	0.305	0.7989
2	2	0.5453	0.4547	6.9890	3.7405	0.3581	0.143	0.199	0.6998
2	1	0.6981	0.3019	5.9740	2.9921	0.3581	0.1358	0.198	0.6918
2	2	0.7341	0.2659	5.7790	2.8825	0.3581	0.163	0.172	0.6933
2	2	0.7294	0.2706	4.8395	2.6999	0.3581	0.143	0.186	0.6871
2	1	0.5530	0.4470	4.6690	2.4958	0.3580	0.136	0.182	0.6758
3	1	0.7983	0.2017	3.9092	1.9895	0.3581	0.135	0.149	0.6698
3	1	0.7889	0.2111	3.8241	1.8870	0.3580	0.163	0.149	0.6694
3	2	0.8500	0.1500	2.8755	1.6092	0.3581	0.135	0.0392	0.6652

Table 2 Shows the effect of mean repair rate and the mean vacation rate on both customer's parameter of interest and the service provider parameter of interest. First, it is observed from the table that as the number of servers that are being repaired increase in number, there is a decrease in the mean waiting time in the queue, the length of the queue. Every arriving customer do not like to wait longer than necessary and so he/she will be interested in joining a queuing system such as in table 2. On the contrary the service provider is not very happy as we observe from the table that the traffic intensity is decreasing. Not just this, it is observed an increase in the proportion of idle time of the servers and the server's active moment decreases correspondingly. Customers only can wait on this queue if only there is no alternative. It is seen from column 7 of table 2 that as the rate at which the server get fixed for service increases, the probability that the servers are working irrespective of the number of customer in the queue almost maintain a constant value. It can also be seen that the probability that the system is under repair decrease as many more servers are being fixed.

Conclusion

In general, as the mean breakdown rate and the server vacation period increase, we see a corresponding increase in the length of the queue, waiting time in the queue and the utilization factor. In contrast we see a decrease in the idle time of the server. This is so because the remaining active sever is being over worked.

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