CHAPTER ONE

FIRST -ORDER PARTIAL DIFFERENTIAL EQUATIONS.

1.1 DERIVATION OF PARTIAL DIFFERENTIAL EQUATIONS.

Consider the family of surfaces

$$f(x, y, u, a, b) = 0$$

where a and b are constants and u is dependent on x and y $(x, y$ are independent variables$)$.

To derive an appropriate partial differential equation (PDE) from $(1.1.4)$ we eliminate the constants a and

Differentiating $(1.1.4)$ wrt x and y we have the following equations :respectively:

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} = 0 \qquad 1.1.5$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} = 0 \qquad 1.1.6$$

Eliminating the constants a and b from $(1.1.4),(1.1.5)$ and $(1.1.6)$ we obtain a general relation

$$F(x, y, u, p, q) = 0 \qquad 1.1.7$$

Eqn$(1.1.7)$ is in general a *first - order PDE* if the number of constants to be eliminated is the same as that of the independent variables and is of *higher order* if the number is greater than the number of the independent variables.

Derivation.

Consider the family of surfaces

$$\phi(f, g) = 0 \qquad 1.1.8$$

where ϕ is an arbitrary differentiable function of f and g that are in turn known differentiable functions of some independent variable x and y with u also a differentiable function of x and y.

Differentiating ϕ wrt x and y we have

$$\frac{\partial \phi}{\partial f}.\frac{\partial f}{\partial x} + \frac{\partial \phi}{\partial f}.\frac{\partial f}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial g}.\frac{\partial g}{\partial x} + \frac{\partial \phi}{\partial g}.\frac{\partial g}{\partial u}.\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial \phi}{\partial f}.\frac{\partial f}{\partial y} + \frac{\partial \phi}{\partial f}.\frac{\partial f}{\partial u}.\frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial g}.\frac{\partial g}{\partial y} + \frac{\partial \phi}{\partial g}.\frac{\partial g}{\partial u}.\frac{\partial u}{\partial y} = 0$$

ie,

$$\left.\begin{aligned} \frac{\partial \phi}{\partial f}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}.p\right) + \frac{\partial \phi}{\partial g}\left(.\frac{\partial g}{\partial x} + \frac{\partial g}{\partial u}p\right) = 0 \\ \frac{\partial \phi}{\partial f}\left(\frac{\partial f}{\partial y} + \frac{\partial f}{\partial u}.q\right) + \frac{\partial \phi}{\partial g}\left(.\frac{\partial g}{\partial y} + \frac{\partial g}{\partial u}q\right) = 0 \end{aligned}\right\} \qquad 1.1.9$$

Eliminating $\frac{\partial \phi}{\partial f}$ and $\frac{\partial \phi}{\partial g}$ we thus have

$$\begin{vmatrix} \frac{\partial f}{\partial x}+\frac{\partial f}{\partial u}.p & \frac{\partial g}{\partial x}+\frac{\partial g}{\partial u}.p \\ \frac{\partial f}{\partial y}+\frac{\partial f}{\partial u}.q & \frac{\partial g}{\partial y}+\frac{\partial g}{\partial u}.q \end{vmatrix}=0 \qquad (1.1.10)$$

Eqn$(1.1.10)$ is equivalent to

$$P.p+Q,q=R \qquad (1.1.11)$$

where

$$P=\frac{\partial(f,g)}{\partial(y,u)},Q=\frac{\partial(f,g)}{\partial(x,u)} \text{ and } R=\frac{\partial(f,g)}{\partial(x,y)} \qquad (1.1.12)$$

Eqn$(1.1.12)$ is first-order differential equation.

Example.

Eliminate a and b from the following families of surfaces to obtain a *PDE*.

$$(x-a)^2+(y-b)^2+u^2=d^2 \qquad (i)$$

Solution

Differentiating (i) partially wry x and y yeilds

$$2(x-a)+2u\frac{\partial u}{\partial x}=0, ie,\ (x-a)+up=0 \qquad (ii)$$

$$2(y-b)+2u\frac{\partial u}{\partial y}=0\ ie,\ (y-b)+uq=0 \qquad (iii)$$

Eliminate a and b from $(i),(ii)$ and (iii) yields

$$(-up)^2+(-uq)^2+u^2=d^2 \qquad (iv)$$

ie,

$$(p^2+q^2+1)u^2=d^2 \qquad (v)$$

Eqn(v) is first-order differential equation.

2 Form a PDE from the family of integral surfaces

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{u^2}{c^2}=1 \qquad (i)$$

Solution

Differentiating (i) partially wry x yeilds

$$\frac{2x}{a^2}+\frac{2u}{c^2}\frac{\partial u}{\partial x}=0 \Rightarrow u\frac{\partial u}{\partial x}=-\frac{c^2}{a^2}x \qquad (ii)$$

Differentiating (i) partially wry y yeilds

$$\frac{2y}{b^2}+\frac{2u}{c^2}\frac{\partial u}{\partial y}=0 \Rightarrow u\frac{\partial u}{\partial y}=-\frac{c^2}{b^2}y \qquad (iii)$$

On differentiating (ii) partially wry y or (iii) partially wry x yeilds

$$u\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial u}{\partial y}\frac{\partial u}{\partial x}=0 \qquad (iv)$$

This is a second-order *PDE*.

1.2 SOLUTION OF LAGRANGES LINEAR EQUATION.

The general partial differential equation

$$P.p + Q.q = R \qquad (1.1.13)$$

where $P, Q,$ and R are functions of $x,$ and y is refered to as the Lagranges Linear Equation.

Theorem 1.1

Given eqn $(1.1.13)$ in which

$$\left.\begin{aligned} f(x,y,u)=0 \\ g(x,y,u)=0 \end{aligned}\right\} \qquad (1.1.14)$$

constitute the integral curves of the simultaneous ordinary differential equations $(ODEs)$

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{du}{R} \qquad (1.1.15)$$

Then the general solution of $(1.1.13)$ is given as

$$F(f,g)=0 \qquad (1.1.16)$$

where F is an arbitrary differentiable function. Further $w(x,y,u)=c$ is any solution of $(1.1.13)$ and if first -order derivatives of f, g and w are all continuous then the solution $w-c=0$ is contained in the general solution of $(1.1.16)$.

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Differentiating the relationship $(1.1.14)$ yields

$$\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial u}du=0$$

$$\frac{\partial g}{\partial x}dx+\frac{\partial g}{\partial y}dy+\frac{\partial g}{\partial u}du=0$$

ie,

$$\frac{dx}{\frac{\partial(f,g)}{\partial(y,u)}}=\frac{dy}{\frac{\partial(f,g)}{\partial(x,u)}}=\frac{du}{\frac{\partial(f,g)}{\partial(x,y)}} \qquad (1.1.17)$$

Since $(1.1.15)$ determines the integral curves of $(1.1.16)$ then we have from $(1.1.17)$

$$\frac{P}{\frac{\partial(f,g)}{\partial(y,u)}}=\frac{Q}{\frac{\partial(f,g)}{\partial(x,u)}}=\frac{R}{\frac{\partial(f,g)}{\partial(x,y)}} \qquad (1.1.18)$$

Now considering any functional relation $(1.1.16)$ when F is differentiable we have

$$\left.\begin{aligned} \frac{\partial F}{\partial f}\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial u}.p\right)+\frac{\partial F}{\partial g}\left(.\frac{\partial g}{\partial x}+\frac{\partial g}{\partial u}.p\right)=0 \\ \frac{\partial F}{\partial f}\left(\frac{\partial f}{\partial y}+\frac{\partial f}{\partial u}.q\right)+\frac{\partial F}{\partial g}\left(.\frac{\partial g}{\partial y}+\frac{\partial g}{\partial u}.q\right)=0 \end{aligned}\right\} \qquad (1.1.19)$$

Eliminating $\frac{\partial F}{\partial f}$ and $\frac{\partial F}{\partial g}$ from the above yields

$$\frac{\partial(f,g)}{\partial(y,u)}.p+\frac{\partial(f,g)}{\partial(x,u)}.q=\frac{\partial(f,g)}{\partial(x,y)} \qquad (1.1.20)$$

Comparing $(1.1.13)$ and $(1.1.20)$ we have that

$$P.p+Q.q=R \qquad (1.1.21)$$

showing that $(1.1.11)$ is a solution of $(1.1.8)$. Thus, $(1.1.11)$ is a general solution of $(1.1.8)$.

Consider any solution $w(x,y,u)=c$.

Differentiating partially we have the following:

$$\left.\begin{aligned}\frac{\partial w}{\partial x}+\frac{\partial w}{\partial u}.p=0\\ \frac{\partial w}{\partial y}+\frac{\partial w}{\partial u}.q=0\end{aligned}\right\} \qquad (1.1.22)$$

It therefore follows that,

$$\left.\begin{aligned}p=-\,\frac{\partial w}{\partial x}\Big/\frac{\partial w}{\partial u}\\ q=-\,\frac{\partial w}{\partial y}\Big/\frac{\partial w}{\partial u}\end{aligned}\right\} \qquad (1.1.23)$$

On substituting p and q into $(1.1.8)$ we obtain

$$P\frac{\partial w}{\partial x}+Q\frac{\partial w}{\partial y}+R\frac{\partial w}{\partial u}= \qquad (1.1.24)$$

and in view of the relation $(1.1.13)$ and $(1.1.24)$ we have

$$\frac{\partial(f,g)}{\partial(y,u)}\cdot\frac{\partial w}{\partial x}+\frac{\partial(f,g)}{\partial(x,u)}\cdot\frac{\partial w}{\partial y}+\frac{\partial(f,g)}{\partial(x,y)}\cdot\frac{\partial w}{\partial x}=0 \qquad (1.1.25)$$

ie,

$$J=\frac{\partial(f,g,w)}{\partial(x,y,u)}=0 \qquad (1.1.26)$$

Since the partial derivatives of f,g and w are supposedly continuous, the vanishing of the Jacobian J in $(1.1.26)$ implies a functional relation of the form $w=\phi(f,g)$. Hence, $w-c=\phi(f,g)-c=G(f,g)$, say. Therefore, the solution $w-c=0$ is contained in the general solution $(1.1.11)$. This completes the proof of the theorem.

$$\frac{P_1}{\Delta_1}=\frac{P_2}{\Delta_2}=\frac{P_3}{\Delta_3}=\cdots\cdots=\frac{P_m}{\Delta_m}=\frac{R}{\Delta} \qquad (1.32)$$

Considering the arbitrary relation $F\left(f_1, f_2, f_3, \cdots f_m\right)=0$ and differentiating partially wrt $x_j\left(j=1(1)m\right)$ we have

$$\left.\begin{aligned}
&\frac{\partial F}{\partial f_1}\left(a_{11}-b_1p_1\right)+\frac{\partial F}{\partial f_2}\left(a_{21}-b_2p_1\right)+\cdots\cdots+\cdots\cdots+\frac{\partial F}{\partial f_m}\left(a_{m1}-b_mp_1\right)=0\\
&\frac{\partial F}{\partial f_1}\left(a_{12}-b_1p_2\right)+\frac{\partial F}{\partial f_2}\left(a_{22}-b_2p_2\right)+\cdots\cdots+\cdots\cdots+\frac{\partial F}{\partial f_m}\left(a_{m2}-b_mp_2\right)=0\\
&\cdot\\
&\cdot\\
&\frac{\partial F}{\partial f_1}\left(a_{1m}-b_1p_m\right)+\frac{\partial F}{\partial f_2}\left(a_{2m}-b_2p_m\right)+\cdots\cdots+\cdots\cdots+\frac{\partial F}{\partial f_m}\left(a_{mm}-b_mp_m\right)=0
\end{aligned}\right\} \qquad (1.33)$$

Eliminating $\frac{\partial F}{\partial f_j}$ $\left(j=1(1)m\right)$ among the relations in (1.33) we have

$$\begin{vmatrix}
a_{11}-b_1p_1 & a_{21}-b_2p_1 & \cdot & \cdot & \cdot & a_{m1}-b_mp_1\\
a_{12}-b_1p_2 & a_{22}-b_2p_2 & \cdot & \cdot & \cdot & a_{m2}-b_mp_2\\
a_{13}-b_1p_3 & a_{23}-b_2p_3 & \cdot & \cdot & \cdot & a_{m3}-b_mp_3\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
a_{1m}-b_1p_m & a_{2m}-b_2p_m & \cdot & \cdot & \cdot & a_{mm}-b_mp_m
\end{vmatrix}=0 \qquad (1.34)$$

The determinant in (1.34) may be expressed as the sum of 2^m determinants of which many will vanish due to symmetry and so be left with

$$\Delta-p_1\Delta_1-p_2\Delta_2-p_3\Delta_3-\cdots\cdots-p_m\Delta_m=0 \qquad (1.35)$$

From (1.32) and (1.35) we have

$$R-p_1P_1-p_2P_2-p_3P_3-\cdots\cdots-p_mP_m=0$$

ie,

$$p_1P_1+p_2P_2+p_3P_3+\cdots\cdots+p_mP_m=R \qquad (1.36)$$

which proves that $F=0$ is a general solution of (1.24).

Note:

The system of *ODEs* (1.12) are known as the *Lagranges auxiliary equations*. The curve of intersection of the surfaces (1.11) called *Lagranges lines.*

Examples.

1 Find the general integral curve of

$$\left(y+ux\right)p-\left(x+uy\right)q=x^2-y^2. \qquad (i)$$

Solution.

The integral surfaces are determined by the integral curves of the system of *ODEs*.

$$\frac{dx}{y+ux}=\frac{dy}{-(x+uy)}=\frac{du}{x^2-y^2} \qquad (ii)$$

each of which is equall to

$$\frac{ydx+xdy+du}{y(y+ux)-x(x+uy)+(x^2-y^2)}=\frac{ydx+xdy+du}{0}$$, considering $y, x, 1$ as multiplier. (iii)

This are also each equall to

$$\frac{xdx+ydy-udu}{x(y+ux)-y(x+uy)-u(x^2-y^2)}=\frac{xdx+ydy-udu}{0}$$, considering $x, y, -u$ as multiplier. (iv)

From (iii) we have

$$ydx+xdy+du \Rightarrow 2xy+u=c_1$$

Similarly, (iv) gives

$$xdx+ydy-udu=0\ ,\ ie,\ x^2+y^2-u^2=c_2$$

Hence, the general solution is

$$F\left(2xy+u, x^2+y^2-u^2\right)=0$$

2 Obtain a general integral of $yzp_1+xzp_2+xyp_3+xyz=0$.

Solution

The standard form of the PDE is given as

$$yup_1+xup_2+xyp_3=-xyz \qquad (i)$$

The corresponding auxiliary equations are

$$\frac{dx}{yz}=\frac{dy}{xz}=\frac{dz}{yx}=\frac{du}{-xyz} \qquad (ii)$$

We have that each of the ratio is equal to the following:

$$\frac{xdx+du}{xyz-xyz}= \ \Rightarrow xdx+du=0\ \ ie,\ \ x^2+2u=c_1 \qquad (iii)$$

$$\frac{ydy+du}{xyz-xyz}= \ \Rightarrow ydy+du=0\ \ ie,\ \ y^2+2u=c_2 \qquad (iv)$$

$$\frac{zdz+du}{xyz-xyz}= \ \Rightarrow zdz+du=0\ \ ie,\ \ z^2+2u=c_3 \qquad (v)$$

We thus have the following:

$$x^2-y^2=c_1',\ x^2-z^2=c_2' \text{ and } x^2+2u=c_1$$

A general integral is therefore given as $x^2+2u=\phi\left(x^2-y^2, x^2-z^2\right)$.

1.2 PARTICULAR INTEGRALS OF LAGRANGE'S EQUATIONS.

Consider the equation

$$P \cdot p + Q \cdot q = R \qquad (1.37)$$

We observe that the integral surfaces of (1.37) are generated from the integral curves of the system

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{du}{R} \qquad (1.38)$$

Assuming that these integral curves are given by

$$\left.\begin{array}{l} f(x, y, u) = a \\ g(x, y, u) = b \end{array}\right\} \qquad (1.39)$$

In order to determine the particular integral of (1.37) passing through a given curve

$$\left.\begin{array}{l} h_1(x, y, u) = 0 \\ \quad \text{or } \; x = x(t), y = y(t), u = u(t) \\ h_2(x, y, u) = 0 \end{array}\right\} \qquad (1.40)$$

in which t is a parameter we eliminate x, y, u between (1.40) and (1.39) . The elimant will therefore be of the form $\phi(a,b) = 0$ and so the required particular integral will be.

$$\phi(f, g) = 0 \qquad (1.41)$$

Examples

1 Determine the particular integral of the *PDE*

$(x-y)p + (y-x-u)q = u$ that passes through the point $u = 1, x^2 + y^2 = 1$.

Solution.

The corresponding auxiliary equations are

$$\frac{dx}{x-y} = \frac{dy}{y-x-u} = \frac{du}{u} \qquad (i)$$

$$= \frac{dx+dy+du}{x-y+y-x-u+u} = \frac{d(x+y+u)}{0} \text{ , taking } (1,1,1 \text{ as multipliers}) \qquad (ii)$$

$$= \frac{dx-dy+du}{x-y-y+x+u+u} = \frac{d(x-y+u)}{2(x-y+u)} \text{ , taking } (1,-1,1 \text{ as multipliers}) \qquad (iii)$$

From (ii) we have

$$dx + dy + du = 0 \Rightarrow d(x+y+u) = 0 \text{ , } ie, \quad x + y + u = a \qquad (iv)$$

From the secod relation we have

$$\frac{du}{u} = \frac{dx-dy+du}{x-y-y+x+u+u} = \frac{d(x-y+u)}{2(x-y+u)} \qquad (v)$$

ie, $\ln u = \frac{1}{2}\ln(x-y+u)+\ln b \Rightarrow 2\ln u = \ln(x-y+u)+\ln b = \ln b(x-y+u)$

ie, $u^2 = b(x-y+u) \Rightarrow b = \frac{u^2}{x-y+u} \qquad (vi)$

We observe that the given curve can be written in the form

$$x=\cos\vartheta, y=\sin\vartheta, u=1 \qquad (vii)$$

Substituting (vii) into(v) (vii) yields

$$\cos\vartheta+\sin\vartheta+1=a \qquad (viii)$$

$$\frac{1}{\cos\vartheta-\sin\vartheta+1}=b \;\Rightarrow \cos\vartheta-\sin\vartheta+1=\frac{1}{b} \qquad (ix)$$

$\Rightarrow$ $(\cos\vartheta+\sin\vartheta)^2+(\cos\vartheta-\sin\vartheta)=(a-1)^2+\left(\frac{1}{b}-1\right)^2$

ie, $\cos^2\vartheta+\sin^2\vartheta+2\cos\vartheta\sin\vartheta+\cos^2\vartheta+\sin^2\vartheta-2\cos\vartheta\sin\vartheta=a^2-2a+1^2+\frac{1}{b^2}-\frac{2}{b}+1$

ie, $2(\cos^2\vartheta+\sin^2\vartheta)=a^2-2a+1^2+\frac{1}{b^2}-\frac{2}{b}+1$

ie, $a^2-2a+\frac{1}{b^2}-\frac{2}{b} \;=0$

Thus, the particular integral surface is given as

$$(x+y+u)^2-2(x+y+u)-\frac{2(x-y+u)}{u^2}+\frac{4(x-y+u)}{u^4}=0$$

2 Determine the solution of the differential equation

$(u+2a)xp+(xu+2yu+2ay)q=u(a+u)$ that passes through the cuce $y=0, u^3+x(a+u)^2=0.$

Solution

The auxiliary equation corresponding to the DE is

$$\frac{dx}{(u+2a)x}=\frac{dy}{xu+2yu+2ay}=\frac{du}{u(a+u)} \qquad (i)$$

Taking the first and third ratios yields

$$\frac{dx}{(u+2a)x}=\frac{du}{u(a+u)} \qquad (ii)$$

$$\Rightarrow \frac{dx}{x}=\frac{(u+2a)du}{u(a+u)}$$

ie, $\ln x = \int\frac{(u+2a)du}{u(a+u)}$

Resolving the integrand on the rhs into partial fraction gives

$$\frac{(u+2a)}{u(a+u)}=\frac{A}{u}+\frac{B}{a+u}=\frac{(A+B)u+Aa}{u(a+u)}$$

ie, $A+B=1, A=2 \Rightarrow A=2, B=-1$

hence, $\frac{(u+2a)}{u(a+u)}=\frac{2}{u}-\frac{1}{a+u}$ *ie,* $\int\frac{(u+2a)du}{u(a+u)}=\int\left(\frac{2}{u}-\frac{1}{a+u}\right)du=2\ln u-\ln(a+u)+c_1$

$\Rightarrow \ln x=\ln\left(\frac{c_1u^2}{a+u}\right)$ *ie,* $x=\frac{c_1u^2}{a+u}$, $\left[c_1=\frac{(a+u)x}{u^2}\right]$

From the second and third ratios we have

$$\frac{dy}{xu+2yu+2ay}=\frac{du}{u(a+u)}$$

ie, $$\frac{dy}{\frac{c_1u^3}{a+u}+2yu+2ay}=\frac{du}{u(a+u)}$$

ie, $$\frac{dy}{\frac{c_1u^3}{a+u}+2y(u+a)}=\frac{du}{u(a+u)}$$

ie, $$\frac{dy}{\frac{c_1u^3}{(a+u)^2}+2y}=\frac{du}{u}$$

ie, $$\frac{dy}{du}=\frac{\frac{c_1u^3}{(a+u)^2}+2y}{u}$$

ie, $$\frac{dy}{du}-\frac{2y}{u}=\frac{c_1u^2}{(a+u)^2} \qquad (iv)$$

This is a first order ODE of the form $y'+p(x)y=f(x)$ which admits an integrating factor $\mu=e^{\int pdx}$.
Hence (iv) has the integrating factor u^{-2}.

ie, $$\left(yu^{-2}\right)'=\frac{c_1}{(a+u)^2}$$

$$\Rightarrow \quad yu^{-2}=\int\frac{c_1}{(a+u)^2}du=-\frac{c_1}{(a+u)}+c_2 \qquad (v)$$

ie,

$$\Rightarrow \quad y=\int\frac{c_1}{(a+u)^2}du=-\frac{c_1u^2}{(a+u)}+c_2u^2 \qquad (vi)$$

Setting $y=0$ in (vi) yields

$$c_2 = \frac{c_1}{(a+u)} \quad ie, \quad u = \frac{c_1}{c_2} - a \qquad (vii)$$

We recall from the second initial condition that

$$u^3 + x(a+u)^2 = 0$$

ie, $$u^3 + \frac{c_1 u^2}{a+u}(a+u)^2 = 0 \Rightarrow u^3 + c_1 u^2 (a+u) = 0$$

ie,

ie, $$u + c_1(a+u) = 0 \qquad (viii)$$

Eliminating u from (vii) and $(viii)$ yields

$$u + c_1(a+u) = 0$$

ie, $$\frac{c_1}{c_2} - a + c_1\left(a + \frac{c_1}{c_2} - a\right) = 0$$

ie, $$\frac{c_1}{c_2} - a + \frac{c_1{}^2}{c_2} = 0$$

$$\Rightarrow (c_1 - ac_2) + c_1{}^2 = 0$$

The required integral surface is thus

$$\frac{(a+u)x}{u^2} - a\left[\frac{y}{u^2}\frac{(a+u)x}{u^2(a+u)}\right] + \frac{(a+u)^2 x^2}{u^4}{}^2 = 0$$

ie, $$x(a+u)u^2 - a(x+y)u^2 + (a+u)^2 x^2 = 0$$

1.3 GENERAL METHOD FOR THE SOLUTION OF FIRST-ORDER PARTIAL DIFFERENTIAL EQUATIONS.

The main methods of solution for the first-order PDE are the ones due to Charpits and Jacobi.

1.3.1 CHARPIT'S METHOD

Given the *PDE*

$$F\left(x,y,u,p,q\right)=0 \qquad (1.2.1)$$

Since u is a function of both x and y we thus have

$$du = pdx + qdy \qquad (1.2.2)$$

If we have another function

$$F\left(x,y,u,p,q,a\right)=0 \qquad (1.2.3)$$

it will be possible to evaluate p and q from the two equations $(1.2.1)$ and $(1.2.2)$ in the form $p=\phi\left(x,y,u,a\right)$ and $q=\psi\left(x,y,u,a\right)$.

Substituting these values into $(1.2.2)$ renders it directly integrable or integrable using some weighting function and the integral which is of the form $f\left(x,y,u,a\right)=b$ will be a solution of the original *PDE* $(1.2.1)$. For this solution gives:

$$\left.\begin{aligned} &f_x dx + f_y dy + f_u du = 0 \\ \text{or}\quad &\frac{f_x}{-f_u}dx + \frac{f_y}{-f_u}dy - du = 0 \end{aligned}\right\} \qquad (1.2.4)$$

Comparing $(1.2.4)$ with $(1.2.2)$ we have

$$\left.\begin{aligned} \frac{f_x}{-f_u} &= p = \phi \\ \frac{f_y}{-f_u} &= q = \psi \end{aligned}\right\} \qquad (1.2.5)$$

From $f\left(x,y,u,a\right)=b$ treating $u=u\left(x,y\right)$ we have

$$\left. f_x + f_u \cdot p = 0,\ \ f_y + f_u \cdot q = 0\right\} \qquad (1.2.6)$$

$(1.2.6)$ implies

$$\left.\begin{aligned} &p=-\frac{f_x}{f_u}, q=-\frac{f_y}{f_u} \\ ie,\quad &p=\phi \text{ and } q=\psi \end{aligned}\right\} \qquad (1.2.7)$$

Since $p=\phi$ and $q=\psi$ satisfy $(1.2.1)$ it thus implies that $f\left(x,y,u,a\right)=b$ is a solution of $(1.2.1)$. Since this solution contains two arbitrary constants, it is therefore a complete solution of $(1.2.1)$. The problem now therefore is to determine the function $(1.2.3)$refered to a the auxiliary function. In doing this we observe that the quantities u, p, q substituted into $(1.2.1)$ $(1.2.3)$ satisfy them identically. As a matter of fact the partial derivatives of F and G with respect to u,x and y must vanish.

$$\left.\begin{aligned}\frac{\partial F}{\partial x}+\frac{\partial F}{\partial u}\cdot p+\frac{\partial F}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial F}{\partial q}\frac{\partial q}{\partial x}=0\\ \frac{\partial G}{\partial x}+\frac{\partial G}{\partial u}\cdot p+\frac{\partial G}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial G}{\partial q}\frac{\partial q}{\partial x}=0\end{aligned}\right\}\qquad(1.2.8)$$

$$\left.\begin{aligned}\frac{\partial F}{\partial y}+\frac{\partial F}{\partial u}\cdot q+\frac{\partial F}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial F}{\partial q}\frac{\partial q}{\partial y}=0\\ \frac{\partial G}{\partial y}+\frac{\partial G}{\partial u}\cdot q+\frac{\partial G}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial G}{\partial q}\frac{\partial q}{\partial y}=0\end{aligned}\right\}\qquad(1.2.9)$$

Eliminating $\frac{\partial p}{\partial x}$ in $(1.2.8)$ we have

$$\frac{\partial(F,G)}{\partial(x,p)}+p\cdot\frac{\partial(F,G)}{\partial(u,p)}+\frac{\partial p}{\partial x}\cdot\frac{\partial(F,G)}{\partial(q,p)}=0\quad(1.2.10)$$

Similarly, eliminating $\frac{\partial q}{\partial y}$ in $(1.2.9)$ we have

$$\frac{\partial(F,G)}{\partial(y,q)}+q\cdot\frac{\partial(F,G)}{\partial(u,q)}+\frac{\partial q}{\partial y}\cdot\frac{\partial(F,G)}{\partial(p,q)}=0\quad(1.2.11)$$

where

$$\frac{\partial(x,y)}{\partial(s,t)}=\frac{\partial x}{\partial s}\frac{\partial y}{\partial t}-\frac{\partial x}{\partial t}\frac{\partial y}{\partial s}\qquad(1.2.12)$$

Recalling that

$$\frac{\partial q}{\partial x}=\frac{\partial}{\partial x}(q)=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial y}(p)=\frac{\partial p}{\partial y}\qquad(1.2.13)$$

we thus have from $(1.2.11)$ and $(1.2.12)$ that

$$\left(\frac{\partial F}{\partial x}+p\cdot\frac{\partial F}{\partial u}\right)\frac{\partial G}{\partial p}+\left(\frac{\partial F}{\partial y}+q\cdot\frac{\partial F}{\partial u}\right)\frac{\partial G}{\partial q}+\left(-p\cdot\frac{\partial F}{\partial p}-q\cdot\frac{\partial F}{\partial q}\right)\frac{\partial G}{\partial u}+\left(-\frac{\partial F}{\partial p}\right)\frac{\partial G}{\partial x}+\left(-\frac{\partial F}{\partial q}\right)\frac{\partial G}{\partial y}=0\quad(1.2.14)$$

This is a linear differential equation of order 1 that must be satisfied by (1.44). Its integrals are integrals of the Lagranges auxiliary equations

$$\frac{dp}{\frac{\partial F}{\partial x}+p\cdot\frac{\partial F}{\partial u}}=\frac{dq}{\frac{\partial F}{\partial y}+q\cdot\frac{\partial F}{\partial u}}=\frac{du}{-p\cdot\frac{\partial F}{\partial p}-q\cdot\frac{\partial F}{\partial q}}=\frac{du}{-\frac{\partial F}{\partial p}}=\frac{du}{-\frac{\partial F}{\partial q}}\qquad(1.2.15)$$

Eqns$(1.2.15)$ are known as Charpit's auxiliary equations. Any integral of $(1..2.15)$ involving p or q or both is taken for the required second relation $(1.2.3)$. In fact the simplest relation of these is taken as$(1.2.3)$ On obtaining $(1.2.3)$ p and q are determined from $(1.2.1)-(1.2.3)$ and the values substituted into $(1.2.2)$ which on integration we obtain the required complete solution of the given differential equation.

1.3.2 JACOBI'S METHOD,

In the last section we discussed the Charpit's method for solving a PDE involving two independent variables x_1 and x_2 (say). The present method (Jacobi's) is quite similar. It is expedient here to recall the following very important theoerem in differential calculus:

Theorem 1.2

If the functions $\psi_j(x_1, x_2, x_3)$, $(j = 1(1)3)$ posess continuous partial first derivatives in x_j , $j = 1(1)3$ then

$$\psi_1 \; dx_1 + \psi_2 dx_2 + \psi_3 dx_3 \qquad (1.2.16)$$

is an exact differential equation iff

$$\frac{\partial \psi_2}{\partial x_3} - \frac{\partial \psi_3}{\partial x_2} = 0 \;, \; \frac{\partial \psi_2}{\partial x_1} - \frac{\partial \psi_1}{\partial x_2} = 0 \;, \; \frac{\partial \psi_1}{\partial x_3} - \frac{\partial \psi_3}{\partial x_1} = 0 \qquad (1.2.17) \;.$$

Suppose we have a differential equation

$$f(x, y, u, p, q) = 0 \qquad (1.2.18)$$

explicitly involving the independent variable *u*. We shall prove that $(1.2.18)$ can be transformed into another differential equation with a new dependent variable which does not explicitly occur and the number of independent variables increased by unity in the process.

We shall rename the variables as follows:

$$x = x_1, y = x_e, u = x_3$$

and introduce a new variable $v = v(x, y, u)$ $\qquad (1.2.19)$

we now consider the relation

$$v(x, y, u) = 0 \qquad (1.2.20)$$

By assuming $p_1 = \frac{\partial v}{\partial x_1}, p_2 = \frac{\partial v}{\partial x_2}, p_3 = \frac{\partial v}{\partial x_3}$, $(1.2.20)$ yields

$$\left.\begin{aligned} &\frac{\partial v}{\partial x} + \frac{\partial v}{\partial u}\frac{\partial u}{\partial x} = 0 \\ &\frac{\partial v}{\partial y} + \frac{\partial v}{\partial u}\frac{\partial u}{\partial y} = 0 \end{aligned}\right\} \qquad (1.2.21)$$

ie, $p = -\frac{p_1}{p_3}$ and $q = -\frac{p_2}{p_3}$

Thus, $v = 0$ will be a solution to $(1.2.18)$ iff

$$f\left(x_1, x_2, x_3, -\frac{p_1}{p_3}, -\frac{p_2}{p_3}\right) = 0 \qquad (1.2.22)$$

Eqn$(1.2.22)$ is an equation of the form

$$G(x_1, x_2, x_3, p_1, p_2, p_3) = 0 \qquad (1.2.23)$$

Clearly, this is a *PDE* in three independent variables x_1, x_2, x_3 that does not explicitly involve the dependent variable v which ends the proof.

This method applies to *PDE* of the form $(1.2.23)$ whose central idea is to construct two more auxiliary relations of the form

$$G_2\left(x_1, x_2, x_3, p_1, p_2, p_3, a\right) = 0 \qquad (1.2.24)$$

$$G_3\left(x_1, x_2, x_3, p_1, p_2, p_3, b\right) = 0 \qquad (1.2.25)$$

$$p_j = \psi_j\left(x_1, x_2, x_3, a, b\right),\ \left(j = 1(1)3\right) \qquad (1.2.26)$$

and such that $p_1dx_1 + p_2dx_2 + p_3dx_3$ becomes exact DE when $p_j = \psi_j$.

Whenever such function G_2, G_3 can be determined then there exists $\phi\left(x_1, x_2, x_3, a, b\right)$ such that

$$\left.\begin{aligned}\frac{\partial\phi}{\partial x_1} &= \psi_1\\ \frac{\partial\phi}{\partial x_2} &= \psi_2\\ \frac{\partial\phi}{\partial x_3} &= \psi_3\end{aligned}\right\} \qquad (1.2.27)$$

then with $p_j = \phi_j$ the DE $p_1dx_1 + p_2dx_2 + p_3dx_3 - dv = 0$ becomes $d\phi - dv = 0$ which then yields

$$\phi - v = A \qquad (1.2.28)$$

Observe that from $(1.2.28)$ we get back $(1.2.27)$

$$\left.\begin{aligned}&p_1 = \frac{\partial\phi}{\partial x_1}, p_2 = \frac{\partial\phi}{\partial x_2}, p_3 = \frac{\partial\phi}{\partial x_3}\\ ie,\quad &p_1 = \frac{\partial\phi}{\partial x_1} = \psi_1, p_2 = \frac{\partial\phi}{\partial x_2} = \psi_2, p_3 = \frac{\partial\phi}{\partial x_3} = \psi_3\end{aligned}\right\} \qquad (1.2.29)$$

Since by hypothesis $p_j = \psi_j$ constitute a solution $(1.2.23),(1.2.24),(1.2.25)$ for p_1, p_2, p_3 we observe that $v = \phi - A$ is a solution of $(1.2.23)$ which contains three arbitrary constants a, b, c therefore it is a complete integral of $(1.2.23)$.

If the original PDE is $(1.2.18)$ we identify $(1.2.22)$ and $(1.2.23)$ so that $v = \phi - A$ is a solution of $(1.2.22)$. Hence, $v = 0$ $(\phi = A)$ is a solution of $(1.2.18)$. This implies that $\phi = A$ gives an $A-$parameter family of complete integrals of $(1.2.18)$ with a and b arbitrary constants.

1.3.2.1 DETERMINATION OF THE FUNCTIONS G_2 & G_3.

Suppose the functions G_2 & G_3 are such that we can solve for p_1, p_2, p_3 from$(1.2.23),(1.2.24)$ and $(1.2.25)$ in $(1.2.26)$. Then they become identities if p_j are replaced with ψ_j so that their partial derivatives wrt x_j vanish independently. Hence, from $(1.2.24)$ and $(1.2.25)$ we have

$$\left.\begin{aligned}\frac{\partial G_2}{\partial x_1}+\frac{\partial G_2}{\partial p_1}\frac{\partial p_1}{\partial x_1}+\frac{\partial G_2}{\partial p_2}\frac{\partial p_2}{\partial x_1}+\frac{\partial G_2}{\partial p_3}\frac{\partial p_3}{\partial x_1}=0\\ \frac{\partial G_3}{\partial x_1}+\frac{\partial G_3}{\partial p_1}\frac{\partial p_1}{\partial x_1}+\frac{\partial G_3}{\partial p_2}\frac{\partial p_2}{\partial x_1}+\frac{\partial G_3}{\partial p_3}\frac{\partial p_3}{\partial x_1}=0\end{aligned}\right\} \qquad (1.2.30)$$

$$\left.\begin{aligned}\frac{\partial G_2}{\partial x_2}+\frac{\partial G_2}{\partial p_1}\frac{\partial p_1}{\partial x_2}+\frac{\partial G_2}{\partial p_2}\frac{\partial p_2}{\partial x_2}+\frac{\partial G_2}{\partial p_3}\frac{\partial p_3}{\partial x_2}=0\\ \frac{\partial G_3}{\partial x_2}+\frac{\partial G_3}{\partial p_1}\frac{\partial p_1}{\partial x_2}+\frac{\partial G_3}{\partial p_2}\frac{\partial p_2}{\partial x_2}+\frac{\partial G_3}{\partial p_3}\frac{\partial p_3}{\partial x_2}=0\end{aligned}\right\} \qquad (1.2.31)$$

$$\left.\begin{aligned}\frac{\partial G_2}{\partial x_3}+\frac{\partial G_2}{\partial p_1}\frac{\partial p_1}{\partial x_3}+\frac{\partial G_2}{\partial p_2}\frac{\partial p_2}{\partial x_3}+\frac{\partial G_2}{\partial p_3}\frac{\partial p_3}{\partial x_3}=0\\ \frac{\partial G_3}{\partial x_3}+\frac{\partial G_3}{\partial p_1}\frac{\partial p_1}{\partial x_3}+\frac{\partial G_3}{\partial p_2}\frac{\partial p_2}{\partial x_3}+\frac{\partial G_3}{\partial p_3}\frac{\partial p_3}{\partial x_3}=0\end{aligned}\right\} \qquad (1.2.32)$$

Eliminating $\frac{\partial p_1}{\partial x_1}$ from $(1.2.30)$, $\frac{\partial p_2}{\partial x_2}$ from $(1.2.31)$ and $\frac{\partial p_3}{\partial x_3}$ from $(1.2.32)$ we obtain

$$\left.\begin{aligned}\frac{\partial(G_2,G_3)}{\partial(x_1,p)}+\frac{\partial(G_2,G_3)}{\partial(p_2,p_1)}\frac{\partial p_2}{\partial x_1}+\frac{\partial(G_2,G_3)}{\partial(p_3,p_1)}\frac{\partial p_3}{\partial x_1}=0\\ \frac{\partial(G_2,G_3)}{\partial(x_{21},p)}+\frac{\partial(G_2,G_3)}{\partial(p_3,p_3)}\frac{\partial p_3}{\partial x_2}+\frac{\partial(G_2,G_3)}{\partial(p_1,p_2)}\frac{\partial p_1}{\partial x_2}=0\\ \frac{\partial(G_2,G_3)}{\partial(x_3,p_3)}+\frac{\partial(G_2,G_3)}{\partial(p_1,p_3)}\frac{\partial p_1}{\partial x_3}+\frac{\partial(G_2,G_3)}{\partial(p_2,p_3)}\frac{\partial p_2}{\partial x_3}=0\end{aligned}\right\} \qquad (1.2.33)$$

Recall that

$$\frac{\partial(G_2,G_3)}{\partial(x_k,p_j)}=-\frac{\partial(G_2,G_3)}{\partial(x_j,p_k)} \qquad (1.2.34)$$

Using $(1.2.34)$ in $(1.2.33)$ yields

$$\frac{\partial(G_2,G_3)}{\partial(x_1,p_1)}+\frac{\partial(G_2,G_3)}{\partial(x_2,p_2)}+\frac{\partial(G_2,G_3)}{\partial(x_3,p_3)}+\frac{\partial(G_2,G_3)}{\partial(p_2,p_3)}\left(\frac{\partial p_2}{\partial x_3}-\frac{\partial p_3}{\partial x_2}\right)+\frac{\partial(G_2,G_3)}{\partial(p_3,p_1)}\left(\frac{\partial p_3}{\partial x_1}-\frac{\partial p_1}{\partial x_3}\right)$$
$$+\frac{\partial(G_2,G_3)}{\partial(p_1,p_2)}\left(\frac{\partial p_1}{\partial x_2}-\frac{\partial p_2}{\partial x_1}\right)=0$$

ie,

$$\left.\begin{aligned}&\frac{\partial(G_2,G_3)}{\partial(p_2,p_3)}\cdot L+\frac{\partial(G_2,G_3)}{\partial(p_3,p_1)}\cdot M+\frac{\partial(G_2,G_3)}{\partial(p_1,p_2)}\cdot N=-(G_2,G_3) \text{ where } L=\left(\frac{\partial p_2}{\partial x_3}-\frac{\partial p_3}{\partial x_2}\right),\\ &M=\left(\frac{\partial p_3}{\partial x_1}-\frac{\partial p_1}{\partial x_3}\right), N=\left(\frac{\partial p_1}{\partial x_2}-\frac{\partial p_2}{\partial x_1}\right), (G_2,G_3)=\frac{\partial(G_2,G_3)}{\partial(x_1,p_1)}+\frac{\partial(G_2,G_3)}{\partial(x_2,p_2)}+\frac{\partial(G_2,G_3)}{\partial(x_3,p_3)}\end{aligned}\right\} \qquad (1.2.35)$$

Similar computation gives

$$\frac{\partial(G_3,G_1)}{\partial(p_2,p_3)}\cdot L+\frac{\partial(G_3,G_1)}{\partial(p_3,p_1)}\cdot M+\frac{\partial(G_3,G_1)}{\partial(p_1,p_2)}\cdot N=-(G_3,G_1) \qquad (1.2.36)$$

$$\frac{\partial(G_1,G_2)}{\partial(p_2,p_3)}\cdot L+\frac{\partial(G_1,G_2)}{\partial(p_3,p_1)}\cdot M+\frac{\partial(G_1,G_2)}{\partial(p_1,p_2)}\cdot N=-(G_1,G_2) \qquad (1.2.37)$$

Suppose noew that the solutions $p_j=\psi_j$ make the expression $p_1dx_1+p_2dx_2+p_3dx_3=0$ and exact differential then $\Rightarrow L=0, M=0$ and $N=0$ identically. Then from Eqn $(1.2.35),(1.2.36)$ and $(1.2.37)$ we get that $(G_2,G_3)=0,\ (G_3,G_1)=0$ and $(G_1,G_2)=0.$

Hence, $Z=G_2$ and $Z=G_3$ are two solutions of the *PDE*, $(Z,G_1)=0$

ie,
$$\left.\begin{aligned}&\frac{\partial(Z,G_1)}{\partial(x_1,p_1)}+\frac{\partial(Z,G_1)}{\partial(x_2,p_2)}+\frac{\partial(Z,G_1)}{\partial(x_3,p_3)}=0\\&\frac{\partial Z}{\partial x_1}\frac{\partial G_1}{\partial p_1}-\frac{\partial Z}{\partial p_1}\frac{\partial G_1}{\partial x_1}+\frac{\partial Z}{\partial x_2}\frac{\partial G_1}{\partial p_2}-\frac{\partial Z}{\partial p_2}\frac{\partial G_1}{\partial x_2}+\frac{\partial Z}{\partial x_3}\frac{\partial G_1}{\partial p_3}-\frac{\partial Z}{\partial p_3}\frac{\partial G_1}{\partial x_3}=0\end{aligned}\right\} \qquad (1.2.38)$$

But we must have

$$(G_2,G_3)=0 \qquad (1.2.39)$$

Observe that (1.79) is a first order *PDE* in the independent variable $x_j, p_j\ \ (j=1(1)3)$ with corresponding auxiliary equations

ie,
$$\frac{dx_1}{\frac{\partial G_1}{\partial p_1}}=\frac{dx_2}{\frac{\partial G_1}{\partial p_2}}=\frac{dx_3}{\frac{\partial G_1}{\partial p_3}}=\frac{dp_1}{-\frac{\partial G_1}{\partial x_1}}=\frac{dp_2}{-\frac{\partial G_1}{\partial x_2}}=\frac{dp_3}{-\frac{\partial G_1}{\partial x_3}}=\frac{dZ}{0} \qquad (1.2.40)$$

The coupled *ODEs* above are the *Jacobi's* auxiliary differential equations.

1.3.1 SUCCESS OF JACOBI'S METHOD

We show here that if $G_2=0$ and $G_3=0$ are two independent integrals of the eqn $(1.2.39)$ and are such that (i) $(G_2,G_3)=0$ and (ii) p_1,p_2,p_3 are solvable from $(1.2.23),(1.2.24),(1.2.25)$ $(1.2.26)$ then these equations will render the expression $p_1dx_1+p_2dx_2+p_3dx_3$ an exact differential.

First, we note that $Z=c$ is an integral of $(1.2.39)$ so $Z=G_2$ and $Z=G_3$ are two solutions of $(1.2.38)$. Thus, we have $(G_2,G_1)=0$ and $(G_3,G_1)=0.$

Consequent on the hypothesis $(G_2,G_3)=0$ the equations in $(1.2.35)-(1.2.37)$ give

$$\left.\begin{aligned}&\frac{\partial(G_2,G_3)}{\partial(p_2,p_3)}\cdot L+\frac{\partial(G_2,G_3)}{\partial(p_3,p_1)}\cdot M+\frac{\partial(G_2,G_3)}{\partial(p_1,p_2)}\cdot N=0\\&\frac{\partial(G_3,G_1)}{\partial(p_2,p_3)}\cdot L+\frac{\partial(G_3,G_1)}{\partial(p_3,p_1)}\cdot M+\frac{\partial(G_3,G_1)}{\partial(p_1,p_2)}\cdot N=0\\&\frac{\partial(G_1,G_2)}{\partial(p_2,p_3)}\cdot L+\frac{\partial(G_1,G_2)}{\partial(p_3,p_1)}\cdot M+\frac{\partial(G_1,G_2)}{\partial(p_1,p_2)}\cdot N=0\end{aligned}\right\} \qquad (1.2.41)$$

This is a system of linear homogeneous equations in the unknowns L, M and N with the coefficient determin

$$\Delta = \begin{vmatrix} \frac{\partial(G_2,G_3)}{\partial(p_2,p_3)} & \frac{\partial(G_2,G_3)}{\partial(p_3,p_1)} & \frac{\partial(G_2,G_3)}{\partial(p_1,p_2)}. \\ \frac{\partial(G_3,G_1)}{\partial(p_2,p_3)} & \frac{\partial(G_3,G_1)}{\partial(p_3,p_1)} & \frac{\partial(G_3,G_1)}{\partial(p_1,p_2)} \\ \frac{\partial(G_1,G_2)}{\partial(p_2,p_3)} & \frac{\partial(G_1,G_2)}{\partial(p_3,p_1)} & \frac{\partial(G_1,G_2)}{\partial(p_1,p_2)} \end{vmatrix} \quad \Big\}(1.2.41)$$

in which

$$J = \frac{\partial(G_1,G_2,G_3)}{\partial(p_1,p_2,p_3)} = \begin{vmatrix} \frac{\partial G_1}{\partial p_1} & \frac{\partial G_1}{\partial p_2} & \frac{\partial G_1}{\partial p_3} \\ \frac{\partial G_2}{\partial p_1} & \frac{\partial G_2}{\partial p_2} & \frac{\partial G_2}{\partial p_3} \\ \frac{\partial G_3}{\partial p_1} & \frac{\partial G_3}{\partial p_2} & \frac{\partial G_3}{\partial p_3} \end{vmatrix} \qquad (1.2.42)$$

$\Rightarrow \Delta = AdjJ = J^2$

Recall that from our hypothesis p_1, p_2, p_3 are solvable from $(1.2.35)-(1.2.37) \Rightarrow J \neq 0$ *ie*, $\Delta \neq 0$. Hence, the system $(1.2.40)$ gives $L = 0,\ M = 0,$ and $N = 0 \Rightarrow p_1 dx_1 + p_2 dx_2 + p_3 dx_3$ is an exact differential equation for all $p_j = \psi_j$. Here lie the success of the Jacobi's method.

Examples

1 Solve the *PDE*: $p^2 + q^2 - 2px - 2qy + 2xy = 0.$

Solution.

The corresponding Charpit's auxiliary *DE* is

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial u}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial u}} = \frac{du}{-p\frac{\partial f}{\partial x} - q\frac{\partial f}{\partial y}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dF}{0} \qquad (i)$$

ie,

$$\frac{dp}{-2p+2y} = \frac{dq}{-2q+2x} = \frac{dx}{-(2p-2x)} = \frac{dy}{-(2q-2y)} \qquad (ii)$$

$$\Rightarrow \qquad \frac{dp+dq}{-2p+2y-2q+2x} = \frac{dx+dy}{-(2p-2x)-(2q-2y)} \qquad (iii)$$

ie,

$$\frac{dp+dq}{-2(p+q)+2(x+y)} = \frac{dx+dy}{-2(p+q)+2(x+y)} \qquad (iv)$$

$$\Rightarrow \qquad d(p+q) = d(x+y) \qquad (v)$$

ie,

$$\left.\begin{aligned} &\Rightarrow \quad p+q = x+y+\alpha \\ &\text{or} \quad (p-x) = (y-q)+\alpha \end{aligned}\right\} \qquad (vi)$$

Observe that the differential equation may be expressed as

$$(p-x)^2+(y-q)^2=(x-y)^2 \qquad (vii)$$

Using (vi) in (vii) yields

$$(y-q)^2+2\alpha(y-q)+\alpha^2+(y-q)^2=(x-y)^2 \qquad (viii)$$

ie, $$2(y-q)^2+2\alpha(y-q)+\alpha^2=(x-y)^2$$

ie, $$2(y-q)^2+2\alpha(y-q)+\alpha^2-(x-y)^2=0$$

ie, $$(y-q)=\frac{-2\alpha\pm\sqrt{-4\alpha^2+8(x-y)^2}}{4}$$

ie, $$\left.\begin{aligned}(y-q)&=\frac{-\alpha-\sqrt{2(x-y)^2-\alpha^2}}{2}\\ \text{or}\quad (y-q)&=\frac{-\alpha+\sqrt{2(x-y)^2-\alpha^2}}{2}\end{aligned}\right\} \qquad (ix)$$

Considering the positive sign only we have

$$\left.\begin{aligned}(y-q)&=\frac{-\alpha+\sqrt{2(x-y)^2-\alpha^2}}{2}\\ \text{ie}\quad q&=y+\frac{\alpha-\sqrt{2(x-y)^2-\alpha^2}}{2}\end{aligned}\right\} \qquad (x)$$

From (vi) we have that

$$\begin{aligned}p&=(y-q)+(x+\alpha)\\ &=(x+\alpha)+\frac{\sqrt{2(x-y)^2-\alpha^2}-\alpha}{2}\\ &=x+\frac{\sqrt{2(x-y)^2-\alpha^2}+\alpha}{2}\end{aligned} \qquad (xi)$$

Recall that

$$\begin{aligned}du&=pdx+qdy\\ &=\left(x+\frac{\alpha+\sqrt{2(x-y)^2-\alpha^2}}{2}\right)dx+\left(y+\frac{\alpha-\sqrt{2(x-y)^2-\alpha^2}}{2}\right)dy\\ &=xdx+ydy+\frac{\alpha+\sqrt{2(x-y)^2-\alpha^2}}{2}dx+\frac{\alpha-\sqrt{2(x-y)^2-\alpha^2}}{2}dy\\ &=xdx+ydy+\frac{1}{2}\left(\alpha+\sqrt{2(x-y)^2-\alpha^2}\right)(dx-dy)\end{aligned} \qquad (xii)$$

Integrating (xii) yields

$$=\frac{1}{2}\left(x^2+y^2\right)+\frac{1}{2}\alpha(x-y)+\frac{1}{2}\int\sqrt{2(x-y)^2-\alpha^2}\,(dx-dy) \qquad (xiii)$$

To compute the integral in $(xiii)$ we set $\sqrt{2}(x-y)=\vartheta$

$$\therefore \qquad (dx-dy)=\frac{1}{\sqrt{2}}d\vartheta$$

ie,
$$\frac{1}{2}\int\sqrt{2(x-y)^2-\alpha^2}\,(dx-dy)=\frac{1}{2\sqrt{2}}\int\sqrt{\vartheta^2-\alpha^2}\,d\vartheta$$

$$=\frac{1}{2\sqrt{2}}\left[\frac{\vartheta}{2}\sqrt{\vartheta^2-\alpha^2}-\frac{\alpha^2}{2}\ln\left(\vartheta+\sqrt{\vartheta^2-\alpha^2}\right)\right]$$

$$=\frac{1}{4\sqrt{2}}\left[\sqrt{2}(x-y)\sqrt{2(x-y)^2-\alpha^2}-\frac{\alpha^2}{2}\ln\left(\sqrt{2}(x-y)+\sqrt{2(x-y)^2-\alpha^2}\right)\right]$$

Hence, the required complete integral is gives as

$$2u=x^2+y^2+\alpha(x-y)\ \ \frac{1}{2\sqrt{2}}\left[\sqrt{2}(x-y)\sqrt{2(x-y)^2-\alpha^2}-\frac{\alpha^2}{2}\ln\left(\sqrt{2}(x-y)+\sqrt{2(x-y)^2-\alpha^2}\right)\right]$$

2 Determine the integral surface of

$(y+uq)^2=u^2\left(1+p^2+q^2\right)$ circumscribed about the surface $2y=x^2-u^2$.

Solution

The equation in the standard form is given by

$$f(x,y,u,p,q)=(y+uq)^2-u^2\left(1+p^2+q^2\right) \qquad (i)$$

The corresponding Charpit's auxiliary DE is

$$\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}=\frac{du}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial u}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial u}}=\frac{dF}{0} \qquad (ii)$$

ie,
$$\frac{dx}{2pu^2}=\frac{dy}{2(y+uq)u-2qu}=\frac{du}{-2p^2u+2yu}=\frac{dp}{-2(y+uq)-2u\left(1+p^2+q^2\right)}$$

$$=\frac{dq}{-2\left\{(y+uq)-qu\left(1+p^2+q^2\right)-2q(y+uq)q\right\}} \qquad (iii)$$

ie,
$$\frac{dy}{2yu}=\frac{du}{-2p^2u+2yuq}=\frac{dq}{-2y+2qu\left(p^2+q^2\right)}$$

ie,
$$=\frac{qdu}{-2p^2qu+2yuq^2-2uy+2qu^2\left(p^2+q^2\right)}=\frac{d(qu)}{-2yu}=\frac{dy+d(qu)}{0}$$

$$\Rightarrow dy+d(qu)=0$$

ie, $\quad y+qu=a$

ie, $\quad q=\dfrac{a-y}{u} \qquad (iv)$

Substituting (iv) into (i) yields

$$\left(y+\left(\frac{a-y}{u}\right)u\right)^2-u^2\left(1+p^2+\left(\frac{a-y}{u}\right)^2\right)=0$$

ie, $\quad y^2+2y(a-y)+(a-y)^2-u^2-u^2p^2-(a-y)^2=0$

ie, $\quad u^2+u^2p^2-2y(a-y)-y^2=0$

ie, $\quad p^2=\dfrac{2ay-u^2-y^2}{u^2}\Rightarrow p=\dfrac{\pm\sqrt{2ay-u^2-y^2}}{u} \qquad (v)$

$\therefore \quad pdx+qdy-dz=0$ becomes

$$\frac{\pm\sqrt{2ay-u^2-y^2}}{u}dx+\frac{a-y}{u}dy-du=0$$

ie, $\quad (a-y)dy\pm\sqrt{2ay-u^2-y^2}dx-udu=0$

ie, $\quad \pm\sqrt{2ay-u^2-y^2}dx+\frac{1}{2}d\left(ay-y^2-u^2\right)=0$

ie, $\quad \pm 2\sqrt{\psi}dx+\frac{1}{2}d\psi=0$

ie, $\quad \pm dx+\frac{1}{2}\psi^{-1/2}d\psi=0 \qquad (vi)$

Integrating (vi) yields

$$\pm x+\psi=b \qquad (vii)$$

ie,

$$\left.\begin{aligned} x+\sqrt{2ay-u^2-y^2}&=b_1\\ -x+\sqrt{2ay-u^2-y^2}&=b_2\end{aligned}\right\} \qquad (vii)$$

These give complete integral of (i) which may be combined as

$$(x-b_1+u)(x-b_2-u)=0$$

$\because$ b_1 and b_2 are arbitrary we may replace b_2 by $-b_1$ and b_1 by b to get

$$(x-b)^2-u^2=0$$

$$(x-b)^2-\left(2ay-u^2-y^2\right)=0 \qquad (viii)$$

Denoting the LHS of $(viii)$ by $F(x,y,u,a,b)$ we may also write $H(x,y,u)=x^2-u^2-2y=0$ and suppose the integral surface $F(x,y,u,a,b)=0$ circumscribe $H(x,y,u)=0$.

Therefore, we must have

$$\frac{F_x}{H_x}=\frac{F_y}{H_y}=\frac{F_u}{H_u}$$

$$\frac{2(x-b)}{2x}=\frac{-2(a-y)}{-2}=\frac{2u}{-2u}=-1$$

or $\quad x-b=-x,\ y-a=1$

$\Rightarrow \quad x=\frac{b}{2},\ y=a+1$

Substituting the values of x, y into $H=0$ and $F=0$ gives

$$\frac{b^2}{4}-u^2-2(a+1)=0$$

$$\left.\begin{aligned} u^2&=\frac{b^2}{4}-2(a+1)\\ \frac{b^2}{4}-\left[2a(a+1)-(a+1)^2-u^2\right]&=0\end{aligned}\right\} \qquad (ix)$$

and

ie, $\quad \frac{b^2}{4}-\left(2a^2+2a-a^2-2a-1-u^2\right)=0$

$$\frac{b^2}{4}-a^2+1+u^2=0$$

$$u^2=-\left(\frac{b^2}{4}-a^2+1\right)=a^2-1-\frac{b^2}{4} \qquad (x)$$

Eliminating u from (ix) and (x) gives

$$\frac{b^2}{4}-2(a+1)=a^2-1-\frac{b^2}{4}$$

ie, $\quad \frac{b^2}{2}=a^2-1+2(a+1)=(a+1)^2$

ie, $\quad b=\pm\sqrt{2}(a+1) \qquad (xi)$

If $b=\sqrt{2}(a+1)$, the integral surface is

$$\left(x-\sqrt{2}(a+1)\right)^2-\left(2ay-u^2-y^2\right)=0 \qquad (xii)$$

Differentiating partially wrt a we obtain

$$-2\sqrt{2}\left(x-\sqrt{2}-\sqrt{2}a\right)-2y=0$$

$\Rightarrow \quad x-\sqrt{2}(1+a)=-\frac{y}{\sqrt{2}} \qquad (xiii)$

From (xii) and $(xiii)$ we have

$$\left(-\frac{y}{\sqrt{2}}\right)^2=-\left[2y\left(\frac{x}{\sqrt{2}}+\frac{y}{2}+1\right)-u^2-y^2\right]=0$$

ie, $\quad \frac{y^2}{2}=-\frac{2yx}{\sqrt{2}}-\frac{2y^2}{2}-2y+u^2+y^2=0$

ie, $\quad 2u^2+y^2-2\sqrt{2}xy+4y=0 \qquad (xiv)$

This is the particular integral surface circumscribing the given surface.

Similarly, if we take $b=-\sqrt{2}\left(a+1\right)$ in $\left(xi\right)$ we obtain

$$2u^2+y^2+2\sqrt{2}xy+4y=0 \qquad (xv)$$

Combining $\left(xiv\right)$ and $\left(xv\right)$ gives

$$\left(2u^2+y^2-2\sqrt{2}xy+4y\right)\left(2u^2+y^2+2\sqrt{2}xy+4y\right)=0$$

ie,

$$\left(2u^2+y^2+4y\right)^2=8xy \qquad (xvi)$$

CHAPTER TWO

PARTIAL DIFFERENTIAL EQUATIONS OF SECOND AND HIGHER ORDERS.

2.1 LINEAR EQUATIONS.

The most general linear $mth-$order Partial Differential Equations $(PDEs)$ is of the form

$$A_0 \frac{\partial^m u}{\partial x^m} + A_1 \frac{\partial^m u}{\partial x^{m-1}\partial y} + A_2 \frac{\partial^m u}{\partial x^{m-2}\partial y^2} + ...+ B_1 \frac{\partial^{m-1} u}{\partial x^{m-1}} + B_2 \frac{\partial^{m-1} u}{\partial x^{m-2}\partial y} + ...$$

$$+.............M \frac{\partial u}{\partial x} + N \frac{\partial u}{\partial y} + Cu = f(x,y) \qquad (2.1)$$

in which A_k, B_k, M, N, C are constants or functions of x and y.

From equation (2.1), a constant coefficient PDE is thus given as

$$\left(a_0 \frac{\partial^m u}{\partial x^m} + a_1 \frac{\partial^m u}{\partial x^{m-1}\partial y} + a_2 \frac{\partial^m u}{\partial x^{m-2}\partial y^2} + ... + a_m \frac{\partial^m u}{\partial y^m}\right)$$

$$+\left(b_0 \frac{\partial^{m-1} u}{\partial x^{m-1}} + b_1 \frac{\partial^{m-1} u}{\partial x^{m-2}\partial y} + ... + b_{m-1} \frac{\partial^{m-1} u}{\partial y^{m-1}}\right)$$

$$+...............\left(k_0 \frac{\partial u}{\partial x} + k_1 \frac{\partial u}{\partial y}\right) + lu = f(x,y) \qquad (2.2)$$

in which $a_i \; i = 0(1)m, b_j \; j = 0(1)m, k_0, k_1$ and l, are constants.

Setting $D^p = \dfrac{\partial^p}{\partial x^p}$ and $D'^r = \dfrac{\partial^r}{\partial y^r}$ $\qquad (2.3)$

then (2.2) becomes:

$$\left.\begin{aligned} &\left[\left(a_0 D^m + a_1 D^{m-1} D' + a_2 D^{m-2} D'^2 + ... + a_m D'^m\right) + \left(b_0 D^{m-1} + b_1 D^{m-2} D' + + ... + b_{m-1} D'^{m-1}\right)\right] u \\ &\qquad + \left[\left(k_0 D + k_1 D'\right) + l\right] u = f(x,y) \\ &\text{or} \\ &\qquad F(D, D') u = f(x,y) \end{aligned}\right\} \qquad (2.4)$$

in which $F(D, D')$ is a differential operator of order *m*.

The correspondind homogeneous differential equation $($reduced equation$)$ to (2.4) is given by

$$F(D, D')u = 0 \qquad (2.5)$$

Definition 2.1

The differential operator $F(D, D')$ is said to be *reducible* if it can be decomposed into factors of the form $(\alpha D + \beta D' + \gamma)$ in which α, β and γ are all constants. Otherwise it is *irreducible.*

2.1 METHOD OF SOLUTION

The solution of (2.4) is analogous to that of an $m-$order Ordinary Differential Equation (ODE) which comprises of a complimentary function (CF) that contains m arbitrary constants and a particular integral (PI) that contains no arbitrary constant. In this case the complimentary function is the solution of (2.5) and the particular integral the solution of (2.4).

2.2.1 Complimentary Functions

In order to obtain the complimentary function corresponding to the solution of (2.5) we recall this theorem from elementary caculus:

Theorem 2.1

If the differential operator $F(D,D')$ the general solution of (2.5) ie,

$$F(D,D')u = (\alpha D + \beta D' + \gamma)^m u = 0 \qquad (2.6)$$

where m is a positive integer is given as

$$\left.\begin{aligned} u &= \exp\left(-\frac{\gamma}{\alpha}x\right)\sum_{r=1}^{m} x^{m-1}\phi_m(\beta x - \alpha y) \qquad \alpha \neq 0 \\ &\text{and} \\ u &= \exp\left(-\frac{\gamma}{\beta}y\right)\sum_{r=1}^{m} y^{m-1}\phi_m(\beta x - \alpha y) \qquad \beta \neq 0 \end{aligned}\right\} \qquad (2.7)$$

in which the functions ϕ_r are sufficiently differentiable arbitrary functions.

Proof

We shall assume that $\alpha \neq 0$ and prove by induction.

For $m = 1$ the equation becomes:

$$\left.\begin{aligned} &(\alpha D + \beta D' + \gamma)u = 0 \ ie,\ \alpha\frac{\partial u}{\partial x} + \beta\frac{\partial u}{\partial y} + \gamma u = 0 \\ &\text{or} \qquad \alpha\frac{\partial u}{\partial x} + \beta\frac{\partial u}{\partial y} = -\gamma u \end{aligned}\right\} (i)$$

This is a first-order PDE with the corresponding Lagranges auxiliary equation as

$$\frac{dx}{\alpha} = \frac{dy}{\beta} = \frac{du}{-\gamma u} \qquad (ii)$$

ie,

$$\beta dx - \alpha dy = 0 \text{ or } \beta x - \alpha y = c \ \ (c \text{ a constant}) \qquad (iii)$$

Also, we have

$$\frac{du}{u} = -\frac{\gamma}{\alpha}dx$$

ie,

$$\ln u = -\frac{\gamma}{\alpha}x + k \qquad (iv)$$

ie,

$$ue^{\left(-\gamma/_{\alpha}x\right)} = c \qquad (v)$$

Hence, a general solution is

$$ue^{\left(-\gamma/_{\alpha}x\right)} = \phi(\beta x - \alpha y) \qquad (vi)$$

where ϕ is a differentiable function. This proves the theorem for $m = 1$.

We then assume the theorem to be true for some $m = p$ and prove that it is true for $m = p+1$.

ie, we assume that

$$(\alpha D + \beta D' + \gamma)^p u = 0 \qquad (vii)$$

Observe that

$$(\alpha D + \beta D' + \gamma)^{p+1} u = 0 = (\alpha D + \beta D' + \gamma)^p w \qquad (viii)$$

where $w = (\alpha D + \beta D' + \gamma)u$

But by our hypothesis

$$w = \exp\left(-\frac{\gamma}{\alpha}x\right)\sum_{r=1}^{p} x^{r-1}\phi_r(\beta x - \alpha y) \qquad \alpha \neq 0 \qquad (ix)$$

or

$$(\alpha D + \beta D' + \gamma)u = \exp\left(-\frac{\gamma}{\alpha}x\right)\sum_{r=1}^{p} x^{m-1}\phi_r(\beta x - \alpha y) \qquad \alpha \neq 0 \qquad (ix)$$

ie,

$$\alpha\frac{\partial u}{\partial x} + \beta\frac{\partial u}{\partial y} = -\gamma u + \exp\left(-\frac{\gamma}{\alpha}x\right)\sum_{r=1}^{p} x^{r-1}\phi_r(\beta x - \alpha y) \qquad \alpha \neq 0 \qquad (x)$$

This is again a first-order linear partial differential equation with the corresponding Lagranges auxiliary equation given as

$$\frac{dx}{\alpha} = \frac{dy}{\beta} = \frac{du}{-\gamma u + \exp\left(-\frac{\gamma}{\alpha}x\right)\sum_{r=1}^{p} x^{r-1}\phi_r(\beta x - \alpha y)} \qquad (xi)$$

in which again from the first two equalities we have

$$\beta dx - \alpha dy = 0 \text{ or } \beta x - \alpha y = c \;\; (c \text{ a constant}) \qquad (xii)$$

Again, we also havefrom the first and third equalities

$$\frac{dx}{\alpha} = \frac{du}{-\gamma u + \exp\left(-\frac{\gamma}{\alpha}x\right)\sum_{r=1}^{p} x^{r-1}\phi_r(c)} \qquad (xiii)$$

or

$$\frac{du}{dx} + \frac{\gamma}{\alpha}u = \exp\left(-\frac{\gamma}{\alpha}x\right)\sum_{r=1}^{p} x^{r-1}\phi_r(c) \qquad (xiv)$$

ie,

$$\exp\left(\frac{\gamma}{\alpha}x\right)\frac{du}{dx} + \exp\left(\frac{\gamma}{\alpha}x\right)\frac{\gamma}{\alpha}u = \frac{1}{\alpha}\sum_{r=1}^{p} x^{r-1}\phi_r(c) \qquad (xv)$$

$$\Rightarrow \qquad \left[\exp\left(\frac{\gamma}{\alpha}x\right)u\right]' = \frac{1}{\alpha}\sum_{r=1}^{p} x^{r-1}\phi_r(c) \qquad (xvi)$$

ie,

$$u\exp\left(\frac{\gamma}{\alpha}x\right) = \int \frac{1}{\alpha}\sum_{r=1}^{p} x^{r-1}\phi_r(c)\,dx \qquad (xvii)$$

$$= \frac{1}{\alpha}\sum_{r=1}^{p}\frac{x^r}{r}\,\phi_r(c) \qquad (xviii)$$

The general solution is therefore

$$u\exp\left(\frac{\gamma}{\alpha}x\right) - \frac{1}{\alpha}\sum_{r=1}^{p}\frac{x^r}{r}\,\phi_r(c) + c' = \psi(\beta x - \alpha y) \qquad (xix)$$

in which ψ is an arbitrary differentiable function. This general solution may also be written in the form

$$u = \exp\left(-\frac{\gamma}{\alpha}x\right)\sum_{r=1}^{p+1} x^{r-1}\psi_r(\beta x - \alpha y) \qquad (xix)$$

which is the theorem for $m = p+1$

This completes the induction and hence the proof of the theorem.

We note that if the operator $F(D,D')$ is reducible it will be seen that

$$F(D,D')e^{(\alpha x+\beta y)} = F(\alpha,\beta)e^{(\alpha x+\beta y)} \qquad (2.8)$$

Therefore it follows that $u = \exp(\alpha x + \beta y)$ is a solution of $F(D,D')u = 0$ if

$$F(\alpha,\beta) = 0 \qquad (2.9)$$

In general, $F(\alpha,\beta) = 0$ gives different pairs of solutions (α_j, β_j). This way we obtain different solutions $c_j\exp(\alpha_j x + \beta_j y)$ where c_j are constants. Obviously the linear combination $\sum_{j=1}^{m} c_j\exp(\alpha_j x + \beta_j y)$ is also a solution. Indeed, the most general solution is of this form.

Examples.

1 Obtain the solution to the *DE*

$$\frac{\partial^2 u}{\partial x^2} - a^2\frac{\partial^2 u}{\partial y^2} = 0$$

Solution

The given *PDE* is of the form

$$\left(D^2 - a^2 D'^2\right)u = 0$$

ie,

$$(D - aD')(D + aD')u = 0$$

The general solution is

$$\phi_1(-ax - y) + \phi_2(-ax - y)$$

where ϕ_1 and ϕ_2 are arbitrary differentiable functions

2 Obtain the solution to the PDE

$$\frac{\partial u}{\partial x}+\frac{\partial^2 u}{\partial x\partial y}-6\frac{\partial u}{\partial y}$$

Solution

Observe that $F\left(D,D'\right)=\left(D+DD'-6D\right)=0$

$$F\left(a,b\right)=a+ab-6b=0$$

ie, $\left(1+b\right)a=6b$

$\Rightarrow$ $a=\dfrac{6b}{1+b},\ b\neq 1$

$\therefore$ $u=\exp\left(\dfrac{6b}{1+b}x+by\right)$ is a solution.

The most general solution therefore is

$$u=\sum_{r=1}^{\infty}A_r\exp\left(\frac{6b_r}{1+b_r}x+b_r y\right).$$

2.2.2 Particular Integrals

To determine the particular integral $\left(P.I\right)$ of eqn $\left(2.5\right)$

ie,

$$F\left(D,D'\right)u=f\left(x,y\right)$$

we shall employ the following two methods:

Method I

If the operator $F\left(D,D'\right)$ is a reducible operator then the Particular Integral is of the form

$$\left.\begin{aligned}&\frac{1}{\left(\alpha_1 D+\beta_1 D'+\gamma_1\right)}\cdot\frac{1}{\left(\alpha_2 D+\beta_2 D'+\gamma_2\right)}\cdots\cdots\cdots\frac{1}{\left(\alpha_m D+\beta_m D'+\gamma_m\right)}f\left(x,y\right)\\&\qquad=\prod_{j=1}^{m}\frac{1}{\left(\alpha_j D+\beta_j D'+\gamma_j\right)}f\left(x,y\right)\end{aligned}\right\}\qquad\left(2.10\right)$$

We start the implimentation of the inversion operation $\left(2.10\right)$ from the last factor on the right as

$$\frac{1}{\left(\alpha_m D+\beta_m D'+\gamma_m\right)}f\left(x,y\right)=G\left(x,y\right)\text{ say}\qquad\left(2.11\right)$$

ie,

$$\left(\alpha_m D+\beta_m D'+\gamma_m\right)G\left(x,y\right)=f\left(x,y\right)\qquad\left(2.12\right)$$

$$\Rightarrow\qquad\alpha_m\frac{\partial G}{\partial x}+\beta_m\frac{\partial G}{\partial y}=f-\gamma_m G\qquad\left(2.13\right)$$

This is Lagranges linear equation with the corresponding auxiliary equations

$$\frac{dx}{\alpha_m}=\frac{dy}{\beta_m}=\frac{dG}{f-\gamma_m G}\qquad\left(2.14\right)$$

From the first two relation we obtain

$$\left.\begin{aligned}&\beta_m dx-\alpha_m dy=0\\ &ie,\ \beta_m x-\alpha_m y=c\end{aligned}\right\}\qquad(2.15)$$

Similarly, we have that

$$\frac{dG}{f-\gamma_m G}=\frac{dx}{\alpha_m}\Rightarrow\frac{dG}{dx}=\frac{f-\gamma_m G}{\alpha_m}$$

ie,

$$\frac{dG}{dx}+\frac{\gamma_m}{\alpha_m}G=\frac{f}{\alpha_m},\ \alpha_m\neq 0\qquad(2.16)$$

This is a first order ODE with an integrating factor (IF) $e^{\int\left(\frac{\gamma_m}{\alpha_m}\right)dx}=e^{\frac{\gamma_m}{\alpha_m}x}$

ie, $$\left(e^{\frac{\gamma_m}{\alpha_m}x}G\right)=\int\frac{f}{\alpha_m}e^{\frac{\gamma_m}{\alpha_m}x}dx,\ \alpha_m\neq 0\qquad(2.16)$$

ie, $$G=e^{-\frac{\gamma_m}{\alpha_m}x}\int\frac{f}{\alpha_m}e^{\frac{\gamma_m}{\alpha_m}x}dx=\frac{1}{\alpha_m}e^{-\frac{\gamma_m}{\alpha_m}x}\int e^{\frac{\gamma_m}{\alpha_m}x}f(x,y)dx,\ \alpha_m\neq 0\qquad(2.17)$$

Similarly,we have

ie, $$G=\frac{1}{\beta_m}e^{-\frac{\gamma_m}{\alpha_m}y}\int e^{\frac{\gamma_m}{\alpha_m}y}f(x,y)dy=\psi(x,y)\text{ say},\ \beta_m\neq 0\qquad(2.18)$$

Observe that no arbitrary constant is introduced because PI does not contain arbitrary constants. It therefore follows that

$$\frac{1}{(\alpha_m D+\beta_m D'+\gamma_m)}f(x,y)=\phi(x,y)\qquad(2.19)$$

This way we operate from the remaining factors from right to the first on the left in turn to finally obtain the PI

Method II

Decomposing the operator $\frac{1}{F(D,D')}$ into partial fractions as

$$\frac{1}{F(D,D')}=\frac{A_1}{(\alpha_1 D+\beta_1 D'+\gamma_1)}+\frac{A_2}{(\alpha_2 D+\beta_{m2} D'+\gamma_2)}+..........+\frac{A_m}{(\alpha_m D+\beta_m D'+\gamma_m)}$$

$$=\sum_{j=1}^{m}\frac{A_j}{(\alpha_j D+\beta_j D'+\gamma_j)}\qquad(2.20)$$

we then perform the inverse operation term-wise to obtain the required PI as demostrated in the following steps:

$$\frac{A_1}{(\alpha_1 D+\beta_1 D'+\gamma_1)}f(x,y)=G(x,y)\qquad(2.21)$$

with the corresponding auxiliary equation

$$\frac{dx}{\alpha_1}=\frac{dy}{\beta_1}=\frac{dG}{f-\gamma_1 G}\qquad(2.22)$$

From the first two relation we obtain

$$\left.\begin{aligned}&\beta_1 dx-\alpha_1 dy=0\\&ie,\ \beta_1 x-\alpha_1 y=c\end{aligned}\right\}\quad(2.23)$$

Similarly, we have that

$$\frac{dG}{f-\gamma_1 G}=\frac{dx}{\alpha_1}\Rightarrow\frac{dG}{dx}=\frac{f-\gamma_1 G}{\alpha_1}$$

ie,

$$\frac{dG}{dx}+\frac{\gamma_1}{\alpha_1}G=\frac{f}{\alpha_1},\ \alpha_1\neq 0\qquad(2.24)$$

This is a first order ODE with an integrating factor (IF) $\exp\left(\int\frac{\gamma_1}{\alpha_1}dx\right)=\exp\left(\frac{\gamma_1}{\alpha_1}x\right)$

ie, $$\exp\left(\frac{\gamma_1}{\alpha_1}x\right)G=A_1\int\frac{f}{\alpha_1}\exp\left(\frac{\gamma_1}{\alpha_1}x\right)dx,\ \alpha_1\neq 0\qquad(2.25)$$

ie, $$G=A_1\exp\left(-\frac{\gamma_1}{\alpha_1}x\right)\int\frac{f}{\alpha_m}\exp\left(\frac{\gamma_1}{\alpha_1}x\right)dx$$

$$=\frac{A_1}{\alpha_1}\exp\left(-\frac{\gamma_1}{\alpha_1}x\right)\int\exp\left(\frac{\gamma_1}{\alpha_1}x\right)f(x,y)dx,\ \alpha_1\neq 0\qquad(2.26)$$

Similarly,we have

ie, $$G=\frac{A_1}{\beta_1}\exp\left(-\frac{\gamma_1}{\alpha_1}y\right)\int\exp\left(\frac{\gamma_1}{\alpha_1}y\right)f(x,y)dy=\psi(x,y)\text{ say, }\beta_m\neq 0\qquad(2.27)$$

The expression for the PI is therefore given nas

$$\frac{A_1}{\alpha_1}\exp\left(-\frac{\gamma_1}{\alpha_1}x\right)\int\exp\left(\frac{\gamma_1}{\alpha_1}x\right)f(x,y)dx+\frac{A_2}{\alpha_2}\exp\left(-\frac{\gamma_2}{\alpha_2}x\right)\int\exp\left(\frac{\gamma_2}{\alpha_2}x\right)f(x,y)dx+\ldots\ldots\ldots\ldots+$$

$$\ldots\ldots\ldots\ldots\ldots\ldots+\frac{A_m}{\alpha_m}\exp\left(-\frac{\gamma_m}{\alpha_m}x\right)\int\exp\left(\frac{\gamma_m}{\alpha_m}x\right)f(x,y)dx,\quad\alpha_j\neq 0\qquad(2.28)$$

or

$$\frac{A_1}{\beta_1}\exp\left(-\frac{\gamma_1}{\beta_1}y\right)\int\exp\left(\frac{\gamma_1}{\beta_1}y\right)f(x,y)dx+\frac{A_2}{\beta_2}\exp\left(-\frac{\gamma_2}{\beta_2}x\right)\int\exp\left(\frac{\gamma_2}{\beta_2}x\right)f(x,y)dx+\ldots\ldots\ldots\ldots+$$

$$\ldots\ldots\ldots\ldots\ldots\ldots+\frac{A_m}{\beta_m}\exp\left(-\frac{\gamma_m}{\beta_m}x\right)\int\exp\left(\frac{\gamma_m}{\beta_m}x\right)f(x,y)dx,\quad\beta_j\neq 0\qquad(2.28)$$

ie,

$$\left.\begin{aligned}&\sum_{j=1}^{m}\frac{A_j}{\alpha_j}\exp\left(-\frac{\gamma_j}{\alpha_j}x\right)\int\exp\left(\frac{\gamma_j}{\alpha_j}x\right)f(x,y)dx,\alpha_j\neq 0\\&\text{or }\sum_{j=1}^{m}\frac{A_j}{\beta_j}\exp\left(-\frac{\gamma_j}{\beta_j}x\right)\int\exp\left(\frac{\gamma_j}{\beta_j}x\right)f(x,y)dx,\beta_j\neq 0\end{aligned}\right\}\quad(2.29)$$

Examples

1 Obtain the solution of the *PDE*

$$\frac{\partial^2 u}{\partial x^2}-6\frac{\partial^2 u}{\partial x\partial y}+9\frac{\partial^2 u}{\partial y^2}=\text{Tan}(3x+y)$$

Solution

In operator form the PDE is expressible as

$$\left(D^2-6DD'+9D'^2\right)u=\text{Tan}(3x+y) \qquad (i)$$

ie, $\quad (D-3D')^2 u=\text{Tan}(3x+y) \qquad (ii)$

The corresponding homogeneous equation is

$$(D-3D')^2 u=0 \qquad (iii)$$

with the corresponding complimentary function

$$u_c=\phi_1(-3x-y)+x\phi_2(3x-y) \qquad (iv)$$

The PI is given as

$$\frac{1}{(D-3D')^2}\text{Tan}(3x+y) \qquad (v)$$

$$=\frac{1}{(D-3D')}\left[\frac{1}{1}\exp\left(-\frac{0}{1}x\right)\int\exp\left(\frac{0}{1}x\right)\text{Tan}(3x+y)\ dx\right]$$

$$=\frac{1}{(D-3D')}\int\text{Tan}(3x+y)\ dx=\frac{1}{(D-3D')}\int\text{Tan}(c)\ dx$$

where $c=-3x-y$

ie,

$$PI=\frac{1}{(D-3D')}x\,\text{Tan}\,c=\frac{x^2}{2}\text{Tan}\,c \qquad (vi)$$

The general solution of the PDE is therefore

$$u(x,y)=\phi_1(-3x-y)+x\phi_2(3x-y)+\frac{x^2}{2}\text{Tan}(3x+y)$$

2 Solve the PDE

$$\left(4D^2-4DD'+D'^2\right)u=16\ln(x+2y)$$

Solution

The corresponding homogeneous equation is given as

$$\left(4D^2-4DD'+D'^2\right)u=0 \qquad (i)$$

ie, $\quad (2D-D')^2 u=0 \qquad (ii)$

The complimentary function is given as

$$u_c=\phi_1(-x-2y)+x\phi_2(-x-2y) \qquad (iii)$$

and the PI is given as

$$\frac{1}{(2D-D')^2}.16\ln(x+2y) \qquad (iv)$$

$$=\frac{1}{(2D-D')}\left[\frac{1}{2}\exp\left(-\frac{0}{2}x\right)\int\exp\left(-\frac{0}{2}x\right)16\ln(c)\,dx\right], c=-x-2y \qquad (v)$$

$$=\frac{1}{(2D-D')}.8x\ln(x+2y) \qquad (vi)$$

$$=8\left[\frac{1}{2}\exp\left(-\frac{0}{2}x\right)\int\exp\left(\frac{0}{2}x\right)x\ln(c)\right] \qquad (vii)$$

$$=4\int x\ln(c)=2x^2\ln(x+2y) \qquad (viii)$$

ie,

$$PI=2x^2\ln(x+2y) \qquad (ix)$$

The solution to the PDE is therefore given as

$$u(x,y)=\phi_1(-x-2y)+x\phi_2(-x-2y)+\ 2x^2\ln(x+2y)$$

2.2.3 Some Special Cases.

We recall that the particular integral of (2.6) is given as

$$u_p(x,y)=\frac{1}{F(D,D')}.f(x,y) \qquad (2.30)$$

This is determined almost the same way as that of *ODEs*.

The inverse operator may be expanded using the Binomial Theorem and thereafter performing the integration $D^{-1},(D')^{-1}$ nwith respect to x and y respectively. The PI corresponding to certain special functions may be obtained by much shorter method than the general method.

In this section we note the following pertinent rules:

Case I :

$$\frac{1}{F(D,D')}.e^{ax+by}=\frac{1}{F(a,b)}.e^{ax+by} \text{ provided } F(a,b)\neq 0$$

Case II :

$$\frac{1}{F(D,D')}.e^{ax+by}\phi(x,y)=e^{ax+by}\frac{1}{F(D+a,D'+b)}.\phi(x,y),\ \phi(x,y) \text{ is arbitrary.}$$

Case III :

If $F(a,b)=0$ in *Case I*, then the PI is obtained as follow:

$$\frac{1}{F(D,D')}.e^{ax+by}=\frac{1}{F(D,D')}.e^{ax+by}.1=e^{ax+by}\frac{1}{F(D+a,D'+b)}.$$

and then apply case II.

Case IV :

$$\frac{1}{F(D,D')}.\text{Cos}(ax+by)=\frac{1}{F(D^2,DD',D'^2)}.\text{Cos}(ax+by)$$

$$=\frac{1}{F(-a^2,-ab,-b^2)}.\text{Cos}(ax+by), \text{ provided } F(a^2,ab,b^2)\neq 0$$

If $F(a^2,ab,b^2)=0$ this case fails. We then compute the PI by considering the real and imaginary parts of

In this case we apply the Binomial theorem to the inverse operator and then operate on $x^m y^n$.
These methods are evidently shorter ways of obtaining the respective *PIs.*

Examples :

1 Solve the *PDE*

$$\left(D^2 - D'^2 - 3D + 3D'\right)u = xy + e^{x+2y}.$$

Solution

Observe that the given differential equation may be put in the form

$$\left(D - D'\right)\left(D + D' - 3\right)u = xy + e^{x+2y}.$$

The complimentary function is given as

$$\phi_1\left(-x - y\right) + e^{3x}\phi_2\left(x - y\right)$$

The particular integral is given as

$$\frac{1}{\left(D - D'\right)\left(D + D' - 3\right)}\left[xy + e^{x+2y}\right]$$

$$= \frac{1}{\left(D - D'\right)\left(D + D' - 3\right)}\left[xy\right] + \frac{1}{\left(D - D'\right)\left(D + D' - 3\right)}\left[e^{x+2y}\right].$$

$$\frac{1}{\left(D - D'\right)\left(D + D' - 3\right)}\left[xy\right]$$

$$= -\frac{1}{3D}\left(1 - \frac{D'}{D}\right)^{-1}\left(1 - \frac{D + D'}{3}\right)^{-1}\left[xy\right]$$

$$= -\frac{1}{3D}\left(1 + \frac{D'}{D} + \frac{D'^2}{D^2} + ...\right)\left(1 + \frac{D + D'}{3} + \frac{\left(D + D'\right)^2}{9}\right)\left[xy\right]$$

$$= -\frac{1}{3D}\left(1 + + \frac{D + D'}{3} + \frac{2DD'}{9} + \frac{D'}{D} + \frac{DD'}{9} + \frac{D'}{3}\right)\left[xy\right]$$

$$= -\frac{1}{3D}\left(1 + \frac{D}{3} + \frac{DD'}{3} + \frac{D'}{D} + \frac{2D'}{3}\right)\left[xy\right]$$

$$= -\frac{1}{3D}\left(xy + \frac{y}{3} + \frac{1}{3} + \frac{x^2}{2} + \frac{2x}{3}\right)$$

$$= -\frac{1}{3}\left(\frac{x^2 y}{2} + \frac{xy}{3} + \frac{x}{3} + \frac{x^3}{6} + \frac{x^2}{3}\right)$$

and

$$\frac{1}{\left(D - D'\right)\left(D + D' - 3\right)}\left[e^{x+2y}\right]$$

$$= \frac{1}{\left(D + D' - 3\right)}.\frac{1}{\left(1 - 2\right)}\left[e^{x+2y}\right] = -\frac{1}{\left(D + D' - 3\right)}.\left[e^{x+2y}\right]$$

$$= -\left[e^{x+2y}\right]\frac{1}{\left(D + 1 + D' + 2 - 3\right)}.1 = -\left[e^{x+2y}\right]\frac{1}{\left(D + D'\right)}.1$$

$$= -\left[e^{x+2y}\right]\frac{1}{D'}\left(1+\frac{D}{D'}\right)^{-1}.1 = -\left[e^{x+2y}\right]\frac{1}{D'}.1$$

$$= -ye^{x+2y}.$$

Thus, the general solution is

$$u(x,y) = \phi_1(-x-y) + e^{3x}\phi_2(x-y) - ye^{x+2y} - \frac{1}{3}\left(\frac{x}{3}+\frac{xy}{3}++\frac{x^2}{3}+\frac{x^2y}{2}+\frac{x^3}{6}\right)$$

2 Solve the *PDE*

$$\left(D^2 - DD' + D' - 1\right)u = \text{Cos}(x+2y) + e^y.$$

Solution

Observe that the *PDE* is of the form

$$(D-1)(D-D+1)u = \text{Cos}(x+2y) + e^y.$$

The reduced DE is

$$(D-1)(D-D'+1)u = 0$$

$$CF = e^x\phi_1(-y) + e^{-x}\phi_2(-x-y)$$

$$PI = \frac{1}{\left(D^2 - DD' + D' - 1\right)}\left[\text{Cos}(x+2y) + e^y\right]$$

$$\frac{1}{\left(D^2 - DD' + D' - 1\right)}\left[\text{Cos}(x+2y)\right]$$

$$= \frac{1}{\left(-1^2 - (2)(-1) + D' - 1\right)}\left[\text{Cos}(x+2y)\right] = \frac{1}{(D')}\left[\text{Cos}(x+2y)\right]$$

$$= \frac{1}{2}\text{Sin}(x+2y).$$

$$\frac{1}{\left(D^2 - DD' + D' - 1\right)}\left[e^y\right] = e^y\frac{1}{\left(D^2 - D(D'+1) + (D'+1) - 1\right)}[1]$$

$$= e^y\frac{1}{\left(D^2 - DD' - D + D'\right)}[1] = -e^y\frac{1}{D}\left[1 - \left(\frac{D'}{D} + D - D'\right)\right]^{-1}[1]$$

$$= -e^y\frac{1}{D}[1] = -xe^y$$

Thus,

$$u(x,y) = e^x\phi_1(-y) + e^{-x}\phi_2(-x-y) + \frac{1}{2}\text{Sin}(x+2y) - xe^y.$$

3 Obtain the solution to the *PDE*

$$\left(D^2 - D'\right)u = xe^{ax+a^2y}.$$

Solution

The reduced equation is

$$\left(D^2 - D'\right)u = 0.$$

The operator $D^2 - D'$ is irreducible. Hence,

$$F(a,b) = a^2 - b = 0 \Rightarrow b = a^2.$$

Hence, $u = xe^{ax+a^2y}$ is a solution of $F(D^2, D') = 0$ has the complimentary function

$$u_c = \sum_{r=1}^{\infty} A_r e^{a_r x + a_r^2 y}$$

$$PI = \frac{1}{D^2 - D'}\left[xe^{ax+a^2y}\right] = e^{ax+a^2y}\cdot\frac{1}{D^2 - D'}.[x]$$

$$= e^{ax+a^2y}\cdot\frac{1}{(D+a)^2 - (D'+a^2)}.[x] = e^{ax+a^2y}\cdot\frac{1}{(D^2 + 2aD + a^2 - D' - a^2)}.[x]$$

$$= e^{ax+a^2y}\cdot\frac{1}{(D^2 + 2aD - D')}.[x] = e^{ax+a^2y}\cdot\frac{1}{(D^2 + 2aD)}\left(1 - \frac{D'}{D^2 + 2aD}\right)^{-1}.[x]$$

$$= e^{ax+a^2y}\cdot\frac{1}{(D^2 + 2aD)}\left(1 + \frac{D'}{D^2 + 2aD} + \frac{D'^2}{(D^2 + 2aD)^2} + ...\right).[x]$$

$$= e^{ax+a^2y}\cdot\frac{1}{(D^2 + 2aD)}.[x] = e^{ax+a^2y}\cdot\frac{1}{2aD}\left(1 + \frac{D}{2a}\right)^{-1}.[x]$$

$$= e^{ax+a^2y}\cdot\frac{1}{2aD}\left(1 - \frac{D}{2a} - \frac{D^2}{4a^2} - ....\right).[x]$$

$$= e^{ax+a^2y}\cdot\frac{1}{2aD}\left(x - \frac{1}{2a}\right).[x]$$

$$= e^{ax+a^2y}\cdot\frac{1}{2a}\left(\frac{x^2}{2} - \frac{x}{2a}\right) = \frac{1}{4a}e^{ax+a^2y}\left(x^2 - \frac{x}{a}\right)$$

$$= \frac{x}{4a^2}(ax - 1)e^{ax+a^2y}.$$

Hence, the general solution of the PDE is

$$u(x,y) = \sum_{r=1}^{\infty} A_r e^{a_r x + a_r^2 y} + \frac{x}{4a^2}(ax - 1)e^{ax+a^2y}.$$

CHAPTER THREE

SECOND – ORDER DIFFERENTIAL EQUATIONS II

3.1 PARTIAL DIFFERENTIAL EQUATIONS OF THE CAUCHY-EULER TYPE

Equations of the of the Cuachy-Euler type are the PDEs of the form

$$F(xD, yD')u = f(x, y) \qquad (3.1)$$

where F is a polynomial in the indeterminate xD and yD'.

In this case we make the following transformations:

$$s = \ln x,\ t = \ln y,\ \vartheta = \frac{\partial}{\partial s} \text{ and } \phi = \frac{\partial}{\partial t} \qquad (3.2)$$

It is therefore immediate from (3.2) that

$$\left.\begin{aligned} (xD)u = \vartheta u,\ (x^2D^2)u = \vartheta(\vartheta-1)u \text{ and } (x^3D^3)u = \vartheta(\vartheta-1)(\vartheta-2)u \\ (yD')u = \phi u,\ (y^2D'^2)u = \phi(\phi-1)u \text{ and } (y^3D'^3)u = \phi(\phi-1)(\phi-2)u \end{aligned}\right\} \qquad (3.3)$$

Substituting (3.3) into (3.1) transforms it into linear equation with constant coefficients with ϑ and ϕ as the new independent variables.

Examples.

Transform the following PDE to linear form

$$\left(x^2D^2 - 4xyDD' + 4y^2D'^2 + 4yD' + xD\right)u = x^2y. \quad (i)$$

Observe that the given PDE is of Cauchy-Euler type. We then define the following transformation:

$$s = \ln x,\ t = \ln y,\ \vartheta = \frac{\partial}{\partial s} \text{ and } \phi = \frac{\partial}{\partial t} \qquad (ii)$$

Using (ii) in (i) we obtain

$$\left[\vartheta(\vartheta-1) - 4\vartheta\phi + 4\phi(\phi-1) + 4\phi + \vartheta\right]u = e^{2s}e^{t} = e^{2s+t}.$$

ie, $\quad \left(\vartheta^2 - 4\vartheta\phi + 4\phi^2\right)u = e^{2s+t}. \qquad (iii)$

$\Rightarrow \quad (\vartheta - 2\phi)^2 u = e^{2s+t}. \qquad (iv)$

This is a linear DE with constant coefficients.

$$\begin{aligned} CF &= \phi_1(-2s-t) + s\phi_2(-2s-t) \\ &= \psi_1(2s+t) + \text{s}\psi_1(2s+t) \qquad (v) \end{aligned}$$

$$PI = \frac{1}{(\vartheta-2\phi)^2}\left[e^{2s+t}\right] = e^{2s+t} \cdot \frac{1}{(\vartheta-2\phi)^2}[1]$$

$$= e^{2s+t} \cdot \frac{1}{\vartheta^2}\left(1-\frac{2\phi}{\vartheta}\right)^{-2}[1] = e^{2s+t} \cdot \frac{1}{\vartheta^2}[1]$$

$$= e^{2s+t}\left(\frac{s^2}{2}+\alpha s+\beta\right)$$

The general solution is therefore,

$$u = \psi_1(2s+t) + s\psi_1(2s+t) + e^{2s+t}\left(\frac{s^2}{2}+\alpha s+\beta\right)$$

ie, $$u = \psi_1(2\ln x+\ln y) + \ln x\ \psi_1(2\ln x+\ln y) + e^{2\ln x+\ln y}\left(\frac{1}{2}(\ln x)^2+\alpha\ln x+\beta\right)$$

$$= \psi_1(\ln x^2 y) + \ln x\ \psi_1(2\ln x+\ln y) + \left(\frac{1}{2}(\ln x)^2 + A\ln x\right)x^2 y$$

Example

$$\frac{1}{x^2}\frac{\partial^2 u}{\partial x^2} - \frac{1}{x^3}\frac{\partial u}{\partial x} = \frac{1}{y^2}\frac{\partial^2 u}{\partial y^2} - \frac{1}{y^3}\frac{\partial u}{\partial y}. \qquad (i)$$

Suppose $s = \frac{x^2}{2}$ and $t = \frac{y^2}{2}$ $\qquad (ii)$

Then

$$\left.\begin{aligned}
&\frac{\partial u}{\partial x} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} = x\frac{\partial u}{\partial s} \text{ or } \frac{\partial u}{\partial s} = \frac{1}{x}\frac{\partial u}{\partial x}\\
&\frac{\partial^2 u}{\partial s^2} = \frac{\partial}{\partial s}\frac{\partial u}{\partial s} = \frac{1}{x}\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial}{\partial x}\right) = \frac{1}{x^2}\frac{\partial^2}{\partial x^2} - \frac{1}{x^3}\frac{\partial}{\partial x}\\
&\frac{1}{x^2}\frac{\partial^2}{\partial x^2} - \frac{1}{x^3}\frac{\partial}{\partial x} = \frac{\partial^2 u}{\partial s^2}
\end{aligned}\right\} \qquad (iii)$$

Similarly,

$$\frac{1}{y^2}\frac{\partial^2}{\partial y^2} - \frac{1}{y^3}\frac{\partial}{\partial y} = \frac{\partial^2 u}{\partial t^2} \qquad (iv)$$

Thus the given PDE is transformed into

$$\frac{\partial^2 u}{\partial s^2} = \frac{\partial^2 u}{\partial t^2} \text{ or } (\vartheta^2 - \phi^2)u = 0 \qquad (v)$$

where $\vartheta = \frac{\partial}{\partial s}$ and $\phi = \frac{\partial}{\partial t}$

$\Rightarrow \qquad (\vartheta-\phi)(\vartheta+\phi) = 0$

Hence,

$$u = \varphi_1(-s-t) + \varphi_2(s-t)$$

$$= \varphi_1\left(-\frac{x^2+y^2}{2}\right) + \varphi_2\left(\frac{x^2-y^2}{2}\right)$$

$$= \psi_1(x^2+y^2) + \psi_2(x^2-y^2)$$

3.2 SECOND-ORDER *PDE* WITH VARIABLE COEFFICIENTS.

Definition.

A partial differential equation with variable coefficients is that which contains atleast one of the partial derivative of the second order and none higher than the second. This is simplified if we consider the case of two independent variables.

We shall define the following:

$$\left.\begin{aligned} p=\frac{\partial u}{\partial x}, q=\frac{\partial u}{\partial y}, r=\frac{\partial^2 u}{\partial x^2}=\frac{\partial p}{\partial x}, s=\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) \\ =\frac{\partial p}{\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial q}{\partial x}, t=\frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial q}{\partial y} \end{aligned}\right\} \qquad (3.4)$$

Our discussion shall be limited to that of the variable coefficients which are of first degree in r, s, t

ie, $$Rr+Ss+Tt=V \qquad (3.5)$$

in which R, S, T and V are in general functions of Rx, y, p, q and u.

This will be illustrated by examples solvable by inspection.

Example.

1 Solve $s=2x+2y$

Solution

The PDE is given by

$$\frac{\partial^2 u}{\partial x\partial y}=2x+2y \qquad (i)$$

Integrating wrt y we have

$$\frac{\partial u}{\partial x}=2xy+y^2+h(x) \qquad (ii)$$

Finally, integrating wrt x yields

$$u(x,y)=x^2y+xy^2+\int h(x)dx+g(y) \qquad (iii)$$

ie, $$u(x,y)=x^2y+xy^2+\phi(x)+g(y) \qquad (iv)$$

2 Solve $xr+p=9x^2y^2$.

The PDE is given by

$$x\frac{\partial^2 u}{\partial x^2}+p=9x^2y^2 \qquad (i)$$

ie, $$\frac{\partial p}{\partial x}+\frac{1}{x}p=9xy^2 \qquad (ii)$$

The DE in (ii) has an integrating factor (IF) x

ie, $$(xp)'=9xy^2 \qquad (iii)$$

ie, $$xp=\int 9x^2y^2dx=3x^3y^2+f(y) \qquad (iv)$$

ie, $$xp=\int 9x^2y^2dx=3x^3y^2+f(y) \qquad (v)$$

ie, $$\frac{\partial u}{\partial x} = 3x^2y^2 + \frac{1}{x}h(y)$$

$\therefore$ $$u(x,y) = x^3y^2 + \int \frac{1}{x}h(y)dx$$

ie, $$u(x,y) = x^3y^2 + h(y)\ln x + \alpha(y)$$

3 Solve $s - t = \frac{x^2}{y}\cdot$

Solution

The DE is $$\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x} = \frac{x^2}{y} \qquad (i)$$

Integrating with respect to y and treating x as a constant and conversely yields

$$p - q = -\frac{x}{y} + f(x) \qquad (ii)$$

This is Lagranges linear equation with auxiliary equation

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{du}{f - x/y} \qquad (iii)$$

From the first two ratios we obtain

$$-dx - dy = 0 \qquad (iv)$$

ie, $$x + y = c \qquad (v)$$

From the first last ratios we have

$$du = f(x)dx - \frac{x}{y}dx = f(x)dx - \frac{x}{c-x}dx$$

$$= f(x)dx + \left(1 - \frac{c}{c-x}\right)dx \qquad (vi)$$

Integrating we have

$$u = \int f(x)dx + \int\left(1 - \frac{c}{c-x}\right)dx = \int f(x)dx + x + c\ln(c-x) + \beta(y)$$

$$= \phi(x) + x + (x+y)\ln y$$

The general solution is therefore

$$u = \phi(x) + (x+y)\ln y + F(x,y).$$

We note that (3.5) is a second - order quasilinear PDE. It is linear if it can be put in the form

$$Rr + Ss + Tt + Pp + Uu = V \qquad (3.6)$$

in which R, S, T, P, U and V are functions of x and y.

(a) $$\frac{\partial^2 u}{\partial x^2} + x\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = e^{xy}\,\mathrm{Sin}\,u$$

(b) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + xy\frac{\partial u}{\partial y}u = x + y$$

Observe that (a) is a second order quasilinear PDE while (b) is a linear second-order PDE.

3.3 MONGE'S METHOD.

In this section we shall discuss the Monge's general method of solving

$$Rr + Ss + Tt = V \qquad (3.7)$$

in which R, S, T and V are functions of x, y, u, p and q with r, s and t retaining their ususl definitions. *ie*,

$$r = \frac{\partial^2 u}{\partial x^2}, s = \frac{\partial^2 u}{\partial x \partial y} \text{ and } t = \frac{\partial^2 u}{\partial y^2} \qquad (3.8)$$

From (3.7) we recall that

$$dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy = rdx + sdy \qquad (3.9)$$

$$dq = \frac{\partial q}{\partial x} dx + \frac{\partial q}{\partial y} dy = sdx + tdy \qquad (3.10)$$

From (3.9) we have

$$r = \frac{dp - sdy}{dx} \text{ and } t = \frac{dq - sdx}{dy} \qquad (3.11)$$

Substituting (3.11) into (3.7) yields

$$R\left(\frac{dp - sdy}{dx}\right) + Ss + T\left(\frac{dq - sdx}{dy}\right) = V \qquad (3.12)$$

or $\qquad Rdpdy - Rs(dy)^2 + Ssdxdy + Tdqdx - Ts(dx)^2 - Vdxdy = 0$

ie, $\qquad (Rdpdy - Vdxdy + Tdqdx) - \left(Rs(dy)^2 - Ssdxdy + Ts(dx)^2\right) = 0$

ie, $\qquad (Rdpdy - Vdxdy + Tdqdx) - s\left(R(dy)^2 - Sdxdy + T(dx)^2\right) = 0 \qquad (3.13)$

If there exists a relation between x, y, u, p and q such that the terms in parenthesis in (3.11) vanish indep endently then it satisfies both (3.13) and (3.7). It therefore follows that

$$R(dy)^2 - Sdxdy + T(dx)^2 = 0 \qquad (3.14)$$

$$Rdpdy - Vdxdy + Tdqdx = 0 \qquad (3.15)$$

These are refered to as the Monge's subsidiary equations.

We now assume that (3.14) is resolvable into factors thus;

$$\left.\begin{aligned} dy - m_1 dx = 0 \\ dy - m_2 dx = 0 \end{aligned}\right\} \qquad (3.16)$$

The first equation in (3.16) combined with (3.13) and with $du = pdx + qdy$ will yield an integral of the form $g_1 = a$ and $h_1 = b$ in which a and b are arbitrary constants. Then a relation of the type

$$h_1 = f_1(g_1) \qquad (3.17)$$

where f_1 is arbitrary will be an integral. This is called an intermediate (first) integral.

Similarly, second equation in (3.16) combined with (3.13) will give another intermediate integral of the type

$$h_2 = f_2(g_2) \qquad (3.18)$$

in which f_2 is also arbitrary.

Solving (3.17) and (3.18) we obtain p and q in terms of x, y and u. These values of p and q are then substituted in $du = pdx + qdy$ which on integration yields the required solution.

We however here note that if $(3.16a)$ is a perfect square it is convinient in some cases to compute only one intermediate integral and integrate it with the help of Lagrange's method to get the complete solution.

Examples.

1 Solve

$$r + (a+b)s + abt = xy \qquad (i)$$

Solution.

We recall that

$$dp = rdx + sdy \text{ and } dq = sdx + tdy \qquad (ii)$$

ie,

$$r = \frac{dp - sdy}{dx} + \text{ and } t = \frac{dq - sdx}{dy} \qquad (iii)$$

Substituting (iii) into (i) we have

$$\frac{dp - sdy}{dx} + (a+b)s + ab\left(\frac{dq - sdx}{dy}\right) = xy$$

ie, $$dpdy - s(dy)^2 + (a+b)sdxdy + abdqdx - sab(dx)^2 - xydxdy = 0$$

ie, $$(dpdy - xydxdy + abdqdx) - s\left((dy)^2 - (a+b)dxdy + ab(dx)^2\right) = 0 \qquad (iv)$$

The Monge's subsidiary equation are thus;

$$dpdy - xydxdy + abdqdx = 0 \qquad (v)$$

$$(dy)^2 - (a+b)dxdy + ab(dx)^2 = 0 \qquad (vi)$$

Considering (vi) in the form

$$\left[(dy)^2 - adxdy\right] + \left[ab(dx)^2 - bdxdy\right] = 0 \qquad (vii)$$

we may have

$$(dy)^2 - adxdy = 0 \qquad (viii)$$

$$ab(dx)^2 - bdxdy = 0 \qquad (ix)$$

which gives respectively

$$dy - adx = 0 \quad \Rightarrow y - ax = c_1 \qquad (x)$$

$$dy - bdx = 0 \quad \Rightarrow y - bx = c_2 \qquad (xi)$$

Substituting (x) into (iv) we obtain

$$adpdx + abdqdx - xa(c_1 + ax)(dx)^2 = 0$$

ie, $dp + bdq - x(c_1 + ax)dx = 0 \qquad (xii)$

Integrating (xii) yields

$$p + bq = \left(c_1 \frac{x^2}{2} + a\frac{x^3}{3}\right) + A$$

ie, $p + bq = (y - ax)\frac{x^2}{2} + \frac{1}{3}ax^3 + A \qquad (xiii)$

Therefore, the first integral is

$$p + bq + \frac{1}{6}ax^3 - \frac{1}{2}x^2 y = f_1(y - ax) \qquad (xiv)$$

Similarly, the other intermediary integral is

$$p + aq + \frac{1}{6}bx^3 - \frac{1}{2}x^2 y = f_2(y - bx) \qquad (xv)$$

From (xiv) and (xv) we have

$$p(b-a) + \frac{1}{6}(b^2 - a^2)x^3 - \frac{1}{2}(b-a)x^2 y = bf_2(y - bx) - af_1(y - ax)$$

ie,

$$p = \frac{1}{2}x^2 y - \frac{1}{6}(b+a)x^3 + \frac{1}{b-a}(bf_2 - af_1) \qquad (xvi)$$

Similarly, we have

$$q(b-a) - \frac{1}{6}(b-a)x^3 = f_1(y-ax) - f_2(y-bx)$$

ie, $q = \frac{1}{6}x^3 + \frac{1}{b-a}\left[f_1(y-ax) - f_2(y-bx)\right] \qquad (xvii)$

$$du = pdx + qdy$$

$$= \left[\frac{1}{2}x^2 y - \frac{1}{6}(b+a)x^3 + \frac{1}{b-a}(bf_2 - af_1)\right]dx + \left[\frac{1}{6}x^3 + \frac{1}{b-a}\left[f_1(y-ax) - f_2(y-bx)\right]\right]dy$$

ie, $u = \int\left[\frac{1}{2}x^2 y - \frac{1}{6}(b+a)x^3 + \frac{1}{b-a}(bf_2 - af_1)\right]dx + \int\left[\frac{1}{6}x^3 + \frac{1}{b-a}\left[f_1(y-ax) - f_2(y-bx)\right]\right]dy$

$$= \frac{1}{6}x^3 y - \frac{1}{24}(b+a)x^4 + \frac{1}{24}x^4 + \frac{1}{b-a}\int(bf_2 - af_1)dx + \frac{1}{b-a}\int\left[f_1(y-ax) - f_2(y-bx)\right]dy$$

ie, $u = \frac{1}{6}x^3 y - \frac{1}{24}(b+a)x^4 + \frac{1}{24}x^4 + \phi_1(y-ax) + \phi_2(y-ax).$

2 Solve

$$t - r\sec^4 y = 2q\tan y. \qquad (i)$$

$$dp = rdx + sdy \text{ and } dq = sdx + tdy \qquad (ii)$$

ie, $r = \frac{dp - sdy}{dx} +$ and $t = \frac{dq - sdx}{dy} \qquad (iii)$

Substituting (iii) into (i) we have

$$\frac{dq - sdx}{dy} - \left(\frac{dp - sdy}{dx}\right)\sec^4 y = 2q\tan y.$$

ie, $\left(dqdx - dpdy\sec^4 y - 2q\tan ydxdy\right) - s\left((dx)^2 - (dy)^2\sec^4 y\right) = 0 \qquad (iv)$

The Monge's subsidiary equations are

$$dqdx - dpdy\sec^4 y - 2q\tan ydxdy = 0 \qquad (v)$$

$$(dx)^2 - (dy)^2\sec^4 y = 0 \qquad (vi)$$

Observe that (vi) is of the form

$$\left(dx - dy\sec^2 y\right)\left(dx + dy\sec^2 y\right) = 0 \qquad (vii)$$

ie,

$$dx - dy\sec^2 y = 0, dx + dy\sec^2 y = 0 \qquad (viii)$$

Substituting the first of $(viii)$ into (v) we have

$$dqdy\sec^2 y - dpdy\sec^4 y - 2q\tan y\sec^2 y(dy)^2 = 0$$

ie, $dq - dp\sec^2 y - 2q\tan ydy = 0$

ie, $dq\cos^2 y - dp - 2q\tan y\sin ydy = 0 \qquad (ix)$

ie, $p - q\cos^2 y = f_1(x - \tan y) \qquad (x)$

Similarly, the second of $(viii)$ and (v) give

$$p + q\cos^2 y = f_2(x + \tan y) \qquad (xi)$$

ie, $p = \frac{1}{2}\left[f_1(x - \tan y) + f_2(x + \tan y)\right] \qquad (xii)$

and

ie, $q = \frac{1}{2}\left[f_2(x + \tan y) - f_1(x - \tan y)\right]\sec^2 \mathrm{y} \qquad (xiii)$

$\therefore$ $du = \frac{1}{2}\left[\left[f_1(x - \tan y) + f_2(x + \tan y)\right]dx + \left[f_2(x + \tan y) - f_1(x - \tan y)\right]\sec^2 ydy\right]$

$$= \frac{1}{2}\left[dx - dy\sec^2 y\right]f_2(x + \tan y) + \frac{1}{2}\left[dx + dy\sec^2 y\right]f_1(x - \tan y)$$

ie, $u = \phi_1(x + \tan y) + \phi_2(x - \tan y)$

Exercise

Prove that the solution to the *PDE* $q^2r - 2pqrs + p^2t = 0$ is given as the intersection between the planes $u = c, y + xf(c) = \phi(c)$.

3.4 GENERAL FORM OF SECOND-ORDER *PDE* WITH VARIABLE COEFFICIENTS ADMITTING A FIRST INTEGRAL AND ITS SOLUTIONS.

In section 3.3 we saw that a relation of the form

$$h = f(g) \tag{3.19}$$

in which g and h are differentiable functions of x, y, u, p and q and f an arbitrary differentiable function is called a first (intermediate) integral of a second-order *PDE* if the latter is obtained by eliminating f and f' from (3.19) together with the relation obtained by differentiating (3.19) partially wrt x and y. We now discuss the general form of second-order *PDE* if admitting first integral and its method of solution due to Monge.

Differentiating (3.19) partially wrt x and y yields

$$\frac{\partial h}{\partial x} + \frac{\partial h}{\partial u}\cdot p + \frac{\partial h}{\partial p}\cdot r + \frac{\partial h}{\partial q}\cdot s = f'(g)\left(\frac{\partial g}{\partial x} + \frac{\partial g}{\partial u}\cdot p + \frac{\partial g}{\partial p}\cdot r + \frac{\partial g}{\partial q}\cdot s\right) \tag{3.20}$$

$$\frac{\partial h}{\partial y} + \frac{\partial h}{\partial u}\cdot q + \frac{\partial h}{\partial p}\cdot s + \frac{\partial h}{\partial q}\cdot t = f'(g)\left(\frac{\partial g}{\partial y} + \frac{\partial g}{\partial u}\cdot q + \frac{\partial g}{\partial p}\cdot s + \frac{\partial g}{\partial q}\cdot t\right) \tag{3.21}$$

Eliminating $f'(g)$ between (3.20) and (3.21) yields

$$Rr + Ss + Tt + U\left(rt - s^2\right) = V \tag{3.22}$$

where

$$\left.\begin{aligned} R &= \frac{\partial(g,h)}{\partial(p,y)} + \frac{\partial(g,h)}{\partial(p,u)}\cdot q,\ S = \frac{\partial(g,h)}{\partial(q,y)} + \frac{\partial(g,h)}{\partial(q,u)}\cdot q + \frac{\partial(g,h)}{\partial(u,p)}\cdot p + \frac{\partial(g,h)}{\partial(x,p)} \\ T &= \frac{\partial(g,h)}{\partial(x,q)} + \frac{\partial(g,h)}{\partial(u,q)}\cdot p, \quad U = \frac{\partial(g,h)}{\partial(p,q)} \\ V &= \frac{\partial(g,h)}{\partial(y,u)}\cdot p + \frac{\partial(g,h)}{\partial(u,x)}\cdot q + \frac{\partial(g,h)}{\partial(y,x)} \end{aligned}\right\} \tag{3.23}$$

Hence, (3.22) is the most general form of second-order PDE that possesses a first (intermediate) integral. We thus proceed as in Monge's method for solving equations of this kind by determining the first integral. Recall that

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy = rdx + sdy \tag{3.24}$$

and

$$dq = \frac{\partial q}{\partial x}dx + \frac{\partial q}{\partial y}dy = sdx + tdy \tag{3.25}$$

ie,

$$r = \frac{dp - sdy}{dx} \text{ and } t = \frac{dq - sdx}{dy} \tag{3.26}$$

Putting (3.26) into (3.22) we have

$$R\left(\frac{dp - sdy}{dx}\right) + Ss + T\left(\frac{dq - sdx}{dy}\right) + U\left(\frac{dp - sdy}{dx}\right)\left(\frac{dq - sdx}{dy}\right) - Us^2 = V$$

ie,

$$Rdpdy - Rs(dy)^2 + Ssdxdy + Tdqdx - Ts(dx)^2 + U\left(dpdq - sdpdx - sdqdy + s^2dxdy\right) - Vdxdy = 0$$

ie,

$$\left(Rdpdy + Tdqdx + Udpdq - Vdxdy\right) - s\left(R(dy)^2 + Udpdx + Udqdy - Sdxdy + T(dx)^2\right) = 0 \quad (3.27)$$

Monge's subsidiary equations are:

$$\left.\begin{aligned} M &= Rdpdy + Tdqdx + Udpdq - Vdxdy = 0 \\ N &= R(dy)^2 + Udpdx + Udqdy - Sdxdy + T(dx)^2 = 0 \end{aligned}\right\} \quad (3.27b)$$

In view of the presence of the terms $Udpdx$ and $Udqdy$ N cannot be factorized . We may however try to factorize

$$N + \lambda N = 0 \qquad (3.28)$$

where λ is an undetermined multiplier.

ie,

$$R(dy)^2 + Udpdx + Udqdy - Sdxdy + T(dx)^2 + \lambda\left(Rdpdy + Tdqdx + Udpdq - Vdxdy\right) = 0 \quad (3.29)$$

Suppose this has factors

$$\left(Rdy + mTdx + \kappa Udp\right) + \lambda\left(dy + \frac{1}{m}dx + \frac{\lambda}{\kappa}dq\right) = 0 \qquad (3.30)$$

Comparing (3.29) and (3.30) we obtain

$$\frac{R}{m} + mT = -(S + \lambda V) \qquad (3.31)$$

$$\kappa = m \qquad (3.32)$$

$$\frac{R\lambda}{\kappa} = U \qquad (3.33)$$

Eliminating κ and m from (3.31) through (3.33) we observe that λ satisfies the quadratic equation

$$\lambda^2(UV + RT) + \lambda US + U^2 = 0 \qquad (3.34)$$

Recall that (3.34) has in general two roots λ_1, λ_2. Putting $\lambda = \lambda_1$ and $\kappa = m = \frac{R\lambda_1}{U}$ in (3.30) we have

$$\left(Udy + \lambda_1 Tdx + \lambda_1 Udp\right)\left(Udx + R\lambda_1 dy + \lambda_1 Udq\right) = 0 \qquad (3.35)$$

Similarly, replacing λ with λ_2 we have

$$\left(Udy + \lambda_2 Tdx + \lambda_2 Udp\right)\left(Udx + R\lambda_2 dy + \lambda_2 Udq\right) = 0 \qquad (3.36)$$

We now obtain two integrals of the form $g_1 = a_1$ and $h_1 = b_1$ by solving the pair $\lambda = \lambda_1$ and $\kappa = m = \frac{R\lambda_1}{U}$ and integrals of the type $g_2 = a_2$ and $h_2 = b_2$ obtained from solving the pairs (λ_1, λ_2). Hence, we get the two integrals of the type $h_1 = f_1(g_1)$ and $h_2 = f_2(g_2)$ where f_1 and f_2 are arbitrary. These are solved to determine p and q as functions of x, y and u thereafter substituting into $du = pdx + qdy$ which when integrated gives the complete solution.

In implementing this procedure we note the following:

1 If (3.34) has double roots, it is only possible to obtain one integral of the form $h_1 = f_1(g_1)$ which can be obtained from either $g_1 = a_1$ or $h_1 = b_1$ to give the values of p and q to render $du = pdx + qdy$ integrable.

2 Since $\lambda_1 = \lambda_2$ we get a more general solution by taking liner relation between $g_1$and h_1 in the form $g_1 = mh_1 + n$ and integrate by Lagrange's method.

3 If the first integral $h_1 = f_1(g_1)$ and $h_2 = f_2(g_2)$ and unsolvable for p and q then one of the first integrals $h_1 = f_1(g_1)$ may be combined with $g_2 = a_2$ or $h_2 = b_2$ to determine the values of p and q and then integrating $du = pdx + qdy$ to obtain the complete solution $($integral$)$.

Examples

1 Solve the differential equation

$$u\left(1+q^2\right)r - 2pqus + u\left(1+p^2\right)t - u^2\left(s^2 - rt\right) + 1 + q^2 + p^2 = 0 \qquad (i)$$

Solution

From the general *PDE*

$$Rr + Ss + Tt + U\left(rt - s^2\right) = V \qquad (ii)$$

we have

$$R = u\left(1+q^2\right),\ S = -2pqu,\ T = u\left(1+p^2\right), U = u^2, V = -\left(1+q^2+p^2\right) = 0 \qquad (iii)$$

Substituting into the λ-equation

$$\lambda^2\left(UV - RT\right) + \lambda SU + U^2 = 0 \qquad (iv)$$

we have

$$\lambda^2\left\{-u^2\left(1+q^2+p^2\right) - u^2\left(1+q^2\right)\left(1+p^2\right)\right\} - 2\lambda pqu^3 + u^4 = 0$$

ie, $$\lambda^2 p^2 q^2 - 2\lambda pqu + u^2 = 0 \qquad (v)$$

ie, $$\left(\lambda pq - u\right)^2 = 0$$

$\Rightarrow$ $$\lambda_1 = \lambda_2 = \frac{u}{pq} \qquad (vi)$$

The intermediate integral is thus given as

$$\left.\begin{aligned} Udy + \lambda_1 Tdx + \lambda_1 Udp = 0 \\ Udx + \lambda_2 Rdy + \lambda_2 Udq = 0 \end{aligned}\right\} \qquad (vii)$$

ie, $$u^2 dy + \frac{u^2}{pq}\left(1+p^2\right)dx + \frac{u^3}{pq}dp = 0$$

$$u^2 dx + \frac{u^2}{pq}\left(1+q^2\right)dy + \frac{u^3}{pq}dq = 0$$

ie, $$\left.\begin{aligned} pqdy + \left(1+p^2\right)dx + udp = 0 \\ pqdx + \left(1+q^2\right)dy + udq = 0 \end{aligned}\right\} \qquad (viii)$$

Also, we have

$$du = pdx + qdy = 0 \qquad (ix)$$

From the $(viiia)$ and (ix) we have

$$dx + udp + pdu = 0\left(ie, viiia - p\times ix\right)$$

ie, $$dx + d\left(up\right) = 0$$

Integrating gives

$$x + up = a \qquad (x)$$

Similarly, from the $(viiib)$ and (ix) we have

$$dy + udq + qdu = 0\left(ie, viiib - q\times ix\right)$$

ie, $$dy + d\left(uq\right) = 0$$

Integrating gives

$$y + uq = b \qquad (xi)$$

From (x)

$$\left.\begin{aligned} p &= \frac{a-x}{u} \\ &\text{and from } (xi) \\ q &= \frac{b-y}{u} \end{aligned}\right\} \qquad (xii)$$

$\therefore$ $$du = \frac{a-x}{u}dx + \frac{b-y}{u}dy$$

ie,

$$udu = \left(a-x\right)dx + \left(b-y\right)dy$$

$$\frac{u^2}{2} = ax - \frac{x^2}{2} + by - \frac{y^2}{2}$$

ie, $$u^2 + \left(x-a\right)^2 + \left(y-b\right)^2 = A$$

is the required solution.

By note 2 we can find a more general solution of the given *PDE*. Hence, we assume

$$\left.\begin{aligned} pu + x &= m\left(qu + y\right) + n \\ \text{or} \quad \left(p - mq\right)u &= my - x + n \end{aligned}\right\} \qquad (xiii)$$

which is a Lagrange's linear equationwith corresponding auxiliary equation given as

$$\frac{dx}{u} = \frac{dy}{-mu} = \frac{du}{my - x + n} = \frac{\frac{x}{u}dx + \frac{y}{u}dy + du}{n}$$

From the first two we have

$$\frac{dx}{u} = \frac{dy}{-mu} \Rightarrow mdx + dy = 0 \Rightarrow y + my = c_1$$

From first and last we have

$$\frac{dx}{u} = \frac{\frac{x}{u}dx + \frac{y}{u}dy + du}{n}$$

or $\qquad ndx = xdx + ydy + udu = \frac{1}{2} d\left(x^2 + y^2 + u^2\right)$

Integrating we have

$$x^2 + y^2 + u^2 - 2nx = c_2$$

The general solution is thus

$$x^2 + y^2 + u^2 - 2nx = f\left(y + mx\right).$$

2 Determine the general solution of the differential equation

$$ar + bs + ct + e\left(rt - s^2\right) = h \qquad (i)$$

where a, b, c, e, h are constants.

Solution

We consider the equation

$$Rr + Ss + Tt + U\left(rt - s^2\right) = V \qquad (ii)$$

Comparing (i) and (ii) we have

$$R = a, S = b, T = c, U = e \text{ and } V = h \qquad (iii)$$

But the λ-equation is in general given as

$$\lambda^2\left(UV + RT\right) + \lambda SU + U^2 = 0 \qquad (iv)$$

ie,

$$\lambda^2\left(ac + eh\right) + \lambda be + e^2 = 0 \qquad (v)$$

Foe convinience we set $\lambda m + e = 0$ in (v) to obtain

$$m^2 - bm + ac + eh = 0 \qquad (vi)$$

We assume further that (vi) admitts roots m_1 and m_2 .

The first system of integrals is

$$\left.\begin{aligned} cdx + edp - m_1 dy &= 0 \\ ady + edq - m_2 dx &= 0 \end{aligned}\right\} \qquad (vii)$$

An intermediate integral is

$$cx + ep - m_1 y = f_1\left(ay + eq - m_2 x\right) \qquad (viii)$$

The second system of integral is given by

$$ady + edq - m_2 dx = 0$$

ie, $\qquad ay + eq - m_2 x =$ constant

and

$$cdx + edp - m_2 dy = 0 \Rightarrow cx + ep - m_2 y = \text{constant}$$

Therefore the other intermediate integral is

$$cx + ep - m_2 y = f_2\left(ay + eq - m_2 x\right) \qquad (ix)$$

Clearly, p and q can not be easily solvedfrom the above intermediate integrals. Therefore we combine any particular integral of the second with the general integral of the firstsystem.

ie,

$$cx + ep - m_2 y = A \qquad (x)$$

From $(viii)$ we obtain

$$\left.\begin{aligned} f_1(ay + eq - m_2 x)\ cx + ep &= (m_2 - m_1)y + A \\ ay + eq &= -m_2 x + \psi\{(m_2 - m_1)y + A\} \end{aligned}\right\} \qquad (xi)$$

where ψ is an inverse functionn of ϕ. Using the values of p and q from (x) and (xi) in the general relation $du = pdx + qdy$ we thus have

$$\begin{aligned} edu &= (A + m_2 y - cx)dx + \left[m_2 x + \psi\{(m_2 - m_1)y + A\} - ay\right]dy \\ &= Adx - cxdx + m_2(xdy + ydx) - aydy + \psi\{(m_2 - m_1)y + A\}dy \end{aligned}$$

Integration gives

$$eu = Ax - \frac{1}{2}cx^2 + 2m_2 xy - \frac{1}{2}ay^2 + F\{(m_2 - m_1)y + A\} + B$$

where $F(m_2 - m_1)y + A = \int \psi\{(m_2 - m_1)y + A\}dy$

CHAPTER FOUR

BOUNDARY VALUE PROBLEMS.

4.1 BOUNDARY CONDITIONS AND BOUNDARY VALUE PROBLEMS.

If a second-order differential equation

$$F\left(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}\right) = 0 \qquad 4.1$$

is to be solved within a specified region R of space in which the values of the dependent variables u are specified at the boundary ∂R then the resulting problem is refered to as a *boundary value problem.* These boundaries need not enclose a finite volume. In this case one of the boundaries may be at infinity.

A PDE in which one of the independent variables is time, the value of the dependent variable and often its time derivatives at some instant of time, $t = 0$ (say) may be given. These type of conditions are called *initial conditions*. Hence, the term *boundary* and *initial* condtions will be used as appropriate.

We shall concern ourselves here primarily with two ntypes of boundary conditions that arise frequently in the description of physical phenomena and which we encounter frequently in many applications:

(a) Dirichlet Conditions; where the dependent variable u is specified at each pointof a boundary in a region. For example at the end of a rectangular region.

$$R : a \le x \le b, c \le y \le d.$$

(b) Cauchy Condition; if one of the independent variables is time (t) and the values of both u and $\frac{\partial u}{\partial t}$ are specified on the boundary at time $t = 0$(at some initial time) then this condition is refered to as *cauchy* type.

In applied Mathematics, Physics and Engineering, *PDE*s generally arise from the mathematical formulation of the *real – life* physical problems. Often, boundary conditions are imposed on the dependent variables and certain of its derivatives. The process of determining a *PDE* subject to the imposed boundary condition is solving a boundary value proble (BVP). It is initial value problem if initial conditions are imposed on the differential equation.

3.2 *METHOD OF SEPERATION OF VARIABLE*.

This is perhaphs the oldest and commonest method of solving a partial differential equation.

Given the unknown function

$$u = u\left(x_1, x_2, x_3, x_4, \cdots x_{m-1}, x_m\right) \qquad (4\cdot 2)$$

we shall on the onset make some fundamental assumptions thus:

that

$$u\left(x_1, x_2 \cdots\cdots x_{m-1}, x_m\right) = X_1\left(x_1\right)\cdot X_2\left(x_2\right)\cdot X_3\left(x_3\right)\cdots X_{m-1}\left(x_{m-1}\right)\cdot X_m\left(x_m\right) \qquad (4\cdot 3)$$

in which

$$X_k = X_k\left(x_k\right) \qquad (4\cdot 4)$$

a function of a single independent variable.

On substituting $(4\cdot 3)$ into $(4\cdot 1)$ and simplifying we obtain ordinary differential equations$(ODEs)$ in the unknown functions $X_k\left(k = 1(1)m\right)$. Some of the boundary conditions of the original *PDE* will give rise to corresponding boundary conditions to be satisfied by some of the functions $X_k\left(k = 1(1)m\right)$. We will therefore have to solve m uncoupled ordinary differential equations some of which may be *BVPs* or *IVPs*. These particular solutions X_k are then used to constitute the most general solution of the original *PDE*.

Consider the *PDE* in two independent variables x and y in the form

$$Rr + Ss + Tt + Pp + Qq + Uu = V \qquad (4\cdot 5)$$

Suppose the solution of $(4\cdot 5)$ is given as

$$u = X(x)\cdot Y(y) \qquad (4\cdot 6)$$

in which X and Y are functions of x and y respectively and u is the dependent variable. Substituting $(4\cdot 6)$ into $(4\cdot 5)$ and simplifying we obtain

$$\frac{1}{X} f(D)\cdot X(x) = \frac{1}{Y}\phi(D')\cdot Y(y) \qquad (4\cdot 7)$$

where $f(D)$ and $\phi(D')$ are quadratic functions of $D = \frac{\partial}{\partial x}$ and $D' = \frac{\partial}{\partial y}$ respectively. We observe that the lhs of $(4\cdot 7)$ is a function of x only while the rhs is a function of y only and the two can not be equal except each is equal to a constant $-\lambda$(say).

We thus have

$$\left.\begin{aligned} f(D)\cdot X(x) &= \lambda X \\ \phi(D')\cdot Y(y) &= \lambda Y \end{aligned}\right\} \qquad (4\cdot 8)$$

The solution of $(4\cdot 5)$ therefore reduces to the solution of $(4\cdot 8)$.

The usefulness of the solutions of *PDE* is quite limited because of the difficulty in choosing the appropriate arbitrary functions that will satisfy the imposed boundary conditions. This is however eliminated for some class of *PDEs* $(linear)$ by certain techniques one of which is based on the principle of superposition of solutions. This states that

"If each of the m functions $z_k\ (k=1(1)m)$ satisfies a linear PDE then an arbitrary linear combination

$$Z=\alpha_1 z_1+\alpha_2 z_2+\alpha_1 z_1+\alpha_2 z_2+\cdots\cdots+\alpha_1 z_1+\alpha_2 z_2=\sum_{j=1}^{m}\alpha_j z_j \qquad (4\cdot 9)$$

where $\alpha_k\ (k=1(1)m)$are constants also satisfies the differential equation".The combination of the method of seperation of variables and the superposition of solution is usually known as *Fourier* method.

Example

1 Solve by the method of seperation of variables the differential equation

$$\frac{\partial^2 u}{\partial x^2}-2\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0$$

Solution

Setting $u(x,y)=X(x)\cdot Y(y)\neq 0 \qquad (i)$

into the differential equation we have

$$X''\cdot Y-2X'\cdot Y+Y'X=0 \qquad (ii)$$

Dividing through by $u(x,y)$ by vitue of (i) yields

$$\frac{X''}{X}-2\frac{X'}{X}+\frac{Y'}{Y}=0 \qquad (iii)$$

ie,

$$\frac{1}{X}(X''-2X')=-\frac{Y'}{Y} \qquad (iv)$$

We observe here that the lhs and rhs of (iv) are functions of x and y respectively. For this equation to be valid each side must be independently equal to a constant $\lambda(\text{say})$. The implication of this yields the following uncoupled ordinary differential equation:

$$\left.\begin{aligned} X''-2X'-\lambda X &= 0 \\ Y'+\lambda Y &= 0 \end{aligned}\right\} \quad (v)$$

ie,

$$\left.\begin{aligned} (D^2-2D-\lambda)X &= 0 \\ (D'+\lambda)Y &= 0 \end{aligned}\right\} \quad (vi)$$

The solution of the ordinary differential equations in (vi) above are given as

$$\left.\begin{array}{l} X(x) = A\exp\left(1+\sqrt{1+\lambda}\right)x + B\exp\left(1-\sqrt{1+\lambda}\right)x \\ \text{and} \quad Y(y) = C\exp(-\lambda y) \end{array}\right\} \quad (vii)$$

By virtue of (i) and (vii) therefore we have

$$u(x,y) = \left(D\exp\left(1+\sqrt{1+\lambda}\right)x + E\exp\left(1-\sqrt{1+\lambda}\right)x\right)\exp(-\lambda y)$$

where $D = AC$ and $E = BC$ are arbitrary constants of integration.

2 Determine the solution to the $3-D$ wave equation

$$c^2\nabla^2 u = \frac{\partial^2 u}{\partial t^2}$$

by method of seperation of variables.

Solution.

Assuming the unknown function t is seperable and of the form

$$u(x,y,z,t) = X(x)\cdot Y(y)\cdot Z(z)\cdot T(t) \neq 0 \qquad (i)$$

then the partial differential equation yields

$$c^2\left(X''YZT + Y''XZT + Z''XYT\right) = \ddot{T}\ XYZ \qquad (ii)$$

ie,

$$c^2\left(\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z}\right) = \frac{\ddot{T}}{T} \qquad (iii)$$

$\Rightarrow$

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = \frac{1}{c^2}\frac{\ddot{T}}{T} \qquad (iv)$$

This equation is true only if each of the component parts is equal to a constant.

ie,

$$\frac{X''}{X} = -p^2, \frac{Y''}{Y} = -q^2, \frac{Z''}{Z} = -r^2, \ \frac{1}{c^2}\frac{\ddot{T}}{T} = -s^2 \qquad (v)$$

This yields the following uncoupled ordinary differential equations:

$$\left.\begin{array}{l} X'' + p^2X = 0 \\ Y'' + q^2Y = 0 \\ Z'' + r^2Z = 0 \\ \ddot{T} + c^2s^2T = 0 \end{array}\right\} \quad (vi)$$

with solutions

$$\left.\begin{array}{l} X_p(x) = A_p \operatorname{Cos} px + B_p \operatorname{Sin} px \\ Y_q(y) = C_q \operatorname{Cos} qy + D_q \operatorname{Sin} qy \\ Z_r(z) = E_r \operatorname{Cos} rz + F_r \operatorname{Sin} rz \\ T_s(t) = P_s \operatorname{Cos}(cs)t + Q_s \operatorname{Sin}(cs)t \end{array}\right\} \quad (vii)$$

Since the parameters p,q,r and s are dependent by virture of (iv) we may express $T(t)$ as

$$T_{pqr}(t) = G_{pqr}\,\mathrm{Cos}\left(\sqrt{p^2+q^2+r^2}\right)t + Q_s\,\mathrm{Sin}\left(\sqrt{p^2+q^2+r^2}\right)t \qquad (viii)$$

Hence by vitue of (i) and (vii) we thus have that

$$u_{pqr}(x,y,t,t) = X_p(x)Y_q(y)Z_r(z)T_{pqr}(t) \qquad (ix)$$

The most general solution is thus given as

$$u_{pqr}(x,y,t,t) = \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{r=1}^{\infty} u_{pqr}(x,y,t,t) \qquad (x)$$

in which the function $u_{pqr}(x,y,t,t)$ are as defined in (vii) and (ix).

4.3 *SOLUTION OF* 3-*D LAPLACE's EQUATION IN CURVILINEAR COORDINATE SYSTEM*.

(I) Cylindrical; (r,ϑ,z)

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \vartheta^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

(II) Spherical; (r,ϑ,ϕ)

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \vartheta^2} + \frac{Cot\vartheta}{r^2}\frac{\partial u}{\partial \vartheta} + \frac{1}{r^2\,\mathrm{Sin}^2\,\vartheta}\frac{\partial^2 u}{\partial \phi^2} = 0$$

In this section we will solve the problem for the shperical coordinate system. The solution for the cylindrical coordinate follows the same procedure.

The corresponding differential equation is given by

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \vartheta^2} + \frac{Cot\vartheta}{r^2}\frac{\partial u}{\partial \vartheta} + \frac{1}{r^2\,\mathrm{Sin}^2\,\vartheta}\frac{\partial^2 u}{\partial \phi^2} = 0 \qquad (i)$$

Assume the unknown function u is seperable in the form

$$u(r,\vartheta,\phi) = R(r)\cdot\Theta(\vartheta)\cdot\Phi(\phi) \neq 0 \qquad (ii)$$

Substitution of (ii) into (i) and dividing through the resuly by $u(r,\vartheta,\phi)$ yields

$$\frac{R''}{R} + \frac{2}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{Cot\vartheta}{r^2}\frac{\Theta'}{\Theta} + \frac{1}{r^2\,\mathrm{Sin}^2\,\vartheta}\frac{\Phi''}{\Phi} = 0 \qquad (iii)$$

ie,

$$\left(\frac{R''}{R} + \frac{2}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{Cot\vartheta}{r^2}\frac{\Theta'}{\Theta}\right)r^2\,\mathrm{Sin}^2\,\vartheta = -\frac{\Phi''}{\Phi} \qquad (iv)$$

Observe that the lhs of (iv) are functions of r and ϑ while the rhs is a function of ϕ only. This can only be valid if each side is a constant m^2, say. Therefore, we have that

$$\Phi'' + m^2\Phi = 0 \qquad (v)$$

$$\frac{1}{R}\left(r^2\;R'' + 2rR'\right) + \frac{1}{\Theta}\left(\Theta'' + Cot\vartheta\Theta'\right) = \frac{m^2}{\mathrm{Sin}^2\,\vartheta} \qquad (vi)$$

ie,

$$\frac{1}{\Theta}\left(\Theta'' + Cot\vartheta\Theta'\right) - \frac{m^2}{\mathrm{Sin}^2\,\vartheta} = -\frac{1}{R}\left(r^2\;R'' + 2rR'\right) \qquad (vii)$$

Eqn (vii) is true if only each side is a constant $-l(l+1)$.This condition gives rise to the following uncoup-

uncoupled ordinary differential equations:

$$r^2\ R''+2rR'\ -l(l+1)\ R=0 \qquad (viii)$$

$$\Theta''+Cot\vartheta\Theta'+\left\{l(l+1)-\frac{m^2}{\text{Sin}^2\vartheta}\right\}\Theta=0 \qquad (ix)$$

Substititing $\text{Cos}\,\vartheta=\mu$ in (ix) yeilds

$$\left(1-\mu^2\right)\frac{d^2\Theta}{d\mu^2}-2\mu\frac{d\Theta}{d\mu}+\left\{l(l+1)-\frac{m^2}{1-\mu^2}\right\}\Theta=0 \qquad (x)$$

Eqn (x) is associated Legendre differential equation.

Solving Eqns $(v),(viii)$ and (x) in standard form we obtain

$$\Phi_m(\phi)=A_m\,\text{Cos}\,m\phi+B_m\,\text{Sin}\,m\phi \qquad (xi)$$

$$R_l(r)=C_lr^l+\frac{D_l}{r^{l+1}} \qquad (xii)$$

and

$$\Theta_{ml}(\vartheta)=E_{ml}P_l^m(\text{Cos}\,\vartheta)+F_{ml}Q_l^m(\text{Cos}\,\vartheta) \qquad (xiii)$$

The general solution of the PDE is therefore

$$u(r,\vartheta,\phi)=\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\left(A_m\,\text{Cos}\,m\phi+B_m\,\text{Sin}\,m\phi\right)\left(C_lr^l+\frac{D_l}{r^{l+1}}\right)\left(E_{ml}P_l^m(\text{Cos}\,\vartheta)+F_{ml}Q_l^m(\text{Cos}\,\vartheta)\right) \qquad (xiv)$$

The arbitrary constants are chosen in a manner that the solution is bounded. This implies that $F_{ml}=0$ $\because Q_l^m(\text{Cos}\,\vartheta)\to\infty$ as $\vartheta\to 0$. Consequently the general solution is

$$u(r,\vartheta,\phi)=\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}E_{ml}P_l^m(\text{Cos}\,\vartheta)\left(A_m\,\text{Cos}\,m\phi+B_m\,\text{Sin}\,m\phi\right)\left(C_lr^l+\frac{D_l}{r^{l+1}}\right) \qquad (xv)$$

A solution of the problem in the form $(xi),(xii)$ and $(xiii)$ are called *sperical harmonics* while the solution (xi) and $(xiii)$ called *plane harmonics*.

3 Determine the potential outside and inside a spherical surface kept at a fixed distribution of electrical potential of the form $u=f(\vartheta)$ assuming that the space inside and outside the sphere is free of charge.

Solution.

In potential theory it is known that the potential u satisfies the Laplace equation $\nabla^2u=0$ in (r,ϑ,ϕ).

$$ie,\ \frac{\partial^2u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2u}{\partial\vartheta^2}+\frac{Cot\vartheta}{r^2}\frac{\partial u}{\partial\vartheta}+\frac{1}{r^2\,\text{Sin}^2\vartheta}\frac{\partial^2u}{\partial\phi^2}=0. \qquad (i)$$

In veiw of spherical symmetry, u is independent of ϕ.

$$ie,\ \frac{\partial u}{\partial\phi}=\frac{\partial^2u}{\partial\phi^2}=0. \qquad (ii)$$

By vitue of (ii) the governing equation (i) reduces to

$$ie,\ \frac{\partial^2u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2u}{\partial\vartheta^2}+\frac{Cot\vartheta}{r^2}\frac{\partial u}{\partial\vartheta}=0. \qquad (iii)$$

Assume the unknown function u is seperable in the form

$$u(r,\vartheta,\phi)=R(r)\cdot\Theta(\vartheta)\neq 0 \qquad (iv)$$

Substitution of (ii) into (i) and dividing through the resuly by $u(r,\vartheta,\phi)$ yields

$$\frac{R''}{R}+\frac{2}{r}\frac{R'}{R}+\frac{1}{r^2}\frac{\Theta''}{\Theta}+\frac{Cot\vartheta}{r^2}\frac{\Theta'}{\Theta}=0 \qquad (v)$$

ie,

$$\frac{1}{\Theta}\left(\Theta''+Cot\vartheta\Theta'\right)+\frac{1}{R}\left(r^2R''+2rR'\right)=0 \qquad (vi)$$

$$\frac{1}{\Theta}\left(\Theta''+Cot\vartheta\Theta'\right)=-\frac{1}{R}\left(r^2R''+2rR'\right) \qquad (vii)$$

Observe that the lhs of (vii) is a function of ϑ while the rhs is a function of r only. This can only be valid if each side is a constant $-l(l+1)$, say. Therefore, we have the following uncoupled ODEs:

$$r^2R''+2rR'-l(l+1)R=0 \qquad (viii)$$

$$\Theta''+Cot\vartheta\Theta'+l(l+1)\Theta=0 \qquad (ix)$$

Solving $(viii)$ and (ix).

We set $\mu=\text{Cos}\,\vartheta$ in (ix)

$$\Rightarrow d\mu=-\text{Sin}\,\vartheta d\vartheta=-\sqrt{1-\mu^2}\,d\vartheta$$

$$\therefore \frac{d\mu}{d\vartheta}=-\sqrt{1-\mu^2}$$

But $$\frac{d\Theta}{d\vartheta}=\frac{d\mu}{d\vartheta}\frac{d\Theta}{d\mu}=-\sqrt{1-\mu^2}\frac{d\Theta}{d\mu}$$

$$\therefore \frac{d^2\Theta}{d\vartheta^2}=\frac{d}{d\vartheta}\left(\frac{d\Theta}{d\vartheta}\right)=\left(-\sqrt{1-\mu^2}\frac{d}{d\mu}\right)\left(-\sqrt{1-\mu^2}\frac{d\Theta}{d\mu}\right)=\left(1-\mu^2\right)\frac{d^2\Theta}{d\mu^2}-\mu\frac{d\Theta}{d\mu} \qquad (x)$$

Hence, (ix) transforms to

$$\left(1-\mu^2\right)\frac{d^2\Theta}{d\mu^2}-2\mu\frac{d\Theta}{d\mu}+l(l+1)\Theta=0 \qquad (xi)$$

This is the Lagendre DE with solution $P_l(\mu)$. Similarly, the solution of $(viii)$ is obtained by assuming

$$R(r)=r^{\alpha} \qquad (xii)$$

giving the solutions

$$R(r)=A_lr^l+\frac{B_l}{r^{l+1}} \qquad (xiii)$$

$$\Theta(\mu)=C_lP_l(\mu)+D_lQ_l(\mu) \qquad (xiv)$$

ie,

$$\Theta(\vartheta)=C_lP_l(\text{Cos}\,\vartheta)+D_lQ_l(\text{Cos}\,\vartheta) \qquad (xv)$$

Suppose the sphere is of radius a. Then, we have

$$u(a,\vartheta)=f(\vartheta) \qquad (xvi)$$

Also, the potential u remains bounded everywhere $\Rightarrow u\prec\infty$ as $\vartheta\to 0$.

$$\because\ Q_l(\text{Cos}\,\vartheta)\to\infty \text{ as } \vartheta\to 0 \Rightarrow D_l=0$$

$$\therefore\ \Theta(\vartheta)=C_lP_l(\text{Cos}\,\vartheta) \qquad (xvii)$$

Hence,

$$u_m(r,\vartheta)=\left(A_lr^l+\frac{B_l}{r^{l+1}}\right)C_lP_l(\text{Cos}\,\vartheta)$$

The most general solution is therefore given as

$$u(r,\vartheta)=\sum_{l=0}^{\infty}\left(E_lr^l+\frac{F_l}{r^{l+1}}\right)P_l(\text{Cos}\,\vartheta) \qquad (xviii)$$

Potential Outside the Sphere.

We also recall from potential theory that $u=0$ as $r\to\infty\Rightarrow E_l=0$.

Therefore, solution for $r\succ 0$ (outside the sphere) is given as

$$u(r,\vartheta)=\sum_{l=0}^{\infty}\frac{F_l}{r^{l+1}}P_l(\text{Cos}\,\vartheta) \qquad (xix)$$

Setting $r=a$ and applying $u(a,\vartheta)=f(\vartheta)$ we have

$$\sum_{l=0}^{\infty}\frac{F_l}{a^{l+1}}P_l(\text{Cos}\,\vartheta)=f(\vartheta)=\sum_{l=0}^{\infty}\frac{F_l}{a^{l+1}}P_l(\mu) \qquad (xx)$$

This is the Legendre series. To determine the coefficients F_l therefore we multiply (xx) by $P_m(\mu)$ and integrate the result in $-1\le\mu\le 1$ to obtain

$$\int_{-1}^{1}\sum_{l=0}^{\infty}\frac{F_l}{a^{l+1}}P_l(\mu)P_m(\mu)\,d\mu=\int_{-1}^{1}f(\vartheta)P_m(\mu)\,d\mu$$

ie, $$\frac{F_m}{a^{m+1}}\int_{-1}^{1}P_m^{\,2}(\mu)\,d\mu=\int_{-1}^{1}f(\vartheta)P_m(\mu)\,d\mu$$

ie, $$\frac{F_m}{a^{m+1}}\cdot\frac{2}{2m+1}=\int_{-1}^{1}f(\vartheta)P_m(\mu)\,d\mu$$

$\therefore$ $$F_m=\frac{2m+1}{2}\cdot a^{m+1}\int_{0}^{\pi}f(\vartheta)P_m(\text{Cos}\,\vartheta)\text{Sin}\,\vartheta d\vartheta \qquad (xxi)$$

Therefore, the required potential is given by (xix) with cosfficient as given by (xxi).

Potential inside the sphere.

From potential theory $u\prec\infty$ as $r\to 0\Rightarrow F_l=0$.

$\therefore$ $$u(r,\vartheta)=\sum_{l=0}^{\infty}E_lr^lP_l(\text{Cos}\,\vartheta) \qquad (xxii)$$

By virtue of the condition on the surface of the sphere $(u(a,\vartheta)=f(\vartheta))$ we thus have

$$u(a,\vartheta)=\sum_{l=0}^{\infty}E_lr^lP_l(\text{Cos}\,\vartheta)=f(\vartheta) \qquad (xxiii)$$

$\therefore$ $$\int_{-1}^{1}\sum_{l=0}^{\infty}E_la^lP_l(\text{Cos}\,\vartheta)P_m(\text{Cos}\,\vartheta)\text{Sin}\,\vartheta d\vartheta=\int_{-1}^{1}f(\vartheta)P_m(\text{Cos}\,\vartheta)\text{Sin}\,\vartheta d\vartheta$$

ie, $$E_m a^m \cdot \frac{2}{2m+1} = \int_{-1}^{1} f(\vartheta) P_m(\text{Cos}\,\vartheta)\text{Sin}\,\vartheta d\vartheta$$

$$\therefore \quad E_m = \frac{2m+1}{2a^m}\int_{-1}^{1} f(\vartheta) P_m(\text{Cos}\,\vartheta)\text{Sin}\,\vartheta d\vartheta \qquad (xxiv)$$

Therefore, inside the sphere the potential is given as

$$u(a,\vartheta) = \sum_{l=0}^{\infty} E_l r^l P_l(\text{Cos}\,\vartheta) = f(\vartheta) \qquad (xxv)$$

where the coefficients E_l are as given in $(xxiv)$.

Exercise

Determine the steady-state temperature of a semi-circulr plate of radius a whose circumference is maintained at temperature T_0 and the base at $T = 0$.

H int :

This is a Laplaceequation in polar coordinate (r,ϑ) with boundary conditions:

$$T(r,0) = 0 = T(r,\pi); 0 \le a \le r$$
$$T(a,\vartheta) = T_0, T \prec \infty \text{ as } r \to 0$$

Solution

$$T(r,\vartheta) = \frac{4T_0}{\pi}\sum_{m=1}^{\infty}\frac{1}{2m-1}\left(\frac{r}{2}\right)^{2m-1}\text{Sin}(2m-1)\vartheta.$$

4.4 *SOLUTION OF THE* $3-D$ *WAVE EQUATIONS IN CURVILINEAR COORDINATE SYSTEM*.

(I) Cylindrical; (r,ϑ,z)

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \vartheta^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}.$$

(II) Spherical; (r,ϑ,ϕ)

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \vartheta^2} + \frac{Cot\vartheta}{r^2}\frac{\partial u}{\partial \vartheta} + \frac{1}{r^2\,\text{Sin}^2\,\vartheta}\frac{\partial^2 u}{\partial \phi^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}.$$

In this section we will solve the problem for the cylindrical coordinate system, the the shperical case follows the same procedure.

Solution.

We recall that the governing eqution in the coordinate system (r,ϑ,z) is given as

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \vartheta^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}. \qquad (i)$$

Assuming a seperable solution of the form

$$u(r,\vartheta,z,t) = R(r)\Theta(\vartheta)Z(z)T(t) \ne 0 \qquad (ii)$$

and dividing through by u we have

$$\frac{1}{R}\left(R'' + \frac{1}{r}R'\right) + \frac{\Theta''}{r^2\Theta} + \frac{Z''}{Z} = \frac{1}{c^2}\frac{\ddot{T}}{T} \qquad (iii)$$

lhs of (iii) is a function of r and ϑ while the *rhs* is a function of t. The equation is only true if they are both constant say $-p^2$.

ie,

$$\ddot{T}=c^2p^2T \qquad (iv)$$

and $$\frac{1}{R}\left(R''+\frac{1}{r}R'\right)+\frac{\Theta''}{r^2\Theta}+\frac{Z''}{Z}=-p^2 \qquad (v)$$

ie, $$\frac{1}{R}\left(R''+\frac{1}{r}R'\right)+\frac{1}{r^2\Theta}\Theta''+p^2=-\frac{Z''}{Z}=s^2 \qquad (vi)$$

$\Rightarrow$ $$Z''+s^2Z=0 \qquad (vii)$$

$$\frac{1}{R}\left(r^2R''+rR'\right)+\left(p^2-s^2\right)r^2=-\frac{\Theta''}{\Theta}=\alpha^2 \qquad (viii)$$

Eqn $(viii)$ results in the following uncoupled *ODE*s:

$$\Theta''+\alpha^2\Theta=0 \qquad (ix)$$

$$r^2R''+rR'+\left(\beta^2r^2-\alpha^2\right)R=0 \qquad (x)$$

where $\beta^2=p^2-s^2$.

Eqn (x) is the Bessel's differential equation.

We thus have the following solutions:

$$T(t)=A_p\,\mathrm{Cos}(cpt)+B_p\,\mathrm{Sin}(cpt) \qquad (xi)$$

$$Z(z)=C_s\,\mathrm{Cos}(sz)+D_s\,\mathrm{Sin}(sz) \qquad (xii)$$

$$\Theta(\vartheta)=E_\alpha\,\mathrm{Cos}(\alpha\vartheta)+F_\alpha\,\mathrm{Sin}(\alpha\vartheta) \qquad (xiii)$$

$$R(r)=G_{ps\alpha}J(\beta r)+H_{ps\alpha}Y(\beta r) \qquad (xiv)$$

The general solution is therefore given by

$$u(r,\vartheta,z,t)=\sum_{p=0}^{\infty}\sum_{s=0}^{\infty}\sum_{\alpha=0}^{\infty}u_{ps\alpha}(r,\vartheta,z,t) \qquad (xv)$$

in which $u_{ps\alpha}$ is as defined in (xi) through (xiv).

In practical application $u\prec\infty$ everywhere including $r=0.\Rightarrow H_{ps\alpha}=0\ \because Y(\beta r)\to\infty$ as $r\to 0$.

Therefore, the finite solution is given by

$$u(r,\vartheta,z,t)=\sum_{p=0}^{\infty}\sum_{s=0}^{\infty}\sum_{\alpha=0}^{\infty}G_{ps\alpha}J(\beta r)\left\{A_p\,\mathrm{Cos}(cpt)+B_p\,\mathrm{Sin}(cpt)\right\}\left\{C_s\,\mathrm{Cos}(sz)+D_s\,\mathrm{Sin}(sz)\right\}\times$$
$$\left\{E_\alpha\,\mathrm{Cos}(\alpha\vartheta)+F_\alpha\,\mathrm{Sin}(\alpha\vartheta)\right\}$$

4 Obtain the solution of the transverse vibration of a thin membrane bounded by a circle of radius a described by the function $u(r,\vartheta,t)$ satisfying the wave equation $\nabla^2u=c^{-2}u_{xx}$ satisfying the conditions: $u(a,\vartheta,t)=0, u(r,\vartheta,0)=f(r,\vartheta), u_t(r,\vartheta,0)=\phi(r,\vartheta)$.

Solution.

The initial boundary value problem is represented by

$$\left.\begin{aligned} &\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \vartheta^2}+\frac{\partial^2 u}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}.\\ &u(a,\vartheta,t)=0, \qquad -\pi\le\vartheta\le\pi, t\ge 0\\ &u(r,\vartheta,0)=f(r,\vartheta), u_t(r,\vartheta,0)=\phi(r,\vartheta).\ 0\le r\le a, -\pi\le\vartheta\le\pi \end{aligned}\right\} \quad (i)$$

Assuming a seperable solution of the form

$$u(r,\vartheta,t)=R(r)\Theta(\vartheta)T(t)\neq 0 \qquad (ii)$$

and dividing through by u we have

$$\frac{1}{R}\left(R''+\frac{1}{r}R'\right)+\frac{\Theta''}{r^2\Theta}=\frac{1}{c^2}\frac{\ddot{T}}{T}=-\lambda^2 \qquad (iii)$$

$$\ddot{T}+c^2\lambda^2 T=0 \qquad (iv)$$

$$\frac{1}{R}\left(R''+\frac{1}{r}R'\right)+\frac{\Theta''}{r^2\Theta}+\lambda^2=0 \qquad (v)$$

$$\frac{1}{R}\left(R''+\frac{1}{r}R'\right)+\lambda^2=-\frac{\Theta''}{r^2\Theta} \qquad (vi)$$

ie,

$$\frac{1}{R}\left(r^2R''+rR'\right)+r^2\lambda^2=-\frac{\Theta''}{\Theta}=m^2 \qquad (vii)$$

Hence, we have

$$\Theta''+m^2\Theta=0 \qquad (viii)$$

$$R''+\frac{1}{r}R'+\left(\lambda^2-\frac{m^2}{r^2}\right)R=0 \qquad (ix)$$

The solutions of (iv) and $(viii)$ are respectively

$$T(t)=A_\lambda \operatorname{Cos}(c\lambda t)+B_\lambda \operatorname{Sin}(c\lambda t) \qquad (x)$$

$$\Theta(\vartheta)=C_\lambda \operatorname{Cos}(m\vartheta)+D_\lambda \operatorname{Sin}(m\vartheta) \qquad (xi)$$

Eqn (ix) is the standard Bessel's differential equation withe solution

$$R(r)=E_\lambda J_m(r\lambda)+F_\lambda Y_m(r\lambda) \qquad (xii)$$

Since solution must remain finite everywhere, we observe that $Y_m(r\lambda)\to\infty$as $r\to 0\Rightarrow F_\lambda=0$

$\therefore \qquad R(r)=E_\lambda J_m(r\lambda) \qquad (xii)$

Thus,

$$u(r,\vartheta,t)=J_m(r\lambda)\left\{A_\lambda{}'\operatorname{Cos}(c\lambda t)+B_\lambda{}'\operatorname{Sin}(c\lambda t)\right\}\left\{C_\lambda \operatorname{Cos}(m\vartheta)+D_\lambda \operatorname{Sin}(m\vartheta)\right\} \qquad (xiii)$$

in which $A_\lambda{}'=A_\lambda E_\lambda$ and $B_\lambda{}'=B_\lambda E_\lambda$.

Recall that

$$u(a,\vartheta,t)=0\ ;\ -\pi\le\vartheta\le\pi, t\ge 0$$

$$\Rightarrow R(a)\Theta(\vartheta)T(t)=0 \quad ie, R(a)=0 \because \Theta(\vartheta)T(t)=0\Rightarrow u(r,\vartheta,t)=0 \text{ trivially}$$

$$\Rightarrow J_m(\lambda a)=0 \qquad (xiv)$$

This is an eigenvalue problem with infinite solutions.

Thus, suppose $\lambda_k\ (k=1,2,3\cdots)$ are the positive roots of (xiv) then the general solution becomes

$$u(r,\vartheta,t)=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}J_m(\lambda_k r)\left\{A_\lambda'\operatorname{Cos}(c\lambda_k t)+B_\lambda'\operatorname{Sin}(c\lambda_k t)\right\}\left\{C_\lambda\operatorname{Cos}(m\vartheta)+D_\lambda\operatorname{Sin}(m\vartheta)\right\} \qquad (xv)$$

Axisymmetric solutions.

This is the case where u is independent of ϑ.

ie,

$$u(r,\vartheta,t)=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}J_0(\lambda_k r)\left\{A_\lambda'\operatorname{Cos}(c\lambda_k t)+B_\lambda'\operatorname{Sin}(c\lambda_k t)\right\} \qquad (xvi)$$

in which λ_k are the positive roots of $J_0(\lambda_k r)=0$. In view of the boundary condition we have

$$u(r,\vartheta,0)=\sum_{k=0}^{\infty}A_\lambda' J_0(\lambda_k r)=f(r) \qquad (xvii)$$

This is Fourier-Bessel series. To obtain the coefficients A_λ' we have

$$\int_0^a rJ_0(\lambda_j r)f(r)\,dr=\int_0^a\sum_{k=0}^{\infty}A_\lambda' rJ_0(\lambda_j r)J_0(\lambda_k r)\,dr$$

ie,

$$\int_0^a\sum_{k=0}^{\infty}A_\lambda' rJ_0(\lambda_j r)J_0(\lambda_k r)\,dr=\int_0^a rJ_0(\lambda_j r)f(r)\,dr$$

$$\Rightarrow \qquad A_j'\int_0^a rJ_0^2(\lambda_j r)\,dr=\int_0^a rJ_0(\lambda_j r)f(r)\,dr$$

But $$\int_0^a rJ_p^2(\lambda_j r)\,dr=\frac{a^2}{2}\left[J_p'^2(\lambda_j a)+\left(1-\frac{p^2}{a^2\lambda_j^2}\right)J_p^2(\lambda_j a)\right] \qquad (xviii)$$

Recall also that $J_p'^2(\lambda_j a)=J_{p+1}^2(\lambda_j a)$

ie, $$\int_0^a rJ_0^2(\lambda_j r)\,dr=\frac{a^2}{2}J_0'^2(\lambda_j a)=\frac{a^2}{2}J_1^2(\lambda_j a)$$

$$\Rightarrow \qquad A_j'\int_0^a rJ_0^2(\lambda_j r)\,dr=\frac{a^2}{2}J_1^2(\lambda_j a)A_j'=\int_0^a rJ_0(\lambda_j r)f(r)\,dr$$

$$\therefore \qquad A_j'=\frac{2}{a^2J_1^2(\lambda_j a)}\int_0^a rJ_0(\lambda_j a)f(r)\,dr \qquad (xix)$$

From the initial condition we have

$$\left.\frac{\partial u}{\partial t}\right|_{t=0}=g(r)=\sum_{k=0}^{\infty}c\lambda_k B_k' J_0(\lambda_k r) \qquad (xx)$$

As in the above, we therefore have

$$c\int_0^a \sum_{k=0}^{\infty} \lambda_k B_k{}' J_0\left(\lambda_j r\right) J_0\left(\lambda_k r\right) dr = \int_0^a J_0\left(\lambda_j r\right) g\left(r\right) dr$$

ie,

$$c\lambda_j B_j{}' \int_0^a J_0{}^2\left(\lambda_j r\right) dr = \int_0^a J_0\left(\lambda_j r\right) g\left(r\right) dr$$

ie,

$$B_j{}' = \frac{\int_0^a J_0\left(\lambda_j r\right) g\left(r\right) dr}{c\lambda_j \int_0^a J_0{}^2\left(\lambda_j r\right) dr}$$

$$B_j{}' = \frac{2}{c\lambda_j a^2 J_1{}^2\left(\lambda_j a\right)} \int_0^a J_0\left(\lambda_j r\right) g\left(r\right) dr \qquad \left(xxi\right)$$

Therefore, $\left(xvi\right)$ is the solution for radially symmetric wave with coefficients defined in $\left(xix\right)$ and $\left(xxi\right)$.