19 A TWO-STEP HYBRID BLOCK FALKNER-TYPE METHOD FOR SOLVING GENERAL SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

Umaru Mohammed[1*], Garba Jamiu[2] and **Alhassan Alhassan[3]**
[1]*Department of Mathematics, Yusuf Maitama Sule University, Kano, Nigeria*
[1,2,3]*Department of Mathematics, Federal University of Technology, Minna*
Email: [1]umaru.mohd@futminna.edu.ng,[2]garbajamiu90@gmail.com

Abstract
One distinct family of methods for the numerical approximation of general and special second order ordinary differential equation is the Falkner-type methods which consists of a couple of rational formulas, one to follow the solution and the order to follow the derivative. In this paper, we explore this method by introducing a number of off-step points in order to increase the number of function evaluation in the derivation process of a two-step Falkner-type method through the interpolation and collocation technique. The two main Falkner formulas and the additional ones to complete the block procedure are obtained from a continuous formulation. The basic properties of the proposed method were investigated and found to be zero-stable and of order $p=9$ *which implies convergence.The performance of the new method was shown through some numerical examples and found to have higher accuracy than the existing methods considered in the literature.*

Keywords: *Falkner-type method, two-step, off-step points, block method.*

1. Introduction

Differential equation of the form

$$y''(x)=f\left(x,y(x),y'(x)\right),\ y(a)=y_0,\ y'(a)=y'_0 \tag{1}$$

where $x\in[a,b],\ y:[a,b]\to\mathbb{R}$ and $f:[a,b]\times\mathbb{R}\to\mathbb{R}$ are sufficiently differentiable functions; is usually used to model numerous problems such as chemical kinetics, orbital dynamics, circuit and control theory and Newton's second law of motion. However, in most cases, the differential equations so formed for these real life problems often do not have analytical solution. Therefore one of the possible ways to tackle this problem is to consider a discrete domain rather a continuous one. Hence for practical purposes such as engineering, a numerical approximation to the solution is often sufficient. Although it is possible to integrate (1) by reducing it to a first-order system and applying one of the methods available for such systems, it however, seems natural to employ numerical methods to integrate the problem directly as this result to more efficiency of the method (Ramos *et al.*, 2016, Mohammed *et al.*, 2010, Mohammed *et al.*, 2019, Badmus and Yahaya, 2009, Awoyemi, 2001). Scholars have proposed numerous numerical methods for approximating initial value problems such as (1); these methods range from discrete schemes (Lambert, 1973; Butcher, 2008; Fatunla, 1988) to predictor corrector methods (Onuman;yi*et al.*, 1994; Fatunla 1994; Awoyemi, and Idowu, 2005; Areo and Adeniyi, 2013; Omar and Kuboye, 2015; Ndanusa and Tafida, 2016) and then block methods ((Badmus and Yahaya, 2009; Jator and Li, 2012; Mohammed, 2011; Mohammed and Adeniyi, 2014; Badmus, *et al.*, 2015; Akinfenwa, *et al.*, 2013; Omar and Adeyeye, 2016; Akinfenwa*et al.*, 2017).

One distinct family of methods for the numerical approximation of (1) is the Falkner-type methods (see Falkner, 1936) which can be written in the form:

$$y_{n+1}=y_n+hy'_n+h^2\sum_{j=0}^{k-1}\beta_j\nabla^j f_n \tag{2}$$

$$y'_{n+1}=y'_n+h^2\sum_{j=0}^{k-1}\gamma_j\nabla^j f_n \tag{3}$$

where h is the step-size, y_n and y'_n are numerical approximations to the theoretical solution and its derivative at the grid point $x_n=a+nh;\ n=0,1,2,3,...,N,\ h=\frac{(b-a)}{N},\ f_n=f\left(x_n,y_n,y'_n\right)$ and $\nabla^j f_n$ is the standard notation for the backward differences.

There exist similar implicit Falkner formulas (see Collatz, 1966) given by

$$y_{n+1}=y_n+hy'_n+h^2\sum_{j=0}^{k}\beta_j\nabla f_{n+1} \tag{4}$$

$$y'_{n+1}=y'_n+h^2\sum_{j=0}^{k}\gamma_j\nabla^j f_{n+1} \tag{5}$$

We note that the formulas given in (4) and (5) are the Adams-Bashforth and Adams-Moulton methods respectively for solving the problem

$$\left(y'(x)\right)' = f\left(x, y(x), y'(x)\right)$$

which are used to obtain the values of the first derivatives.

The usual and unusual implementation of these methods have been considered in the literature. For instance, in molecular dynamics, when the acceleration at time only depends on position and not on velocity, the direct integration methods are usually implemented in a semi-implicit formulation. This is the case for the well known Velocity Verlet algorithm (Swope, *et al.*, 1982). This method uses the one-step explicit method in (2) to compute the positions.

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2} y''_n \qquad (6)$$

and the one-step implicit method in (5) to update the velocities

$$y'_{n+1} = y'_n + \frac{h^2}{2}\left(4y''_n + y''_{n+1}\right) \qquad (7)$$

Beeman (1976) proposed the modification of the Verlet family of methods for the calculation of velocities. The method used to compute the positions at time $x_n + h$ is the following two-step explicit method given in (2)

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{6}\left(4f_n - f_{n-1}\right) \qquad (8)$$

while the formula to update the velocities is the following two-step method

$$y'_{n+1} = hy'_n + \frac{h}{6}\left(2f_{n+1} + 5f_n - f_{n-1}\right) \qquad (9)$$

In this paper, we present the hybrid-block form of the Falkner formulas where generalized 6 off-step points are considered within $0 \le x \le 2$ in order to increase the number of function evaluation.

2. Development of the method

In developing the new 2-step Falkner type computational method for solving general and special second order differential equation in (1), we shall consider the power series as a basis function in the form

$$y(x) = \sum_{j=0}^{r+s-1} a_j x^j \qquad (2)$$

on the partition

$a = x_0 < x_1 < \ldots < x_n < x_{n+1} < \ldots < x_N = b$ of the interval of integration $[a,b]$, with a constant step size h, given by $h = x_{n+1} - x_n$; $n = 0,1,\ldots,N-1$. a_j's are unknown coefficients to be determined r and s are numbers of interpolation and collocation points respectively. We impose that the interpolating function (2) coincides with the analytical solution at the end point x_n $(r=1)$ to obtain the equation

$$y(x_n) = y_n \qquad (3)$$

Also if the function (2) satisfies the differential equation (1), we demand that in order to obtain the Falkner type method, we collocate the first derivative of (2) at x_n to obtain the following equation

$$y'(x_n) = y'_n \qquad (4)$$

and collocate its second derivative at the grid points (0,1,2) and at the carefully selected off-step points

$v = \left(\frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{6}{5}, \frac{7}{5}, \frac{9}{5}\right)$ to obtain

$$y''\left(x_{n+vj}\right) = f_{n+vj} \qquad (5)$$

We emphasize that equations (3), (4) and (5) lead to a system of eleven equations which is solved using the matrix inversion method to obtain a_j's. The proposed 2-step Falkner method is constructed by substituting the values of a_j's into equation (2) and then simplified to obtain the continuous representation of the method in the form

$$y(x) = y_n + xy'_n + \beta_{n+j}(x) f_{n+j} + \beta_{n+v}(x) f_{n+v} \qquad (6)$$

Differentiating (6) we get the first derivative of the continuous scheme as $y'(x) = y'_n + \beta'_{n+j}(x) f_{n+j} + \beta'_{n+v}(x) f_{n+v}$ (7)

Evaluating (6) and (7) at $\left(\frac{2}{5},\frac{3}{5},\frac{4}{5},1,\frac{6}{5},\frac{7}{5},\frac{9}{5},2\right)$ respectively, we obtain the 2-step block hybrid Falkner-type method defined in general matrix form as

$$A_1Y_m = A_0Y_{m-2} + h\left(B_0Y'_{m-2}\right) + h^2\left(C_0F_{m-2} + C_1F_m\right) \tag{8}$$

and $B_1Y'_m = B_{0,1}Y'_{m-2} + h\left(D_0F_{m-2} + D_1F_m\right)$ (9)

where Y_m, Y'_m and F_m are vectors defined as

$$Y_m = \left[y_{n+\frac{2}{5}}, y_{n+\frac{3}{5}}, y_{n+\frac{4}{5}}, y_{n+1}, y_{n+\frac{6}{5}}, y_{n+\frac{7}{5}}, y_{n+\frac{9}{5}}, y_{n+2}\right]^T, Y_{m-2} = \left[y_{n-\frac{8}{5}}, y_{n-\frac{7}{5}}, y_{n-\frac{6}{5}}, y_{n-1}, y_{n-\frac{4}{5}}, y_{n-\frac{3}{5}}, y_{n-\frac{1}{5}}, y_n\right]^T,$$

$$Y'_m = \left[y'_{n+\frac{2}{5}}, y'_{n+\frac{3}{5}}, y'_{n+\frac{4}{5}}, y'_{n+1}, y'_{n+\frac{6}{5}}, y'_{n+\frac{7}{5}}, y'_{n+\frac{9}{5}}, y'_{n+2}\right]^T, Y'_{m-2} = \left[y'_{n-\frac{8}{5}}, y'_{n-\frac{7}{5}}, y'_{n-\frac{6}{5}}, y'_{n-1}, y'_{n-\frac{4}{5}}, y'_{n-\frac{3}{5}}, y'_{n-\frac{1}{5}}, y'_n\right]^T,$$

$$F_m = \left[f_{n+\frac{2}{5}}, f_{n+\frac{3}{5}}, f_{n+\frac{4}{5}}, f_{n+1}, f_{n+\frac{6}{5}}, f_{n+\frac{7}{5}}, f_{n+\frac{9}{5}}, f_{n+2}\right]^T, F_{m-2} = \left[f_{n-\frac{8}{5}}, f_{n-\frac{7}{5}}, f_{n-\frac{6}{5}}, f_{n-1}, f_{n-\frac{4}{5}}, f_{n-\frac{3}{5}}, f_{n-\frac{1}{5}}, f_n\right]^T$$

$$A_1 = \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix} A_0 = \begin{pmatrix} 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1 \end{pmatrix}$$

$$B_0 = \begin{pmatrix} 0&0&0&0&0&0&0&\frac{2}{5}\\ 0&0&0&0&0&0&0&\frac{3}{5}\\ 0&0&0&0&0&0&0&\frac{4}{5}\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&\frac{6}{5}\\ 0&0&0&0&0&0&0&\frac{7}{5}\\ 0&0&0&0&0&0&0&\frac{9}{5}\\ 0&0&0&0&0&0&0&2 \end{pmatrix} B_1 = \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix} B_{0,1} = \begin{pmatrix} 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1 \end{pmatrix}$$

$$C_1 = \begin{pmatrix} \frac{1687517}{6615000} & -\frac{103178}{165375} & \frac{64924}{70875} & -\frac{500114}{590625} & \frac{27131}{56700} & -\frac{343162}{2480625} & \frac{17422}{1488375} & -\frac{233509}{99225000}\\ \frac{7819983}{15680000} & -\frac{87609}{78400} & \frac{458217}{280000} & -\frac{131949}{87500} & \frac{190497}{224000} & -\frac{481221}{1960000} & \frac{20323}{980000} & -\frac{326673}{78400000}\\ \frac{41098}{55125} & -\frac{2346656}{1488375} & \frac{835568}{354375} & -\frac{47392}{21875} & \frac{259556}{212625} & -\frac{174752}{496125} & \frac{663776}{22325625} & -\frac{74078}{12403125}\\ \frac{672065}{677376} & -\frac{129025}{63504} & \frac{8065}{2592} & -\frac{34123}{12096} & \frac{76975}{48384} & -\frac{58295}{127008} & \frac{29515}{762048} & -\frac{15809}{2032128}\\ \frac{303507}{245000} & -\frac{15234}{6125} & \frac{16929}{4375} & -\frac{15066}{4375} & \frac{6879}{3500} & -\frac{17334}{30625} & \frac{1462}{30625} & -\frac{2349}{245000}\\ \frac{4278239}{2880000} & -\frac{5720897}{1944000} & \frac{2998849}{648000} & -\frac{3656723}{900000} & \frac{3690337}{1555200} & -\frac{2168887}{3240000} & \frac{82663}{1458000} & -\frac{1475929}{129600000}\\ \frac{6187023}{3136000} & -\frac{749169}{196000} & \frac{212139}{35000} & -\frac{7236783}{1400000} & \frac{685017}{224000} & -\frac{4374}{6125} & \frac{173151}{1960000} & -\frac{1275021}{78400000}\\ \frac{2605}{1176} & -\frac{50650}{11907} & \frac{3830}{567} & -\frac{358}{63} & \frac{22825}{6804} & -\frac{2770}{3969} & \frac{5210}{35721} & -\frac{521}{31752} \end{pmatrix}$$

$$C_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1116326}{37209375} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{238897}{4900000} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{7534328}{111628125} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{65717}{762048} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{643}{6125} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{4509617}{36450000} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1581093}{9800000} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{6436}{35721} \end{pmatrix}$$

$$D_1 = \begin{pmatrix} \frac{1544381}{1323000} & -\frac{151783}{59535} & \frac{1038277}{283500} & -\frac{56543}{16875} & \frac{22867}{12150} & -\frac{268871}{496125} & \frac{203741}{4465125} & -\frac{181751}{19845000} \\ \frac{1938789}{1568000} & -\frac{73931}{31360} & \frac{397701}{112000} & -\frac{261603}{80000} & \frac{2947}{1600} & -\frac{416457}{784000} & \frac{35123}{784000} & -\frac{70533}{7840000} \\ \frac{203789}{165375} & -\frac{135104}{59535} & \frac{262046}{70875} & -\frac{55936}{16875} & \frac{11302}{6075} & -\frac{265792}{496125} & \frac{201472}{4465125} & -\frac{22469}{2480625} \\ \frac{417875}{338688} & -\frac{3476125}{1524096} & \frac{276025}{72576} & -\frac{11041}{3456} & \frac{28625}{15552} & -\frac{270125}{508032} & \frac{205175}{4572288} & -\frac{9157}{1016064} \\ \frac{60399}{49000} & -\frac{557}{245} & \frac{13221}{3500} & -\frac{1917}{625} & \frac{97}{50} & -\frac{3303}{6125} & \frac{277}{6125} & -\frac{2223}{245000} \\ \frac{1066583}{864000} & -\frac{355201}{155520} & \frac{4935427}{1296000} & -\frac{6758129}{2160000} & \frac{817369}{388800} & -\frac{597737}{1296000} & \frac{515627}{11664000} & -\frac{115493}{1296000} \\ \frac{1875717}{1568000} & -\frac{65259}{31360} & \frac{367173}{112000} & -\frac{182979}{80000} & \frac{1971}{1600} & \frac{158679}{784000} & \frac{143379}{784000} & -\frac{141669}{7840000} \\ \frac{13025}{35721} & -\frac{26875}{11907} & \frac{8425}{2268} & -\frac{79}{27} & \frac{875}{486} & -\frac{275}{3969} & \frac{13025}{35721} & \frac{1529}{31752} \end{pmatrix}$$

$$D_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{8385473}{89302500} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{367201}{3920000} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2092474}{22325625} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{428425}{4572288} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{11481}{122500} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5463689}{58320000} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{370593}{3920000} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{13393}{142884} \end{pmatrix}$$

4. Analysis of the method

4.1. Local truncation error and order of accuracy.

Following the definition of Fatunla (1991) and Lambert (1973), we define the local truncation error associated with the conventional forms of (8) and (9) to be the linear difference operators

$$L\left[y\left(x_n\right);h\right]=\sum_{j=0}^{k}\left(\alpha_j y\left(x_n+jh\right)-h\beta y'\left(x_n\right)-h^2\gamma_{vj} f\left(x_n+jvh\right)\right) \qquad 10)$$

and

$$L\left[y'\left(x_n\right);h\right]=\sum_{j=0}^{k}\left(h\bar{\beta}_j y'\left(x_n+jvh\right)-h^2\bar{\gamma}_{vj} hf\left(x_n+jvh\right)\right) \qquad (11)$$

respectively. Assuming that $y\left(x_n\right)$ and $y'\left(x_n\right)$ are sufficiently differentiable, we can expand the terms in (10) and (11) as Taylor series about the point x_n to obtain the expression

$$L\left[y\left(x_n\right);h\right]=C_0 y\left(x_n\right)+C_1 hy'\left(x_n\right)+\ldots+C_q h^q y^{(q)}\left(x_n\right)+\ldots \qquad (12)$$

and

$$L\left[y'\left(x_n\right);h\right]=\bar{C}_0 y'\left(x_n\right)+\bar{C}_1 hy''\left(x_n\right)+\ldots+\bar{C}_q h^q y^{(q+1)}\left(x_n\right)+\ldots \qquad (13)$$

respectively;

where the constants C_q and $\bar{C}_q$ $q=0,1,\ldots$ are given as follows

$$\left.\begin{aligned}
C_0 &= \sum_{j=0}^{k}\alpha_j \\
C_1 &= \sum_{j=1}^{k} j\alpha_j \\
C_2 &= \frac{1}{2!}\sum_{j=1}^{k}\left(j\right)^2\alpha_j-\sum_{j=0}^{k}\beta_j \\
&\vdots \\
C_q &= \frac{1}{\text{q}!}\sum_{j=1}^{k}\left(j\right)^q\alpha_j-\frac{1}{\left(q-2\right)!}\sum_{j=1}^{k} j^{q-1}\beta_j
\end{aligned}\right\} \qquad (13)$$

$q=2,3,\ldots$

$$\left.\begin{aligned}
\bar{C}_0 &= \sum_{j=0}^{k}\bar{\alpha}_j \\
\bar{C}_1 &= \sum_{j=1}^{k} j\bar{\alpha}_j-\sum_{j=0}^{k}\bar{\beta}_j \\
\bar{C}_2 &= \frac{1}{2!}\sum_{j=1}^{k}\left(j\right)^2\bar{\alpha}_j-\sum_{j=1}^{k} j\bar{\beta}_j \\
&\vdots \\
\bar{C}_q &= \frac{1}{\text{q}!}\sum_{j=1}^{k}\left(j\right)^q\bar{\alpha}_j-\frac{1}{\left(q-1\right)!}\sum_{j=1}^{k} j^{q-1}\bar{\beta}_j
\end{aligned}\right\} \qquad (14)$$

According to Henrici (1962), we say the methods (8) and (9) are of order p if $C_0=C_1=\ldots C_p=C_{p+1}=0,\ C_{p+2}\neq 0$ and C_{p+2} is the error constant and $C_{p+2}h^{p+2}y^{(p+2)}\left(x_n\right)$ the principal truncation error at the point x_n.

From our analysis, the block method (8) and (9) have a uniform order $p=9$ with relative small error constants

$$C_{p+2}=C_{11}=\begin{pmatrix} \frac{671233}{106589355468750}, & \frac{3737673}{336875000000000}, & \frac{7614548}{479652099609375}, & \frac{541157}{26195400000000}, \\ \frac{3348}{131591796875}, & \frac{151458853}{5011875000000000}, & \frac{13630599}{336875000000000}, & \frac{14027}{306977343750} \end{pmatrix}^T$$

$$\bar{C}_{p+2}=\bar{C}_{11}=\begin{pmatrix} \frac{1687517}{69767578125000}, & \frac{29203}{1225000000000}, & \frac{208892}{8720947265625}, & \frac{34123}{1428840000000}, \\ \frac{2293}{95703125000}, & \frac{2168887}{91125000000000}, & \frac{173151}{6125000000000}, & \frac{521}{22325625000} \end{pmatrix}^T$$

4.2. Zero Stability

Zero-stability is concerned with the stability of the difference system in the limit as h tends to zero (Akinfenwa*et al., 2018).* Thus, as $h \to 0$, the method (8) tends to the difference system

$$A^{(1)}Y_m - A^{(0)}Y_{m-2} = 0 \tag{14}$$

whose first characteristic polynomial $\rho(\lambda)$ is given by

$$\rho(\lambda) = \left|\lambda A^{(1)} - A^{0}\right| \tag{15}$$

Definition (Zero-stability): The block method (8) is said to be zero stable if the roots of the first characteristic polynomial $\rho(\lambda)$ satisfies $|\lambda_j| \le 1,\ j=1,2,3,\ldots$ and for those roots with $|\lambda_j|=1$, the multiplicity must not exceed 2 (Fatunla, 1991).

$$\begin{gathered}\rho(\lambda) = \lambda^7(\lambda - 1) = 0 \\ \lambda = \{0,0,0,0,0,0,0,1\}\end{gathered} \tag{16}$$

Therefore, our method (8) is zero stable since is satisfies $|\lambda_j| \le 1.$

4.2. Consistency: The block method (8) is consistent if it has order of accuracy $p>1.$ According to Henrici the method is convergent, since the necessary and sufficient condition for convergence is for the method to be zero-stable and consistent.

5. Numerical Examples

We consider various problems of the type (1) to test the performance of the two-step block hybrid Falkner-type method and the errors obtained from the solutions are compared with some of its kinds in the literature. For the purpose of comparative analysis, the following notations are adopted.

- $HFBM_{2,1}$: 2-step, one off-step hybrid block Falkner-type method by Nicholas (2019)
- $HFBM_{2,2}$: 2-step, 2 off-step hybrid block Falkner-type method by Nicholas (2019)
- $HFBM_{2,4.}$: 2-step, one off-grid hybrid block Falkner-type method by Nicholas (2019)
- $HFBM_{2,4.}$: 4-step, two off-grid hybrid block Falkner-type method by Nicholas (2019)
- BFM_6: Block Falkner method for k=6 by Ramos *et al.*, (2016)

Problem 1. (*Source:* Ramos *et al.* (2016))

Consider the non-linear homogeneous problem given by

$$y'' = x(y')^2,\ \ y(0)=1,\ y'(0)=\frac{1}{2},\ 0 \le x \le 1$$

with the exact solution $y(x) = 1 + \frac{1}{2}\ln\left(\frac{2+x}{2-x}\right)$

Table 5.1: comparison of absolute errors for problem 1

X	Numerical solution	James *et al.* (2013) h=0.1	BFM_6 h=0.05	Mohammad and Zurni (2017), h=0.05	$HFBM_{22}$ h=0.1	New Method h=0.1
0.1	1.0500417292784912678	$1.110*10^{-15}$	$3.114*10^{-12}$	$2.220*10^{-16}$	$2.000*10^{-12}$	$4.112*10^{-19}$
0.2	1.1003353477310755792	$5.995*10^{-15}$	$6.660*10^{-12}$	$2.220*10^{-16}$	$3.000*10^{-12}$	$1.388*10^{-18}$
0.3	1.1511404359364668019	$2.554*10^{-14}$	$9.833*10^{-12}$	$6.661*10^{-16}$	$6.000*10^{-12}$	$3.305*10^{-18}$
0.4	1.2027325540540821840	$7.105*10^{-14}$	$2.173*10^{-11}$	$1.110*10^{-15}$	$9.000*10^{-11}$	$6.972*10^{-18}$
0.5	1.2554128118829953275	$1.157*10^{-13}$	$3.570*10^{-11}$	$4.440*10^{-16}$	$1.400*10^{-11}$	$1.412*10^{-17}$
0.6	1.3095196042031116868	$1.199*10^{-13}$	$4.859*10^{-11}$	$8.881*10^{-16}$	$2.200*10^{-11}$	$2.860*10^{-17}$
0.7	1.3654437542713961096	$6.857*10^{-13}$	$1.310*10^{-10}$	$1.554*10^{-15}$	$3.500*10^{-12}$	$5.932*10^{-17}$
0.8	1.4236489301936016784	$3.475*10^{-12}$	$2.313*10^{-10}$	$4.440*10^{-15}$	$5.900*10^{-11}$	$1.284*10^{-16}$
0.9	1.4847002785940514471	$1.222*10^{-11}$	$3.286*10^{-10}$	$8.660*10^{-16}$	$1.010*10^{-10}$	$2.943*10^{-16}$
1.0	1.5493061443340541188	$7.728*10^{-11}$	$1.335*10^{-09}$	$1.266*10^{-14}$	-	$7.267*10^{-16}$

Problem 2. (*Source:* Ramos *et al.* (2016))
Consider a linear homogeneous problem given by

$$y'' = y',\ \ y(0) = 0,\ y'(0) = -1,\ 0 \le x \le 1$$

with the exact solution $y(x) = 1 - e^{-x}$

Table 5.2: comparison of absolute errors for problem 2

x	Kayode and Adeyeye. (2013), h=0.1	BFM_6 h=0.1	Mohammad and Zurni (2017), h=0.01	$HFBM_{,4}$ h=0.1	New Method h=0.1
0.2	$8.171*10^{-07}$	$2.427*10^{-11}$	$1.388*10^{-16}$	$2.000*10^{-12}$	$5.556*10^{-19}$
0.3	$3.103*10^{-06}$	$4.001*10^{-11}$	$3.331*10^{-16}$	$1.000*10^{-12}$	$1.178*10^{-19}$
0.4	$6.569*10^{-06}$	$5.746*10^{-11}$	$4.996*10^{-16}$	$1.010*10^{-12}$	$1.930*10^{-18}$
0.5	$1.143*10^{-05}$	$7.741*10^{-11}$	$7.772*10^{-16}$	$1.400*10^{-11}$	$3.093*10^{-18}$
0.6	$1.796*10^{-05}$	$9.517*10^{-11}$	$1.332*10^{-15}$	$2.100*10^{-11}$	$4.457*10^{-18}$
0.7	$2.644*10^{-05}$	$1.221*10^{-10}$	$1.776*10^{-15}$	$3.000*10^{-12}$	$6.372*10^{-18}$
0.8	$3.722*10^{-05}$	$1.604*10^{-10}$	$2.887*10^{-15}$	$4.000*10^{-11}$	$8.582*10^{-18}$
0.9	$5.067*10^{-05}$	$2.013*10^{-10}$	$3.775*10^{-15}$	$5.000*10^{-11}$	$1.152*10^{-17}$
1.0	$5.255*10^{-05}$	$2.466*10^{-10}$	$5.107*10^{-15}$	-	$1.489*10^{-17}$

Problem 3. (*Source:* Adediran and Ogundare,(2015))
Consider a highly stiff initial value problem given by

$$y'' = -100y' - 1000y,\ \ y(0) = 1,\ y'(0) = -1,\ 0 \le x \le 1$$

with the exact solution $y(x) = e^{-x}$

Table 5.3: comparison of absolute errors for problem 3 with h=0.1

x	Numerical solution	Adediran and Ogundare. (2015)	Mohammad and Zurni (2017)	New Method
0.1	0.90483741803595957330	$2.050*10^{-11}$	$1.055*10^{-14}$	$1.400*10^{-19}$
0.2	0.81873075307798185950	$4.390*10^{-11}$	$1.776*10^{-14}$	$8.300*10^{-19}$
0.3	0.74081822068171786593	$6.550*10^{-11}$	$2.342*10^{-14}$	$1.400*10^{-19}$
0.4	0.67032004603563930116	$8.380*10^{-11}$	$2.798*10^{-14}$	$4.200*10^{-19}$
0.5	0.60653065971263342286	$9.860*10^{-10}$	$3.131*10^{-14}$	$7.400*10^{-19}$
0.6	0.54881163609402643237	$1.100*10^{-10}$	$3.397*10^{-14}$	$2.600*10^{-19}$
0.7	0.49658530379140951345	$1.190*10^{-10}$	$3.564*10^{-14}$	$1.250*10^{-18}$
0.8	0.44932896411722159065	$1.240*10^{-10}$	$3.675*10^{-14}$	$7.800*10^{-19}$
0.9	0.40656965974059911030	$1.280*10^{-10}$	$3.730*10^{-14}$	$1.580*10^{-18}$
1.0	0.36787944117144232048	$1.300*10^{-10}$	$3.741*10^{-14}$	$1.120*10^{-18}$

Table 5.4: comparison of absolute errors for problem 4 with h=0.05

x	Numerical solution	Adeyefa (2017)	$HFBM_{4,2}$	$HFBM_{4,3}$	New Method
0.1	0.90483741803595957228	$3.240*10^{-15}$	$1.365*10^{-16}$	$8.130*10^{-18}$	$8.800*10^{-19}$
0.2	0.81873075307798185656	$1.794*10^{-14}$	$7.844*10^{-17}$	$7.680*10^{-18}$	$2.110*10^{-18}$
0.3	0.74081822068171786286	$6.910*10^{-14}$	$3.090*10^{-16}$	$2.745*10^{-17}$	$3.210*10^{-18}$
0.4	0.67032004603563929664	$2.372*10^{-13}$	$2.400*10^{-16}$	$1.074*10^{-17}$	$4.100*10^{-18}$
0.5	0.60653065971263341881	$7.810*10^{-13}$	$4.203*10^{-16}$	$9.310*10^{-18}$	$4.790*10^{-18}$

Problem 4. Dynamic Problem (*Source:* Nicholas (2019))

A 10kg mass is attached to a spring having a constant of 140N/m. The mass is started in motion from the equilibrium position with an initial value of 1m/sec in upward direction and with an applied external force $F(t)=0.5\sin(t)$. The resulting equation due to air resistance $90y'N$ is given as

$$y''+9y'+14y=\frac{1}{2}\text{sint},\ \ y(0)=0,\ y'(0)=-1,\ 0\le x\le 0.1$$

with the exact solution $y(t)=-\frac{9}{50}e^{-2t}+\frac{99}{500}e^{-7t}-\frac{9}{500}\cos(t)+\frac{13}{500}\sin(t)$

Table 5.5: Numerical solution of problem 4 with h=0.001

x	Numerical solution
0.01	-0.00956088919464984680416649694081838 40
0.02	-0.018285603224426365353666983008451928
0.03	-0.026233952945284610061734300806231079
0.04	-0.033461640772405912220290869637185496
0.05	-0.040020539768239219056775516866554309
0.06	-0.045958953836075225438313904734123363
0.07	-0.051321860297053846359402482277050722
0.08	-0.056151136042101352630537395135211578
0.09	-0.060485768369730929574669608793358429
0.10	-0.064362051545524582478780919666828961

Table 5.6: comparison of absolute errors for problem 3 with h=0.1

x	$HFBM_{2,1}$	$HFBM_{2,2}$	$HFBM_{2,4}$	New Method
0.01	$1.304*10^{-10}$	$4.500*10^{-13}$	$1.700*10^{-13}$	$2.567*10^{-31}$
0.02	$3.323*10^{-10}$	$1.000*10^{-13}$	$4.000*10^{-13}$	$8.954*10^{-31}$
0.03	$6.448*10^{-10}$	$6.000*10^{-13}$	$2.000*10^{-15}$	$1.855*10^{-30}$
0.04	$1.003*10^{-09}$	$1.500*10^{-12}$	$7.130*10^{-13}$	$3.081*10^{-30}$
0.05	$1.438*10^{-09}$	$9.000*10^{-12}$	$1.000*10^{-15}$	$4.525*10^{-30}$
0.06	$1.899*10^{-09}$	$1.400*10^{-12}$	$4.000*10^{-13}$	$6.144*10^{-30}$
0.07	$2.412*10^{-09}$	$2.001*10^{-12}$	$1.010*10^{-12}$	$7.901*10^{-30}$
0.08	$2.933*10^{-09}$	$1.500*10^{-12}$	$4.000*10^{-13}$	$9.762*10^{-30}$
0.09	$3.489*10^{-09}$	$1.600*10^{-12}$	$5.000*10^{-13}$	$1.170*10^{-29}$
0.10	$4.041*10^{-09}$	$1.400*10^{-12}$	$3.000*10^{-13}$	$1.368*10^{-29}$

Problem 5. Van Der Pol Oscillator (*Source:* Mohammed *et al.*,(2019))

$$y''-2\xi\left(1-y^2\right)y'-y=0,\ \ y(0)=0,\ y'(0)=0.5,\ \ \xi=0.025,\ \ 0\le x\le 1$$

This problem has no exact solution, our result is however validated usingRunge-Kutta (RK45) and compared with Mohammed *et al.,* (2019).

Table 5.7: Result for the Van Der Pol Oscillator Problem with h=0.1

x	RK(5)	4(2019)	New Method
0.0	0	0	0
1.0	0.431051	0.431051	0.43105
2.0	0.47631	0.476309	0.47636
3.0	0.076077	0.076076	0.076241
4.0	-0.41546	-0.41546	-0.41532
5.0	-0.53857	-0.53857	-0.53866

6.0	-0.16135	-0.16134	-0.16167
7.0	0.386024	0.386025	0.38573
8.0	0.595231	0.59523	0.59530
9.0	0.254655	0.254653	0.25509
10.0	-0.34157	0.34158	-0.34110

6. Conclusion

In this paper, we have developed a modified 2-step hybrid block linear multistep method of Falkner type to solve initial value problem of general and special second order ordinary differential equations. Our method is found to be zero stable, consistent and convergent. The numerical results show that our method is computationally reliable and gave better accuracy than the existing methods found in the literature.

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