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CONTINUOUS FORM OF MULTISTEP METHODS FOR SOLVING FIRST ORDER ORDINARY DIFFERENTIAL EQUATION

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ABSTRACT

The study aims to develop the theory of numerical methods used for the numerical solution of first order ordinary differential equations (ODEs). The linear multistep backward differentiation formulae (BDF) was reformulated for applications in the continuous form. The suggested approach eliminates requirement for a starting value and its speed proved to be up when computations with the block discrete schemes were used. The test problem was solved with the proposed numerical method and obtained numerical and analytical solutions were compared.

Key words: Block methods, Self-starting integration scheme, First order ordinary differential equations.

1. INTRODUCTION

Many scientific and engineering problems are described using apparatus of ordinary differential equations (ODEs), where the analytic solution is unknown. Much research has been done by the scientific community on developing numerical methods which can provide an approximate solution of the original ODE. In recent years many review articles and books have appeared on numerical methods for integrating ODEs (Feldman *et al.*, 2001; Butcher, 2008), in particular in stiff cases e.g. (Ibáñez *et al.*, 2009). Stiff problems are very common problems in many fields of the applied sciences: control theory, biology, chemical engineering processes, electrical networks, fluid dynamics, plastic deformation etc.

Most of numerical methods for solving initial value problems (IVPs) for ODEs will become unbearably slow when the ODEs are stiff. The most popular multistep methods families for stiff ODEs are formed by the backward differentiation formulae (BDF or Gear methods) methods, Rosenbrock methods, implicit or diagonally implicit Runge-Kutta methods (Jiaxiang *et al.*, 1995; Ibáñez *et al.*, 2009).

In this paper we are suggested a construction of block multistep BDF method, it is self-starting and can be applied for the numerical solution of IVPs (Cauchy problem) for first-order ODEs. Development of linear multistep methods (LMMs) for solving ODEs can be generated using different methods. The collocation technique for a construction a set of implicit BDF formulas was used.

Block methods for solving ODEs have initially been proposed by Milne (Milne, 1953). The Milne's idea of proceeding in blocks was developed by Rosser (Rosser, 1967) for Runge-Kutta method. Also block methods are discussed and developed by many researchers (Feldman *et al.*, 2001; Ibrahim *et al.*, 2007; Majid *et al.*, 2007; Akinfenwa *et al.*, 2011; Muhammad *et al.*, 2012). The method of collocation and interpolation of the power series approximate solution to generate continuous LMM has been adopted by many researchers among them are (Fatunla, 1991; Houwen *et al*., 1991; Awoyemi 1991; Jiaxiang *et al.*, 1995).

The paper is presented as follows: In section 2, we discuss the basic idea behind the algorithm and obtain a continuous representation $Y(x)$ for the exact solution $y(x)$ which is used to generate members of the block method for solving IVPs. In section 3, the stability analysis and convergence of proposed method were showed. Finally, we present some numerical result and concluding remarks.

2. DERIVATION OF THE CONTINUOUS BLOCK BDF METHOD

Backward differentiation formula (BDF) is one of the most popular class of implicit methods for the solution of stiff ODEs. We are concerned with the numerical solution of IVPs for first-order ODEs of the form

$$y' = f(x, y) \tag{1}$$

and given initial condition $y(a) = y_0$, $a \le x \le b$.

The aim is to construct a set of implicit BDF formulas described as r-point block BDF methods using previously computed solution values and their derivatives, where r denotes the block size to produce solution values at a block of time steps $x_{n+1}, ..., x_{n+i}, \ldots, x_{n+r}$. Each application of a r-point block method simultaneously produces r new values: $y_{n+1}, ..., y_{n+i}, \ldots, y_{n+r}$ at the time discretization points $x_{n+1}, ..., x_{n+i}, \ldots, x_{n+r}$.

If r denotes the block size and h is the step size, then block size in time is $r \cdot h$. Let m=0, 1, 2, … represent the block number and let $N = m \cdot r$, then the k-block, r-point method can be written in the following general form (Fatunla, 1991; Ibrahim *et al.*, 2007; Majid *et al.*, 2007):

$$A^{(0)} Y_m = \sum_{i=1}^{r} A^{(i)} Y_{m-i} + h \sum_{i=0}^{r} B^{(i)} F_{m-i}\ , \tag{2}$$

where $Y_m = (y_{n+1}, ..., y_{n+i}, \ldots, y_{n+r})^{\mathrm{T}}$, $F_m = (f_{n+1}, ..., f_{n+i}, \ldots, f_{n+r})^{\mathrm{T}}$, $A^{(i)}$ and $B^{(i)}$ are r by r coefficient matrices, i=0, 1, …,k.

In the explicit r-point block method, the interval $a \le x \le b$ is divided into series of blocks with each block containing r points. Each application of the formulae generates a block of r new equally spaced solution values simultaneously. The computational tasks at each point within a block are sufficiently independent and considered as separate task. See Figure 1 for details.

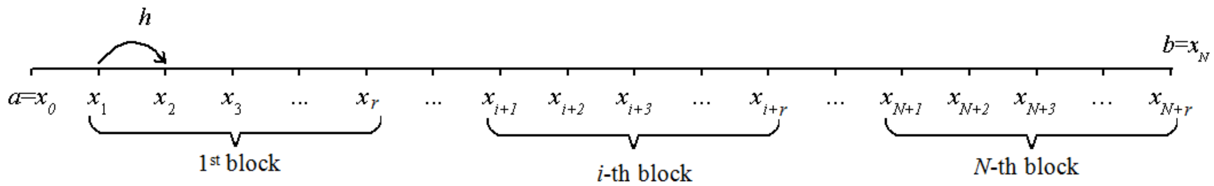

Fig. 1. A partition of the integration interval into blocks

Development of LMMs for solving ODEs can be generated using methods such as Taylor's series, numerical integration, and collocation method, which are restricted by an assumed order of convergence. In study (Majid *et al.*, 2007) the method of generating functions in the form of infinite series was used in order to obtain a useful recurrence relation for coefficient of interpolating polynomial which interpolates function $f(x, y)$ from Equation (1) at r-points.

In continues to the research presented in publications (Yahaya *et al.*, 2009; Mohammed *et al.*, 2010) was suggested using the collocation method to construction of the block multistep BDF method. We proceed by assuming that the exact solution $y(x)$ is locally represented in the range $[x_0, x_0 + r \cdot h]$ by the continuous solution $Y(x)$ of the form

$$Y(x) = \sum_{j=0}^{r} b_j \varphi_j(x) \tag{3}$$

where b_j are unknown coefficients to be determined and $\varphi_j(x)$ are polynomial basis function of degree j=0, 1, ... , r. The block multistep BDF method with $\varphi_j(x) = x^j$ by imposing the following collocation relations was construct

$$Y(x_{n+j}) = y_{n+j},\ j = 0,1, ..., r, \text{ and } Y'(x_{n+r}) = f_{n+r}, \tag{4}$$

where y_{n+j} is the approximation for the exact solution $y(x_{n+j})$, $f_{n+r} = f(x_{n+r}, y_{n+r})$ and n=0, r, ... , $N-r$ is the grid index, $N = (b-a)/h$. It should be noted that Equation (4) leads to a system of equations

$$\begin{bmatrix} x_n^0 & x_n & x_n^2 & \cdots & x_n^r \\ x_{n+1}^0 & x_{n+1} & x_{n+1}^2 & \cdots & x_{n+1}^r \\ x_{n+2}^0 & x_{n+2} & x_{n+2}^2 & \cdots & x_{n+2}^r \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n+r-1}^0 & x_{n+r-1} & x_{n+r-1}^2 & \cdots & x_{n+r-1}^r \\ 0 & 1 & 2x_{n+r} & \cdots & rx_{n+r}^r \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \cdots \\ b_{r-1} \\ b_r \end{bmatrix} = \begin{bmatrix} y_n \\ y_{n+1} \\ y_{n+2} \\ \cdots \\ y_{n+r-1} \\ f_{n+r} \end{bmatrix}, \tag{5}$$

In the matrix (5) the superscript $j = 0, 1, 2, ..., r$ denotes the degree. The system (5) must be solved to obtain the coefficients b_j, j=0, 1, ... , r, which are substituted into (3) and after some algebraic computation, our continuous representation yields the form

$$Y(x) = \sum_{j=0}^{r} \alpha_j(x) y_{n+j} + h\beta_r(x) f_{n+r}\ , \tag{6}$$

where $\alpha_j(x)$ and $\beta_r(x)$ are continuous coefficients. The method (6) is then used to generate the multistep BDF. Coefficients of multistep BDF (r=6) was defined in the study (Akinfenwa, 2013). A normal block form (Akinfenwa, 2013) can be obtain with left multiplying the matrices $\boldsymbol{A}^{(1)}$, $\boldsymbol{A}^{(0)}$, $\boldsymbol{B}^{(0)}$ with the matrix inverse $[\boldsymbol{A}^{(0)}]^{-1}$. In the case of r=6 the block method based on the Equation (2) can be written in the normalized form:

$$\begin{bmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \\ y_{n+6} \end{bmatrix} = \begin{bmatrix} 0&0&0&0&0&1 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&1 \end{bmatrix} \begin{bmatrix} y_{n-5} \\ y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} +$$

$$+h \begin{bmatrix} \frac{4277}{1440} & -\frac{2641}{480} & \frac{4991}{720} & -\frac{3649}{720} & \frac{959}{480} & \frac{3541}{30240} \\ \frac{33}{10} & -\frac{203}{45} & \frac{287}{45} & -\frac{71}{15} & \frac{169}{90} & \frac{160}{1323} \\ \frac{105}{32} & -\frac{651}{160} & \frac{567}{80} & -\frac{393}{80} & \frac{309}{160} & \frac{1017}{7840} \\ \frac{148}{45} & -\frac{62}{15} & \frac{344}{45} & -\frac{196}{45} & \frac{28}{15} & \frac{902}{6615} \\ \frac{105}{32} & -\frac{1175}{288} & \frac{1075}{144} & -\frac{175}{48} & \frac{665}{288} & \frac{7415}{42336} \\ \frac{33}{10} & -\frac{21}{5} & \frac{39}{5} & -\frac{21}{5} & \frac{33}{10} & \frac{158}{245} \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \\ f_{n+6} \end{bmatrix}. \tag{7}$$

3. STABILITY ANALYSIS

The block method (2) is zero-stable (Fatunla, 1991) provided the root R_j, j=1, 2, … r of the first characteristic polynomial $\rho(R)$ specified as

$$\rho(R) = \det\left|\sum_{i=0}^{r} A^{(i)} R^{r-i}\right| = 0\,, \tag{8}$$

satisfies $|R_j|\leq 1$ and for those roots with $|R_j|=1$, the multiplicity must not exceed 1.

Thus, as step $h\to 0$, the method (2) tends to the difference system $\boldsymbol{A}^{(0)}Y_{m+1}-\boldsymbol{A}^{(1)}Y_m=0$ whose first characteristic polynomial $\rho(R)$ is given by

$$\rho(R) = \det(R\boldsymbol{A}^{(0)} - \boldsymbol{A}^{(1)}) = R^{r-1}(1-R)=0. \tag{9}$$

From the definition (8) it follows at once that the block method (7) is zero-stable, since from equation (9) satisfies $|R_j|\le 1$, j=1, 2, … r, and the multiplicity is equal to 1 for root with $|R_j|=1$. And as the block method is zero-stable by Henrici (Henrici, 1962) the block method is convergent.

Continuous form is evaluated at various (or several distinct) points to obtained scheme (6) above. Let r=6, after some manipulations we obtain a continuous form of solution (1)

$$\begin{aligned}
y(x)=&\left(\begin{array}{l}1-\frac{21509}{8820}\frac{x-x_n}{h}+\frac{58997}{26460}\left(\frac{x-x_n}{h}\right)^2-\frac{2351}{2352}\left(\frac{x-x_n}{h}\right)^3+\\ +\frac{4975}{21168}\left(\frac{x-x_n}{h}\right)^4-\frac{979}{35280}\left(\frac{x-x_n}{h}\right)^5+\frac{137}{105840}\left(\frac{x-x_n}{h}\right)^6\end{array}\right)y_n+\\
&+\left(\begin{array}{l}\frac{290}{49}\frac{x-x_n}{h}-\frac{2505}{294}\left(\frac{x-x_n}{h}\right)^2+\frac{688}{147}\left(\frac{x-x_n}{h}\right)^3-\\ -\frac{1451}{1176}\left(\frac{x-x_n}{h}\right)^4+\frac{23}{147}\left(\frac{x-x_n}{h}\right)^5-\frac{3}{398}\left(\frac{x-x_n}{h}\right)^6\end{array}\right)y_{n+1}+\\
&+\left(\begin{array}{l}-\frac{355}{49}\frac{x-x_n}{h}+\frac{8257}{588}\left(\frac{x-x_n}{h}\right)^2-\frac{2683}{294}\left(\frac{x-x_n}{h}\right)^3+\\ +\frac{131}{49}\left(\frac{x-x_n}{h}\right)^4-\frac{107}{294}\left(\frac{x-x_n}{h}\right)^5+\frac{11}{588}\left(\frac{x-x_n}{h}\right)^6\end{array}\right)y_{n+2}+\\
&+\left(\begin{array}{l}\frac{2740}{441}\frac{x-x_n}{h}-\frac{17299}{1323}\left(\frac{x-x_n}{h}\right)^2+\frac{1394}{147}\left(\frac{x-x_n}{h}\right)^3-\\ -\frac{16087}{5292}\left(\frac{x-x_n}{h}\right)^4+\frac{391}{882}\left(\frac{x-x_n}{h}\right)^5-\frac{127}{5292}\left(\frac{x-x_n}{h}\right)^6\end{array}\right)y_{n+3}+\\
&+\left(\begin{array}{l}-\frac{635}{196}\frac{x-x_n}{h}+\frac{2083}{294}\left(\frac{x-x_n}{h}\right)^2-\frac{12793}{2352}\left(\frac{x-x_n}{h}\right)^3+\\ +\frac{4393}{2352}\left(\frac{x-x_n}{h}\right)^4-\frac{683}{2352}\left(\frac{x-x_n}{h}\right)^5+\frac{13}{784}\left(\frac{x-x_n}{h}\right)^6\end{array}\right)y_{n+4}+\\
&+\left(\begin{array}{l}\frac{194}{245}\frac{x-x_n}{h}-\frac{2599}{1470}\left(\frac{x-x_n}{h}\right)^2+\frac{206}{147}\left(\frac{x-x_n}{h}\right)^3-\\ -\frac{197}{392}\left(\frac{x-x_n}{h}\right)^4+\frac{121}{1470}\left(\frac{x-x_n}{h}\right)^5-\frac{29}{5880}\left(\frac{x-x_n}{h}\right)^6\end{array}\right)y_{n+5}+\\
&+h\left(\begin{array}{l}\frac{10}{147}\frac{x-x_n}{h}+\frac{137}{882}\left(\frac{x-x_n}{h}\right)^2-\frac{25}{196}\left(\frac{x-x_n}{h}\right)^3+\\ +\frac{85}{1764}\left(\frac{x-x_n}{h}\right)^4-\frac{5}{588}\left(\frac{x-x_n}{h}\right)^5+\frac{1}{1764}\left(\frac{x-x_n}{h}\right)^6\end{array}\right)f_{n+6}.
\end{aligned} \tag{10}$$

4. NUMERICAL EXPERIMENT

In this section, proposed method (7) above was tested with IVP our written Maple code.

Consider the linear problem

$$y' = 100(\sin(x)-y),\ y(0)=1,\ 0\le x\le 1. \tag{11}$$

which have analytical solution

$$y(x) = \frac{\sin x - 0.01\cos x + 0.01e^{-100x}}{1.001}. \tag{12}$$

The example task (11) was solved with our method (7) on the interval $0 \le x \le 1$ with step *h*=0.01. The numerical and analytical solutions and difference between them (absolute errors) are presented in Table 1.

Table 1. Numerical and analytical solutions and absolute error.

x	Numerical solution	Analytical solution	Absolute Error
0.1	0.0899207167	0.0899202414	4.75E-07
0.2	0.1888478298	0.1888497821	1.95E-06
0.3	0.2859328150	0.2859382479	5.43E-06
0.4	0.3801693118	0.3801697154	4.04E-07
0.5	0.4706002066	0.4706026527	2.45E-06
0.6	0.5563280111	0.5563334839	5.47E-06
0.7	0.6365064915	0.6365056148	8.77E-07
0.8	0.7103177127	0.7103179920	2.79E-07
0.9	0.7770303486	0.7770331066	2.76E-06
1.0	0.8359823521	0.8359843633	2.01E-06

5. CONCLUSION

The block method has been proposed and implemented as a self-starting method for the solution of first-order ordinary differential equations (1). The proposed method is accurate. It is a necessary to define the order and error constants of the method. The convergence and zero-stability properties of method make it suitable for numerical solution of stiff problems. The process produces some schemes which are combined in order to form an accurate and efficient block method for parallel or sequential solution of ordinary differential equations (ODEs).

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