Block Unification of Multi-Step Methods for the Solution of Boundary Layer Flow

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Abstract

We develop a class of block unification multi-step method (BUMM) which are used as boundary value methods for the numerical integration of third order boundary value problems in ordinary differential equations resulting from boundary layer flow. The method solves the problem directly instead of converting it to a system of first order ordinary differential equations before solving. The block unification multi-step methods are constructed using Chebyshev polynomials as basis function and employing interpolation and collocation method.The basic properties of the methods are investigated and numerical experiments are given to show the performance of the methods.

INTRODUCTION

In understanding physical phenomena mathematical models are developed in science, engineering and technology to help understand these physical phenomena. The mathematical models are expressed in equations in which a function and its derivatives play significant roles. An equation that contains some derivatives of an unknown function of one or more several variables is called a differential equation. These equations arise not only in fields like physical science but also in fields like operation research, psychology, medicine, economics, engineering, etc, ranging from models that describe neural works, acoustic wave propagation in relaxing media, draining and coating flow problems to the deflection of a curved beam that has a constant or varying cross section and as such faster and more accurate numerical methods are required.

Steady flow of viscous incompressible fluids has attracted considerable attention in recent years due to its crucial role in numerous engineering applications. Numerical analysts encounter actually a wide variety of challenges in obtaining suitable algorithms for computing flow and heat transfer of viscous fluids (Bataller, 2010). Boundary layer flow problems of third order and third order ordinary differential equations have been discussed in many papers in recent years. Examples of such papers are (Abdullah *et al* 2013; 2013) who had developed a fifth order block method using constant step size with shooting technique to solve third order non-linear boundary value problems and developed a fourth order two-point block method for solving non-linear third order boundary value problems. The combination of the standard adomian decomposition method and a finite difference scheme, while taking note of their respective advantages and disadvantages, was used to solve the Blasius problem in Akdi and Sedra (2014). This way the coupled method offset the limitations of the individual methods. Aminikhah and Kazemi (2016) used quartic b-splines approximations to construct the numerical solution to Blasius equation. Collocation approximation was applied in deriving schemes that were applied as a block method to solve special third order initial value problems in Olabode (2009). Jator (2008) used a continuous linear multistep method to generate multiple finite difference methods that were assembled into a single block matrix that was used to solve third order BVPs. Jator (2009) presented Multiple Finite Difference Methods obtained from a linear multistep method of step 4, these were used to solve third order boundary value problems directly.A family of three step hybrid methods independent of first and second derivative components using Taylor approach were proposed to solve special third order ODEs in Jikantoro *et al* (2018), These were all done without reducing the ODEs to equivalent systems of first order ODEs.Ahmed (2017) used the variational iteration method to get numerical solutions to third order ordinary boundary value problems after reducing them to a system of first order ODEs.

In this paper, third order ordinary differential equations resulting from boundary layer flow such as Blasius, Sakiadis and Falkner-Skan are considered.

METHODOLOGY

In this section, the construction of the block unification multistep method through the interpolation and collocation approach is discussed, which will be used to produce several discrete schemes for solving boundary layer flow.

The starting point is to construct the block unification multi-step method (BUMM) which has the form

$$U(x)=\alpha_v(x)y_{n+v}+\alpha_{v-1}(x)y_{n+v-1}+\alpha_0(x)y_n+h^3\sum_{j=0}^{k}\beta_j(x)f_{n+j}+h^3\beta_w(x)f_{n+w}, \qquad (1)$$

Where $v=\begin{cases}\dfrac{k}{2} & for\ \ even\ \ k\\ \dfrac{k+1}{2} & for\ \ odd\ \ k\end{cases}$

$\alpha_0(x),\alpha_{v-1}(x),\alpha_v,\beta_j,\beta_w$ are continuous coefficients and v is chosen to be half the step number so that the formula derived from (1) satisfies the root condition.

The main and additional methods are then obtained by evaluating (1) at x_{n+j} where $j=1(1)2v,\ j\neq v-1,v$ to obtain the formula of the following form:

$$y_{n+j}+\alpha_v y_{n+v}+\alpha_{v-1}y_{n+v-1}+\alpha_0 y_n=h^3\sum_{i=o}^{k}\beta_i f_{n+i}+h^3\beta_w f_{n+w} \qquad (2)$$

The first and second derivative formulas for (1) are used to generate additional methods by evaluating $U'(x)$ and $U''(x)$ at $x_{n+j},\ j=0(1)k$. The construction of (1) is discussed in the following theorem.

Theorem 2.1 *Let* $T_j(x),\ j=0(1)(k+3)$ *be the Chebyshev Polynomial used as basis function and W a vector given by* $W=(y_n,y_{n+v-1},y_{n+v},f_n,f_{n+1},...,f_k)^T$ *where T is the transpose. Consider the matrix V defined as*

$$V=\begin{pmatrix} T_0(x_n) & T_1(x_n) & \cdots & T_{k+3}(x_n)\\ T_0(x_{n+v-1}) & T_1(x_{n+v-1}) & \cdots & T_{k+3}(x_{n+v-1})\\ T_0(x_{n+v}) & T_1(x_{n+v}) & \cdots & T_{k+3}(x_{n+v})\\ T_0'''(x_n) & T_1'''(x_n) & \cdots & T_{k+3}'''(x_n)\\ T_0'''(x_{n+1}) & T_1'''(x_{n+1}) & \cdots & T_{k+3}'''(x_{n+1})\\ \vdots & \vdots & \vdots & \vdots\\ T_0'''(x_{n+k}) & T_1'''(x_{n+k}) & \cdots & T_{k+3}'''(x_{n+k}) \end{pmatrix}$$

and obtained by replacing the jth column of V by the vector W and let (2) satisfy

$U(x_{n+j})=y_{n+j}\ \ j=0,v-1,v$ and $j=0,v-2,v-1,v$

$$U'''(x_{n+j})=f_{n+j} \qquad j=0(1)k \qquad (3)$$

then the continuous representation (1) is equivalent to

$$U(x) = \sum_{j=0}^{k+3} \frac{\det(V_j)}{\det(V)} T_j(x) \tag{4}$$

Proof The basis function for (1) is taken as

$$\begin{cases} \alpha_j(x) = \sum_{i=0}^{k+3} \alpha_{i+1,j} T_i(x), & j = 0, v-1, v \\ h^3 \beta_j(x) = \sum_{i=0}^{k+3} h^3 \beta_{i+1.j} T_i(x), & j = 0(1)k \end{cases} \tag{5}$$

where $\alpha_{i+1,j}, h^3\beta_{i+1,j}$ are coefficients to be determined.

Inserting (5) into (1) gives

$$U(x) = \sum_{i=0}^{k+3} \alpha_{i+1,v} T_i(x) y_{n+v} + \sum_{i=0}^{k+3} \alpha_{i+1,v-1} T_i(x) y_{n+v-1} + \sum_{i=0}^{k+3} \alpha_{i+1,0} T_i(x) y_n +$$
$$h^3 \sum_{j=0}^{k} \sum_{i=0}^{k+3} \beta_{i+1,j} T_i(x) f_{n+j} + h^3 \sum_{i=1}^{k+3} \beta_{i+1,w} T_i(x) f_{n+w},$$

Simplified to

$$U(x) = \sum_{i=0}^{k+3} \left\{ \alpha_{i+1,v} y_{n+v} + \alpha_{i+1,v-1} y_{n+v-1} + \alpha_{i+1,0} y_n + \sum_{j=0}^{k} h^3 \beta_{i+1,j} f_{n+j} + h^3 \beta_{i+1,w} f_{n+w} \right\} T_i(x),$$

expressed in the form

$$U(x) = \sum_{i=0}^{k+3} \eta_i T_i(x) \tag{6}$$

Imposing conditions (3) on (6), a system of (k+4) equations is obtained which could be expressed in the form

$VH = W$ where

$H = (\eta_0, \eta_1, \eta_2, \ldots, \eta_{k+3})^T$ is a vectors of (k+4) undetermined coefficients.

The elements of H are found using the Cramer's rule

$$\eta_i = \frac{\det(V_j)}{\det(V)}, \; j = 0(1)(k+3)$$

where V_j is obtained by replacing the jth column of V by W. Using the newly found elements of H, (6) is re-written as

$$U(x)=\sum_{i=0}^{k+3}\frac{\det(V_j)}{\det(V)}T_i(x)$$

Evaluating the BUMM (1) at $x_{n+i,} i=1,\ldots,v-2,v+1,\ldots,k$ and using it to obtain the first derivative formulae

given by

$$U'(x)=\frac{1}{h}\left(\alpha'_v(x)y_{n+v}+\alpha'_{v-1}(x)y_{n+v-1}+\alpha'_0(x)y_n+h^3\sum_{j=0}^{k}\beta'_j(x)f_{n+j}+h^3\beta'_w(x)f_{n+w},\right) \quad (7)$$

effectively applied by imposing

$$U'(a)=y'_0, U'(b)=y'_N$$

to produce derivative formulae of the form (7).

The second derivative formula is also obtained from (1). This is given by

$$U''(x)=\frac{1}{h^2}\left(\alpha''_v(x)y_{n+v}+\alpha''_{v-1}(x)y_{n+v-1}+\alpha''_0(x)y_n+h^3\sum_{j=0}^{k}\beta''_j(x)f_{n+j}+h^3\beta''_w(x)f_{n+w},\right) \quad (8)$$

effectively imposed by applying

$$U''(a)=y''_0, U''(b)=y''_N$$

to generate (8)

Specification of Methods

To derive an implicit three step method with one off-grid point, the following specifications were considered, r = 3, s= 5, k = 3, v = $\frac{7}{3}$, to give the continuous form as:

$$y(x)=\alpha_0 y_n+\alpha_1 y_{n+1}+\alpha_2 y_{n+2}+h^3[\beta_0 f_n+\beta_1 f_{n+1}+\beta_2 f_{n+2}+\beta_{\frac{7}{3}} f_{n+\frac{7}{3}}+\beta_3 f_{n+3}] \quad (9)$$

Evaluating equation (9) at points $x=x_{n+3}, x=x_{n+\frac{7}{3}}$ gives

$$y_{n+3}=y_n-3y_{n+1}+3y_{n+2}+\frac{1}{140}h^3 f_n+\frac{37}{80}h^3 f_{n+1}+\frac{13}{20}h^3 f_{n+2}-\frac{81}{560}h^3 f_{n+\frac{7}{3}}+\frac{1}{40}h^3 f_{n+3}$$

$$y_{n+\frac{7}{3}}=\frac{2}{9}y_n-\frac{7}{9}y_{n+1}+\frac{14}{9}y_{n+2}+\frac{137}{76545}h^3 f_n+\frac{2911}{29160}h^3 f_{n+1}+\frac{139}{1215}h^3 f_{n+2}-\frac{1067}{22680}h^3 f_{n+\frac{7}{3}}+\frac{169}{43740}h^3 f_{n+3}$$

For $n=0(3)(N-3)$

The first derivative formulae are

$$hy'_n = -\frac{3}{2}y_n + 2y_{n+1} - \frac{1}{2}y_{n+2} + \frac{167}{2940}h^3 f_n + \frac{577}{1680}h^3 f_{n+1} - \frac{101}{420}h^3 f_{n+2} + \frac{793}{3920}h^3 f_{n+\frac{7}{3}} - \frac{11}{420}h^3 f_{n+3}$$

$$hy'_{n+1} = -\frac{1}{2}y_n + \frac{1}{2}y_{n+2} - \frac{11}{1470}h^3 f_n - \frac{173}{1120}h^3 f_{n+1} + \frac{1}{105}h^3 f_{n+2} - \frac{27}{1568}h^3 f_{n+\frac{7}{3}} + \frac{1}{336}h^3 f_{n+3}$$

$$hy'_{n+2} = \frac{1}{2}y_n - 2y_{n+1} + \frac{3}{2}y_{n+2} + \frac{13}{2940}h^3 f_n + \frac{367}{1680}h^3 f_{n+1} + \frac{27}{140}h^3 f_{n+2} - \frac{351}{3920}h^3 f_{n+\frac{7}{3}} + \frac{1}{140}h^3 f_{n+3}$$

$$hy'_{n+\frac{7}{3}} = \frac{5}{6}y_n - \frac{8}{3}y_{n+1} + \frac{11}{6}y_{n+2} + \frac{680}{107163}h^3 f_n + \frac{310459}{816480}h^3 f_{n+1} + \frac{25679}{51030}h^3 f_{n+2} - \frac{38813}{211680}h^3 f_{n+\frac{7}{3}}$$
$$+ \frac{3865}{244944}h^3 f_{n+3}$$

$$hy'_{n+3} = \frac{3}{2}y_n - 4y_{n+1} + \frac{5}{2}y_{n+2} + \frac{9}{980}h^3 f_n + \frac{2393}{3360}h^3 f_{n+1} + \frac{89}{84}h^3 f_{n+2} - \frac{27}{1568}h^3 f_{n+\frac{7}{3}} + \frac{39}{560}h^3 f_{n+3} \quad (11)$$

for $n = 0(3)(N-3)$

And the second derivative formulae are

$$h^2 y''_n = y_n - 2y_{n+1} + y_{n+2} - \frac{389}{1260}h^3 f_n - \frac{227}{240}h^3 f_{n=1} + \frac{53}{60}h^3 f_{n+2} - \frac{81}{112}h^3 f_{n+\frac{7}{3}} + \frac{17}{180}h^3 f_{n+3}$$

$$h^2 y''_{n+1} = y_n - 2y_{n+1} + y_{n+2} + \frac{53}{2520}h^3 f_n + \frac{1}{12}h^3 f_{n=1} - \frac{11}{40}h^3 f_{n+2} + \frac{27}{140}h^3 f_{n+\frac{7}{3}} - \frac{1}{45}h^3 f_{n+3}$$

$$h^2 y''_{n+2} = y_n - 2y_{n+1} + y_{n+2} + \frac{1}{180}h^3 f_n + \frac{39}{80}h^3 f_{n=1} + \frac{49}{60}h^3 f_{n+2} - \frac{27}{80}h^3 f_{n+\frac{7}{3}} + \frac{1}{36}h^3 f_{n+3}$$

$$h^2 y''_{n+\frac{7}{3}} = y_n - 2y_{n+1} + y_{n+2} + \frac{407}{68040}h^3 f_n + \frac{29}{60}h^3 f_{n=1} + \frac{641}{648}h^3 f_{n+2} - \frac{71}{420}h^3 f_{n+\frac{7}{3}} + \frac{29}{1215}h^3 f_{n+3}$$

$$h^2 y''_{n+3} = y_n - 2y_{n+1} + y_{n+2} + \frac{1}{540}h^3 f_n + \frac{31}{60}h^3 f_{n=1} + \frac{79}{120}h^3 f_{n+2} + \frac{81}{140}h^3 f_{n+\frac{7}{3}} + \frac{11}{45}h^3 f_{n+3} \quad (12)$$

for $n = 0(3)(N-3)$

CONVERGENCE OF THE METHOD

Here the convergence of the method is established. The equation (1) is evaluated at x_{n+1}, x_{n+2},…, x_{n+v-2}, x_{n+v+1},…, $x_{n+\omega}$, x_{n+2v} to give

$$y_{n+1} + \alpha_v^{(1)} y_{n+v} + \alpha_{v-1}^{(1)} y_{n+v-1} + \alpha_0^{(1)} y_0 = h^3 \sum_{i=0}^{k} \beta_i^{(1)} f_{n+i} + h^3 \beta_\omega^{(1)} f_{n+\omega}$$

$$y_{n+2} + \alpha_v^{(2)} y_{n+v} + \alpha_{v-1}^{(2)} y_{n+v-1} + \alpha_0^{(2)} y_0 = h^3 \sum_{i=0}^{k} \beta_i^{(2)} f_{n+i} + h^3 \beta_\omega^{(2)} f_{n+\omega}$$

$$\vdots$$

$$y_{n+v-2} + \alpha_v^{(v-2)} y_{n+v} + \alpha_{v-1}^{(v-2)} y_{n+v-1} + \alpha_0^{(v-2)} y_0 = h^3 \sum_{i=0}^{k} \beta_i^{(v-2)} f_{n+i} + h^3 \beta_\omega^{(v-2)} f_{n+\omega} \quad (13)$$

$$y_{n+v+1} + \alpha_v^{(v+1)} y_{n+v} + \alpha_{v-1}^{(v+1)} y_{n+v-1} + \alpha_0^{(v+1)} y_0 = h^3 \sum_{i=0}^{k} \beta_i^{(v+1)} f_{n+i} + h^3 \beta_\omega^{(v+1)} f_{n+\omega}$$

$$\vdots$$

$$y_{n+\omega} + \alpha_v^{(\omega)} y_{n+v} + \alpha_{v-1}^{(\omega)} y_{n+v-1} + \alpha_0^{(\omega)} y_0 = h^3 \sum_{i=0}^{k} \beta_i^{(\omega)} f_{n+i} + h^3 \beta_\omega^{(\omega)} f_{n+\omega}$$

$$y_{n+k} + \alpha_v^{(k)} y_{n+v} + \alpha_{v-1}^{(k)} y_{n+v-1} + \alpha_0^{(k)} y_0 = h^3 \sum_{i=0}^{k} \beta_i^{(k)} f_{n+i} + h^3 \beta_\omega^{(k)} f_{n+\omega}$$

$U'(x)$ is evaluated at x_{n+j} $j = 0(1)k$ and $x_{n+\omega}$ to give

$$hy_n' + \alpha_v'^{(0)} y_{n+v} + \alpha_{v-1}'^{(0)} y_{n+v-1} + \alpha_0'^{(0)} y_n = h^3 \sum_{i=0}^{k} \beta_i'^{(0)} f_{n+i} + h^3 \beta_\omega'^{(0)} f_{n+\omega}$$

$$hy_n' + \alpha_v'^{(1)} y_{n+v} + \alpha_{v-1}'^{(1)} y_{n+v-1} + \alpha_0'^{(1)} y_n = h^3 \sum_{i=0}^{k} \beta_i'^{(1)} f_{n+i} + h^3 \beta_\omega'^{(1)} f_{n+\omega}$$

$$\vdots \qquad (14)$$

$$hy_n' + \alpha_v'^{(\omega)} y_{n+v} + \alpha_{v-1}'^{(\omega)} y_{n+v-1} + \alpha_0'^{(\omega)} y_n = h^3 \sum_{i=0}^{k} \beta_i'^{(\omega)} f_{n+i} + h^3 \beta_\omega'^{(\omega)} f_{n+\omega}$$

$$hy_n' + \alpha_v'^{(k)} y_{n+v} + \alpha_{v-1}'^{(k)} y_{n+v-1} + \alpha_0'^{(k)} y_n = h^3 \sum_{i=0}^{k} \beta_i'^{(k)} f_{n+i} + h^3 \beta_\omega'^{(k)} f_{n+\omega}$$

And also evaluate $U''(x)$ to give

$$hy_n'' + \alpha_v''^{(0)} y_{n+v} + \alpha_{v-1}''^{(0)} y_{n+v-1} + \alpha_0''^{(0)} y_n = h^3 \sum_{i=0}^{k} \beta_i''^{(0)} f_{n+i} + h^3 \beta_\omega''^{(0)} f_{n+\omega}$$

$$hy_n'' + \alpha_v''^{(1)} y_{n+v} + \alpha_{v-1}''^{(1)} y_{n+v-1} + \alpha_0''^{(1)} y_n = h^3 \sum_{i=0}^{k} \beta_i''^{(1)} f_{n+i} + h^3 \beta_\omega''^{(1)} f_{n+\omega}$$

$$\vdots \qquad (15)$$

$$hy_n'' + \alpha_v''^{(\omega)} y_{n+v} + \alpha_{v-1}''^{(\omega)} y_{n+v-1} + \alpha_0''^{(\omega)} y_n = h^3 \sum_{i=0}^{k} \beta_i''^{(\omega)} f_{n+i} + h^3 \beta_\omega''^{(\omega)} f_{n+\omega}$$

$$hy_n'' + \alpha_v''^{(k)} y_{n+v} + \alpha_{v-1}''^{(k)} y_{n+v-1} + \alpha_0''^{(k)} y_n = h^3 \sum_{i=0}^{k} \beta_i''^{(k)} f_{n+i} + h^3 \beta_\omega''^{(k)} f_{n+\omega}$$

All the equations in (13) to (15) are of order $O(h^{k+5})$ and can be compactly written in matrix form by introducing the following notations. Let A be a $3N \times 3N$ matrix defined by

$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$ where A_{ij} are $N \times N$ matrices given as

$$
A_{11}=\left(\begin{array}{cccccccc}
 & \alpha'^{(0)}_{v-1} & \alpha'^{(0)}_{v} & & & \alpha'^{(0)}_{0} & & \\
 & \alpha''^{(0)}_{v-1} & \alpha''^{(0)}_{v} & & & \alpha''^{(0)}_{0} & & \\
1 & \alpha^{(1)}_{v-1} & \alpha'^{(1)}_{v} & & & \alpha^{(1)}_{0} & & \\
1 & \alpha^{(2)}_{v-1} & \alpha'^{(2)}_{v} & & & \alpha^{(2)}_{0} & & \\
 & \vdots & \vdots & & & \vdots & & \\
1 & \alpha^{(v-1)}_{v-1} & \alpha^{(v-1)}_{v} & & & \alpha^{(v-1)}_{0} & & \\
1 & \alpha^{(v)}_{v-1} & \alpha^{(v)}_{v} & & & \alpha^{(v)}_{0} & & \\
 & \vdots & \vdots & & & \vdots & & \\
1 & \alpha^{(k)}_{v-1} & \alpha^{(k)}_{v} & & & \alpha^{(k)}_{0} & & \\
 & & & & & \alpha'^{(0)}_{v-1} & \alpha'^{(0)}_{v} & \alpha'^{(0)}_{0} \\
 & & & 1 & & \alpha^{(1)}_{v-1} & \alpha^{(1)}_{v} & \alpha^{(1)}_{0} \\
 & & & & & & \vdots & \vdots \\
 & & & 1 & & \alpha^{(v-1)}_{v-1} & \alpha^{(v-1)}_{v} & \alpha^{(0)}_{0} \\
 & & & 1 & & \alpha^{(v)}_{v-1} & \alpha^{(v)}_{v} & \alpha^{(v)}_{0} \\
 & & & & & & \ddots & \ddots \\
 & & & & 1 & \alpha^{(k)}_{v-1} & \alpha^{(k)}_{v} & \alpha^{(k)}_{0}
\end{array}\right)
$$

$$
A_{21}=\left(\begin{array}{cccccc}
\alpha'^{(1)}_{v-1} & \alpha'^{(1)}_{v} & \alpha'^{(1)}_{0} & & & \\
\alpha'^{(2)}_{v} & \alpha'^{(2)}_{v} & \alpha'^{(2)}_{0} & & & \\
\vdots & \vdots & \vdots & & & \\
\alpha'^{(k)}_{v-1} & \alpha'^{(k)}_{v} & \alpha'^{(k)}_{0} & & & \\
 & & & \alpha'^{(1)}_{v-1} & \alpha'^{(1)}_{v} & \alpha'^{(1)}_{0} \\
 & & & \ddots & \ddots & \ddots \\
 & & & \alpha'^{(k)}_{v-1} & \alpha'^{(k)}_{v} & \alpha'^{(2v)}_{0}
\end{array}\right)
$$

$$
A_{31}=\left(\begin{array}{cccccc}
\alpha''^{(1)}_{v-1} & \alpha''^{(1)}_{v} & \alpha''^{(1)}_{0} & & & \\
\alpha''^{(2)}_{v} & \alpha''^{(2)}_{v} & \alpha''^{(2)}_{0} & & & \\
\vdots & \vdots & \vdots & & & \\
\alpha''^{(k)}_{v-1} & \alpha''^{(k)}_{v} & \alpha''^{(k)}_{0} & & & \\
 & & & \alpha''^{(1)}_{v-1} & \alpha''^{(1)}_{v} & \alpha''^{(1)}_{0} \\
 & & & \ddots & \ddots & \ddots \\
 & & & \alpha''^{(k)}_{v-1} & \alpha''^{(k)}_{v} & \alpha''^{(2v)}_{0}
\end{array}\right)
$$

A_{12}, A_{13}, A_{23}, A_{32}, are $N \times N$ null matrices and A_{22}, A_{33} are $N \times N$ identity matrices. Similarly, another matrix B which is a $3N \times 3N$ matrix defined as

$$
B=\begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix}
$$

Where B_{ij} are $N \times N$ matrices given as

$$B_{11}=\begin{pmatrix}
\beta_1'^{(0)} & \beta_2'^{(0)} & \cdots & \beta_k'^{()} & & & & & & \\
\beta_1''^{()} & \beta_2''^{()} & \cdots & \beta_k''^{()} & & & & & & \\
\beta_1^{(1)} & \beta_2^{(1)} & \cdots & \beta_k^{(1)} & & & & & & \\
\vdots & \vdots & \cdots & \vdots & & & & & & \\
\beta_1^{(v-2)} & \beta_2^{(v-2)} & \cdots & \beta_k^{(v-2)} & & & & & & \\
\beta_1^{(v+1)} & \beta_2^{(v+1)} & \cdots & \beta_k^{(v+1)} & & & & & & \\
\vdots & \vdots & \cdots & \vdots & & & & & & \\
\beta_1^{(k)} & \beta_2^{(k)} & \cdots & \beta_k^{(k)} & & & & & & \\
 & & & & \beta_0'^{(0)} & \beta_1'^{(0)} & \cdots & \beta_k'^{(0)} & & \\
 & & & & \beta_0'^{(0)} & \beta_1'^{(1)} & \cdots & \beta_k'^{(1)} & & \\
 & & & & \vdots & \vdots & \cdots & \vdots & & \\
 & & & & \beta_0^{(v-2)} & \beta_1^{(v-2)} & \cdots & \beta_k^{(v-2)} & & \\
 & & & & \beta_0^{(v+1)} & \beta_1^{(v+1)} & \cdots & \beta_k^{(v+1)} & & \\
 & & & & \vdots & \vdots & \cdots & \vdots & & \\
 & & & & \beta_0^{(k)} & \beta_1^{(k)} & \vdots & \beta_k^{(k)} & & \\
 & & & & & \ddots & \ddots & \ddots & & \\
 & & & & & & \beta_0^{(k)} & \beta_1^{(k)} & \cdots & \beta_k^{(k)}
\end{pmatrix}$$

$$B_{21}=\begin{pmatrix}
\beta_1'^{(1)} & \beta_2'^{(1)} & \cdots & \beta_k'^{(1)} & & & & & & \\
\vdots & \vdots & \cdots & \vdots & & & & & & \\
\beta_1'^{(k)} & \beta_2'^{(k)} & \cdots & \beta_k'^{(1)} & & & & & & \\
 & & & & \beta_0'^{(1)} & \beta_1'^{(1)} & \cdots & \beta_k'^{(1)} & & \\
 & & & & \vdots & \vdots & \cdots & \vdots & & \\
 & & & & \beta_0'^{(k)} & \beta_1'^{(k)} & \cdots & \beta_k'^{(k)} & & \\
 & & & & & & \ddots & \ddots & & \\
 & & & & & & \beta_0'^{(k)} & \beta_1'^{(k)} & \cdots & \beta_k'^{(k)}
\end{pmatrix}$$

$$B_{31}=\begin{pmatrix}
\beta_1''^{(1)} & \beta_2''^{(1)} & \cdots & \beta_k''^{(1)} & & & & & & \\
\vdots & \vdots & \cdots & \vdots & & & & & & \\
\beta_1''^{(k)} & \beta_2''^{(k)} & \cdots & \beta_k''^{(1)} & & & & & & \\
 & & & & \beta_0''^{(1)} & \beta_1''^{(1)} & \cdots & \beta_k''^{(1)} & & \\
 & & & & \vdots & \vdots & \cdots & \vdots & & \\
 & & & & \beta_0''^{(k)} & \beta_1''^{(k)} & \cdots & \beta_k''^{(k)} & & \\
 & & & & & & \ddots & \ddots & & \\
 & & & & & & \beta_0''^{(k)} & \beta_1''^{(k)} & \cdots & \beta_k''^{(k)}
\end{pmatrix}$$

B_{12}, B_{13}, B_{22}, B_{23}, B_{32}, B_{33} are $N\times N$ null matrices

And then the following vectors are defined

$$\overline{Y}=(y_{n+1},\ldots,y_{n+k},hy'_{n+1},\ldots,hy'_{n+k},h^2y''_{n+1},\ldots,h^2y''_{n+k})^T$$

$$Y=(y(x_{n+1}),\ldots,y(x_{n+k}),hy'(x_{n+1}),\ldots,hy'(x_{n+k}),h^2y''(x_{n+1}),\ldots,h^2y''(x_{n+k}))^T$$

$$F=(f_{n+1},\ldots,f_{n+2v},hf'_{n+1},\ldots,hf'_{n+k},h^2f''_{n+1},\ldots,h^2f''_{n+k})^T$$

$L(h) = (l_1, \ldots, l_N, l'_1, \ldots, l'_N, l''_1, \ldots, l''_N)^T$

$C = (\beta_0'^{(0)} h^3 f_0 - hy'_0, \beta_0''^{(0)} h^3 f_0 - hy''_0, \beta_0^{(0)} h^3 f_0 - y_0, \beta_0^{(1)} h^3 f_0, \ldots, \beta_0^{(v-2)} h^3 f_0, \beta_0^{(v+1)} h^3 f_0, \ldots,$
$\beta_0^{(k)} h^3 f_0, 0, \ldots, 0, \beta_0'^{(1)} h^3 f_0 - \alpha_0'^{(1)} y_0, \beta_0'^{(k)} h^3 f_0 - \alpha_0'^{(k)} y_0, 0, \ldots 0, \beta_0''^{(1)} h^3 f_0 - \alpha_0''^{(1)} y_0,$
$\beta_0''^{(k)} h^3 f_0 - \alpha_0''^{(k)} y_0, 0, \ldots 0)^T$

With $L(h)$ representing the local truncation error vector at the point x_n of the methods (12) to (14).

Theorem 4.1: Let (y_i, y'_i, y''_i) be an approximation to the solution vector $(y(x_i), y'(x_i), y''(x_i))$ for the third order ordinary equations from boundary layer flow. If $e_i = |y(x_i) - y_i|, e'_i = |y'(x_i) - y'_i|, e''_i = |y''(x_i) - y''_i|$, where the exact solution given by the vector $(y(x), y'(x), y''(x))$ is several times differentiable and if $\|E\| = \|Y - \bar{Y}\|$, then the BVMs are said to be convergent of order $k+2$ which implies that
$\|E\| = O(h^{k+2})$, where k is the step number.

Proof: Consider the exact form of the system formed from (13) to (15) given by

$$PY - h^3 QF(Y) + C + L(h) = 0 \qquad (16)$$

where $L(h)$ is the truncation error vector obtained from the formulae (13) to (15). The approximate form of the system is given by

$$P\bar{Y} - h^3 QF(\bar{Y}) + C = 0 \qquad (17)$$

where $\bar{Y}$ is the approximate solution of vector Y.

Subtracting (16) from (17) and letting $E = |\bar{Y} - Y| = (e_1, \ldots e_N, e'_1, \ldots e'_N, e''_1, \ldots e''_N)^T$ and using the mean value theorem, we have the error system

$$(P - h^3 QB)E = L(h) \qquad (18)$$

where B is the Jacobian matrix and its entries $B_{rs}, r, s = 1,2,3,$ are defined as

$$B_{rs} = \begin{pmatrix} \frac{\partial f_1^{(r-1)}}{\partial y_1^{(s-1)}} & \cdots & \frac{\partial f_1^{(r-1)}}{\partial f_N^{(s-1)}} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_N^{(r-1)}}{\partial f_1^{(s-1)}} & \cdots & \frac{\partial f_N^{(r-1)}}{\partial f_N^{(s-1)}} \end{pmatrix}$$

From (17) and $L(h)$

$$E = (P - h^3 QB)^{-1} L(h)$$

$$E = SL(h)$$

$$\|E\| = \|SL(h)\|$$
$$= O(h^{-3}) O(h^{k+5})$$
$$= O(h^{k+2})$$

Which show that the methods are convergent and the global errors are of order $O(h^{k+2})$

Numerical Examples

Here, three numerical examples are considered: Blasius equation, Sakiadis equation and Falkner-Skan equation. All three solutions were compared with solutions using Runge-Kutta method.

Problem 1: Blasius Equation

$$2y''' + yy'' = 0$$

$$y(0) = 0, y'(0) = 0, y'(\infty) = 1$$

Table 1: Comparison of the Solutions from Proposed Methods and Runge-Kutta Method

Proposed Method	Runge-Kutta

X	N	$y''(0)$	$y(x_\infty)$	$y''(x_\infty)$	$y''(0)$	$y(x_\infty)$	$y''(x_\infty)$	N
1.0	9	1.021157329	0.5063049940	0.9381906626	1.021157016	0.506305291	0.93810698	27
2.0	17	0.5442717691	1.051664551	0.3810337080	0.5442717609	1.051664633	0.381033607	51
3.0	25	0.4045496973	1.679698960	0.1689551177	0.4045497078	1.6796990467	0.168955073	75
4.0	33	0.3527462516	2.432249676	0.06202511200	0.3527462779	2.432249926	0.0620251103	99
5.0	41	0.33256595103	3.3170985421	0.0155692563	0.3325659529	3.3170985488	0.0155692560	123

Problem 2: Sakiadis flow

$$2y''' + yy'' = 0$$

$$y(0) = 0, y'(0) = 1, y'(\infty) = 0$$

Table 2: Comparison of the Solutions from Proposed Methods and Runge-Kutta Method

	Proposed Method				Runge-Kutta			
x	N	$y''(0)$	$y(x_\infty)$	$y''(x_\infty)$	$y''(0)$	$y(x_\infty)$	$y''(x_\infty)$	N
1.0	9	-1.062106604	0.4858145149	-0.9021137979	-1.0621056881	0.4858148417	-0.9021137490	27
2.0	17	-0.6214631716	0.8954882570	-0.3357451645	-0.6214629182	0.895488335	-0.3357452060	51
3.0	25	-0.5078781704	1.190534705	-0.1428727781	-0.5078780256	1.190534757	-0.1428727865	75
4.0	33	-0.4687973723	1.377935656	-0.06161582430	-0.4687972558	1.3779357168	-0.0616581740	99
5.0	41	-0.4539702818	1.487355776	-0.02661787579	-0.4539701772	1.487355831	-0.0266178690	123

Problem 3: Falkner-Skan Equation

$$f'''(\eta) + \beta_0 f(\eta) f''(\eta) + \beta\left(1 - f'(\eta)^2\right) = 0$$

$$f(0) = 0, f'(0) = 0, \lim_{\eta\to\infty} f'(\eta) = 1$$

Table 3: Comparison of the Errors from Proposed Methods and Runge-Kutta Method

	Proposed Method			Runge-Kutta Method		
x	N	$y''(x_\infty)$	$y(x_\infty)$	N	$y''(x_\infty)$	$y(x_\infty)$
0.1	9	0.5223955323	0.6065298823	27	0.522394253	0.606530550
0.2	17	0.03825982349	1.510386946	51	0.0382595394	1.510388234
0.3	25	0.0014085063	2.502848721	75	0.0014082032	2.502849911
0.4	33	0.0000245898	3.502571462	99	0.0000245779	3.502571249

Results Discussion

From tables 1 to 3, it is evident that the proposed methods have a good performance compared with the existing Runge-Kutta method.

CONCLUSION

In this paper, BUMMs have been proposed using the boundary value technique to solve boundary layer flow problems in ordinary differential equations. This has been done by applying the method directly to the differential equations. The convergence of this class of methods was carried out and numerical examples were given. The efficiency of the methods was given in the Tables 1, 2 and 3. In all three tables, the accuracy of the results can be comparable as the proposed methods have a good performance in comparison to the Runge-Kutta method.

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