# CHAPTER ONE

# **FIRST -ORDER PARTIAL DIFFERENTIAL EQUATIONS.**<br>
1.1 DERIVATION OF PARTIAL DIFFERENTIAL EQUATIONS.<br>
Consider the family of surfaces<br>  $f(x, y, u, a, b) = 0$ <br>
where a and h are constants and u is dependent on x and y (x, y are ind

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$$
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$$

1.1 DERIVATION OF PARTIAL DIFFERENTIAL EQUATIONS.<br>Consider the family of surfaces<br> $f(x, y, u, a, b) = 0$ <br>where a and b are constants and u is dependent on x and y (x, y are independent of  $a$  and  $b$  are constants and u is depe  $(x, y$  are independent variables) ndent variables).<br>we eliminate the

 $( PDE )$  from  $(1.1.4)$  $(1.1.4)$ Consider the family of surfaces<br>  $f(x, y, u, a, b) = 0$ <br>
where *a* and *b* are constants and *u* is dependent on *x* and *y*  $(x, y$  are independent variables).<br>
To derive an appropriate partial differential equation *(PDE)* from  $f(x, y, u, a, b) = 0$ <br>where *a* and *b* are constants and *u* is dependent on *x* and *y* (*x*, *y* are independent v<br>To derive an appropriate partial differential equation (*PDE*) from (1.1.4) we elir<br>Differentiating (1.1.4) *b* are constants and *u* is de<br> **n** appropriate partial differe<br>  $\log (1.1.4)$  wrt *x* and *y* we<br>  $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} = 0$ 

where *a* and *b* are constants and *u* is dependent on *x* and *y* (*x*, *y* are independent variances).  
\nTo derive an appropriate partial differential equation (*PDE*) from (1.1.4) we eliminate the con-  
\nDifferentiating (1.1.4) wrt *x* and *y* we have the following equations:respectively:  
\n
$$
\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} = 0
$$
\n1.1.5  
\n
$$
\frac{\partial f}{\partial y} + \frac{\partial f}{\partial u} \frac{\partial u}{\partial y} = 0
$$
\n1.1.6  
\nEliminating the constants *a* and *b* from (1.1.4), (1.1.5) and (1.1.6) we obtain a general relation  
\n
$$
F(x, y, u, p, q) = 0
$$
\n1.1.7  
\nFor (1.1.7) is in general a first-order PDE if the number of constants to be eliminated is the

$$
F(x, y, u, p, q) = 0 \tag{1.1.7}
$$

 $(1.1.7)$ Eliminating the constants *a* and *b* from (1.1.4),(1.1.5) and (1.1.6) we obtain a general relation<br>  $F(x, y, u, p, q) = 0$  1.1.7<br>
Eqn (1.1.7) is in general a *first - order PDE* if the number of constants to be eliminated is t of the independent variables and is of *higher order* if the number is greater than the number of the *f* and *b* from  $(1.1)$ <br>= 0<br>*first - order PDE*<br> $\frac{1}{2}$  and is of *highe higher order* independent variables.

#### . *Derivation*

Consider the family of surfaces

Derivation.

\nConsider the family of surfaces

\n
$$
\phi(f, g) = 0
$$
\nWhere  $\phi$  is an arbitrary differentiable function of  $f$  and  $g$  that are in the form  $f(x) = \phi(x)$ .

where  $\phi$  is an arbitrary differentiable function of f and g that are in turn known differentiable functions of some independent variable x and y with u also a differentiable function of x and y.  $f$  and  $g$ <br>a differ *x* and *y* with u also a differentiable function of *x* and *y* with u also a differentiable function of *x* and *y* we have

Differentiating  $\phi$  wrt x and y we have

where 
$$
\phi
$$
 is an arbitrary differentiable function of  $f$  and  $g$  that are in turn  
of some independent variable  $x$  and  $y$  with  $u$  also a differentiable function  
Differentiating  $\phi$  wrt  $x$  and  $y$  we have  

$$
\frac{\partial \phi}{\partial f} \cdot \frac{\partial f}{\partial x} + \frac{\partial \phi}{\partial f} \cdot \frac{\partial u}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial g} \cdot \frac{\partial g}{\partial x} + \frac{\partial \phi}{\partial g} \cdot \frac{\partial g}{\partial u} \cdot \frac{\partial u}{\partial x} = 0
$$

$$
\frac{\partial \phi}{\partial f} \cdot \frac{\partial f}{\partial y} + \frac{\partial \phi}{\partial f} \cdot \frac{\partial u}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial g} \cdot \frac{\partial g}{\partial y} + \frac{\partial \phi}{\partial g} \cdot \frac{\partial g}{\partial u} \cdot \frac{\partial u}{\partial y} = 0
$$

$$
\frac{\partial \phi}{\partial f} \left( \frac{\partial f}{\partial x} + \frac{\partial f}{\partial u} \cdot p \right) + \frac{\partial \phi}{\partial g} \left( \frac{\partial g}{\partial x} + \frac{\partial g}{\partial u} p \right) = 0
$$
*ie,*
$$
\frac{\partial \phi}{\partial f} \left( \frac{\partial f}{\partial y} + \frac{\partial f}{\partial u} \cdot q \right) + \frac{\partial \phi}{\partial g} \left( \frac{\partial g}{\partial y} + \frac{\partial g}{\partial u} q \right) = 0
$$
[Eliminating  $\frac{\partial \phi}{\partial f}$  and  $\frac{\partial \phi}{\partial g}$  we thus have

, *ie*

Eliminating  $\frac{dy}{dx}$  and  $\frac{dy}{dx}$  we thus have

$$
\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u} \cdot p \quad \frac{\partial g}{\partial x} + \frac{\partial g}{\partial u} \cdot p
$$
\n
$$
\frac{\partial f}{\partial y} + \frac{\partial f}{\partial u} \cdot q \quad \frac{\partial g}{\partial y} + \frac{\partial g}{\partial u} \cdot q
$$
\nEqn(1.1.10) is equivalent to

 $(1.1.10)$  $\begin{aligned}\n\beta y & \partial u^T \\
\text{Eqn(1.1.10)} & \text{is equivalent} \\
P.p + Q, q = R\n\end{aligned}$ 

$$
\begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial u}{\partial u} \end{vmatrix}
$$
  
\n
$$
\begin{vmatrix} 0 & \frac{\partial u}{\partial u} \end{vmatrix}
$$
  
\n
$$
\begin{vmatrix} P \cdot p + Q, q = R \end{vmatrix}
$$
  
\n
$$
\begin{vmatrix} 1.1.11 \end{vmatrix}
$$

where

$$
P \cdot p + Q, q = R \qquad (1.1.11)
$$
  
where  

$$
P = \frac{\partial (f, g)}{\partial (y, u)}, Q = \frac{\partial (f, g)}{\partial (x, u)} \text{ and } R = \frac{\partial (f, g)}{\partial (x, y)} \qquad (1.1.12)
$$
  
Eqn (1.1.12) is first-order differential equation.

 $(1.1.12)$ 2) is fir<br> $a$  and  $b$ 

. *Example*

Eliminate  $a$  and  $b$  from the following familie from the following families of surfaces to obtain a *PDE*.<br>
<sup>2</sup> + (y-b)<sup>2</sup> + u<sup>2</sup> = d<sup>2</sup> (i)

and *b* from the following families of surfaces to obtain  
\n
$$
(x-a)^2 + (y-b)^2 + u^2 = d^2
$$
\n(*i*)  
\n
$$
(i)
$$
 partially wry *x* and *y* yields

*Solution*

Differentiating  $(i)$  partially wry x and y yeilds

Differentiating (i) partially wry x and y yields  
\n
$$
2(x-a)+2u \frac{\partial u}{\partial x} = 0, ie, (x-a)+up = 0
$$
 (ii)  
\n
$$
2(y-b)+2u \frac{\partial u}{\partial y} = 0 ie, (y-b)+uq = 0
$$
 (iii)

$$
2(y - b) + 2u \frac{\partial u}{\partial y} = 0 \text{ ie}, (y - b) + uq = 0 \qquad \text{(iii)}
$$
  
Eliminate *a* and *b* from (*i*), (*ii*) and (*iii*) yields

$$
\frac{\partial y}{\partial y}
$$
  
and *b* from (*i*), (*ii*) and (*iii*) yields  

$$
(-up)^2 + (-uq)^2 + u^2 = d^2
$$
 (*iv*)

, *ie*

$$
(-up)^{2} + (-uq)^{2} + u^{2} = d^{2}
$$
 (*iv*)  
*ie*,  

$$
(p^{2} + q^{2} + 1)u^{2} = d^{2}
$$
 (*v*)  
Eqn(*v*) is first-order differential equation.

 $(v)$ *v*

2 Form a PDE from the family of integral surfaces<br> $x^2$   $y^2$   $u^2$ 

Eqn(v) is first-order differential equation.  
\n2 Form a PDE from the family of integral surfaces  
\n
$$
\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{u^2}{c^2} = 1
$$
\n*Solution*  
\nDifferentiating (i) partially wry x yields

*Solution*

Differentiating  $(i)$  partially wry x yeilds

Solution  
\nDifferentiating (i) partially wry x yields  
\n
$$
\frac{2x}{a^2} + \frac{2u}{c^2} \frac{\partial u}{\partial x} = 0 \implies u \frac{\partial u}{\partial x} = -\frac{c^2}{a^2} x
$$
 (ii)  
\nDifferentiating (i) partially wry y yields

Differentiating  $(i)$  partially wry y yeilds

$$
a^{2} + c^{2} \partial x = 0 \implies u \partial x = a^{2} \quad (u)
$$
  
itating (i) partially wry y yields  

$$
\frac{2y}{b^{2}} + \frac{2u}{c^{2}} \frac{\partial u}{\partial y} = 0 \implies u \frac{\partial u}{\partial y} = -\frac{c^{2}}{b^{2}} y \qquad (iii)
$$
  
rentiating (ii) partially wry y or (iii) partially wry x

On differentiating  $(ii)$  partially wry y or  $(iii)$  partially wry x yeilds

$$
b^{2} \quad c^{2} \quad \partial y \qquad \partial y \qquad b^{2}
$$
\nOn differentiating (ii) partially wry y or (iii) partially wr  
\n
$$
u \frac{\partial^{2} u}{\partial y \partial x} + \frac{\partial u}{\partial y} \frac{\partial u}{\partial x} = 0
$$
 (iv)

This is a second-order PDE.

#### 1.2 SOLUTION OF LAGRANGES LINEAR EQUATION.

The general partial differential equation

1.2 SOLUTION OF LAGRANGES LINEAR EQUATION.  
The general partial differential equation  

$$
P.p + Q.q = R
$$
(1.1.13)  
where P O and R are functions of x and y is referred to as the Lagranges Linear Eqs

wh SOLUTION OF LAGRANGES LINEAR EQUATION.<br>
e general partial differential equation<br>  $P.p + Q.q = R$  (1.1.13)<br>
ere  $P,Q$ , and R are functions of x, and y is refered to as the Lagranges Linear Equation.<br>
Exerem 1.1 1.1 *Theorem*  $P \cdot p + Q \cdot q = R$ <br>where  $P \cdot Q$ , and  $R$  are functi<br>Theorem 1.1<br>Given eqn (1.1.13) in which d *R* are functi<br>13) in which<br> $f(x, y, u) = 0$ **LUTION OF LAGRANGES LINEA**<br>
eneral partial differential equation<br>  $P \cdot p + Q \cdot q = R$ <br>  $P \cdot Q$ , and R are functions of x, and y<br>  $p \cdot m$  1.1

 $(1.1.13)$ 

.13) in which  
\n
$$
f(x, y, u) = 0
$$
 (1.1.14)  
\n $g(x, y, u) = 0$  (1.1.14)  
\nintegral curves of the simultaneous ordinary differential equations (OD)  
\n
$$
\frac{dx}{P} = \frac{dy}{Q} = \frac{du}{R}
$$
(1.1.15)

( ) constitute the integral curves of the simultaneous ordinary differential equations *ODEs*

$$
g(x, y, u) = 0
$$
\nconstitute the integral curves of the simultaneous ordinary differential equations (OD)

\n
$$
\frac{dx}{P} = \frac{dy}{Q} = \frac{du}{R}
$$
\nThen the general solution of (1.1.13) is given as

\n
$$
F(f, g) = 0
$$
\nwhere *E* is an arbitrary differentiable function. Further, *W*(*x*, *y*, *u*) = *a* is any solution of the equation.

\n
$$
F(f, g) = 0
$$
\nThus, *E* is an arbitrary differentiable function.

 $(1.1.13)$ 

$$
F(f,g) = 0 \tag{1.1.16}
$$

 $(x, y, u) = c$  is any solution of  $(1.1.13)$ From the general solution of (1.1.13) is given as<br>  $F(f, g) = 0$  (1.1.16)<br>
where F is an arbitrary differentiable function. Further  $w(x, y, u) = c$  is any solution of (1.1.13) and if first<br>
corder derivatives of f, g and w are a Then the general solution of (1.1.13) is given as<br>  $F(f, g) = 0$  (1.1.1<sup>o</sup><br>
where *F* is an arbitrary differentiable function. Further  $w(x, y, u) = c$  is any solution<br>
-order derivatives of *f*, *g* and *w* are all continuous th *P Q R*<br>
he general solution of (1.1.13) is given as<br>  $F(f, g) = 0$ <br> *F* is an arbitrary differentiable function. Further  $w(x, y, u) = c$ <br>
derivatives of *f g* and *w* are all continuous than the solution *w* ation of (1.1.13) is given as<br> *g*) = 0<br> *f g* and *w* are all continuous then the solution  $w - c$ <br> *f g* and *w* are all continuous then the solution  $w - c$ = (1.1.16)<br>
is any solution of (1.1<br>  $-c = 0$  is contained in solution of  $(1.1.16)$ . -order derivatives of f, g and w are all continuous then the solution  $w - c = 0$  is contained in the general  $F(f,$ <br>where F is an arbitra-order derivatives of<br>solution of (1.1.16).<br>Proof order derivatives of  $f$ ,  $g$  and  $w$  are all continuos obtained solution of (1.1.16).<br>Proof<br>Differentiating the relationship (1.1.14) yields

#### Pr *oof*

 $(1.1.14)$ 

solution of (1.1.16).  
\nProof  
\nDifferentiating the relationship (1.1.14) yields  
\n
$$
\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial u} du = 0
$$
\n
$$
\frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy + \frac{\partial g}{\partial u} du = 0
$$
\n*ie,*\n
$$
\frac{dx}{\partial (f,g)} = \frac{dy}{\partial (f,g)} = \frac{du}{\partial (f,g)}
$$
\n
$$
\frac{\partial (f,g)}{\partial (y,u)} = \frac{\partial f}{\partial (x,u)} = \frac{\partial u}{\partial (x,y)}
$$
\nSince (1.1.15) determines the integral curves of (1.1.16) then we have from (1.1.17)  
\n
$$
\frac{P}{\partial (f,g)} = \frac{Q}{\partial (f,g)} = \frac{R}{\partial (f,g)}
$$
\n(1.1.18)

, *ie*

 $(1.1.15)$  determines the integral curves of  $(1.1.16)$  then we have from  $(1.1.17)$ 

Since (1.1.15) determines the integral curves of (1.1.16) then we have from (1.1.1)  
\n
$$
\frac{P}{\partial(y,u)} = \frac{Q}{\partial(f,g)} = \frac{R}{\frac{\partial(f,g)}{\partial(x,u)}} = \frac{R}{\frac{\partial(f,g)}{\partial(x,y)}}
$$
\nNow considering any functional relation (1.1.16) when F is differentiable we have  
\n
$$
\frac{\partial F}{\partial f} \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u} \cdot p\right) + \frac{\partial F}{\partial g} \left(\frac{\partial g}{\partial x} + \frac{\partial g}{\partial u} \cdot p\right) = 0
$$
\n(1.1.19)

 $(1.1.16)$ *F*

$$
\frac{\partial (f,g)}{\partial (y,u)} \frac{\partial (f,g)}{\partial (x,u)} \frac{\partial (f,g)}{\partial (x,y)} \qquad (11.116)
$$
\n
$$
\frac{\partial F}{\partial f} \left( \frac{\partial f}{\partial x} + \frac{\partial f}{\partial u} \cdot p \right) + \frac{\partial F}{\partial g} \left( \frac{\partial g}{\partial x} + \frac{\partial g}{\partial u} \cdot p \right) = 0
$$
\n
$$
\frac{\partial F}{\partial f} \left( \frac{\partial f}{\partial y} + \frac{\partial f}{\partial u} \cdot q \right) + \frac{\partial F}{\partial g} \left( \frac{\partial g}{\partial y} + \frac{\partial g}{\partial u} \cdot q \right) = 0
$$
\n
$$
(1.1.19)
$$

Eliminating  $\frac{a}{\sigma}$  and  $\frac{a}{\sigma}$  from the above yields  $\frac{F}{f}$  and  $\frac{\partial F}{\partial g}$  $\frac{F}{f}$  and  $\frac{\partial F}{\partial g}$ <br>*g*)  $\frac{\partial f}{\partial g}$  $\frac{\partial F}{\partial f}$  and  $\frac{\partial F}{\partial g}$  from the abov  $\frac{\partial F}{\partial f}$  and  $\frac{\partial F}{\partial g}$  from the abov

Eliminating 
$$
\frac{\partial F}{\partial f}
$$
 and  $\frac{\partial F}{\partial g}$  from the above yields  
\n
$$
\frac{\partial (f,g)}{\partial (y,u)} \cdot p + \frac{\partial (f,g)}{\partial (x,u)} \cdot q = \frac{\partial (f,g)}{\partial (x,y)}
$$
\nComparing (1.1.13) and (1.1.20) we have that  
\n $P \cdot p + Q \cdot q = R$  (1.1.21)  
\nshowing that (1.1.11) is a solution of (1.1.8). Thus, (1.1.11) is a

 $(1.1.13)$  and  $(1.1.20)$ 

$$
P.p + Q.q = R \tag{1.1.21}
$$

 $(1.1.11)$  is a solution of  $(1.1.8)$ . Thus,  $(1.1.11)$  is a general solution of  $(1.1.8)$ Consider any solution  $w(x, y, u) = c$ .  $\partial(y, u) = \partial(x, u) = \partial(x, y)$ <br>
Comparing (1.1.13) and (1.1.20) we have that<br>  $P \cdot p + Q \cdot q = R$  (1.1.21)<br>
showing that (1.1.11) is a solution of (1.1.8). Thus, (1.1.11) is a general solution of (1.1.8).<br>
Consider any solution  $y(x, y, u$ Comparing (1.1.13) and (1.1.20) w<br>  $P.p + Q.q = R$ <br>
showing that (1.1.11) is a solution of<br>
Consider any solution  $w(x, y, u) = c$ .<br>
Differentiating partially we have the Let any solution  $w(x, y, u)$ <br>then any solution  $w(x, y, u)$ <br> $\frac{\partial w}{\partial x} + \frac{\partial w}{\partial u} \cdot p = 0$ 1.11) is<br>
ution *w*<br>
artially<br>  $\frac{w}{dx} + \frac{\partial w}{\partial u}$ 

Differentiating partially we have the following:

showing that (1.1.11) is a solution of (1.1.8). Thus, (1.1.11) is a g. Consider any solution 
$$
w(x, y, u) = c
$$
.

\nDifferentiating partially we have the following:

\n
$$
\frac{\partial w}{\partial x} + \frac{\partial w}{\partial u} \cdot p = 0
$$
\n
$$
\frac{\partial w}{\partial y} + \frac{\partial w}{\partial u} \cdot q = 0
$$
\ntherefore follows that

It therefore follows that,

It therefore follows that,  
\n
$$
p = -\frac{\frac{\partial w}{\partial x}}{\frac{\partial w}{\partial u}}
$$
\n
$$
q = -\frac{\frac{\partial w}{\partial y}}{\frac{\partial w}{\partial u}}
$$
\nOn substituting *p* and *q* into (1.1.8) we obtain  
\n
$$
P \frac{\partial w}{\partial x} + Q \frac{\partial w}{\partial y} + R \frac{\partial w}{\partial u} = (1.1.24)
$$
\n(1.1.24)

 $(1.1.8)$ 

On substituting *p* and *q* into (1.1.8) we obtain  
\n
$$
P \frac{\partial w}{\partial x} + Q \frac{\partial w}{\partial y} + R \frac{\partial w}{\partial u} =
$$
\nand in view of the relation (1.1.13) and (1.1.24) we have  
\n
$$
\frac{\partial (f, g)}{\partial (y, u)} \cdot \frac{\partial w}{\partial x} + \frac{\partial (f, g)}{\partial (y, u)} \cdot \frac{\partial w}{\partial x} + \frac{\partial (f, g)}{\partial (y, v)} \cdot \frac{\partial w}{\partial y} = 0
$$
\n(1.1

 $(1.1.13)$  and  $(1.1.24)$ 

$$
P\frac{\partial w}{\partial x} + Q\frac{\partial w}{\partial y} + R\frac{\partial w}{\partial u} = \qquad (1.1.24)
$$
  
and in view of the relation (1.1.13) and (1.1.24) we have  

$$
\frac{\partial (f,g)}{\partial (y,u)} \cdot \frac{\partial w}{\partial x} + \frac{\partial (f,g)}{\partial (x,u)} \cdot \frac{\partial w}{\partial y} + \frac{\partial (f,g)}{\partial (x,y)} \cdot \frac{\partial w}{\partial x} = 0 \qquad (1.1.25)
$$
  
*ie,*
$$
J = \frac{\partial (f,g,w)}{\partial (x,y,u)} = 0 \qquad (1.1.26)
$$

, *ie*

$$
\frac{\partial (f,g)}{\partial (y,u)} \cdot \frac{\partial w}{\partial x} + \frac{\partial (f,g)}{\partial (x,u)} \cdot \frac{\partial w}{\partial y} + \frac{\partial (f,g)}{\partial (x,y)} \cdot \frac{\partial w}{\partial x} = 0 \qquad (1.1.25)
$$
  
ie,  

$$
J = \frac{\partial (f,g,w)}{\partial (x,y,u)} = 0 \qquad (1.1.26)
$$
  
Since the partial derivatives of f, g, and w are supposedly continuous, the

 $(1.1.26)$  implies a functional relation of the form  $w = \phi(f, g)$ *Since the partial derivatives of f, g and w are supposedly continuous, the vanishing of the Jacobian J in*<br> *f* (1.1.26) implies a functional relation of the form  $w = \phi(f, g)$ . Hence  $w = c = \phi(f, g) = c = G(f, g)$  s  $J = \frac{\partial (f, g, w)}{\partial (x, y, u)} = 0$  (1.1.26)<br>
Since the partial derivatives of f, g and w are supposedly continuous, the vanishing of the Jacobian J in<br>
1.1.26) implies a functional relation of the form  $w = \phi(f, g)$ . Hence,  $w - c = \phi$ Therefore, the solution  $w - c = 0$  is contained in the general solution (1.1.11). This completes the proof of Since the partial derivatives of f, g and w are supposedly continuous, the vanishing of the Jacobian J in (1.1.26) implies a functional relation of the form  $w = \phi(f, g)$ . Hence,  $w - c = \phi(f, g) - c = G(f, g)$ , say.<br>Therefore, the sol the theorem. 26)<br>
he vanishing of the Jacobian<br>  $w-c = \phi(f, g) - c = G(f, g)$ <br>
(1,1,1,1) This completes the r  $x, y, u$ <br>ives<br>ional<br> $w - c$ ortherminal values of the Jacobian *J* in<br>  $-c = φ(f, g) - c = G(f, g)$ , say.  $(y, u)$ <br>es of *f*, *g* and *w* are so<br>nal relation of the form<br> $-c = 0$  is contained if

$$
\frac{P_1}{\Delta_1} = \frac{P_2}{\Delta_2} = \frac{P_3}{\Delta_3} = \dots = \frac{P_m}{\Delta_m} = \frac{R}{\Delta}
$$
\n(1.32)

 $(f_1, f_2, f_3, \dots, f_m) = 0$  and differentiating partially wrt  $x_j$   $(j = 1(1)m)$  $\frac{P_1}{\Delta_1} = \frac{P_2}{\Delta_2} = \frac{P_3}{\Delta_3} = \dots = \frac{P_m}{\Delta_m} = \frac{R}{\Delta}$  (1.32)<br>Considering the arbitrary relation  $F(f_1, f_2, f_3, \dots, f_m) = 0$  and differentiating partially wrt  $x_j$  ( $j = 1(1)m$ we have (1.32)<br>  $\cdot \cdot f_m$ ) = 0 and differentiating partially wrt  $x_j$  ( $j = 1(1)m$ )

$$
\Delta_{1} \quad \Delta_{2} \quad \Delta_{3} \qquad \Delta_{m} \quad \Delta \qquad (1.52)
$$
\nConsidering the arbitrary relation  $F(f_{1}, f_{2}, f_{3}, \cdots, f_{m}) = 0$  and differentiating partially wrt  $x_{j} (j = 1(1)m)$ 

\nwe have

\n
$$
\frac{\partial F}{\partial f_{1}}(a_{11} - b_{1}p_{1}) + \frac{\partial F}{\partial f_{2}}(a_{21} - b_{2}p_{1}) + \cdots + \cdots + \frac{\partial F}{\partial f_{m}}(a_{m1} - b_{m}p_{1}) = 0
$$
\n
$$
\frac{\partial F}{\partial f_{1}}(a_{12} - b_{1}p_{2}) + \frac{\partial F}{\partial f_{2}}(a_{22} - b_{2}p_{2}) + \cdots + \cdots + \frac{\partial F}{\partial f_{m}}(a_{m2} - b_{m}p_{2}) = 0
$$
\n
$$
\vdots
$$
\n
$$
\frac{\partial F}{\partial f_{1}}(a_{1m} - b_{1}p_{m}) + \frac{\partial F}{\partial f_{2}}(a_{2m} - b_{2}p_{m}) + \cdots + \cdots + \frac{\partial F}{\partial f_{m}}(a_{mm} - b_{m}p_{m}) = 0
$$
\n
$$
(1.33)
$$

 $\overline{\phantom{a}}$ 

$$
\frac{\partial F}{\partial f_1}(a_{1m} - b_1 p_m) + \frac{\partial F}{\partial f_2}(a_{2m} - b_2 p_m) + \dots + \dots + \frac{\partial F}{\partial f_m}(a_{mm} - b_m p_m) = 0
$$
\nEliminating  $\frac{\partial F}{\partial f_j}$   $(j = 1(1)m)$  among the relations in (1.33) we have\n
$$
\begin{vmatrix}\na_{11} - b_1 p_1 & a_{21} - b_2 p_1 & \cdots & a_{m1} - b_m p_1 \\
a_{12} - b_1 p_2 & a_{22} - b_2 p_2 & \cdots & a_{m2} - b_m p_2 \\
a_{13} - b_1 p_3 & a_{23} - b_2 p_3 & \cdots & a_{m3} - b_m p_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{1m} - b_1 p_m & a_{2m} - b_2 p_m & \cdots & a_{mm} - b_m p_m\n\end{vmatrix} = 0
$$
\n(1.34)  
\nThe determinant in (1.34) may be expressed as the sum of  $2^m$  determinants of which many will vanish due to symmetry and so be left with

 $(1.34)$ be left with<br>  $h_1 \Delta_1 - p_2 \Delta_2 - p_3 \Delta_3$ to symmetry and so be left with The determinant in (1.34) may b<br>to symmetry and so be left with<br> $\Delta - p_1 \Delta_1 - p_2 \Delta_2 -$ <br>From (1.32) and (1.35) we have  $m^{\Delta}$  $a_{1m} - b_1 p_m \quad a_{2m} - b_2 p_m \quad \cdots \quad a_{mn} - b_m p_m$ <br> *p n* in (1.34) may be expressed as the sum of 2<sup>*m*</sup> determinary<br>  $\Delta - p_1 \Delta_1 - p_2 \Delta_2 - p_3 \Delta_3 - \cdots - p_m \Delta_m = 0$  (1.43) we have

to symmetry and so be left with  
\n
$$
\Delta - p_1 \Delta_1 - p_2 \Delta_2 - p_3 \Delta_3 - \cdots - p_m \Delta_m = 0
$$
\n(1.35)  
\nFrom (1.32) and (1.35) we have  
\n
$$
R - p_1 P_1 - p_2 P_2 - p_3 P_3 - \cdots - p_m P_m = 0
$$

 $(1.32)$  and  $(1.35)$ 

nd so be left with  
\n
$$
\Delta - p_1 \Delta_1 - p_2 \Delta_2 - p_3 \Delta_3 - \cdots - p_m \Delta_m =
$$
\n
$$
R - p_1 P_1 - p_2 P_2 - p_3 P_3 - \cdots - p_m P_m = 0
$$
\n
$$
p_1 P_1 + p_2 P_2 + p_3 P_3 + \cdots + p_m P_m = R
$$
\n
$$
R = 0 \text{ is a general solution of (1.24)}.
$$

, *ie*

From (1.32) and (1.35) we have  
\n
$$
R - p_1 P_1 - p_2 P_2 - p_3 P_3 - \cdots - p_m P_m = 0
$$
\n*i.e,*\n
$$
p_1 P_1 + p_2 P_2 + p_3 P_3 + \cdots + p_m P_m = R
$$
\n
$$
p_m
$$
\n
$$
p_1 P_2 + p_3 P_3 + \cdots + p_m P_m = R
$$
\n(1.36)

 $(1.24)$  $R - p_1 P_1 - p_2 P_2 - p_3 P_3 - \cdots - p_m P_m = 0$ <br>  $i.e,$ <br>  $p_1 P_1 + p_2 P_2 + p_3 P_3 + \cdots + p_m P_m = R$ <br>
which proves that  $F = 0$  is a general solution of (1.24).<br>
Note: Note:

The system of  $ODEs(1.12)$  $p_2P_2 + p_3P_3 + \cdots + p_mP_m = R$  (1.36)<br>= 0 is a general solution of (1.24).<br>(1.12) are known as the *Lagranges auxiliary equations*. The curve of intersection which proves that  $F = 0$  is a general solution of<br>Note:<br>The system of *ODEs* (1.12) are known as the<br>of the surfaces (1.11) called *Lagranges lines*.  $P_1P_2 + P_2P_2 + P_3P_3 + \cdots + P_mP_m = R$  (1.36)<br>at  $F = 0$  is a general solution of (1.24).<br>*DEs* (1.12) are known as the *Lagranges auxiliary equations Lagranges lines* ranges lines.<br>
of<br>  $=x^2 - y^2$ . (*i*)

. *Examples*

1 Find the general integral curve of

access (1.11) called *Lagranges lines.*  
\n
$$
e^{i \text{general integral curve of}} (y + ux) p - (x + uy) q = x^2 - y^2.
$$
 (i)

#### . *Solution*

The integral surfaces are determined by the integral curves of the system of ODEs.

$$
\frac{dx}{y+ux} = \frac{dy}{-(x+uy)} = \frac{du}{x^2 - y^2}
$$
 (*ii*)  
n is equal to  

$$
\frac{dx + xdy + du}{x(x+uy) + (x^2 - y^2)} = \frac{ydx + xdy + du}{0}
$$
, consideri

each of which is equall to

$$
\frac{dx}{y+ux} = \frac{dy}{-(x+uy)} = \frac{du}{x^2 - y^2}
$$
 (*ii*)  
each of which is equal to  

$$
\frac{ydx + xdy + du}{y(y+ux) - x(x+uy) + (x^2 - y^2)} = \frac{ydx + xdy + du}{0}
$$
, considering y, x, 1 as multiplier. (*iii*)  
This are also each equal to  

$$
\frac{xdx + ydy - udu}{x(x+uv) - y(x+uv) - y(x^2 - y^2)} = \frac{xdx + ydy - udu}{0}
$$
, considering x, y, -u as multiplier. (

This are also each equall to

$$
\frac{yax + xay + au}{y(y + ux) - x(x + uy) + (x^2 - y^2)} = \frac{yax + xay + au}{0}
$$
, considering y, x, 1 as multiplier. (iii)  
his are also each equal to  

$$
\frac{xdx + ydy - udu}{x(y + ux) - y(x + uy) - u(x^2 - y^2)} = \frac{xdx + ydy - udu}{0}
$$
, considering x, y, -u as multiplier. (iv)  
from (iii) we have  

$$
ydx + xdy + du \Rightarrow 2xy + u = c_1
$$

From *(iii)* we have

$$
ydx + xdy + du \Rightarrow 2xy + u = c_1
$$

Similarly,  $(iv)$  gives *iv*

From (ii) we have  
\n
$$
ydx + xdy + du \Rightarrow 2xy + u = c_1
$$
  
\nSimilarly, (iv) gives  
\n $xdx + ydy - udu = 0$ , *ie*,  $x^2 + y^2 - u^2 = c_2$   
\nHence the general solution is

Hence, the general solution is

$$
xdx + ydy - udu = 0, \text{ ie, } x^2 + y^2 - u^2 = c
$$
  
Hence, the general solution is  

$$
F(2xy + u, x^2 + y^2 - u^2) = 0
$$

 $= 0$ <br>  $y_1 + xzp_2 + xyp_3$ 2 Obtain a general integral of  $yzp_1 + xzp_2 + xyp_3 + xyz = 0$ . *i*e,  $x + y = u - c_2$ <br>
<sup>2</sup>) = 0<br> *yzp*<sub>1</sub> + *xzp*<sub>2</sub> + *xyp*<sub>3</sub> + *xyz* = 0. a general integral of  $yzp_1 + xzp_2 + x$ <br> *x*d form of the *PDE* is given as<br>  $yup_1 + xup_2 + xyp_3 = -xyz$  (*i*)

*Solution*

The standard form of the *PDE* is given as<br>  $yup_1 + xup_2 + xyp_3 = -xyz$ 

$$
yup_1 + xup_2 + xyp_3 = -xyz \qquad (i)
$$

The corresponding auxiliary equations are

From of the *PDE* is given as  
\n
$$
up_1 + xup_2 + xyp_3 = -xyz \qquad (i)
$$
\nanding auxiliary equations are

\n
$$
\frac{dx}{yz} = \frac{dy}{xz} = \frac{dz}{yx} = \frac{du}{-xyz} \qquad (ii)
$$
\nit each of the ratio is equal to the following:

\n
$$
\frac{xdx + du}{xyz - yyz} = \implies xdx + du = 0 \quad ie, \quad x
$$

We have that each of the ratio is equal to the following:

$$
\frac{dx}{yz} = \frac{dy}{xz} = \frac{az}{yx} = \frac{du}{-xyz}
$$
 (*ii*)  
it each of the ratio is equal to the following:  

$$
\frac{xdx + du}{xyz - xyz} = \Rightarrow xdx + du = 0
$$
 *ie*,  $x^2 + 2u = c_1$  (*iii*)  

$$
xdy + du
$$

We have that each of the ratio is equal to the following:  
\n
$$
\frac{xdx + du}{xyz - xyz} = \Rightarrow xdx + du = 0 \text{ ie, } x^2 + 2u = c_1 \quad \text{(iii)}
$$
\n
$$
\frac{ydy + du}{xyz - xyz} = \Rightarrow ydy + du = 0 \text{ ie, } y^2 + 2u = c_2 \quad \text{(iv)}
$$
\n
$$
\frac{xdz + du}{zdz + du}
$$

$$
xyz - xyz
$$
  
\n
$$
\frac{ydy + du}{xyz - xyz} = \Rightarrow ydy + du = 0 \text{ ie, } y^2 + 2u = c_2 \qquad (iv)
$$
  
\n
$$
\frac{zdz + du}{xyz - xyz} = \Rightarrow zdz + du = 0 \text{ ie, } z^2 + 2u = c_3 \qquad (v)
$$
  
\nWe thus have the following:  
\n
$$
x^2 - y^2 = c_1', x^2 - z^2 = c_2' \text{ and } x^2 + 2u = c_1
$$

following:

$$
\frac{zaz + au}{xyz - xyz} = \Rightarrow zdz + du = 0 \text{ ie, } z^2 + 2u = c_3 \qquad (v)
$$
  
We thus have the following:  

$$
x^2 - y^2 = c_1', x^2 - z^2 = c_2' \text{ and } x^2 + 2u = c_1
$$
  
A general integral is therefore given as  $x^2 + 2u = \phi(x^2 - y^2, x^2 - z^2)$ .

### 1.2 PARTICULAR INTEGRALS OF LAGRANGE'S EQUATIONS.

Consider the equation

Consider the equation  
\n
$$
P \cdot p + Q \cdot q = R
$$
 (1.37)  
\nWe observe that the integral surfaces of (1.37)

 $(1.37)$ Consider the equation<br>  $P \cdot p + Q \cdot q = R$  (1.37)<br>
We observe that the integral surfaces of (1.37) are generated from the integral curves of the system<br>  $\frac{dx}{dt} = dy = du$ 

e equation  
\n
$$
p+Q \cdot q = R
$$
 (1.37)  
\nthat the integral surfaces of (1.37)  
\n $\frac{dx}{P} = \frac{dy}{Q} = \frac{du}{R}$  (1.38)  
\nhat these integral curves are given  
\n $f(x, y, u) = a$  (1.39)

Assuming that these integral curves are given by  $rac{dx}{P} = \frac{dy}{Q}$ <br>Assuming that thes<br> $f(x, y, z)$ 

$$
P \quad Q \quad R
$$
  
Assuming that these integral curves are given by  
 $f(x, y, u) = a$ }\n $g(x, y, u) = b$ }\n  
In order to determine the particular integral of (

In order to determine the particular integral of  $(1.37)$  $f(1.37)$  passing through a given curve

$$
g(x, y, u) = b
$$
  
In order to determine the particular integral of (1.37) passing through  

$$
h_1(x, y, u) = 0
$$
  
or  $x = x(t), y = y(t), u = u(t)$   

$$
h_2(x, y, u) = 0
$$
  
(1.40)  
in which t is a parameter we eliminate x, y, u between (1.40) and (1  
of the form  $\phi(x, b) = 0$  and so the required particular integral will be

in which t is a parameter we eliminate  $x, y, u$  between  $(1.40)$  and  $(1.39)$  $\phi(a,b)$  =  $\begin{cases} i(t), u = u(t) \end{cases}$  (1.40) (1.40)<br>inate x, y, u between (1.40) and (1.39). The elimant will therefore be or  $x = x(t)$ ,  $y = y(t)$ ,  $u = u(t)$ <br>  $h_2(x, y, u) = 0$ <br>
in which t is a parameter we eliminate x, y, u between (1.40) and (1.<br>
of the form  $\phi(a, b) = 0$  and so the required particular integral will be.  $n_2(x, y, u) = 0$ <br>in which *t* is a parameter we eliminate *x*,<br>of the form  $\phi(a, b) = 0$  and so the require<br> $\phi(f, g) = 0$  (1.41 or  $\left(x, \frac{1}{p}a, b\right)$ *x*, *y*<br>
parai<br> *f*, *g* 

 $(f, g) = 0$  (1.41)  $\phi(f,g)$ 

*Examples*

1 Determin e the particular integral of the *PDE*

$$
\phi(f, g) = 0
$$
 (1.41)  
Examples  
1 Determine the particular integral of the *PDE*  

$$
(x - y) p + (y - x - u) q = u
$$
 that passes through the point  $u = 1, x^2 + y^2 = 1$ .  
Solution

. *Solution*

The corresponding auxiliary equations are

$$
(x-y)p + (y-x-u)q = u \text{ that passes through the point } u = 1, x^2 + y^2 = 1.
$$
  
*Solution.*  
The corresponding auxiliary equations are  

$$
\frac{dx}{x-y} = \frac{dy}{y-x-u} = \frac{du}{u} \qquad (i)
$$

$$
= \frac{dx+dy+du}{x-y+y-x-u+u} = \frac{d(x+y+u)}{0}, \text{ taking (1,1,1 as multipliers)} \qquad (ii)
$$

$$
= \frac{dx-dy+du}{x-y-y+x+u+u} = \frac{d(x-y+u)}{2(x-y+u)}, \text{ taking (1,-1,1 as multipliers)} \qquad (ii)
$$

$$
= \frac{dx - dy + du}{x - y - y + x + u + u} = \frac{d(x - y + u)}{2(x - y + u)}, \text{ taking (1, -1, 1 as multipliers)} \qquad (iii)
$$
  
From (ii) we have  

$$
dx + dy + du = 0 \Rightarrow d(x + y + u) = 0, \text{ i}e, \quad x + y + u = a \qquad (iv)
$$

From  $(ii)$  we have

$$
dx + dy + du = 0 \Rightarrow d\left(x + y + u\right) = 0, \text{ i}e, \text{ x} + y + u = a \quad (iv)
$$

From the secod relation we have

From (ii) we have  
\n
$$
dx + dy + du = 0 \Rightarrow d(x + y + u) = 0, i e, x + y + u = a
$$
\nFrom the second relation we have\n
$$
\frac{du}{u} = \frac{dx - dy + du}{x - y - y + x + u + u} = \frac{d(x - y + u)}{2(x - y + u)}
$$
\n(v)

*i.e,* 
$$
\ln u = \frac{1}{2} \ln (x - y + u) + \ln b \Rightarrow 2 \ln u = \ln (x - y + u) + \ln b = \ln b (x - y + u)
$$
*ie,* 
$$
u^{2} = b (x - y + u) \Rightarrow b = \frac{u^{2}}{x - y + u} \qquad (vi)
$$

*i.e,* 
$$
\ln u = \frac{1}{2} \ln (x - y + u) + \ln b \Rightarrow 2 \ln u = \ln (x - y + i e),
$$

$$
u^{2} = b (x - y + u) \Rightarrow b = \frac{u^{2}}{x - y + u} \qquad (vi)
$$
We observe that the given curve can be written in the form  $x = \cos \theta$ ,  $y = \sin \theta$ ,  $u = 1 \qquad (vii)$ .

We observe that the given curve can be written in the form

that the given curve can be written in the form  
\n
$$
x = \cos \theta
$$
,  $y = \sin \theta$ ,  $u = 1$  (vii)  
\n $(vii)$  into (v) (vii) yields

 $(vii)$  into  $(v)$   $(vii)$ We observe that the given curve can be<br>  $x = \cos \theta$ ,  $y = \sin \theta$ ,  $u = 1$ <br>
Substituting (vii) into (v) (vii) yields

We observe that the given curve can be written in the form  
\n
$$
x = \cos \theta
$$
,  $y = \sin \theta$ ,  $u = 1$  (*vii*)  
\nSubstituting *(vii)* into*(v) (vii)* yields  
\n $\cos \theta + \sin \theta + 1 = a$  (*viii*)  
\n
$$
\frac{1}{\cos \theta - \sin \theta + 1} = b \implies \cos \theta - \sin \theta + 1 = \frac{1}{b}
$$
 (*ix*)  
\n
$$
\implies (\cos \theta + \sin \theta)^2 + (\cos \theta - \sin \theta) = (a - 1)^2 + (\frac{1}{b} - 1)^2
$$
\n*ie*,  $\cos^2 \theta + \sin^2 \theta + 2\cos \theta \sin \theta + \cos^2 \theta + \sin^2 \theta - 2\cos \theta \sin \theta = a^2 - 2a + 1^2 + \frac{1}{b^2} - \frac{2}{b} + 1$ 

$$
\cos \theta - \sin \theta + 1
$$
\n  
\n
$$
\Rightarrow \qquad (\cos \theta + \sin \theta)^2 + (\cos \theta - \sin \theta) = (a - 1)^2 + \left(\frac{1}{b} - 1\right)^2
$$
\n  
\ni.e,  $\cos^2 \theta + \sin^2 \theta + 2\cos \theta \sin \theta + \cos^2 \theta + \sin^2 \theta - 2\cos \theta \sin \theta = a^2 - 2a + 1^2 + \frac{1}{b^2} - \frac{2}{b} + 1$ \n  
\ni.e,  $2\left(\cos^2 \theta + \sin^2 \theta\right) = a^2 - 2a + 1^2 + \frac{1}{b^2} - \frac{2}{b} + 1$ 

i.e, 
$$
\cos^2 \theta + \sin^2 \theta + 2\cos \theta \sin \theta + \cos^2 \theta + \sin^2 \theta - 2\cos \theta
$$
  
\ni.e,  $2(\cos^2 \theta + \sin^2 \theta) = a^2 - 2a + 1^2 + \frac{1}{b^2} - \frac{2}{b} + 1$   
\ni.e,  $a^2 - 2a + \frac{1}{b^2} - \frac{2}{b} = 0$   
\nThus, the particular integral surface is given as

*i.e.* 
$$
a^2 - 2a + \frac{1}{b^2} - \frac{2}{b} = 0
$$

Thus, the particular integral surface is given as

*ie,* 
$$
a^2 - 2a + \frac{1}{b^2} - \frac{2}{b} = 0
$$
  
\nThus, the particular integral surface is given as  
\n
$$
(x + y + u)^2 - 2(x + y + u) - \frac{2(x - y + u)}{u^2} + \frac{4(x - y + u)}{u^4} = 0
$$
\n2 Determine the solution of the differential equation

#### 2 Determine the solution of the differential equation

 $(u+2a)$  $(x-$ <br>etermine t<br> $u+2a$ ) xp  $+ 2a$ ) xp +  $(xu + 2yu + 2ay)q = u(a + u)$  that passes through the cuce  $y = 0, u<sup>3</sup> + x(a + u)<sup>2</sup>$  $(u+u)^2 - 2(x+y+u) - \frac{2(x-y+u)}{u^2} + \frac{4(x-y+u)}{u^4} = 0$ <br>
solution of the differential equation<br>  $xu + 2yu + 2ay$   $q = u(a+u)$  that passes through the cuce  $y = 0, u^3 + x(a+u)^2 = 0$ . *Solution*  $d_1(2a)xp + (xu + 2yu + 2ay)q = u(a + u)$  that pas<br> *dx*<br>  $d_2x - 2a(x + 2yu + 2ay) = \frac{du}{u(a + u)}$  (*i*  $(-y + u)^2 - 2(x + y + u) - \frac{2(x - y + u)}{u^2} + \frac{4(x - y + u)}{u^4} = 0$ <br>
ne solution of the differential equation<br>  $+(xu + 2yu + 2ay)q = u(a + u)$  that passes through the cuce  $y = 0, u^3 + x(a + u)^2 = 0$ .

The auxiliary equation corresponding to the  $DE$  is *DE*

$$
(u+2a)xy + (xu+2yu+2ay)q = u(u+u) \text{ that pass}
$$
  
ation  
auxiliary equation corresponding to the *DE* is  

$$
\frac{dx}{(u+2a)x} = \frac{dy}{xu+2yu+2ay} = \frac{du}{u(a+u)}
$$
  
ing the first and third ratios yields  

$$
\frac{dx}{(u+2a)x} = \frac{du}{u(a+u)}
$$
  
(ii)

Taking the first and third ratio s yields

$$
(u+2a)x = xu + 2yu + 2ay = u(a+u)
$$
  
\nTaking the first and third ratios yields  
\n
$$
\frac{dx}{(u+2a)x} = \frac{du}{u(a+u)}
$$
\n
$$
\Rightarrow \frac{dx}{x} = \frac{(u+2a)du}{u(a+u)}
$$
\n*ii.e,*\n
$$
\ln x = \int \frac{(u+2a)du}{u(a+u)}
$$

Resolving the integrand on the rhs into partial fraction gives  
\n
$$
\frac{(u+2a)}{u(a+u)} = \frac{A}{u} + \frac{B}{a+u} = \frac{(A+B)u + Aa}{u(a+u)}
$$
\ni.e,  $A+B=1, A=2 \Rightarrow A=2, B=-1$   
\nhence, 
$$
\frac{(u+2a)}{u(a+u)} = \frac{2}{u} - \frac{1}{a+u}ie, \quad \int \frac{(u+2a)du}{u(a+u)} = \int \left(\frac{2}{u} - \frac{1}{a+u}\right)
$$

$$
\frac{(u+2a)}{u(a+u)} = \frac{A}{u} + \frac{B}{a+u} = \frac{(A+B)u + Aa}{u(a+u)}
$$
  
\n*i.e,*  $A+B=1, A=2 \Rightarrow A=2, B=-1$   
\nhence,  $\frac{(u+2a)}{u(a+u)} = \frac{2}{u} - \frac{1}{a+u}$  *ie,*  $\int \frac{(u+2a)du}{u(a+u)} = \int \left(\frac{2}{u} - \frac{1}{a+u}\right) du = 2 \ln u - \ln(a+u) + c_1$   
\n $\Rightarrow \ln x = \ln \left(\frac{c_1u^2}{a+u}\right)ie$ ,  $x = \frac{c_1u^2}{a+u}$ ,  $\left[c_1 = \frac{(a+u)x}{u^2}\right]$   
\nFrom the second and third ratios we have  
\n
$$
\frac{dy}{xu+2vu+2av} = \frac{du}{u(a+u)}
$$

From the second and third ratios we have ,<br>,<br>,

$$
\frac{dy}{du} = \frac{du}{u} \left( \frac{u}{u} \right)^{u} = \frac{du}{u}
$$
  
From the second and third ratios we have  

$$
\frac{dy}{u} = \frac{du}{u(u+u)}
$$
  
i.e,
$$
\frac{dy}{u} = \frac{du}{u(u+u)}
$$
  
ii.e,
$$
\frac{dy}{u+u} = \frac{du}{u(u+u)}
$$
  
ii.e,
$$
\frac{dy}{u^3} = \frac{du}{u(u+u)}
$$

, *ie* ,<br>,<br>,

$$
\frac{1}{a+u} + 2yu + 2ay
$$
\n
$$
ie,
$$
\n
$$
\frac{dy}{\frac{c_1u^3}{a+u} + 2y(u+a)} = \frac{du}{u(a+u)}
$$
\n
$$
ie,
$$
\n
$$
\frac{dy}{\frac{c_1u^3}{(a+u)^2} + 2y} = \frac{du}{u}
$$
\n
$$
\frac{c_1u^3}{(a+u)^2} + 2y
$$

 $(a+u)$ 

+

3  $\frac{1^{\mu}}{2}+2$ 

 $(a+u)$ 

2 1

+

2

*y*

$$
\frac{1}{a+u} + 2y(u)
$$
  

$$
\frac{dy}{(a+u)^2} + 2y
$$
  

$$
\frac{c_1u^3}{(a+u)^2} + 2y
$$
  

$$
\frac{dy}{du} = \frac{(a+u)^2}{u}
$$
  

$$
\frac{dy}{du} = 2y
$$

=

, *ie*

2 ,<br>,<br>, *ie*,  $\frac{dy}{du} = \frac{\frac{c_1u^3}{(a+u)^2} + 2y}{u}$ <br>*ie*,  $\frac{dy}{du} - \frac{2y}{u} = \frac{c_1u^2}{(a+u)^2}$  (*iv*  $\frac{dy}{du} = \frac{(a+u)^2}{u}$   $\frac{dy}{du} - \frac{2y}{u} = \frac{c_1u^2}{(a+u)}$  $=\frac{\frac{c_1u^2}{(a+u)^2} + 2y}{u}$ <br>-  $\frac{2y}{u} = \frac{c_1u^2}{(a+u)^2}$ 

This is a first order ODE of the form  $y' + p(x)y = f(x)$  which admits an integrating factor  $\mu = e^{\int p \alpha x}$ . Hence  $(iv)$  has the integrating factor  $u^{-2}$ . h<br>e<br>, (*iv*)<br> $p(x)y = f(x)$  which admits an integrating factor  $\mu = e^{\int pdx}$  $\frac{du}{du} - \frac{v}{u} = \frac{1}{(a+u)^2}$ <br>first order ODE of the form y'<br>*iv*) has the integrating factor u This is a first orc<br>
Hence *(iv)* has t<br> *ie*, *(yu*  $\mu$ − (iv)<br>+  $p(x)y = f(x)$  which admits an integrating factor  $\mu = e^{\int pdx}$ .

 $(iv)$ 

i
$$
e
$$

This is a first order ODE of the form y<sup>3</sup>-  
ence (iv) has the integrating factor 
$$
u^{-1}
$$
  

$$
\left( yu^{-2} \right)' = \frac{c_1}{\left( a+u \right)^2}
$$

$$
yu^{-2} = \int \frac{c_1}{\left( a + u \right)^2} du = -\frac{c_1}{\left( a + u \right)^2}
$$

 $\overline{\phantom{a}}$ 

Hence (iv) has the integrating factor 
$$
u^{-2}
$$
.  
\n*i.e.*  $(yu^{-2})' = \frac{c_1}{(a+u)^2}$   
\n $\Rightarrow$   $yu^{-2} = \int \frac{c_1}{(a+u)^2} du = -\frac{c_1}{(a+u)} + c_2$  (v)  
\n*i.e.*  $y = \int \frac{c_1}{(a+u)^2} du = -\frac{c_1u^2}{(a+u)} + c_2u^2$  (vi)

, *ie*

$$
\Rightarrow yu^{-2} = \int \frac{c_1}{(a+u)^2} du = -\frac{c_1}{(a+u)} + c_2 \qquad (v)
$$
  
\n*i.e,*  
\n
$$
\Rightarrow y = \int \frac{c_1}{(a+u)^2} du = -\frac{c_1 u^2}{(a+u)} + c_2 u^2 \qquad (vi)
$$
  
\nSetting y = 0 in (vi) yields

Setting  $y = 0$  in  $(vi)$  yields =

$$
c_2 = \frac{c_1}{(a+u)} i e, \ \ u = \frac{c_1}{c_2} - a \qquad (vii)
$$

We recall from the second initial condition that from the sec<br> $u^3 + x(a+u)$ order in the second initial contract that  $+ x (a + u)^2 = 0$ 

 $x^3 + x(a+u)^2$ 

$$
u^{3} + x(a + u)^{2} = 0
$$
  
i.e, 
$$
u^{3} + \frac{c_{1}u^{2}}{a + u}(a + u)^{2} = 0 \Rightarrow u^{3} + c_{1}u^{2}(a + u) = 0
$$

, *ie*

, *ie* ( ) ( ) ( ) ( ) Eliminating from and yields *u vii viii* 1 0 *u c a u viii* + + =

Eliminating *u* from 
$$
(vii)
$$
 and  $(viii)$  yields  
\n
$$
u + c_1(a+u) = 0
$$
\ni.e, 
$$
\frac{c_1}{c_2} - a + c_1\left(a + \frac{c_1}{c_2} - a\right) = 0
$$
\ni.e, 
$$
\frac{c_1}{c_2} - a + \frac{c_1^2}{c_2} = 0
$$

*i.e.*, 
$$
\frac{c_1}{c_2} - a + \frac{c_1}{c_2} = 0
$$

$$
\Rightarrow (c_1 - ac_2) + c_1^2 = 0
$$

The required integral surface is thus  
\n
$$
\frac{(a+u)x}{u^2} - a \left[ \frac{y}{u^2} \frac{(a+u)x}{u^2 (a+u)} \right] + \frac{(a+u)^2 x^2}{u^4} = 0
$$
\ni.e,  $x(a+u)u^2 - a(x+y)u^2 + (a+u)^2 x^2 = 0$ 

### 1.3 GENERAL METHOD FOR THE SOLUTION OF FIRST-ORDER PARTIAL DIFFERENTIAL EQUATIONS.

The main methods of solution for the first-order PDE are the ones due to Charpits and Jacobi.<br>
1.3.1 CHARPIT'S METHOD<br>
Given the *PDE*<br>  $F(x, y, u, p, q) = 0$  (1.2.1)<br>
Since u is a function of both x and y we thus have

#### 1.3.1 CHARPIT'S METHOD

Given the *PDE*

$$
DE
$$
  
\n $F(x, y, u, p, q) = 0$  (1.2.1)  
\nfunction of both x and y we thus have  
\n $du = pdx + qdy$  (1.2.2)  
\n $u = pdx + qdy$ 

Since  $u$  is a function of both  $x$  and  $y$  we thus have **CHARPIT'S METHOD**<br>the *PDE*<br> $F(x, y, u, p, q) = 0$ <br> $u$  is a function of both x and y<br> $du = pdx + ady$ 

function of both *x* and *y* we thus have  
\n
$$
du = pdx + qdy
$$
 (1.2.2  
\nanother function  
\n $F(x, y, u, p, q, a) = 0$  (1.2.3  
\nuseible to evaluate *p* and *a* from the two equations (1.2.1) and (1.2.2) in the f

If we have another function

$$
F(x, y, u, p, q, a) = 0 \tag{1.2.3}
$$

Since *u* is a function of both *x* and *y* we thus have<br>  $du = pdx + qdy$  (1.2.2<br>
If we have another function<br>  $F(x, y, u, p, q, a) = 0$  (1.2.3<br>
it will be possible to evaluate *p* and *q* from the two equations (1.2.1) and (1.2.2) in  $p = \phi(x, y, u, a)$  and  $q = \psi(x, y, u, a)$ . (1.2.2)<br>  $p = 0$  (1.2.3)<br>
p and q from the two equations (1.2.1) and (1.2.2) in the form we another function<br>  $F(x, y, u, p, q, a) = 0$ <br>
e possible to evaluate p and q f<br>  $y, u, a$  and  $q = \psi(x, y, u, a)$ .<br>
ting these values into (1.2.2)  $F(x, y, u, p, q, a) = 0$  (1.2.3)<br>it will be possible to evaluate p and q from the two equations (1.2.1) and (1.2.2) in the form<br> $p = \phi(x, y, u, a)$  and  $q = \psi(x, y, u, a)$ .<br>Substituting these values into (1.2.2) renders it directly integ  $= 0$ <br>*p* and *q* If we have another function<br> $F(x, y, u, p, q, a) = 0$ <br>it will be possible to evaluate p and q from<br> $p = \phi(x, y, u, a)$  and  $q = \psi(x, y, u, a)$ .

Substituting these values into  $(1.2.2)$  renders it directly integrable or integrable using some weighting function possible to evaluate p and q from the two equations (1.2.1) and (1.2.2) in the form  $y, u, a$ ) and  $q = \psi(x, y, u, a)$ .<br>ing these values into (1.2.2) renders it directly integrable or integrable using some weighting<br>and the inte For this solution gives: these values into (1)<br>the integral which<br>ion gives:<br> $f_x dx + f_y dy + f_u du$ = values into (1.2.2) renders it dir<br>
integral which is of the form  $f(x,$ <br>
gives:<br>  $+ f_y dy + f_u du = 0$ 

$$
\overline{O1}
$$

function and the integral which is of the form 
$$
f(x, y, u, a) = b
$$
 with be a solution of the  
For this solution gives:  

$$
f_x dx + f_y dy + f_u du = 0
$$

$$
-f_u \left(\frac{f_x}{f_y} + \frac{f_y}{f_u}\right) du = 0
$$

$$
-f_u \left(\frac{1.2.4}{f_x}\right) + \frac{1}{f_u} du = 0
$$

$$
-f_u \left(\frac{1.2.4}{f_y}\right) + \frac{1}{f_u} du = 0
$$

$$
f = 0
$$

 $(1.2.4)$  with  $(1.2.2)$ 

or 
$$
\frac{J_x}{-f_u} dx + \frac{J_y}{-f_u} dy - du = 0
$$
\nComparing (1.2.4) with (1.2.2) we have\n
$$
\frac{f_x}{-f_u} = p = \phi
$$
\n
$$
\frac{f_y}{-f_u} = q = \psi
$$
\nFrom  $f(x, y, u, a) = b$  treating  $u = u(x, y)$  we have\n
$$
f_x + f_u \cdot p = 0, \ f_y + f_u \cdot q = 0
$$
\n(1.2.6)\n(1.2.6) implies

 $(x, y, u, a) = b$  treating  $u = u(x, y)$  $-$  5<br>From  $f(x, y, i)$ <br>1.2.6) implie

$$
y, u, a) = b \text{ treating } u = u(x, y) \text{ we have}
$$
  

$$
f_x + f_u \cdot p = 0, \ f_y + f_u \cdot q = 0
$$
} (1.2.6)

$$
f_x + f_u \cdot p = 0, \ f_y + f_u \cdot q = 0
$$
 (1.2.6)  
(1.2.6) implies  

$$
p = -\frac{f_x}{f_u}, q = -\frac{f_y}{f_u}
$$
 (1.2.7)  
i.e,  $p = \phi$  and  $q = \psi$  (1.2.1) it thus implies that  $f(x, y, u, a) = b$  is a solution of (1.2.1). Since  
this solution contains two arbitrary constants, it is therefore a complete solution of (1.2.1). The problem

 $(1.2.1)$  $(1.2.1)$  $(1.2.3)$  refered to a the auxiliary function. In doing this we ie,  $p = \phi$  and  $q = \psi$ <br>
Since  $p = \phi$  and  $q = \psi$  satisfy (1.2.1) it thus implies that  $f(x, y, u, a) = b$  is a solution of (1.2.1). Since<br>
this solution contains two arbitrary constants, it is therefore a complete solution of (1. ie,  $p = \phi$  and  $q = \psi$  ]<br>Since  $p = \phi$  and  $q = \psi$  satisfy (1.2.1) it thus implie<br>this solution contains two arbitrary constants, it is t<br>now therefore is to determine the function (1.2.3) r observe that the quantities u, p, q substituted into  $(1.2.1)$   $(1.2.3)$  satisfy them identically. As a matter of Since  $p = \phi$  and  $q = \psi$  satisfy (1.2.1) it thus implies that  $f(x, y, u, a) = b$  is a solution of (1.2.1). Since<br>this solution contains two arbitrary constants, it is therefore a complete solution of (1.2.1). The problem<br>now th fact the partial derivatives of  $F$  and  $G$  with respect to u,x and y must vanish. *F* is therefore a complete solut the function  $(1.2.3)$  refered to a the auxiliary fi *p*, *q* substituted into  $(1.2.1) (1.2.3)$  satisfy then *F* and *G* with respect to u,x and *y* must vanish.

$$
\frac{\partial F}{\partial x} + \frac{\partial F}{\partial u} \cdot p + \frac{\partial F}{\partial p} \frac{\partial p}{\partial x} + \frac{\partial F}{\partial q} \frac{\partial q}{\partial x} = 0
$$
\n
$$
\frac{\partial G}{\partial x} + \frac{\partial G}{\partial u} \cdot p + \frac{\partial G}{\partial p} \frac{\partial p}{\partial x} + \frac{\partial G}{\partial q} \frac{\partial q}{\partial x} = 0
$$
\n
$$
\frac{\partial F}{\partial y} + \frac{\partial F}{\partial u} \cdot q + \frac{\partial F}{\partial p} \frac{\partial p}{\partial y} + \frac{\partial F}{\partial q} \frac{\partial q}{\partial y} = 0
$$
\n
$$
\frac{\partial G}{\partial y} + \frac{\partial G}{\partial u} \cdot q + \frac{\partial G}{\partial p} \frac{\partial p}{\partial y} + \frac{\partial G}{\partial q} \frac{\partial q}{\partial y} = 0
$$
\nEliminating  $\frac{\partial p}{\partial x}$  in (1.2.8) we have

 $(1.2.8)$ *p x*  $\partial$ 

Eliminating 
$$
\frac{\partial p}{\partial y} + p \cdot \frac{\partial}{\partial p} \frac{\partial p}{\partial y} + \frac{\partial}{\partial q} \frac{\partial}{\partial y} = 0
$$
  
\nEliminating  $\frac{\partial p}{\partial x}$  in (1.2.8) we have  
\n
$$
\frac{\partial (F, G)}{\partial (x, p)} + p \cdot \frac{\partial (F, G)}{\partial (u, p)} + \frac{\partial p}{\partial x} \cdot \frac{\partial (F, G)}{\partial (q, p)} = 0 \quad (1.2.10)
$$
\nSimilarly, eliminating  $\frac{\partial q}{\partial y}$  in (1.2.9) we have  
\n
$$
\frac{\partial (F, G)}{\partial (y, q)} + q \cdot \frac{\partial (F, G)}{\partial (u, q)} + \frac{\partial q}{\partial y} \cdot \frac{\partial (F, G)}{\partial (q, q)} = 0 \quad (1.2.11)
$$

 $(1.2.9)$ *y*  $\partial$ 

$$
\frac{\partial}{\partial(x, p)} + p \cdot \frac{\partial}{\partial(u, p)} + \frac{\partial}{\partial x} \cdot \frac{\partial}{\partial(q, p)} = 0 \quad (1.2.10)
$$
\nSimilarly, eliminating  $\frac{\partial q}{\partial y}$  in (1.2.9) we have\n
$$
\frac{\partial (F, G)}{\partial (y, q)} + q \cdot \frac{\partial (F, G)}{\partial (u, q)} + \frac{\partial q}{\partial y} \cdot \frac{\partial (F, G)}{\partial (p, q)} = 0 \quad (1.2.11)
$$
\nwhere\n
$$
\frac{\partial (x, y)}{\partial (s, t)} = \frac{\partial x}{\partial s} \frac{\partial y}{\partial t} - \frac{\partial x}{\partial t} \frac{\partial y}{\partial s} \qquad (1.2.12)
$$

where

$$
\overline{\partial(y,q)} + q \cdot \overline{\partial(u,q)} + \overline{\partial y} \cdot \overline{\partial(p,q)} = 0 \quad (1.2.11)
$$
  

$$
\frac{\partial(x,y)}{\partial(s,t)} = \frac{\partial x}{\partial s} \frac{\partial y}{\partial t} - \frac{\partial x}{\partial t} \frac{\partial y}{\partial s} \qquad (1.2.12)
$$

Recalling that

Recalling that  
\n
$$
\frac{\partial(x, y)}{\partial(s, t)} = \frac{\partial x}{\partial s} \frac{\partial y}{\partial t} - \frac{\partial x}{\partial t} \frac{\partial y}{\partial s}
$$
\n(1.2.12)  
\nRecalling that  
\n
$$
\frac{\partial q}{\partial x} = \frac{\partial}{\partial x} (q) = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial y} \right) = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} \right) = \frac{\partial}{\partial y} (p) = \frac{\partial p}{\partial y}
$$
\n(1.2.13)  
\nwe thus have from (1.2.11) and (1.2.12) that  
\n
$$
\left( \frac{\partial F}{\partial x} + p \cdot \frac{\partial F}{\partial u} \right) \frac{\partial G}{\partial p} + \left( \frac{\partial F}{\partial y} + q \cdot \frac{\partial F}{\partial u} \right) \frac{\partial G}{\partial a} + \left( -p \cdot \frac{\partial F}{\partial p} - q \cdot \frac{\partial F}{\partial a} \right) \frac{\partial G}{\partial u} + \left( -\frac{\partial F}{\partial p} \right) \frac{\partial G}{\partial x} + \left( -\frac{\partial F}{\partial a} \right) \frac{\partial G}{\partial x}
$$

 $(1.2.11)$  and  $(1.2.12)$ 

$$
\frac{\partial q}{\partial x} = \frac{\partial}{\partial x} \left( q \right) = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial y} \right) = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} \right) = \frac{\partial}{\partial y} \left( p \right) = \frac{\partial p}{\partial y}
$$
(1.2.13)  
we thus have from (1.2.11) and (1.2.12) that  

$$
\left( \frac{\partial F}{\partial x} + p \cdot \frac{\partial F}{\partial u} \right) \frac{\partial G}{\partial p} + \left( \frac{\partial F}{\partial y} + q \cdot \frac{\partial F}{\partial u} \right) \frac{\partial G}{\partial q} + \left( -p \cdot \frac{\partial F}{\partial p} - q \cdot \frac{\partial F}{\partial q} \right) \frac{\partial G}{\partial u} + \left( -\frac{\partial F}{\partial p} \right) \frac{\partial G}{\partial x} + \left( -\frac{\partial F}{\partial q} \right) \frac{\partial G}{\partial y} = 0
$$
(1.2.14)  
This is a linear differential equation of order 1 that must be satisfied by (1.44). Its integrals are integrals of the Lagrange's auxiliary equations

This is a linear differential of the Lagranges auxiliary equations  $\int \partial p \left( \partial y \right)^{1} q^{2} du \int \partial q \left( \frac{P}{\partial p} \right)^{1} \partial q \int \partial u \left( \frac{p^{2}}{2} \right)^{2} du$ <br>differential equation of order 1 that must be satisfied by<br>es auxiliary equations<br> $\frac{dp}{\partial F} = \frac{dq}{\partial F} = \frac{du}{\partial F} = \frac{du}{\partial F} = \frac{du}{\partial F} = \frac{du}{\partial F}$ 

*F F F F F F F F p q p q x u y u p q p q* = = = = + + − − − − ( ) 1.2.15 Eqns 1.2.15 are known as Charpit's auxiliary equations. Any integral of 1..2.15 involving or or both *p q*

 $(1.2.15)$  are known as Charpit's auxiliary equations. Any integral of  $(1..2.15)$  $(1.2.3)$  $\frac{\partial F}{\partial x} + p \cdot \frac{\partial F}{\partial u} = \frac{\partial F}{\partial y} + q \cdot \frac{\partial F}{\partial u} = -p \cdot \frac{\partial F}{\partial p} - q \cdot \frac{\partial F}{\partial q} = -\frac{\partial F}{\partial p} = -\frac{\partial F}{\partial q}$ <br>Eqns (1.2.15) are known as Charpit's auxiliary equations. Any integral of<br>is taken for the required second relation  $(1.2.3)$  $(1.2.3)$  p and q are determined from  $(1.2.1) - (1.2.3)$  and the values substituted into  $(1.2.2)$ (1.2.15)<br>(1..2.15) involving *p* or *q* c<br>on of these is taken as (1.2.3  $\partial x$   $\partial u$   $\partial y$   $\partial u$   $\partial p$   $\partial q$   $\partial p$   $\partial q$ <br>
Eqns (1.2.15) are known as Charpit's auxiliary equations. Any integral of (1..2.15) involving p or q or bot<br>
is taken for the required second relation (1.2.3). In fact the which on integration we obtain the required complete solution of the given differential equation.

#### 1.3.2 JACOBI'S METHOD,

1.3.2 JACOBI'S METHOD,<br>In the last section we discussed the Charpit's method for solving a PDE involvariables  $x_1$  and  $x_2$  (say). The present method (Jacobi's) is quite similar. It is<br>the following very important theog In the last section we discussed the Charpit's method for solving a PDE involving two independent variables  $x_1$  and  $x_2$  (say). The present method (Jacobi's) is quite similar. It is expedient here to recall<br>the following very important theoerem in differential calculus:<br>*Theorem* 1.2<br>If the functions  $\psi_j(x_1, x_2, x_$  expedient here to recall the following very important theoerem in differential calculus:

#### 1.2 *Theorem*

 then Theorem 1.2<br>
If the functions  $\psi_j(x_1, x_2, x_3)$ ,  $(j = 1(1)3)$  posess continuous partial first derivative<br>
then<br>  $\psi_1 dx_1 + \psi_2 dx_2 + \psi_3 dx_3$  (1.2.16)<br>
s an exact differential equation iff

$$
\psi_1 \, dx_1 + \psi_2 \, dx_2 + \psi_3 \, dx_3 \tag{1.2.16}
$$

is an exact differential equation iff

then  
\n
$$
\psi_1 dx_1 + \psi_2 dx_2 + \psi_3 dx_3
$$
\n
$$
\text{is an exact differential equation iff}
$$
\n
$$
\frac{\partial \psi_2}{\partial x_3} - \frac{\partial \psi_3}{\partial x_2} = 0, \quad \frac{\partial \psi_2}{\partial x_1} - \frac{\partial \psi_1}{\partial x_2} = 0, \quad \frac{\partial \psi_1}{\partial x_3} - \frac{\partial \psi_3}{\partial x_1} = 0 \tag{1.2.17}.
$$
\nSuppose we have a differential equation

\n
$$
f(x, y, u, p, q) = 0 \tag{1.2.18}
$$
\nexplicitly involving the independent variable  $u$ . We shall prove that (1.2.18) can be

Suppose we have a differential equation

$$
f(x, y, u, p, q) = 0 \t\t(1.2.18)
$$

explicitly involving the independent variable u. We sh  $\frac{2.18}{\partial x_1}$  = 0 (1.2.17).<br>
(1.2.17).<br>
(1.2.18) can be transformed into another<br>
hich does not explicitly occur and the number of differential equation with a new dependent variable which does not explicitly occur and the number of independent variables increased by unity in the process. differential equation with a new dependent variables increased by u<br>We shall rename the variables as foll<br> $x = x_1, y = x_e, u = x_3$ <br>and introduce a new variable  $y = y(x)$ tion with a new dep<br>
ables increased by<br> *x* the variables as fo<br>  $x = x_1, y = x_e, u = x_0$ <br>
new variable  $y = y(x)$ on with a new dependent variable<br>ples increased by unity in the pro<br>he variables as follows:<br>=  $x_1$ ,  $y = x_e$ ,  $u = x_3$ 

We shall rename the variables as follows:

independent variables increased by unity in the process.  
\nWe shall rename the variables as follows:  
\n
$$
x = x_1, y = x_e, u = x_3
$$
  
\nand introduce a new variable  $v = v(x, y, u)$   
\nwe now consider the relation  
\n $v(x, y, u) = 0$   
\n $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial v}{\partial y}$ 

we now consider the relation

$$
v(x, y, u) = 0 \t\t(1.2.20)
$$

and introduce a new variable 
$$
v = v(x, y, u)
$$
 ]  
\nwe now consider the relation  
\n
$$
v(x, y, u) = 0
$$
\nBy assuming  $p_1 = \frac{\partial v}{\partial x_1}, p_2 = \frac{\partial v}{\partial x_2}, p_3 = \frac{\partial v}{\partial x_3}, (1.2.20)$  yields  
\n
$$
\frac{\partial v}{\partial x} + \frac{\partial v}{\partial u} \frac{\partial u}{\partial x} = 0
$$
\n
$$
\frac{\partial v}{\partial y} + \frac{\partial v}{\partial u} \frac{\partial u}{\partial y} = 0
$$
\ni.e,  $p = -\frac{p_1}{p_3}$  and  $q = -\frac{p_2}{p_3}$   
\nThus,  $v = 0$  will be a solution to (1.2.18) iff

 $(1.2.18)$ Thus,  $v = 0$  w.<br>Eqn  $(1.2.22)$  i =

*i.e,* 
$$
p = -\frac{p_1}{p_3}
$$
 and  $q = -\frac{p_2}{p_3}$   
\nThus,  $v = 0$  will be a solution to (1.2.18) iff  
\n
$$
f\left(x_1, x_2, x_3, -\frac{p_1}{p_3}, -\frac{p_2}{p_3}\right) = 0 \qquad (1.2.22)
$$
\nEqn (1.2.22) is an equation of the form  
\n $G(x_1, x_2, x_3, p_1, p_2, p_3) = 0 \qquad (1.2.23)$   
\nClearly this is a *PDE* in three independent variables x

 $(1.2.22)$ s an equation of the form

an equation of the form  

$$
G(x_1, x_2, x_3, p_1, p_2, p_3) = 0
$$
 (1.2.23)

Eqn (1.2.22) is an equation of the form<br>  $G(x_1, x_2, x_3, p_1, p_2, p_3) = 0$  (1.2.23)<br>
Clearly, this is a *PDE* in three independent variables  $x_1, x_2, x_3$  that does not explicitly involve the depen-<br>
dent variable v which en dent variable  $\nu$  which ends the proof. Clearly, this is a *PDE* in three independent variables  $x_1, x_2, x_3$  that does not explicitly involve the dependent variable  $\nu$  which ends the proof.<br>This method applies to *PDE* of the form (1.2.23) whose central idea

 $(1.2.23)$ *PDE*

relations of the form

relations of the form  
\n
$$
G_2(x_1, x_2, x_3, p_1, p_2, p_3, a) = 0
$$
\n(1.2.24)  
\n
$$
G_2(x_1, x_2, x_3, p_1, p_2, p_3, b) = 0
$$
\n(1.2.25)

relations of the form  
\n
$$
G_2(x_1, x_2, x_3, p_1, p_2, p_3, a) = 0
$$
\n
$$
G_3(x_1, x_2, x_3, p_1, p_2, p_3, b) = 0
$$
\n(1.2.24)  
\n
$$
D_3(x_1, x_2, x_3, p_1, p_2, p_3, b) = 0
$$
\n(1.2.25)

relations of the form  
\n
$$
G_2(x_1, x_2, x_3, p_1, p_2, p_3, a) = 0
$$
\n
$$
G_3(x_1, x_2, x_3, p_1, p_2, p_3, b) = 0
$$
\n
$$
P_j = \psi_j(x_1, x_2, x_3, a, b), (j = 1(1)3)
$$
\nand such that  $p_1 dx_1 + p_2 dx_2 + p_3 dx_3$  becomes exact DE when  $p_j = \psi_j$ .  
\nWhenever such function G, G, can be determined then there exists  $\phi(x, x, x, a, b)$  such that

and such that  $p_1 dx_1 + p_2 dx_2 + p_3 dx_3$  becomes exact DE when  $p_i = \psi_i$ .  $=\psi$ 

 $G_3(x_1, x_2, x_3, p_1, p_2, p_3, b) = 0$  (1.2.25)<br>  $p_j = \psi_j(x_1, x_2, x_3, a, b), (j = 1(1)3)$  (1.2.26)<br>
and such that  $p_1 dx_1 + p_2 dx_2 + p_3 dx_3$  becomes exact DE when  $p_j = \psi_j$ .<br>
Whenever such function  $G_2, G_3$  can be determined then there  $\phi$ 

$$
\frac{\partial \phi}{\partial x_1} = \psi_1
$$
\n
$$
\frac{\partial \phi}{\partial x_2} = \psi_2
$$
\n
$$
\frac{\partial \phi}{\partial x_3} = \psi_3
$$
\nthen with  $p_j = \phi_j$  the DE  $p_1 dx_1 + p_2 dx_2 + p_3 dx_3 - dv = 0$  becomes  $d\phi - dv = 0$  which then give\n
$$
\phi - v = A
$$
\n(1.2.28)\n2.29

 $(1.2.28)$ ields  $\phi - v = A$  (1.2.28)  $\begin{aligned}\n\frac{\partial x_3}{\partial x_3} &\xrightarrow{0} \\
\text{then with } p_j = \phi_j \text{ the DE } p_1 dx_1 + p_2 dx_2 + p_3 dx_3 \\
\phi - v = A\n\end{aligned}$ Observe that from (1.2.28) we get back (1.2.27

$$
\phi - v = A
$$
\n(1.2.28)  
\nObserve that from (1.2.28) we get back (1.2.27)  
\n
$$
p_1 = \frac{\partial \phi}{\partial x_1}, p_2 = \frac{\partial \phi}{\partial x_2}, p_3 = \frac{\partial \phi}{\partial x_3}
$$
\ni.e.,  
\n
$$
p_1 = \frac{\partial \phi}{\partial x_1} = \psi_1, p_2 = \frac{\partial \phi}{\partial x_2} = \psi_2, p_3 = \frac{\partial \phi}{\partial x_3} = \psi_3
$$
\n(1.2.29)  
\nSince by hypothesis  $p_j = \psi_j$  constitute a solution (1.2.23), (1.2.24), (1.2.25) for  $p_1, p_2, p_3$   
\n
$$
y_1 = \phi_1
$$

 $(1.2.23), (1.2.24), (1.2.25)$  $=\psi$  $v = \phi - A$  is a solution of (1.2.23) which contains three arbitrary constants a, b, c therefore it is a complete integral of  $(1.2.23)$ . 1.2.29)<br> $P_1, P_2, P_3$  we observe that  $p_1 = \frac{\partial \phi}{\partial x_1} = \psi_1, p_2 = \frac{\partial \phi}{\partial x_2} = \psi_2, p_3 = \frac{\partial \phi}{\partial x_3} = \psi_3$  <br>hypothesis  $p_j = \psi_j$  constitute a solution (1.2.23),(1.2.24),(1.2.25) for  $p_1, p_2, p_3$  we observe that is a complete is a complete of (1.2.23) which Since by hypothesis<br>  $v = \phi - A$  is a solution<br>
integral of (1.2.23). Since by hypothesis  $p_j = \psi_j$  constitute a solution (1.2.23),  $\nu = \phi - A$  is a solution of (1.2.23) which contains three arbititing the original *PDE* is (1.2.18) we identify (1.2.22) and (1.19) Hence  $\nu = 0$  ( $\phi = A$ ) is a s *ie*,  $p_1 = \frac{\partial \phi}{\partial x_1} = \psi_1, p_2 = \frac{\partial \phi}{\partial x_2} = \psi_2, p_3 = \frac{\partial \phi}{\partial x_3} = \psi_3$  <br>
Since by hypothesis  $p_j = \psi_j$  constitute a solution  $(1.2.23), (1.2.24), (1.2.25)$  for  $v = \phi - A$  is a solution of  $(1.2.23)$  which contains three ar (1.2.24), (1.2.25) for  $p_1$ ,  $p_2$ ,  $p_3$  we observe that<br>itrary constants a, b, c therefore it is a complete<br>.2.23) so that  $v = \phi - A$  is a solution of (1.2.22). for  $p_1$ ,  $p_2$ ,  $p_3$  we obto<br>a, b, c therefore it is a contract is a contract of  $= \phi - A$  is a solution of  $\phi$ 

 $(1.2.18)$  we identify  $(1.2.22)$ *PDE* is  $(1.2.18)$  we identify  $(1.2.22)$  and  $(1.2.23)$  so that  $v = \phi - A$  is a solution of  $(1.2.22)$  $(\phi = A)$  is a solution of  $(1.2.18)$ complete integrals of  $(1.2.18)$  with a and b arbitrary constants.  $\nu = \phi - A$  is a solution of (1.2.23) which contains three arbitrary constants a, b, c therefore it is a comple-<br>integral of (1.2.23).<br>If the original *PDE* is (1.2.18) we identify (1.2.22) and (1.2.23) so that  $\nu = \phi - A$  is integral of (1.2.23).<br>If the original *PDE* is (1.2.18) we identify (1.2.22) and (1.2.2<br>Hence,  $v = 0$  ( $\phi = A$ ) is a solution of (1.2.18). This implies the<br>complete integrals of (1.2.18) with a and b arbitrary constants.<br>1 *A* is a solution of (1.2.23) which contains three arbitrary constants *a*,*b*,*c* t of (1.2.23).<br> *v* and (1.2.23) *v* and (1.2.22) and (1.2.23) so that  $v = \phi - A$ <br>  $v = 0$  ( $\phi = A$ ) is a solution of (1.2.18). This implies t dentify<br>*a* and *b*<br> $F$  FING  $\phi$ is a solution of (1.2.23) which contains three arbitrary constants a, b, c therefore it is a con<br>
f (1.2.23).<br>
ginal *PDE* is (1.2.18) we identify (1.2.22) and (1.2.23) so that  $v = \phi - A$  is a solution of (<br>  $= 0$  ( $\phi = A$ )  $\frac{1}{2}$  &  $G_3$ Hence,  $v = 0$  ( $\phi = A$ ) is a solution of (1.2.18). This implies that  $\phi = A$  gives an  $A$  – parameter family of<br>complete integrals of (1.2.18) with a and b arbitrary constants.<br>1.3.2.1 DETERMINATION OF THE FUNCTIONS  $G_2 \& G$ f the original *PDE* is (1.2.18) we identify (1.2.22) and (1.2<br>
Hence,  $v = 0$  ( $\phi = A$ ) is a solution of (1.2.18). This implies t<br>
omplete integrals of (1.2.18) with *a* and *b* arbitrary constant<br>
.3.2.1 DETERMINATION OF is a solution of (1.2.18). This implies that  $\phi = A$ <br>
(1.2.18) with *a* and *b* arbitrary constants.<br>
FION OF THE FUNCTIONS  $G_2 \& G_3$ .<br>  $G_2 \& G_3$  are such that we can solve for  $p_1, p_2, p_3$ <br>
become identities if n are r

1

 $(1.2.23), (1.2.24)$  and  $(1.2.25)$  $(1.2.26)$ . Then they become identities if  $p_i$  are replaced with  $\psi_i$  so that their partial derivatives wrt complete integrals of (1.2.18) with *a* and *b* arbitrary constant 1.3.2.1 DETERMINATION OF THE FUNCTIONS  $G_2 \& G_3$ <br>Suppose the functions  $G_2 \& G_3$  are such that we can solve for in (1.2.26). Then they become identities vanish independently. Hence, from  $(1.2.24)$  and  $(1.2.25)$  we have 1.3.2.1 DETERMINATION OF THE FUNCTIONS  $G_2 \& G_3$ .<br>Suppose the functions  $G_2 \& G_3$  are such that we can solve for  $p_1$ ,  $p$  in (1.2.26). Then they become identities if  $p_j$  are replaced with  $\psi$  vanish independently. H *,*  $p_2$ *,*  $p_3$  *from*(1.2.23),(1.2.24) and (1.2.2),<br> $\psi_j$  so that their partial derivatives wrt  $x_j$ 

$$
\frac{\partial G_2}{\partial x_1} + \frac{\partial G_2}{\partial p_1} \frac{\partial p_1}{\partial x_1} + \frac{\partial G_2}{\partial p_2} \frac{\partial p_2}{\partial x_1} + \frac{\partial G_2}{\partial p_3} \frac{\partial p_3}{\partial x_1} = 0
$$
\n
$$
\frac{\partial G_3}{\partial x_1} + \frac{\partial G_3}{\partial p_1} \frac{\partial p_1}{\partial x_1} + \frac{\partial G_3}{\partial p_2} \frac{\partial p_2}{\partial x_1} + \frac{\partial G_3}{\partial p_3} \frac{\partial p_3}{\partial x_1} = 0
$$
\n
$$
\frac{\partial G_2}{\partial x_2} + \frac{\partial G_2}{\partial p_1} \frac{\partial p_1}{\partial x_2} + \frac{\partial G_2}{\partial p_2} \frac{\partial p_2}{\partial x_2} + \frac{\partial G_2}{\partial p_3} \frac{\partial p_3}{\partial x_2} = 0
$$
\n
$$
\frac{\partial G_3}{\partial x_2} + \frac{\partial G_3}{\partial p_1} \frac{\partial p_1}{\partial x_2} + \frac{\partial G_3}{\partial p_2} \frac{\partial p_2}{\partial x_2} + \frac{\partial G_3}{\partial p_3} \frac{\partial p_3}{\partial x_2} = 0
$$
\n
$$
\frac{\partial G_2}{\partial x_3} + \frac{\partial G_2}{\partial p_1} \frac{\partial p_1}{\partial x_3} + \frac{\partial G_2}{\partial p_2} \frac{\partial p_2}{\partial x_3} + \frac{\partial G_2}{\partial p_3} \frac{\partial p_3}{\partial x_3} = 0
$$
\n
$$
\frac{\partial G_3}{\partial x_3} + \frac{\partial G_3}{\partial p_1} \frac{\partial p_1}{\partial x_3} + \frac{\partial G_3}{\partial p_2} \frac{\partial p_2}{\partial x_3} + \frac{\partial G_3}{\partial p_3} \frac{\partial p_3}{\partial x_3} = 0
$$
\nEliminating  $\frac{\partial p_1}{\partial x_1}$  from (1.2.30),  $\frac{\partial p_2}{\partial x_2}$  from (1.2.31) and  $\frac{\partial p_3}{\partial x_3}$  from (1.2.32) we obtain

 $(1.2.30), \frac{r_{12}}{2}$  from  $(1.2.31)$  and  $\frac{r_{13}}{2}$  from  $(1.2.32)$  $\partial p_1 \partial x_3 \partial p_2 \partial x_3 \partial p_3 \partial x_3$ <br>
1 (1.2.30),  $\frac{\partial p_2}{\partial x_2}$  from (1.2.31) and  $\frac{\partial p_3}{\partial x_3}$ <br>  $\frac{\partial p_3}{\partial x_3}$ ,  $\frac{\partial q_2}{\partial x_2}$  from (1.2.31) and  $\frac{\partial p_3}{\partial x_3}$ *g*<sub>1</sub>  $\partial p_1$   $\partial x_3$   $\partial p_2$   $\partial x_3$   $\partial p_3$   $\partial x_3$   $\partial p_4$ <br> *G*<sub>2</sub>, *G*<sub>3</sub>),  $\frac{\partial p_2}{\partial x_2}$  from (1.2.31) and  $\frac{\partial p_3}{\partial x_3}$ <br> *G*<sub>2</sub>, *G*<sub>3</sub>) +  $\frac{\partial (G_2, G_3)}{\partial (R_2, R_1)} \frac{\partial p_2}{\partial x_1} + \frac{\partial (G_2, G_3)}{\partial (R_2, R_1)} \frac{\partial p$  $\partial x_3$   $\partial p_1$   $\partial x_3$   $\partial p_2$   $\partial x_3$   $\partial p_3$   $\partial x_3$   $\Big)$ <br>
from (1.2.30),  $\frac{\partial p_2}{\partial x_2}$  from (1.2.31) and  $\frac{\partial p_3}{\partial x_3}$  from<br>  $\frac{\partial (G_2, G_3)}{\partial (x_1, x_1)} + \frac{\partial (G_2, G_3)}{\partial (x_2, x_2)} \frac{\partial p_2}{\partial x_2} + \frac{\partial (G_2, G_3)}{\partial (x_1,$ 

$$
\frac{\partial p_1}{\partial x_1} \text{ from } (1.2.30), \frac{\partial p_2}{\partial x_2} \text{ from } (1.2.31) \text{ and } \frac{\partial p_3}{\partial x_3} \text{ from } (1.2.32) \text{ we can}
$$
\n
$$
\frac{\partial (G_2, G_3)}{\partial (x_1, p)} + \frac{\partial (G_2, G_3)}{\partial (p_2, p_1)} \frac{\partial p_2}{\partial x_1} + \frac{\partial (G_2, G_3)}{\partial (p_3, p_1)} \frac{\partial p_3}{\partial x_1} = 0
$$
\n
$$
\frac{\partial (G_2, G_3)}{\partial (x_2, p)} + \frac{\partial (G_2, G_3)}{\partial (p_3, p_3)} \frac{\partial p_3}{\partial x_2} + \frac{\partial (G_2, G_3)}{\partial (p_1, p_2)} \frac{\partial p_1}{\partial x_2} = 0
$$
\n
$$
\frac{\partial (G_2, G_3)}{\partial (x_3, p_3)} + \frac{\partial (G_2, G_3)}{\partial (p_1, p_3)} \frac{\partial p_1}{\partial x_3} + \frac{\partial (G_2, G_3)}{\partial (p_2, p_3)} \frac{\partial p_2}{\partial x_3} = 0
$$
\n
$$
\text{Recall that}
$$
\n
$$
\frac{\partial (G_2, G_3)}{\partial (x_3, p_3)} = -\frac{\partial (G_2, G_3)}{\partial (x_3, p_3)} \frac{\partial p_1}{\partial x_3} + \frac{\partial (G_2, G_3)}{\partial (p_2, p_3)} \frac{\partial p_2}{\partial x_3} = 0
$$
\n
$$
(1.2.34)
$$

Recall that

$$
\frac{\partial (G_2, G_3)}{\partial (x_3, p_3)} + \frac{\partial (G_2, G_3)}{\partial (p_1, p_3)} \frac{\partial p_1}{\partial x_3} + \frac{\partial (G_2, G_3)}{\partial (p_2, p_3)} \frac{\partial p_2}{\partial x_3} = 0
$$
\nRecall that\n
$$
\frac{\partial (G_2, G_3)}{\partial (x_k, p_j)} = -\frac{\partial (G_2, G_3)}{\partial (x_j, p_k)}
$$
\nUsing (1.2.34) in (1.2.33) yields\n
$$
\frac{\partial (G_2, G_3)}{\partial (x_n, p_1)} + \frac{\partial (G_2, G_3)}{\partial (
$$

$$
\frac{\partial(\mathbf{G}_2, \mathbf{G}_3)}{\partial(x_k, p_j)} = -\frac{\partial(\mathbf{G}_2, \mathbf{G}_3)}{\partial(x_k, p_k)}
$$
(1.2.34)  
Using (1.2.34) in (1.2.33) yields  

$$
\frac{\partial(G_2, G_3)}{\partial(x_k, p_1)} + \frac{\partial(G_2, G_3)}{\partial(x_k, p_2)} + \frac{\partial(G_2, G_3)}{\partial(x_k, p_3)} + \frac{\partial(G_2, G_3)}{\partial(p_2, p_3)} \left(\frac{\partial p_2}{\partial x_3} - \frac{\partial p_3}{\partial x_2}\right) + \frac{\partial(G_2, G_3)}{\partial(p_3, p_1)} \left(\frac{\partial p_3}{\partial x_1} - \frac{\partial p_1}{\partial x_3}\right)
$$

$$
+ \frac{\partial(G_2, G_3)}{\partial(p_1, p_2)} \left(\frac{\partial p_1}{\partial x_2} - \frac{\partial p_2}{\partial x_1}\right) = 0
$$
  
ie,  

$$
\frac{\partial(G_2, G_3)}{\partial(p_1, p_2)} \cdot L + \frac{\partial(G_2, G_3)}{\partial(p_1, p_3)} \cdot M + \frac{\partial(G_2, G_3)}{\partial(p_2, p_3)} \cdot N = -(G_2, G_3) \text{ where } L = \left(\frac{\partial p_2}{\partial x} - \frac{\partial p_3}{\partial x}\right),
$$

, *ie*

$$
+\frac{\partial (G_2, G_3)}{\partial (p_1, p_2)} \left( \frac{\partial p_1}{\partial x_2} - \frac{\partial p_2}{\partial x_1} \right) = 0
$$
  
\n*i.e,*  
\n
$$
\frac{\partial (G_2, G_3)}{\partial (p_2, p_3)} \cdot L + \frac{\partial (G_2, G_3)}{\partial (p_3, p_1)} \cdot M + \frac{\partial (G_2, G_3)}{\partial (p_1, p_2)} \cdot N = -(G_2, G_3) \text{ where } L = \left( \frac{\partial p_2}{\partial x_3} - \frac{\partial p_3}{\partial x_2} \right),
$$
  
\n
$$
M = \left( \frac{\partial p_3}{\partial x_1} - \frac{\partial p_1}{\partial x_3} \right), N = \left( \frac{\partial p_1}{\partial x_2} - \frac{\partial p_2}{\partial x_1} \right), (G_2, G_3) = \frac{\partial (G_2, G_3)}{\partial (x_1, p_1)} + \frac{\partial (G_2, G_3)}{\partial (x_2, p_2)} + \frac{\partial (G_2, G_3)}{\partial (x_3, p_3)} \right)
$$
(1.2.35)

Similar computation gives

Similar computation gives  
\n
$$
\frac{\partial (G_3, G_1)}{\partial (p_2, p_3)} \cdot L + \frac{\partial (G_3, G_1)}{\partial (p_3, p_1)} \cdot M + \frac{\partial (G_3, G_1)}{\partial (p_1, p_2)} \cdot N = -(G_3, G_1)
$$
\n
$$
\frac{\partial (G_1, G_2)}{\partial (p_2, p_3)} \cdot L + \frac{\partial (G_1, G_2)}{\partial (p_3, p_1)} \cdot M + \frac{\partial (G_1, G_2)}{\partial (p_1, p_2)} \cdot N = -(G_1, G_2)
$$
\n(1.2.37)  
\nSuppose now that the solutions  $p_j = \psi_j$  make the expression  $p_1 dx_1 + p_2 dx_2 + p_3 dx_3 = 0$  and exact differ-  
\nential then  $\Rightarrow I = 0$  M = 0 and N = 0 identically. Then from Eq. (1.2.35), (1.2.36), and (1.2.37), we get

 $(1.2.35), (1.2.36)$  and  $(1.2.37)$  $(G_2, G_3) = 0$ ,  $(G_3, G_1) = 0$  and  $(G_1, G_2)$  $(1.2.37)$ <br>  $p_1 dx_1 + p_2 dx_2 + p_3 dx_3$ al then  $\Rightarrow$   $L = 0, M = 0$  and  $N = 0$  io<br>  $(Z_2, G_3) = 0, (G_3, G_1) = 0$  and  $(G_1, G_2)$  $p_i = \psi_i$  make the expression  $p_1 dx_1 + p_2 dx_2 + p_3 dx_3 = 0$  and exact differ- $\frac{\partial (G_1, G_2)}{\partial (p_2, p_3)} \cdot L + \frac{\partial (G_1, G_2)}{\partial (p_3, p_1)} \cdot M + \frac{\partial (G_1, G_2)}{\partial (p_1, p_2)} \cdot N = -(G_1, G_2)$  (1.2.37)<br>Suppose noew that the solutions  $p_j = w_j$  make the expression  $p_1 dx_1 + p_2 dx_2 + p_3 dx_3 = 0$  and exact differential then  $\$  $\partial (p_2, p_3)$   $\partial (p_3, p_1)$   $\partial (p_1, p_2)$ <br>
Suppose noew that the solutions  $p_j = \psi_j$  make<br>
ential then  $\Rightarrow L = 0, M = 0$  and  $N = 0$  identica<br>  $(G_2, G_3) = 0$ ,  $(G_3, G_1) = 0$  and  $(G_1, G_2) = 0$ .<br>
Hence  $Z = G$  and  $Z = G$  are two solut  $\frac{\partial (G_1, G_2)}{\partial (p_3, p_1)} \cdot M + \frac{\partial (G_1, G_2)}{\partial (P_1, P_2)}$ <br>that the solutions p<br> $L = 0, M = 0$  and N  $p_2, p_3$   $\partial (p_3, p_1)$   $\partial (p_1, p_2)$ <br>ppose noew that the solutions  $p_j = \psi$ <br>tial then  $\Rightarrow L = 0, M = 0$  and  $N = 0$  i<br> $G_2, G_3$   $= 0$ ,  $(G_3, G_1) = 0$  and  $(G_1, G_2)$  $\vec{r}_3$ ) = 0<br> $\vec{r}_2$   $\vec{r}_3$ ) = 0<br> $\vec{z}$  = *G*  $L + \frac{\partial (G_1, G_2)}{\partial (p_3, p_1)} \cdot M + \frac{\partial (G_1, G_2)}{\partial (p_1, p_2)} \cdot N = -(G_1)$ <br>ew that the solutions  $p_j = \psi_j$  make the ex<br> $\Rightarrow L = 0, M = 0$  and  $N = 0$  identically. Th  $\partial (p_3, p_1)$   $\partial (p_1, p_2)$   $\partial (p_1, p_2)$   $\partial (p_1, p_2)$ <br>
bew that the solutions  $p_j = \psi_j$  make the expression<br>  $\Rightarrow L = 0, M = 0$  and  $N = 0$  identically. Then from<br>  $= 0, (G_3, G_1) = 0$  and  $(G_1, G_2) = 0$ . = 0 and  $(G_1, G_2)$  = 0.<br>3 are two solutions of the *PDE*,  $(Z, G_1)$ nat the solutions  $p_j = \psi_j$  make the expression  $p_1 dx_1 +$ <br>  $= 0, M = 0$  and  $N = 0$  identically. Then from Eqn (1...<br>  $(G_3, G_1) = 0$  and  $(G_1, G_2) = 0$ .<br>
nd  $Z = G_3$  are two solutions of the *PDE*,  $(Z, G_1) = 0$  $L = 0, M = 0$  and  $N = 0$ <br>  $(G_3, G_1) = 0$  and  $(G_1, G_2)$ <br>
and  $Z = G_3$  are two solu<br>  $\frac{G_1}{G_1} + \frac{\partial (Z, G_1)}{\partial (x_1, p_2)} + \frac{\partial (Z, p_3)}{\partial (x_2, p_3)}$ the solutions  $p_j = \psi_j$  make the expression  $p_j$ <br>  $D, M = 0$  and  $N = 0$  identically. Then from Eq. ,  $G_1$   $= 0$  and  $(G_1, G_2) = 0$ .<br>  $Z = G_3$  are two solutions of the *PDE*,  $(Z, G)$  $Z \Rightarrow L = 0, M = 0 \text{ and } N = 0 \text{ id}$ <br>  $Q_2$ ,  $Q_3$ ,  $G_1$  = 0 and  $Q_1$ ,  $G_2$ <br>  $Q_3$  and  $Z = G_3$  are two solutions<br>  $Z, G_1$  +  $\frac{\partial (Z, G_1)}{\partial (x_1, p_2)} + \frac{\partial (Z, G_2)}{\partial (x_2, p_3)}$ e solutions  $p_j = \psi_j$  make the expression  $p_1 dx_1 + p_2 dx_2 + p_3 dx_3$ <br>  $M = 0$  and  $N = 0$  identically. Then from Eqn (1.2.35), (1.2.36)<br>  $G_i$  = 0 and ( $G_i$ ,  $G_2$ ) = 0.<br>  $G_i$  =  $G_3$  are two solutions of the *PDE*, (Z,  $G_1$ ) =  $\Rightarrow$  L = 0, M = 0 and N = 0 identically. T<br>
= 0,  $(G_3, G_1) = 0$  and  $(G_1, G_2) = 0$ .<br>
=  $G_2$  and Z =  $G_3$  are two solutions of the B<br>  $\frac{\partial (Z, G_1)}{\partial (x, n)} + \frac{\partial (Z, G_1)}{\partial (x, n)} + \frac{\partial (Z, G_1)}{\partial (x, n)} = 0$ 

Hence,  $Z = G_2$  a  $=G_2$  and  $Z = G_3$  are two solutions of the PDE,  $(Z, G_1)$ 

$$
(G_2, G_3) = 0, (G_3, G_1) = 0 \text{ and } (G_1, G_2) = 0.
$$
  
\nHence,  $Z = G_2$  and  $Z = G_3$  are two solutions of the *PDE*,  $(Z, G_1) = 0$   
\n
$$
\frac{\partial (Z, G_1)}{\partial (x_1, p_1)} + \frac{\partial (Z, G_1)}{\partial (x_2, p_2)} + \frac{\partial (Z, G_1)}{\partial (x_3, p_3)} = 0
$$
  
\ni.e, 
$$
\frac{\partial Z}{\partial x_1} \frac{\partial G_1}{\partial p_1} - \frac{\partial Z}{\partial p_1} \frac{\partial G_1}{\partial x_1} + \frac{\partial Z}{\partial x_2} \frac{\partial G_1}{\partial p_2} - \frac{\partial Z}{\partial p_2} \frac{\partial G_1}{\partial x_2} + \frac{\partial Z}{\partial x_3} \frac{\partial G_1}{\partial p_3} - \frac{\partial Z}{\partial p_3} \frac{\partial G_1}{\partial x_3} = 0
$$
  
\nBut we must have  
\n $(G_2, G_3) = 0$  (1.2.39)  
\nObserve that (1.70) is a first order *PDE* in the independent variable  $x$ ,  $p = (1-1)(1/2)$  with

But we must have

, *ie*

we must have  
\n
$$
(G_2, G_3) = 0
$$
 (1.2.39)

 $(1.79)$  $\partial x_1 \partial p_1 \partial p_1 \partial x_1$ <br>
But we must have<br>  $(G_2, G_3) = 0$ <br>
Observe that (1.79) is a first o  $(j = 1(1)3)$ ry equations<br>  $\frac{1}{2} = \frac{dx_2}{\partial G} = \frac{dx_3}{\partial G} = \frac{dp_1}{\partial G} = \frac{dp_2}{\partial G} = \frac{dp_3}{\partial G}$  $\partial x_2 \, \partial p_2 \, \partial z_2 \, \partial x_3 \, \partial p_3 \, \partial p_3 \, \partial x_3$  (1.2.39)<br>
rder *PDE* in the independent variable  $x_j$ ,  $p_j$  ( $j = 1(1)3$ ) with corresponding auxiliary equations bserve that (1.79) is a first order *PDE* in the independent variable  $x_j$ ,  $p_j$  ( $j = g$  auxiliary equations<br>
,  $\frac{dx_1}{\frac{\partial G_1}{\partial t}} = \frac{dx_2}{\frac{\partial G_1}{\partial t}} = \frac{dx_3}{\frac{\partial G_1}{\partial t}} = \frac{dp_1}{-\frac{\partial G_1}{\partial t}} = \frac{dp_2}{-\frac{\partial G_1}{\partial t}} = \frac{dp_3}{-\frac$  $\partial p_2$   $\partial p_2$   $\partial x_2$   $\partial x_3$   $\partial p_3$   $\partial p_3$   $\partial x_3$ <br>(1.2)<br>*PDE* in the independent variable  $x_j$ ,  $p_j$  (*j*  $(G_2, G_3) = 0$ <br>
Observe that (1.79) is a first order *PDE* in the independent<br>
ing auxiliary equations<br> *ie*,  $\frac{dx_1}{\partial G_1} = \frac{dx_2}{\partial G_1} = \frac{dx_3}{\partial G_1} = \frac{dp_1}{\partial G_1} = \frac{dp_2}{\partial G_1} = \frac{dp_3}{\partial G_1} = \frac{dZ}{\partial G_1}$ 

(1.2.3  
\nObserve that (1.79) is a first order *PDE* in the independent variable 
$$
x_j
$$
,  $p_j$  ( $j =$   
\ning auxiliary equations  
\n
$$
ie, \qquad \frac{dx_1}{\frac{\partial G_1}{\partial p_1}} = \frac{dx_2}{\frac{\partial G_1}{\partial p_2}} = \frac{dx_3}{\frac{\partial G_1}{\partial p_3}} = \frac{dp_1}{-\frac{\partial G_1}{\partial x_1}} = \frac{dp_2}{-\frac{\partial G_1}{\partial x_2}} = \frac{dp_3}{-\frac{\partial G_1}{\partial x_3}} = \frac{dZ}{0}
$$
(1.2.40)  
\nThe coupled *ODEs* above are the *Jacobi's* auxiliary differential equations.  
\n1.3.1 **SUCCESS OF JACOBI'S METHOD**

The coup

#### 1.3.1 SUCCESS OF JACOBI'S METHOD

The coupled *ODEs* above are the *Jacobi's* auxiliary differential equations.<br>
1.3.1 SUCCESS OF JACOBI'S METHOD<br>
We show here that if  $G_2 = 0$  and  $G_3 = 0$  are two independent integrals of the eqn (1.2.39) and are such<br>
t th how here that if  $G_2 = 0$  and  $G_3 = 0$  are two independent integrals of the eqn (1.2.39) and (*i*)  $(G_2, G_3) = 0$  and (*ii*)  $p_1, p_2, p_3$  are solvable from (1.2.23),(1.2.24),(1.2.25) (1.2.26) ,  $p_3$  are solvable fro<br>  $\int_1^2 dx_1 + p_2 dx_2 + p_3 dx_3$ at coupled *ODEs* above are the *Jacobi's* auxiliary differential equations.<br>  $\therefore$  SUCCESS OF JACOBI'S METHOD<br>  $\Rightarrow$  show here that if  $G_2 = 0$  and  $G_3 = 0$  are two independent integrals of the eqn(1.2.39) and are such<br>
a equations will render the expression  $p_1 dx_1 + p_2 dx_2 + p_3 dx_3$  an exact differential. **FHOD**<br>*p*<sub>2</sub>, *p*<sub>3</sub> are solvable from  $p_1 dx_1 + p_2 dx_2 + p_3 dx_3$ <br>**p**<sub>1</sub> dx  $p_1 dx_1 + p_2 dx_2 + p_3 dx_3$ if  $\alpha = 0$  and  $Z = c$ <br>*Z* = *c* = 21 are two independent integrals are solvable from  $(1.2.23)$ <br>+  $p_2 dx_2 + p_3 dx_3$  an exact d an exact differently and  $Z = G_3$  $G_3 = 0$  are two independent integrals of the eqn (1.2.39) and are such  $p_2$ ,  $p_3$  are solvable from (1.2.23),(1.2.24),(1.2.25) (1.2.26) then these  $p_1 dx_1 + p_2 dx_2 + p_3 dx_3$  an exact differential.<br>al of (1.2.39) so  $Z = G_2$ = 0 and (*ii*)  $p_1$ ,  $p_2$ ,  $p_3$  are solvable<br>ler the expression  $p_1 dx_1 + p_2 dx_2 + p_3$ <br> $Z = c$  is an integral of (1.2.39) so 2<br>= 0 and ( $G_3$ ,  $G_1$ ) = 0.

First, we note that  $Z = c$  is an integral of  $(1.2.39)$  so  $Z = G_2$  and  $Z = G_3$  are two solutions of  $(1.2.38)$  $(G_2, G_1) = 0$  and  $(G_3, G_1)$ that (*i*)  $(G_2, G_3) = 0$  and (*ii*)  $p_1, p_2, p_3$ <br>equations will render the expression  $p_1d$ <br>First, we note that  $Z = c$  is an integral of<br>we have  $(G_2, G_1) = 0$  and  $(G_3, G_1) = 0$ .  $\binom{1}{2} =$ equations will render the expression  $p_1 dx_1 + p_2 dx_2 + p_3 dx_3$  an exact differential.<br>First, we note that  $Z = c$  is an integral of (1.2.39) so  $Z = G_2$  and  $Z = G_3$  are two sol<br>we have  $(G_2, G_1) = 0$  and  $(G_3, G_1) = 0$ .<br>Consequen that  $Z = c$  is an in<br>
,  $G_1$ ) = 0 and  $(G_3, G_4)$ <br>
on the hypothesis (<br>  $\left(\frac{G_2, G_3}{G_2}, \frac{G_4}{G_3}\right)$  .  $L + \frac{\partial (G_2, G_3)}{\partial (R_2, G_4)}$  $(G_2, G_3) = 0$  and *(ii) p*<br>*will render the expressi*<br>*ote that*  $Z = c$  is an inte<br> $G_2, G_1$  = 0 and  $(G_3, G_1)$ <br>*t* on the hypothesis *(G* sion  $p_1 dx_1 + p_2 dx_2 + p_3 dx_3$  an exact differential.<br> *ttegral* of (1.2.39) so  $Z = G_2$  and  $Z = G_3$  are two solutio<br>  $G_1$ ) = 0.<br>  $G_2, G_3$ ) = 0 the equations in (1.2.35) – (1.2.37) give ote that  $Z = c$  is an integral of (1.3<br>  $G_2, G_1$ ) = 0 and  $(G_3, G_1) = 0$ .<br>
at on the hypothesis  $(G_2, G_3) = 0$  t<br>  $\frac{\partial (G_2, G_3)}{\partial (n_1, n_2)} \cdot L + \frac{\partial (G_2, G_3)}{\partial (n_1, n_2)} \cdot M + \frac{\partial}{\partial (n_2, n_1)}$ tegral of (1.2.39) *s*<br> *G*<sub>1</sub>) = 0.<br> *G*<sub>2</sub>, *G*<sub>3</sub>) = 0 the equ<br> *G*<sub>3</sub>) *M* +  $\frac{\partial (G_2, G_1)}{\partial (R_1, R_2)}$ 

we have 
$$
(G_2, G_1) = 0
$$
 and  $(G_3, G_1) = 0$ .  
\nConsequent on the hypothesis  $(G_2, G_3) = 0$  the equations in  $(1.2.35) - (1.2.37)$  give  
\n
$$
\frac{\partial(G_2, G_3)}{\partial(p_2, p_3)} \cdot L + \frac{\partial(G_2, G_3)}{\partial(p_3, p_1)} \cdot M + \frac{\partial(G_2, G_3)}{\partial(p_1, p_2)} \cdot N = 0
$$
\n
$$
\frac{\partial(G_3, G_1)}{\partial(p_2, p_3)} \cdot L + \frac{\partial(G_3, G_1)}{\partial(p_3, p_1)} \cdot M + \frac{\partial(G_3, G_1)}{\partial(p_1, p_2)} \cdot N = 0
$$
\n
$$
\frac{\partial(G_1, G_2)}{\partial(p_2, p_3)} \cdot L + \frac{\partial(G_1, G_2)}{\partial(p_3, p_1)} \cdot M + \frac{\partial(G_1, G_2)}{\partial(p_1, p_2)} \cdot N = 0
$$
\n(1.2.41)

This is a system of linear homogeneous equations in the unknowns L, M and N with the coefficient determin<br> $|\partial(G, G) - \partial(G, G)|$ *L*, *M* and *N* 

m of linear homogeneous equations in the unknowns *L*, *M* and *N* with\n
$$
\frac{\partial(G_2, G_3)}{\partial(p_2, p_3)} = \frac{\partial(G_2, G_3)}{\partial(p_3, p_1)} = \frac{\partial(G_2, G_3)}{\partial(p_1, p_2)}.
$$
\n
$$
\Delta = \frac{\partial(G_3, G_1)}{\partial(p_2, p_3)} = \frac{\partial(G_3, G_1)}{\partial(p_3, p_1)} = \frac{\partial(G_3, G_1)}{\partial(p_1, p_2)} = \frac{\partial(G_1, G_2)}{\partial(p_1, p_2)} = \frac{\partial(G_1, G_2)}{\partial(p_2, p_3)} = \frac{\partial(G_1, G_2)}{\partial(p_3, p_1)} = \frac{\partial(G_1, G_2)}{\partial(p_1, p_2)}
$$

in which

in which  
\n
$$
\overline{\partial(p_2, p_3)} \quad \overline{\partial(p_3, p_1)} \quad \overline{\partial(p_1, p_2)} \Big|
$$
\n
$$
J = \frac{\partial (G_1, G_2, G_3)}{\partial (p_1, p_2, p_3)} = \begin{vmatrix} \frac{\partial G_1}{\partial p_1} & \frac{\partial G_1}{\partial p_2} & \frac{\partial G_1}{\partial p_3} \\ \frac{\partial G_2}{\partial p_1} & \frac{\partial G_2}{\partial p_2} & \frac{\partial G_2}{\partial p_3} \\ \frac{\partial G_3}{\partial p_1} & \frac{\partial G_3}{\partial p_2} & \frac{\partial G_3}{\partial p_3} \end{vmatrix}
$$
\n
$$
\Rightarrow \Delta = AdjJ = J^2
$$
\nRecall that from our hypothesis *p*, *p*, *p*, are solvable from (1 2.35).

2

 $(1.2.35) - (1.2.37)$  $(1.2.40)$  $P_1, P_2, P_3$ e from  $(1.2.35) - (1.2)$ <br>  $p_1 dx_1 + p_2 dx_2 + p_3 dx_3$  $\begin{vmatrix} \frac{\partial G_3}{\partial p_1} & \frac{\partial G_3}{\partial p_2} & \frac{\partial G_3}{\partial p_3} \end{vmatrix}$ <br>  $\Rightarrow \Delta = AdjJ = J^2$ <br>
Recall that from our hypothesis  $p_1, p_2, p_3$  are solvable from  $(1.2.35) - (1.2.37) \Rightarrow J \neq 0$  *ie*,  $\Delta \neq 0$ . Hence, the<br>system  $(1.2.40)$  gives  $I$  $|\partial p_1 \partial p_2 \partial p_3|$ <br>  $\Rightarrow \Delta = AdjJ = J^2$ <br>
Recall that from our hypothesis  $p_1, p_2, p_3$  are solvable from  $(1.2.35) - (1.2.37) \Rightarrow J \neq 0$  *ie*,  $\Delta \neq 0$ . Hence, the<br>
system  $(1.2.40)$  gives  $L = 0$ ,  $M = 0$ , and  $N = 0 \Rightarrow p_1 dx_1 + p_2 dx_2$ all  $p_i = \psi_i$ . Here lie the success of the Jacobi's method.  $|\partial p_1 \quad \partial p_2 \quad \partial p_3|$ <br> *P*ypothesis  $p_1, p_2, p_3$  are solvable from  $(1.2.35) - (1.2.45)$ <br> *L* = 0, *M* = 0, and *N* = 0  $\Rightarrow$   $p_1 dx_1 + p_2 dx_2 + p_3 dx_3$ <br>
e success of the Iacobi's method  $-(1.2.37) \Rightarrow J \neq 0$  ie,  $\Delta \neq 0$ . Hence, the othesis  $p_1$ ,  $p_2$ ,  $p_3$  are solvable from  $(1.2.35)$  –  $(1.2.37) \Rightarrow J =$ <br>= 0,  $M = 0$ , and  $N = 0 \Rightarrow p_1 dx_1 + p_2 dx_2 + p_3 dx_3$  is an exact the su<br> $a^2 + q^2$ system (1.2.40) gives  $L = 0$ ,  $M = 0$ , and  $N = 0 =$ <br>all  $p_j = \psi_j$ . Here lie the success of the Jacobi's me<br>*Examples*<br>1 Solve the *PDE*:  $p^2 + q^2 - 2px - 2qy + 2xy = 0$ .<br>Solution. *Examples*  $=\psi$  $L = 0$ ,  $M = 0$ , and  $N = 0 \Rightarrow p_1 dx_1 + p_2 dx_2$ <br>
ne success of the Jacobi's method.<br>  $+ q^2 - 2px - 2qy + 2xy = 0$ .

. *Solution*

The corresponding Charpit's auxiliary DE is

PDE: 
$$
p^2 + q^2 - 2px - 2qy + 2xy = 0
$$
.  
\n
$$
\text{anding Charpit's auxiliary } DE \text{ is}
$$
\n
$$
\frac{dp}{\frac{\partial f}{\partial x} + p \frac{\partial f}{\partial u}} = \frac{dq}{\frac{\partial f}{\partial y} + q \frac{\partial f}{\partial u}} = \frac{du}{-p \frac{\partial f}{\partial x} - q \frac{\partial f}{\partial y}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dF}{0}
$$
\n
$$
\frac{dp}{-2p + 2y} = \frac{dq}{-2q + 2x} = \frac{dx}{-(2p - 2x)} = \frac{dy}{-(2q - 2y)}
$$
\n(ii)

, *ie*

 $\left\langle \frac{1}{2} \right\rangle$ 

*ie*

$$
\frac{dy}{dx} + p\frac{dy}{du} - \frac{dy}{dy} - q\frac{dy}{dx} - q\frac{dy}{dy} - \frac{dy}{dp} - \frac{dy}{dq}
$$
  
\n*i.e,*  
\n
$$
\frac{dp}{-2p+2y} = \frac{dq}{-2q+2x} = \frac{dx}{-(2p-2x)} = \frac{dy}{-(2q-2y)}
$$
 (*ii*)  
\n
$$
\Rightarrow \frac{dp + dq}{-2p+2y-2q+2x} = \frac{dx+dy}{-(2p-2x)-(2q-2y)}
$$
 (*iii*)  
\n
$$
\frac{dp + dq}{-2p+2y-2q+2x} = \frac{dx+dy}{-(2p-2x)-(2q-2y)}
$$

$$
\frac{dp}{-2p+2y} = \frac{aq}{-2q+2x} = \frac{ax}{-(2p-2x)} = \frac{ay}{-(2q-2y)}
$$
(ii)  

$$
\frac{dp+dq}{-2p+2y-2q+2x} = \frac{dx+dy}{-(2p-2x)-(2q-2y)}
$$
(iii)  

$$
\frac{dp+dq}{-2(p+q)+2(x+y)} = \frac{dx+dy}{-2(p+q)+2(x+y)}
$$
(iv)

⇒ 
$$
\frac{dp + dq}{-2p + 2y - 2q + 2x} = \frac{dx + dy}{-(2p - 2x) - (2q - 2y)}
$$
 (iii)  
\ni.e, 
$$
\frac{dp + dq}{-2(p + q) + 2(x + y)} = \frac{dx + dy}{-2(p + q) + 2(x + y)}
$$
 (iv)  
\n⇒ 
$$
d(p + q) = d(x + y)
$$
 (v)  
\ni.e, 
$$
p + q = x + y + \alpha
$$

$$
\Rightarrow d(p+q) =
$$

, *ie*

$$
u^{(p)} = \frac{-2(p+q)+2(x+y)}{-2(p+q)+2(x+y)}
$$
\n
$$
\Rightarrow \qquad d(p+q) = d(x+y)
$$
\n
$$
v^{(p)}
$$
\n
$$
\Rightarrow \qquad p+q = x+y+\alpha
$$
\n
$$
p+q = x+y+\alpha
$$
\nor\n
$$
(p-x) = (y-q)+\alpha
$$
\n
$$
(vi)
$$

Observe that the differential equation may be expressed as<br>  $(p-x)^2 + (y-q)^2 = (x-y)^2$ 

that the differential equation may be expressed as  
\n
$$
(p-x)^2 + (y-q)^2 = (x-y)^2
$$
\n
$$
(vii)
$$
\n
$$
(vii)
$$
 in  $(vii)$  yields

Using  $(vi)$  in  $(vii)$  yields

$$
(p-x)^{2} + (y-q)^{2} = (x-y)^{2}
$$
 (vii)  
Using (vi) in (vii) yields  

$$
(y-q)^{2} + 2\alpha (y-q) + \alpha^{2} + (y-q)^{2} = (x-y)^{2}
$$
 (viii)  
ie,
$$
2(y-q)^{2} + 2\alpha (y-q) + \alpha^{2} = (x-y)^{2}
$$
  
ie,
$$
2(y-q)^{2} + 2\alpha (y-q) + \alpha^{2} - (x-y)^{2} = 0
$$

$$
(y-q)^{2} + 2\alpha (y-q) + \alpha^{2} + (y-q)^{2} = (x-y)
$$
  
\n*i.e.*\n
$$
2(y-q)^{2} + 2\alpha (y-q) + \alpha^{2} = (x-y)^{2}
$$
  
\n*i.e.*\n
$$
2(y-q)^{2} + 2\alpha (y-q) + \alpha^{2} - (x-y)^{2} = 0
$$
  
\n*i.e.*\n
$$
(y-q) = \frac{-2\alpha \pm \sqrt{-4\alpha^{2} + 8(x-y)^{2}}}{4}
$$

*i.e.* 
$$
(y-q) = \frac{-2\alpha \pm \sqrt{-4\alpha^2 + 8(x-y)^2}}{4}
$$
  
\n*i.e.*  $(y-q) = \frac{-\alpha - \sqrt{2(x-y)^2 - \alpha^2}}{2}$  *(ix)*

*i.e.* 
$$
(y-q) = \frac{-\alpha - \sqrt{2(x-y)^2 - \alpha^2}}{2}
$$
  
or  $(y-q) = \frac{-\alpha + \sqrt{2(x-y)^2 - \alpha^2}}{2}$  *(ix)*

o r

Considering the positive sign only we have

 $(y-q)$ 

$$
\begin{array}{ll}\n\text{Considering the positive sign only we have} \\
\left(y-q\right) = \frac{-\alpha + \sqrt{2(x-y)^2 - \alpha^2}}{2} \\
\text{i.e} & q = y + \frac{\alpha - \sqrt{2(x-y)^2 - \alpha^2}}{2}\n\end{array}\n\right\}\n\tag{x}
$$

2

 $\it ie$ 

From  $(vi)$  we have that

$$
q = y + \frac{\sqrt{y^2 + y^2}}{2}
$$
  
\n) we have that  
\n
$$
p = (y-q) + (x+\alpha)
$$
  
\n
$$
= (x+\alpha) + \frac{\sqrt{2(x-y)^2 - \alpha^2} - \alpha}{2}
$$
  
\n
$$
= x + \frac{\sqrt{2(x-y)^2 - \alpha^2} + \alpha}{2}
$$
  
\nat  
\n
$$
du = pdx + qdy
$$

Recall that

$$
= x + \frac{\sqrt{2(x - y)} - \alpha}{2}
$$
  
\nRecall that  
\n
$$
du = pdx + qdy
$$
\n
$$
= \left(x + \frac{\alpha + \sqrt{2(x - y)^2 - \alpha^2}}{2}\right)dx + \left(y + \frac{\alpha - \sqrt{2(x - y)^2 - \alpha^2}}{2}\right)dy
$$
\n
$$
= xdx + ydy + \frac{\alpha + \sqrt{2(x - y)^2 - \alpha^2}}{2}dx + \frac{\alpha - \sqrt{2(x - y)^2 - \alpha^2}}{2}dy
$$
\n
$$
= xdx + ydy + \frac{1}{2}\left(\alpha + \sqrt{2(x - y)^2 - \alpha^2}\right)(dx - dy) \qquad \text{(xii)}
$$

Integrating  $(xii)$  yields

$$
= xdx + ydy + \frac{1}{2} \left( \alpha + \sqrt{2(x-y)^2 - \alpha^2} \right) (dx - dy)
$$
\n
$$
\text{Integrating } (\text{xii}) \text{ yields}
$$
\n
$$
= \frac{1}{2} (x^2 + y^2) + \frac{1}{2} \alpha (x - y) + \frac{1}{2} \int \sqrt{2(x-y)^2 - \alpha^2} (dx - dy) \qquad (\text{xiii})
$$

*xiii*) we set  $\sqrt{2}(x - y)$ 9  $(-y) = \vartheta$ 

 $\therefore$ 

To compute the integral in 
$$
(xiii)
$$
 we set  $\sqrt{2}(x-y) = 9$   
\n
$$
\therefore \qquad (dx - dy) = \frac{1}{\sqrt{2}} d \theta
$$
\n
$$
i\dot{e}, \qquad \frac{1}{2} \int \sqrt{2(x-y)^2 - \alpha^2} (dx - dy) = \frac{1}{2\sqrt{2}} \int \sqrt{9^2 - \alpha^2} d \theta
$$
\n
$$
= \frac{1}{2\sqrt{2}} \left[ \frac{9}{2} \sqrt{9^2 - \alpha^2} - \frac{\alpha^2}{2} \ln \left( 9 + \sqrt{9^2 - \alpha^2} \right) \right]
$$
\n
$$
= \frac{1}{4\sqrt{2}} \left[ \sqrt{2} (x-y) \sqrt{2(x-y)^2 - \alpha^2} - \frac{\alpha^2}{2} \ln \left( \sqrt{2} (x-y) + \sqrt{2(x-y)^2 - \alpha^2} \right) \right]
$$
\nHence, the required complete integral is gives as

Hence, the required complete integral is gives as

$$
= \frac{1}{4\sqrt{2}} \left[ \sqrt{2} (x - y) \sqrt{2(x - y)^2 - \alpha^2} - \frac{\alpha^2}{2} \ln \left( \sqrt{2} (x - y) + \sqrt{2(x - y)^2 - \alpha^2} \right) \right]
$$
  
Hence, the required complete integral is gives as  

$$
2u = x^2 + y^2 + \alpha (x - y) \frac{1}{2\sqrt{2}} \left[ \sqrt{2} (x - y) \sqrt{2(x - y)^2 - \alpha^2} - \frac{\alpha^2}{2} \ln \left( \sqrt{2} (x - y) + \sqrt{2(x - y)^2 - \alpha^2} \right) \right]
$$
  
2 Determine the integral surface of  

$$
(y + uq)^2 = u^2 (1 + p^2 + q^2)
$$
 circumscribed about the surface  $2y = x^2 - u^2$ .

2 Determine the integral surface of

$$
2\sqrt{2} \text{ L}
$$
  
the integral surface of  

$$
(y+uq)^2 = u^2(1+p^2+q^2) \text{ circumscribed about the surface } 2y = x^2 - u^2.
$$

#### *Solution*

The equation in the standard form is given by

$$
(y+uq) = u^2(1+p^2+q^2)
$$
 circumscribed about the surface  $2y = x^2 - u^2$ .  
*Solution*  
The equation in the standard form is given by  

$$
f(x, y, u, p, q) = (y+uq)^2 - u^2(1+p^2+q^2)
$$
(i)  
The convergence change of logarithm DFG is

The corresponding Charpit's auxiliary DE is

The equation in the standard form is given by  
\n
$$
f(x, y, u, p, q) = (y + uq)^2 - u^2 (1 + p^2 + q^2)
$$
\n
$$
(i)
$$
\nThe corresponding Charpit's auxiliary *DE* is\n
$$
\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{du}{-\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial u}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial u}} = \frac{dF}{0}
$$
\n
$$
i.e,
$$
\n
$$
\frac{dx}{2pu^2} = \frac{dy}{2(y + uq)u - 2qu} = \frac{du}{-2p^2u + 2yu} = \frac{dp}{-2(y + uq) - 2u(1 + p^2 + q^2)}
$$

,

*i.e,*  
\n
$$
\frac{dx}{2pu^2} = \frac{dy}{2(y+uq)u - 2qu} = \frac{du}{-2p^2u + 2yu} = \frac{dp}{-2(y+uq) - 2u(1+p^2+q^2)}
$$
\n
$$
= \frac{dq}{-2\{(y+uq) - qu(1+p^2+q^2) - 2q(y+uq)q\}}
$$
\n*iie,*  
\n
$$
\frac{dy}{2yu} = \frac{du}{-2p^2u + 2yuq} = \frac{dq}{-2y + 2qu(p^2+q^2)}
$$
\n*(iii)*

,

$$
-2\{(y+uq) - qu(1+p+q) - 2q(y+uq)q\}
$$
  
\n*i.e,* 
$$
\frac{dy}{2yu} = \frac{du}{-2p^2u + 2yuq} = \frac{dq}{-2y + 2qu(p^2 + q^2)}
$$
  
\n*ie,* 
$$
= \frac{qdu}{-2p^2qu + 2yuq^2 - 2uy + 2qu^2(p^2 + q^2)} = \frac{d(qu)}{-2yu} = \frac{dy + d(qu)}{0}
$$
  
\n
$$
\Rightarrow dy + d(qu) = 0
$$
  
\n*ie,* 
$$
y + qu = a
$$
  
\n
$$
a - y
$$

$$
\Rightarrow dy + d (qu) = 0
$$

$$
e, \qquad y + qu = a
$$

$$
-2p^{2}qu + 2yuq^{2} - 2uy + 2qu^{2}(p^{2})
$$
  
\n
$$
\Rightarrow dy + d(qu) = 0
$$
  
\ni*e*,  $y + qu = a$   
\ni*e*,  $q = \frac{a - y}{u}$  (iv)

Substituting  $(iv)$  into  $(i)$  yields  $iv$ ) into (*i*)

Substituting (iv) into (i) yields  
\n
$$
\left(y + \left(\frac{a-y}{u}\right)u\right)^2 - u^2 \left(1 + p^2 + \left(\frac{a-y}{u}\right)^2\right) = 0
$$
\n*ie,*\n
$$
y^2 + 2y(a-y) + (a-y)^2 - u^2 - u^2p^2 - (a-y)^2 = 0
$$
\n*ie,*\n
$$
u^2 + u^2p^2 - 2y(a-y) - y^2 = 0
$$

*i.e.* 
$$
u^2 + u^2 p^2 - 2y (a - y) - y^2 = 0
$$

*i.e,*  
\n
$$
y^{2} + 2y(a - y) + (a - y)^{2} - u^{2} - u^{2}p^{2} - (a - y)^{2} = 0
$$
\n*ie,*  
\n
$$
u^{2} + u^{2}p^{2} - 2y(a - y) - y^{2} = 0
$$
\n*ie,*  
\n
$$
p^{2} = \frac{2ay - u^{2} - y^{2}}{u^{2}} \Rightarrow p = \frac{\pm \sqrt{2ay - u^{2} - y^{2}}}{u}
$$
  
\n
$$
\therefore \quad pdx + qdy - dz = 0 \text{ becomes}
$$
  
\n(v)

$$
u^{2} = u
$$
  
\n
$$
pdx + qdy - dz = 0 \text{ becomes}
$$
  
\n
$$
\frac{\pm \sqrt{2ay - u^{2} - y^{2}}}{u} dx + \frac{a - y}{u} dy - du = 0
$$
  
\ni*e*,  $(a - y) dy \pm \sqrt{2ay - u^{2} - y^{2}} dx - u du = 0$   
\ni*e*,  $\pm \sqrt{2ay - u^{2} - y^{2}} dx + \frac{1}{2} d (ay - y^{2} - u^{2}) = 0$ 

*i.e,* 
$$
(a-y)dy \pm \sqrt{2ay - u^2 - y^2}dx - udu = 0
$$
  
\n*i.e,*  $\pm \sqrt{2ay - u^2 - y^2}dx + \frac{1}{2}d(ay - y^2 - u^2)$ 

$$
\frac{du}{u} dx + \frac{dv}{u} dy - du = 0
$$
  
\ni*e*,  $(a - y) dy \pm \sqrt{2ay - u^2 - y^2} dx - u du = 0$   
\ni*e*,  $\pm \sqrt{2ay - u^2 - y^2} dx + \frac{1}{2} d (ay - y^2 - u^2) = 0$   
\ni*e*,  $\pm 2\sqrt{y} dx + \frac{1}{2} dy = 0$ 

$$
\pm 2\sqrt{\psi} dx + \frac{1}{2} d\psi = 0
$$
  
i.e, 
$$
\pm dx + \frac{1}{2} \psi^{-\frac{1}{2}} d\psi = 0
$$

$$
i.e., \t t2\sqrt{\psi}dx + \frac{1}{2}d\psi = 0
$$
  

$$
i.e., \t t2\sqrt{\psi}dx + \frac{1}{2}\psi^{-1/2}d\psi = 0
$$
  
(vi)  
Integrating (vi) yields  

$$
t x + \psi = b
$$
 (vii)

 $(vi)$ *vi*

$$
\pm x + \psi = b \tag{vii}
$$

, *ie*

$$
\pm x + \psi = b \qquad (vii)
$$
  
\ni.e,  
\n
$$
x + \sqrt{2ay - u^2 - y^2} = b_1
$$
\n
$$
-x + \sqrt{2ay - u^2 - y^2} = b_2
$$
\nThese give complete integral of (i) which may be combined as  
\n
$$
(x - b_1 + u)(x - b_2 - u) = 0
$$
  
\n
$$
\therefore b \text{ and } b \text{ are arbitrary, we may replace } b \text{ by } b \text{ and } b \text{ by } b \text{ to get}
$$

These give complete integral of (*i*) which may be combined as<br>  $(x-b_1+u)(x-b_2-u)=0$  $x + \sqrt{2ay - b_2}$ <br>
ese give complete integral of (*i*) which may be combined as<br>  $(x - b_1 + u)(x - b_2 - u) = 0$ <br>  $b_1$  and  $b_2$  are arbitrary we may replace  $b_2$  by  $-b_1$  and  $b_1$  by b

$$
(x - b_1 + u)(x - b_2 - u) = 0
$$

 $(x-b_1+u)(x-b_2-u)=0$ <br>
<sup>2</sup><sub>1</sub> and  $b_2$  are arbitrary we may replace  $b_2$  by  $-b_1$  and  $b_1$  $b_1$  and  $b_2$  are arbitrary we may replace  $b_2$  by  $-b_1$  and  $b_1$  by  $b$  to get −

These give complete integral of (i) which may be combined as  
\n
$$
(x-b_1+u)(x-b_2-u)=0
$$
  
\n $\therefore b_1$  and  $b_2$  are arbitrary we may replace  $b_2$  by  $-b_1$  and  $b_1$  by b to get  
\n $(x-b)^2 - u^2 = 0$   
\n $(x-b)^2 - (2ay - u^2 - y^2) = 0$  (viii)  
\nDenoting the LHS of (viii) by  $F(x, y, u, a, b)$  we may also write  $H(x, y, u) = x^2 - u^2 - 2y$ 

 $(x, y, u, a, b) = 0$  circumscribe  $H(x, y, u)$  $(x-b)^2 - u^2 = 0$ <br>  $(x-b)^2 - (2ay - u^2 - y^2) = 0$  (viii)<br>
Denoting the LHS of (viii) by  $F(x, y, u, a, b)$  we may also write  $H(x, y, u) = x^2 - u^2 - 2y = 0$  and  $(x-b)^2 - (2ay - u^2 - y^2) = 0$  (viii)<br>Denoting the LHS of (viii) by  $F(x, y, u, a, b)$  we may also write  $H(x, y, u)$ <br>suppose the integral surface  $F(x, y, u, a, b) = 0$  circumscribe  $H(x, y, u) = 0$ .<br>Therefore, we must have Therefore, we must have<br> $\frac{F_x}{H} = \frac{F_y}{H} = \frac{F_u}{H}$  $\begin{aligned} xy - u^2 - y^2 &= 0 \quad \text{(viii)} \\ \text{by } F(x, y, u, a, b) \text{ we may also write } H(x, b) \\ F(x, y, u, a, b) &= 0 \text{ circumscribe } H(x, y, u) \end{aligned}$  $=x^2 - u^2 - 2y = 0$  and 0 (viii)<br>
b) we may also write  $H(x, y, u) = x^2 - u^2 - 2$ <br>
= 0 circumscribe  $H(x, y, u) = 0$ . *F F F* surface  $F(x, y, u, a, b)$ <br>have<br> $= \frac{F_y}{H_y} = \frac{F_u}{H_u}$ 

$$
\begin{aligned}\n\text{egral surface } F \left( \text{must have} \right. \\
\frac{F_x}{H_x} = \frac{F_y}{H_y} = \frac{F_u}{H_u}\n\end{aligned}
$$

$$
\frac{2(x-b)}{2x} = \frac{-2(a-y)}{-2} = \frac{2u}{-2u} = -1
$$
  
or  

$$
x-b = -x, y-a = 1
$$

$$
x = \frac{b}{2}, y = a+1
$$

 $\Rightarrow$   $x = \frac{b}{2}$ ,  $y = a + 1$ <br>Substituting the values of *x*, *y* into *H* = 0 and *F* = 0 gives

Substituting the values of x, y into 
$$
H = 0
$$
 and  $F = 0$  gives  
\n
$$
\frac{b^2}{4} - u^2 - 2(a+1) = 0
$$
\n
$$
u^2 = \frac{b^2}{4} - 2(a+1)
$$
\nand\n
$$
\frac{b^2}{4} - \left[2a(a+1) - (a+1)^2 - u^2\right] = 0
$$
\n
$$
(ix)
$$

and 
$$
\frac{b}{4} - \left[2a(a+1) - (a+1)^2 - u^2\right] = 0
$$
  
*i.e,* 
$$
\frac{b^2}{4} - \left(2a^2 + 2a - a^2 - 2a - 1 - u^2\right) = 0
$$

$$
\frac{b^2}{4} - a^2 + 1 + u^2 = 0
$$

$$
u^2 = -\left(\frac{b^2}{4} - a^2 + 1\right) = a^2 - 1 - \frac{b^2}{4}
$$

$$
12. (x)
$$
Eliminating *u* from (*ix*) and (*x*) gives

Eliminating *u* from *(ix)* and *(x)* gives  
\n
$$
\frac{b^2}{4} - 2(a+1) = a^2 - 1 - \frac{b^2}{4}
$$
\n*ie,*  
\n
$$
\frac{b^2}{2} = a^2 - 1 + 2(a+1) = (a+1)^2
$$
\n*ie,*  
\n
$$
b = \pm \sqrt{2}(a+1)
$$
\nIf  $b = \sqrt{2}(a+1)$ , the integral surface is

 $(a+1)$ ie,  $b = \pm \sqrt{2} (a+1)$ <br>If  $b = \sqrt{2} (a+1)$ , the integral surface is  $b = \sqrt{2} (a$ 

If 
$$
b = \sqrt{2}(a+1)
$$
, the integral surface is  
\n
$$
\left(x - \sqrt{2}(a+1)\right)^2 - \left(2ay - u^2 - y^2\right) = 0
$$
\n(xii)

Differentiating partially wrt  $a$  we obtain *a*

$$
(x - \sqrt{2}(a + 1)) - (2ay - a - y') = 0 \qquad (3a)
$$
  
Differentiating partially wrt a we obtain  

$$
-2\sqrt{2}(x - \sqrt{2} - \sqrt{2}a) - 2y = 0
$$

$$
\Rightarrow \qquad x - \sqrt{2}(1 + a) = -\frac{y}{\sqrt{2}} \qquad (xiii)
$$

From  $(xii)$  and  $(xiii)$  we have  $x - \sqrt{x}$ <br>*xii*) and (*xiii* 

From (xii) and (xiii) we have  
\n
$$
\left(-\frac{y}{\sqrt{2}}\right)^2 = -\left[2y\left(\frac{x}{\sqrt{2}} + \frac{y}{2} + 1\right) - u^2 - y^2\right] = 0
$$
\ni.e, 
$$
\frac{y^2}{2} = -\frac{2yx}{\sqrt{2}} - \frac{2y^2}{2} - 2y + u^2 + y^2 = 0
$$
\ni.e, 
$$
2u^2 + y^2 - 2\sqrt{2}xy + 4y = 0 \qquad (xiv)
$$

This is the particular integral surface circumscribing the given surface. This is the particular integral surface circumscribing the Similarly, if we take  $b = -\sqrt{2} (a+1)$  in  $(xi)$  we obtain Similarly, if we take  $b = -\sqrt{2}(a+1)$  in (<br>  $2u^2 + y^2 + 2\sqrt{2}xy + 4y = 0$ integral surface circus<br>  $b = -\sqrt{2} (a+1)$  in (xi retain integral startice encall<br>
a given<br>  $u^2 + y^2 + 2\sqrt{2}xy + 4y = 0$  (*xv* tegral surface circum<br>=  $-\sqrt{2}(a+1)$  in  $(xi)$ that integral surface chemisching<br>the  $b = -\sqrt{2}(a+1)$  in  $(xi)$  we<br>+  $y^2 + 2\sqrt{2}xy + 4y = 0$ 

we take 
$$
b = -\sqrt{2(a+1)}
$$
 in  $(xi)$  we obtain  
\n $2u^2 + y^2 + 2\sqrt{2}xy + 4y = 0$  (xy)

*xv*

$$
2u + y + 2\sqrt{2xy + 4y} = 0
$$
\n(2*2u*) (2*u*) (2*u*

, *ie*

i.e,  
\n
$$
(2u^2 + y^2 + 4y)^2 = 8xy
$$
\n $(xvi)$ 

# CHAPTER TWO

### PARTIAL DIFFERENTIAL EQUATIONS OF SECOND AND HIGHER ORDERS.

#### 2.1 LINEAR EQUATIONS.

 $(PDEs)$  $M$ STEETES.<br> *mth* – order Partial Differential Equations (*PDEs*<br>  $P$ <sub>*m*</sub>...  $\approx$   $\frac{2^m}{n}$ ...  $\approx$   $\frac{2^{m-1} \cdot \cdot \cdot}{n}$ 

2.1 LINEAR EQUATIONS.  
\nThe most general linear *mth* – order Partial Differential Equations (*PDEs*) is of the form  
\n
$$
A_0 \frac{\partial^m u}{\partial x^m} + A_1 \frac{\partial^m u}{\partial x^{m-1} \partial y} + A_2 \frac{\partial^m u}{\partial x^{m-2} \partial y^2} + \dots + B_1 \frac{\partial^{m-1} u}{\partial x^{m-1}} + B_2 \frac{\partial^{m-1} u}{\partial x^{m-2} \partial y} + \dots
$$
\n+
$$
\dots \dots \dots M \frac{\partial u}{\partial x} + N \frac{\partial u}{\partial y} + Cu = f(x, y)
$$
\nin which  $A_k, B_k, M, N, C$  are constants or functions of *x* and *y*.  
\nFrom equation (2.1), a constant coefficient *PDE* is thus given as

+.............. $M \frac{\partial u}{\partial x} + N$ <br>in which  $A_k$ ,  $B_k$ ,  $M$ ,  $N$ ,  $C$  are constand  $(2.1)$ , a constand  $(2.1)$ 2.1), a constant coefficient *PDE* is thus gi<br> $\overline{\partial}^{m} u$ <br> $\overline{\partial}^{m} u$ 

$$
\frac{\partial x}{\partial y} \qquad \frac{\partial y}{\partial y}
$$
\nin which  $A_k, B_k, M, N, C$  are constants or functions of x and y.  
\nFrom equation (2.1), a constant coefficient *PDE* is thus given as\n
$$
\left( a_0 \frac{\partial^m u}{\partial x^m} + a_1 \frac{\partial^m u}{\partial x^{m-1} \partial y} + a_2 \frac{\partial^m u}{\partial x^{m-2} \partial y^2} + \dots + a_m \frac{\partial^m u}{\partial y^m} \right) + \left( b_0 \frac{\partial^{m-1} u}{\partial x^{m-1}} + b_1 \frac{\partial^{m-1} u}{\partial x^{m-2} \partial y} + \dots + b_{m-1} \frac{\partial^{m-1} u}{\partial y^{m-1}} \right) + \dots + \left( k_0 \frac{\partial u}{\partial x} + k_1 \frac{\partial u}{\partial y} \right) + l u = f(x, y)
$$
\nin which  $a_i$   $i = 0(1)m, b_j$   $j = 0(1)m, k_0, k_1$  and l, are constants.

 $(1)$ *m*,  $b_j$   $j = 0(1)$ 

+
$$
+\text{.................}\left(k_0 \frac{\partial u}{\partial x} + k_1 \frac{\partial u}{\partial y}\right) + lu = f(x, y)
$$
\n(2.2)

\nin which  $a_i$   $i = 0(1)m$ ,  $b_j$   $j = 0(1)m$ ,  $k_0$ ,  $k_1$  and  $l$ , are constants.

\nSetting  $D^p = \frac{\partial^p}{\partial x^p}$  and  $D'^r = \frac{\partial^r}{\partial y^r}$ 

\n(2.3)

\nthen (2.2) becomes:

\n
$$
\left[\left(a_0 D^m + a_1 D^{m-1} D' + a_2 D^{m-2} D'^2 + \dots + a_m D'^m\right) + \left(b_0 D^{m-1} + b_1 D^{m-2} D' + \dots + b_{m-1} D'^{m-1}\right)\right]u
$$
\n
$$
\left[\left(a_0 D^m + a_1 D^{m-1} D' + a_2 D^{m-2} D'^2 + \dots + a_m D'^m\right) + \left(b_0 D^{m-1} + b_1 D^{m-2} D' + \dots + b_{m-1} D'^{m-1}\right)\right]u
$$

 $(2.2)$ 

then (2.2) becomes:  
\n
$$
\left[ \left( a_0 D^m + a_1 D^{m-1} D' + a_2 D^{m-2} D'^2 + \dots + a_m D'^m \right) + \left( b_0 D^{m-1} + b_1 D^{m-2} D' + \dots + b_{m-1} D'^{m-1} \right) \right] u
$$
\n
$$
+ \left[ \left( k_0 D + k_1 D' \right) + l \right] u = f(x, y)
$$
\nor\n
$$
F(D, D') u = f(x, y)
$$
\nin which  $F(D, D')$  is a differential operator of order *m*.  
\nThe corresponding homogeneous differential equation (reduced equation) to (2.4) is given by\n
$$
F(D, D') u = 0
$$
\n(2.5)

 $(D, D')$  $\overline{\phantom{a}}$ 

The correspondind homogeneous differential equation (reduced equation) to  $(2.4)$ in which  $F(D, D')$  is a differential operator of order *m*.<br>The correspondind homogeneous differential equation (reduced equation) to (2.4) is given<br> $F(D, D')u = 0$  (2.5)

$$
F(D, D')u = 0 \tag{2.5}
$$

#### 2.1 *Definition*

 $(D, D')$ form  $(\alpha D + \beta D' + \gamma)$  in which  $\alpha, \beta$  and  $\gamma$  are all constants. Otherwise it is *irreducible*. The corresponding homogeneous differential equation (reduced equation) to (2.4) is given by<br>  $F(D, D')u = 0$  (2.5)<br>
Definition 2.1<br>
The differential operator  $F(D, D')$  is said to be *reducible* if it can be decomposed into fact *F*(*D*,*D*<sup>*'*</sup>)*u* = 0<br>*Definition* 2.1<br>The differential operator *F*(*D*,*D'*) is said to be *reducible* if it can be decomposed into:<br>form  $(\alpha D + \beta D' + \gamma)$  in which  $\alpha$ ,  $\beta$  and  $\gamma$  are all constants. Otherwise it is geneous differential equation (re<br>0<br>*F*  $(D, D')$  is said to be *reducible F*(*D*,*D'*)*u* = 0<br>*tion* 2.1<br>*fferential operator <i>F*(*D*,*D'*) is said to be *reducible* if it can be decomposed into<br> $\alpha D + \beta D' + \gamma$ ) in which  $\alpha$ ,  $\beta$  and  $\gamma$  are all constants. Otherwise it is *irreducible*  $\overline{\phantom{a}}$  $F(D, D')u = 0$ <br>2.1<br>mtial operator  $F(D, D')$ <br>+  $\beta D' + \gamma$  in which  $\alpha$ ,

#### 2.1 METHOD OF SOLUTION

 $(2.4)$  is analogous to that of an  $m$  – order Ordinary Differential Equation  $(ODE)$ comprises of a complimentary function  $(CF)$  that contains m arbitrary constants and 2.1 METHOD OF SOLUTION<br>The solution of  $(2.4)$  is analogous to that of an  $m$  – order Ordinary Differential Equation  $(ODE)$  which<br>comprises of a complimentary function  $(CE)$  that contains  $m$  arbitrary constants and a part *c* t of an  $m$  – order Order Order Order Order (*CF*) that contains  $m$ −  $(PI)$  that contains no arbitrary constant. In this case the complimentary function is the solution of  $(2.5)$ and the particular integral the solution of  $(2.4)$ . a particular integral METHOD OF SOLUTION<br>solution of (2.4) is analogous to that of an  $m$  – order Ordinary Differential Equation (*ODE*) which<br>prises of a complimentary function (*CF*) that contains m arbitrary constants and a particular integ The solution of  $(2.4)$  is analogous to that of an comprises of a complimentary function  $(CF)$  the  $(PI)$  that contains no arbitrary constant. In this and the particular integral the solution of  $(2.4)$ .

#### 2.2.1 Complimentary Function s

 $(2.5)$ In order to obtain the complimentary functions<br>
In order to obtain the complimentary function corresponding to the solution of  $(2.5)$  we recall this theorem<br>
from elementary caculus: from elementary caculus: In order to obtain the complimentary function corresponding to the so<br>from elementary caculus:<br>*Theorem* 2.1<br>If the differential operator  $F(D, D')$  the general solution of (2.5) ie, from elementary caculus:<br> *Fheorem* 2.1<br>
If the differential operator  $F(D, D')$  the general solution of (2.5) ie,<br>  $F(D, D')u = (\alpha D + \beta D' + \gamma)^m u = 0$  (2.6<br>
where *m* is a positive integer is given as

#### 2.1 *Theorem*

 $(D, D')$ 

$$
F(D, D')u = (\alpha D + \beta D' + \gamma)^m u = 0
$$
\n(2.6)

where *m* is a positive integer is given as

If the differential operator 
$$
F(D, D')
$$
 the general solution of (2.5) i.e,  
\n
$$
F(D, D')u = (\alpha D + \beta D' + \gamma)^m u = 0
$$
\nwhere *m* is a positive integer is given as  
\n
$$
u = \exp\left(-\frac{\gamma}{\alpha}x\right) \sum_{r=1}^{m} x^{m-1} \phi_m(\beta x - \alpha y) \qquad \alpha \neq 0
$$
\nand  
\n
$$
u = \exp\left(-\frac{\gamma}{\beta}y\right) \sum_{r=1}^{m} y^{m-1} \phi_m(\beta x - \alpha y) \qquad \beta \neq 0
$$
\nin which the functions  $\phi$  are sufficiently differentiable arbitrary functions

$$
u = \exp\left(-\frac{\gamma}{\beta}y\right)\sum_{r=1}^{m}y^{m-1}\phi_m\left(\beta x - \alpha y\right) \qquad \beta \neq
$$

in which the functions  $\phi_r$  are sufficiently differentiable arbitrary functions.

Proof

We shall assume that  $\alpha \neq 0$  and prove by induction.  $\alpha \neq$ 

For  $m = 1$  the equation becomes:

sume that 
$$
\alpha \neq 0
$$
 and prove by induction.

\nthe equation becomes:

\n
$$
(\alpha D + \beta D' + \gamma)u = 0 \text{ i}e, \ \alpha \frac{\partial u}{\partial x} + \beta \frac{\partial u}{\partial y} + \gamma u = 0
$$
\nor

\n
$$
\alpha \frac{\partial u}{\partial x} + \beta \frac{\partial u}{\partial y} = -\gamma u
$$
\nst-order PDE with the corresponding Lagrange's auxiliary equation as

\n
$$
\frac{dx}{dt} = \frac{dy}{dt} = \frac{du}{dt} \qquad \qquad (ii)
$$

This is a first-order PDE with the corresponding Lagranges auxiliary equation as

or 
$$
\alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y} = -\gamma u
$$
  
\ni-order PDE with the corresponding Lagrange's auxiliary equation as  
\n
$$
\frac{dx}{\alpha} = \frac{dy}{\beta} = \frac{du}{-\gamma u}
$$
 (*ii*)  
\n
$$
\beta dx - \alpha dy = 0 \text{ or } \beta x - \alpha y = c
$$
 (*c* a constant) (*iii*)

, *ie*

$$
\beta dx - \alpha dy = 0 \text{ or } \beta x - \alpha y = c \ (c \text{ a constant}) \tag{iii}
$$

Also, we have

*i.e,*  
\n
$$
\beta dx - \alpha dy = 0 \text{ or } \beta x - \alpha y = c
$$
 (*c* a constant)  
\nAlso, we have  
\n
$$
\frac{du}{u} = -\frac{\gamma}{\alpha} dx
$$
\n*ii.e,* 
$$
\ln u = -\frac{\gamma}{\alpha} x + k
$$
 (*iv*)

, *ie*

$$
ue^{(-\frac{y}{a}x)} = c
$$
\ngeneral solution is

Hence, a general solution is

$$
ue^{(-\frac{y}{a})} = c
$$
  
\ngeneral solution is  
\n
$$
ue^{(-\frac{y}{a})} = \phi(\beta x - \alpha y)
$$
 (vi)  
\ns a differentiable function. This proves the theorem for  $m = 1$ .  
\nassume the theorem to be true for some  $m = p$  and prove that it is true for m  
\nsume that

where  $\phi$  is a differentiable function. This proves the theorem for  $m = 1$ .  $\phi$  is a differentiable function. This proves the theorem for  $m =$ 

We then assume the theorem to be true for some  $m = p$  and prove that it is true for  $m = p + 1$ .  $ue^{(-\frac{1}{\alpha})}$  =<br>here  $\phi$  is a differ<br>let then assume that<br>, we assume that *i*)<br> $m = p$ *ie*, we assume that  $= p + 1.$ 

$$
(\alpha D + \beta D' + \gamma)^p u = 0 \qquad (vii)
$$

Observe that

where 
$$
\phi
$$
 is a differentiable function. This proves the theorem for  $m = 1$ .  
\nWe then assume the theorem to be true for some  $m = p$  and prove that it is true for  $m = ie$ , we assume that  
\n
$$
(\alpha D + \beta D' + \gamma)^p u = 0
$$
\n( $vii$ )  
\nObserve that  
\n
$$
(\alpha D + \beta D' + \gamma)^{p+1} u = 0 = (\alpha D + \beta D' + \gamma)^p w
$$
\n
$$
(viii)
$$
\nwhere  $w = (\alpha D + \beta D' + \gamma)u$ 

where  $w = (\alpha D + \beta D' + \gamma)$ 

But by our hypothesis

$$
(\alpha D + \beta D' + \gamma)^{i+1} u = 0 = (\alpha D + \beta D' + \gamma)^{i} w \qquad (viii)
$$
  
where  $w = (\alpha D + \beta D' + \gamma) u$   
But by our hypothesis  

$$
w = \exp\left(-\frac{\gamma}{\alpha}x\right) \sum_{r=1}^{p} x^{r-1} \phi_r (\beta x - \alpha y) \qquad \alpha \neq 0 \qquad (ix)
$$

or

$$
w = \exp\left(-\frac{\gamma}{\alpha}x\right) \sum_{r=1}^{p} x^{r-1} \phi_r (\beta x - \alpha y) \qquad \alpha \neq 0
$$
 (ix)  

$$
(\alpha D + \beta D' + \gamma)u = \exp\left(-\frac{\gamma}{\alpha}x\right) \sum_{r=1}^{p} x^{m-1} \phi_r (\beta x - \alpha y) \qquad \alpha \neq 0
$$
 (ix)  

$$
\alpha \frac{\partial u}{\partial x} + \beta \frac{\partial u}{\partial y} = -\gamma u + \exp\left(-\frac{\gamma}{\alpha}x\right) \sum_{r=1}^{p} x^{r-1} \phi_r (\beta x - \alpha y) \qquad \alpha \neq 0
$$
 (x)

, *ie*

$$
(\alpha D + \beta D' + \gamma)u = \exp\left(-\frac{\gamma}{\alpha}x\right)\sum_{r=1}^{\gamma} x^{m-1}\phi_r(\beta x - \alpha y) \qquad \alpha \neq 0 \qquad (ix)
$$
  
\n
$$
\alpha \frac{\partial u}{\partial x} + \beta \frac{\partial u}{\partial y} = -\gamma u + \exp\left(-\frac{\gamma}{\alpha}x\right)\sum_{r=1}^{\rho} x^{r-1}\phi_r(\beta x - \alpha y) \qquad \alpha \neq 0 \qquad (x)
$$

equation given as  $\frac{d}{dx} + p \frac{d}{dy} = -\gamma u + \exp\left(-\frac{d}{dx}x\right) + \sum_{r=1}^{\infty} \frac{x^r}{r} \frac{\varphi_r(p\cdot x - \alpha y)}{\varphi_r(p\cdot x - \alpha y)}$   $\alpha \neq 0$  (x)<br>
in a first-order linear partial differential equation with the corresponding Lagranges auxis<br>
iven as<br>  $\frac{dx}{\alpha} = \frac{$ 

$$
(\alpha D + \beta D' + \gamma)u = \exp\left(-\frac{\gamma}{\alpha}x\right)\sum_{r=1}^{\gamma} x^{m-1}\phi_r(\beta x - \alpha y) \qquad \alpha \neq 0 \qquad (ix)
$$
  
\n*ie,*  
\n
$$
\alpha \frac{\partial u}{\partial x} + \beta \frac{\partial u}{\partial y} = -\gamma u + \exp\left(-\frac{\gamma}{\alpha}x\right)\sum_{r=1}^{\rho} x^{r-1}\phi_r(\beta x - \alpha y) \qquad \alpha \neq 0 \qquad (x)
$$
  
\nThis is again a first-order linear partial differential equation with the corresponding Lagrange's auxiliary equation given as  
\n
$$
\frac{dx}{\alpha} = \frac{dy}{\beta} = \frac{du}{-\gamma u + \exp\left(-\frac{\gamma}{\alpha}x\right)\sum_{r=1}^{\rho} x^{r-1}\phi_r(\beta x - \alpha y)
$$
\n*(xi)*  
\nin which again from the first two equalities we have  
\n
$$
\beta dx - \alpha dy = 0 \text{ or } \beta x - \alpha y = c \text{ (c a constant)}
$$
\n*(xii)*

in whi ch again from the first two equalities we have

$$
\beta dx - \alpha dy = 0 \text{ or } \beta x - \alpha y = c \text{ (c a constant)}
$$
 (xii)

Again, we also havefrom the first and third equalities

Again from the first two equalities we have

\n
$$
\beta dx - \alpha dy = 0 \text{ or } \beta x - \alpha y = c \text{ (c a constant)}
$$
\n(xii)

\nalso havefrom the first and third equalities

\n
$$
\frac{dx}{\alpha} = \frac{du}{-\gamma u + \exp\left(-\frac{\gamma}{\alpha}x\right)\sum_{r=1}^{p} x^{r-1}\phi_r(c)}
$$
\n
$$
\frac{du}{dx} + \frac{\gamma}{\alpha}u = \exp\left(-\frac{\gamma}{\alpha}x\right)\sum_{r=1}^{p} x^{r-1}\phi_r(c)
$$
\n(xi)

or

$$
\alpha \qquad -\gamma u + \exp\left(-\frac{\gamma}{\alpha}x\right) \sum_{r=1}^{p} x^{r-1} \phi_r(c)
$$
\nor

\n
$$
\frac{du}{dx} + \frac{\gamma}{\alpha} u = \exp\left(-\frac{\gamma}{\alpha}x\right) \sum_{r=1}^{p} x^{r-1} \phi_r(c)
$$
\ni.e,

\n
$$
\exp\left(\frac{\gamma}{\alpha}x\right) \frac{du}{dx} + \exp\left(\frac{\gamma}{\alpha}x\right) \frac{\gamma}{\alpha} u = \frac{1}{\alpha} \sum_{r=1}^{p} x^{r-1} \phi_r(c)
$$
\n(xv)

, *ie*

$$
\frac{du}{dx} + \frac{\gamma}{\alpha} u = \exp\left(-\frac{\gamma}{\alpha} x\right) \sum_{r=1}^{\infty} x^{r-1} \phi_r(c)
$$
\n
$$
\exp\left(\frac{\gamma}{\alpha} x\right) \frac{du}{dx} + \exp\left(\frac{\gamma}{\alpha} x\right) \frac{\gamma}{\alpha} u = \frac{1}{\alpha} \sum_{r=1}^{p} x^{r-1} \phi_r(c)
$$
\n
$$
(xv)
$$

$$
\Rightarrow \qquad \left[\exp\left(\frac{\gamma}{\alpha}x\right)u\right]' = \frac{1}{\alpha}\sum_{r=1}^{p} x^{r-1}\phi_r(c)
$$
\n*(xvi)*\n
$$
u \exp\left(\frac{\gamma}{\alpha}x\right) = \int \frac{1}{\alpha}\sum_{r=1}^{p} x^{r-1}\phi_r(c) dx
$$
\n*(xvii)*

, *ie*

$$
\begin{aligned}\n\left[\exp\left(\frac{\gamma}{\alpha}x\right)u\right] &= \frac{1}{\alpha}\sum_{r=1}^{r}x^{r-1}\phi_r\left(c\right) \\
u\exp\left(\frac{\gamma}{\alpha}x\right) &= \int \frac{1}{\alpha}\sum_{r=1}^{p}x^{r-1}\phi_r\left(c\right)dx \\
&= \frac{1}{\alpha}\sum_{r=1}^{p}\frac{x^r}{r}\phi_r\left(c\right)\n\end{aligned}\n\tag{xvii}
$$

$$
= \frac{1}{\alpha} \sum_{r=1}^{p} \frac{x^r}{r} \phi_r(c)
$$
  
solution is therefore  

$$
\iota \exp\left(\frac{\gamma}{\alpha}x\right) - \frac{1}{\alpha} \sum_{r=1}^{p} \frac{x^r}{r} \phi_r(c) + c' = \psi(\beta x - \alpha y)
$$
(*xix*)

The general solution is therefore

$$
= \frac{1}{\alpha} \sum_{r=1}^{p} \frac{x^r}{r} \phi_r(c)
$$
 (xviii)  
The general solution is therefore  

$$
u \exp\left(\frac{\gamma}{\alpha}x\right) - \frac{1}{\alpha} \sum_{r=1}^{p} \frac{x^r}{r} \phi_r(c) + c' = \psi(\beta x - \alpha y)
$$
 (xix)  
in which  $\psi$  is an arbitrary differentiable function. This general solution may also be w

in which  $\psi$  is an arbitrary differentiable function. This general solution may also be written in the form  $\psi$ 

$$
u \exp\left(\frac{\gamma}{\alpha}x\right) - \frac{1}{\alpha} \sum_{r=1}^{p} \frac{x^r}{r} \phi_r(c) + c' = \psi(\beta x - \alpha y) \qquad (xix)
$$
  
in which  $\psi$  is an arbitrary differentiable function. This general solution may also be written  

$$
u = \exp\left(-\frac{\gamma}{\alpha}x\right) \sum_{r=1}^{p+1} x^{r-1} \psi_r(\beta x - \alpha y) \qquad (xix)
$$
which is the theorem for  $m = p + 1$   
This completes the induction and hence the proof of the theorem.  
We note that if the operator  $F(D, D')$  is reducible it will be seen that

which is the theorem for  $m = p + 1$ 

This completes the induction and hence the proof of the theorem.

 $(D, D')$ ) is reducible it will be seen that

which is the theorem for 
$$
m = p + 1
$$
  
\nThis completes the induction and hence the proof of the theorem.  
\nWe note that if the operator  $F(D, D')$  is reducible it will be seen that  
\n
$$
F(D, D')e^{(\alpha x + \beta y)} = F(\alpha, \beta)e^{(\alpha x + \beta y)}
$$
\n(2.8)  
\nTherefore it follows that  $u = \exp(\alpha x + \beta y)$  is a solution of  $F(D, D')u = 0$  if  
\n
$$
F(\alpha, \beta) = 0
$$
\n[2.9)  
\nIn general,  $F(\alpha, \beta) = 0$  gives different pairs of solutions  $(\alpha_j, \beta_j)$ . This way we obtain di

 $(\alpha x + \beta y)$  is a solution of  $F(D, D')$ 

$$
F(\alpha, \beta) = 0 \tag{2.9}
$$

 $(\alpha, \beta) = 0$  gives different pairs of solutions  $(\alpha_i, \beta_j)$ . Therefore it follows that  $u = \exp(\alpha x + \beta y)$  is a solution of  $F(D, D')u = 0$  if<br>  $F(\alpha, \beta) = 0$ <br>
In general,  $F(\alpha, \beta) = 0$  gives different pairs of solutions  $(\alpha_j, \beta_j)$ . This way we obtain d *F* = 0 gives different pairs of solutions  $(\alpha_i, \beta_i)$ . This way we obtain different solutions

 $(\alpha_i x + \beta_i y)$  where  $c_i$  are constants. Obviously the linear combination  $\sum c_i \exp(\alpha_i x + \beta_i y)$  $F(\alpha, \beta) = 0$ <br>general,  $F(\alpha, \beta) = 0$  gives different pairs of solutions  $(\alpha_j, \beta_j)$ . This way we obtain different solutions<br> $exp(\alpha_j x + \beta_j y)$  where  $c_j$  are constants. Obviously the linear combination  $\sum_{j=1}^{m} c_j exp(\alpha_j x + \beta_j y)$  i a solution. Indeed, the most general solution is of this form.<br> *Examples*.<br>
1 Obtain the solution to the *DE*<br>  $\frac{\partial^2 u}{\partial x^2} - a^2 \frac{\partial^2 u}{\partial y^2} = 0$ *m F*( $\alpha$ , $\beta$ ) = 0 gives different pairs of solutions  $(\alpha_j, \beta_j)$ . This way we obtain different solutions  $c_j \exp(\alpha_j x + \beta_j y)$  where  $c_j$  are constants. Obviously the linear combination  $\sum_{j=1}^{m} c_j \exp(\alpha_j x + \beta_j y)$  is also

. *Examples*

1 Obtain the so

Examples.  
\n1 Obtain the solution to the *DE*  
\n
$$
\frac{\partial^2 u}{\partial x^2} - a^2 \frac{\partial^2 u}{\partial y^2} = 0
$$
\nSolution

Solution

The given *PDE* is of the form

Solution  
\nSolution  
\nThe given *PDE* is of the form  
\n
$$
(D^2 - a^2 D'^2)u = 0
$$
\n
$$
(D - aD')(D + aD')u = 0
$$
\nThe general solution is

ie,

$$
(D - aD')(D + aD')u = 0
$$

The general solution is

$$
(D - aD')(D + aD')u = 0
$$
  
l solution is  

$$
\phi_1(-ax - y) + \phi_2(-ax - y)
$$

 $(D - aD') (D + aD') u = 0$ <br>The general solution is<br> $\phi_1 (-ax - y) + \phi_2 (-ax - y)$ <br>where  $\phi_1$  and  $\phi_2$  are arbitrary differentiable functions

#### 2 Obtain the solution to the PDE

2 Obtain the solution to the PDE  
\n
$$
\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x \partial y} - 6 \frac{\partial u}{\partial y}
$$
\nSolution  
\nObserve that  $F(D, D') = (D + DD' - 6D) =$ 

*Solution*

 $(D, D') = (D + DD' - 6D)$  $\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x \partial y} - 6 \frac{\partial u}{\partial y}$ <br>
Solution<br>
Observe that  $F(D, D') = (D + DD' - 6D) = 0$ Solution<br>
Observe that  $F(D, D') = (D + DD' - F(a, b) = a + ab - 6b = 0$ bserve that  $F(D, D')$ <br>  $F(a,b) = a$ <br>  $(1+b)a = 6$ *F*  $B = (a, b) = a + ab - 6b$ <br>*F*  $(a, b) = a + ab - 6b$ Observe that  $F(D, D') = F(a, b) = a +$ <br>*ie*,  $(1+b)a = 6b$  $\frac{d}{dy} - 6 \frac{\partial u}{\partial y}$ <br>  $y' = (D + DD' - 6D) = 0$ cy cy<br>  $D' = (D + DD' - 6D) = 0$ <br>  $= a + ab - 6b = 0$  $F(D, D') = (D + DD')$ <br>  $(a,b) = a + ab - 6b = 0$ <br>  $(b) = 6b$ 

$$
F(a,b) = a + ab - 6b = 0
$$

 $\ddot{\cdot}$ 

$$
f_{\rm{max}}
$$

$$
F(a,b) = a + ab - 6b = 0
$$
  
\n*ie,*  $(1+b)a = 6b$   
\n $\Rightarrow$   $a = \frac{6b}{1+b}, b \ne 1$   
\n $\therefore$   $u = \exp\left(\frac{6b}{1+b}x + by\right)$  is a solution.  
\nThe most general solution therefore is  
\n
$$
u = \sum_{r=0}^{\infty} A_r \exp\left(\frac{6b_r}{1+b}x + b_r y\right).
$$

The most general solution therefore is

 $(1+b)a = 6b$ 

6

*b*

$$
u = \exp\left(\frac{1}{1+b}x + by\right)
$$
 is a solution.  
The most general solution therefore is  

$$
u = \sum_{r=1}^{\infty} A_r \exp\left(\frac{6b_r}{1+b_r}x + b_r y\right).
$$

2.2.2 Particular Integrals

To determine the particular integral  $(P.I)$  of eqn $(2.5)$  $\Big)$  .<br> $\therefore I$  of eqn (2.5) 2.2.2 Particular integrals<br>
To determine the particular integrals<br>  $\vec{F}(D, D')u = f(x,$ *I*

, *ie*

the particular integral  
the particular integral  

$$
F(D, D')u = f(x, y)
$$

we shall employ the following two methods:

Method I

 $(D, D')$ If the operator  $F(D, D')$  is a reducible operator then the Particular Integral is of the form<br>*F* (*D*,*D'*) is a reducible operator then the Particular Integral is of the form  $\overline{\phantom{a}}$ 

( 1 1 1 ) 1 . *D D* + + ( ) ( ) ( ) ( ) ( ) ( ) 2 2 2 1 1 1 ........... , *f x y* 2.10 1 , We start the implimentation of the inversion operation 2.10 from the last factor on the right *m m m m j j j j D D D D* + + + + *f x y D D* <sup>=</sup> = + + 

 $(2.10)$ as

We start the implementation of the inversion operation (2.10) from the last factor on the  
\n
$$
\frac{1}{(\alpha_m D + \beta_m D' + \gamma_m)} f(x, y) = G(x, y) \text{ say}
$$
\ni.e,  
\n
$$
(\alpha_m D + \beta_m D' + \gamma_m) G(x, y) = f(x, y)
$$
\n(2.12)

, *ie*

$$
\left(\alpha_m D + \beta_m D' + \gamma_m\right) G(x, y) = f(x, y) \tag{2.12}
$$

 $\Rightarrow \qquad \alpha_m \frac{\partial G}{\partial x} + \beta_m \frac{\partial G}{\partial y} = f - \gamma_m G$ (2.13)  $\begin{aligned} \n\mathcal{L}_m D + p_m D + \gamma_m, \\\\ \n\mathcal{L}_m D + \beta_m D' + \gamma_m, G(x, y) \\\\ \n\mathcal{L}_m \frac{\partial G}{\partial x} + \beta_m \frac{\partial G}{\partial y} = f - \gamma_m G \n\end{aligned}$  $\beta + \beta_m D' + \frac{G}{x} + \beta_m \frac{\partial G}{\partial y}$ <br>*x* inear eq.  $\frac{\partial G}{\partial x} + \beta$ <br>ges line<br> $\frac{dx}{\alpha} = \frac{dy}{\beta}$  $(\alpha_m D + \beta_m D' + \gamma_m)$ <br>  $(\alpha_m D + \beta_m D' + \gamma_m) G(x, y) = f(x, y)$ <br>  $\alpha_m \frac{\partial G}{\partial x} + \beta_m \frac{\partial G}{\partial y} = f - \gamma_m G$  $D + \beta_m D' + \gamma_m G(x, y) =$ <br> $\frac{\partial G}{\partial x} + \beta_m \frac{\partial G}{\partial y} = f - \gamma_m G$ 

This is Lagranges linear equation with the corresponding auxiliary equations

$$
\frac{\partial G}{\partial x} + \beta_m \frac{\partial G}{\partial y} = f - \gamma_m G \tag{2.13}
$$
  
nges linear equation with the corresponding auxiliary equations  

$$
\frac{dx}{\alpha_m} = \frac{dy}{\beta_m} = \frac{dG}{f - \gamma_m G} \tag{2.14}
$$

From the first two relation we obtain  
\n
$$
\beta_m dx - \alpha_m dy = 0
$$
\n*ie,*  $\beta_m x - \alpha_m y = c$   
\nSimilarly, we have that  
\n
$$
\frac{dG}{f - \gamma} = \frac{dx}{\alpha} \Rightarrow \frac{dG}{dx} = \frac{f - \gamma_m G}{\alpha}
$$
\n(2.15)

Similarly, we have that

$$
\beta_m x - \alpha_m y = c
$$
  
have that  

$$
\frac{dG}{f - \gamma_m G} = \frac{dx}{\alpha_m} \Rightarrow \frac{dG}{dx} = \frac{f - \gamma_m G}{\alpha_m}
$$

$$
\frac{dG}{dx} + \frac{\gamma_m}{\alpha} G = \frac{f}{\alpha}, \ \alpha_m \neq 0
$$

, *ie*

$$
\frac{d\omega}{f - \gamma_m G} = \frac{dx}{\alpha_m} \Rightarrow \frac{d\omega}{dx} = \frac{f - f_m}{\alpha_m}
$$
  
\n*i.e,*  
\n
$$
\frac{dG}{dx} + \frac{\gamma_m}{\alpha_m} G = \frac{f}{\alpha_m}, \ \alpha_m \neq 0
$$
\n
$$
\text{This is a first order ODE with an integrating factor } (IF) e^{\int \left(\frac{\gamma_m}{\alpha_m}\right) dx} = e^{\frac{\gamma_m}{\alpha_m}x}
$$
\n
$$
\left(\frac{\gamma_m}{\alpha_m}\right)^2 = \frac{\gamma_m}{\alpha_m} \left(\frac{\gamma_m}{\alpha_m}\right)^2
$$
\n
$$
\left(\frac{\gamma_m}{\alpha_m}\right)^2 = \frac{\gamma_m}{\alpha_m} \left(\frac{\gamma_m}{
$$

This is a first order ODE with an integrating factor  $(IF)$  $\frac{m}{m}$   $\left|dx\right|$   $\frac{m}{m}$ *DE* with an integrating factor  $(Hf) e^{-(\alpha_m)^2} = e^{\alpha_m}$ DE with an integrating factor  $(HF) e^{\int \left(\frac{Im}{\alpha_m}\right)dx} = e^{\frac{Im}{\alpha_m}x}$ <br>  $G = \int \frac{f}{\alpha_m} e^{\frac{\gamma_m}{\alpha_m}} dx$   $\alpha \neq 0$  $\gamma_m$   $\gamma_m$   $\gamma$  $\alpha_{m}$   $\alpha$ DDE with an integrating<br> $\left(\frac{y_m}{x_m}\right)$  of  $\left(\frac{y_m}{x_m}\right)$ 

$$
dx' \alpha_m \alpha_m, \alpha_m, \alpha_m \dots
$$
\nThis is a first order ODE with an integrating factor (IF)  $e^{\int \left(\frac{\gamma_m}{\alpha_m}\right) dx} = e^{\frac{\gamma_m}{\alpha_m}x}$ 

\nie,

\n
$$
\left(e^{\frac{\gamma_m}{\alpha_m}x}G\right) = \int \frac{f}{\alpha_m} e^{\frac{\gamma_m}{\alpha_m}x} dx, \alpha_m \neq 0
$$
\ni.e,

\n
$$
G = e^{\frac{-\gamma_m}{\alpha_m}x} \int \frac{f}{\alpha_m} e^{\frac{\gamma_m}{\alpha_m}x} dx = \frac{1}{\alpha_m} e^{\frac{-\gamma_m}{\alpha_m}x} \int e^{\frac{\gamma_m}{\alpha_m}x} f(x, y) dx, \alpha_m \neq 0
$$
\n(2.17

\nSimilarly, we have

$$
e, \qquad \qquad \left| e^{\alpha_m \widetilde{\alpha}} G \right| = \int \frac{J}{\alpha_m} e^{\alpha_m \widetilde{\alpha}} dx, \ \alpha_m \neq 0 \qquad (2.16)
$$
\n
$$
e, \qquad G = e^{\frac{-\gamma_m}{\alpha_m}} \int \frac{f}{\alpha_m} e^{\frac{\gamma_m}{\alpha_m} x} dx = \frac{1}{\alpha_m} e^{\frac{-\gamma_m}{\alpha_m} x} \int e^{\frac{\gamma_m}{\alpha_m} x} f(x, y) dx, \ \alpha_m \neq 0 \qquad (2.17)
$$
\nSimilarly, we have

Similarly,we have

*i.e.* 
$$
G = e^{-\alpha_m} \int \frac{1}{\alpha_m} e^{\alpha_m} dx = \frac{1}{\alpha_m} e^{-\alpha_m} \int e^{\alpha_m} f(x, y) dx, \ \alpha_m \neq 0
$$
(2.17)  
Similarly, we have  
*i.e.* 
$$
G = \frac{1}{\beta_m} e^{-\frac{\gamma_m}{\alpha_m}y} \int e^{\frac{\gamma_m}{\alpha_m}y} f(x, y) dy = \psi(x, y) \text{ say, } \beta_m \neq 0
$$
(2.18)

Observe that no arbitrary constant is introduced because PI does not contain arbitrary constants. It therefore follows that

Observe that no arbitrary constant is introduced because *PI* does not contain arbitrary constants.  
\nIt therefore follows that\n
$$
\frac{1}{(\alpha_m D + \beta_m D' + \gamma_m)} f(x, y) = \phi(x, y)
$$
\n(2.19)

This way we operate from the remaining factors from right to the first on the left in turn to finally obtain the *PI* **This way v**<br>obtain the<br>*Method II* 

De composing the operator  $\frac{1}{\sqrt{(n-1)}}$  into partial fractions as ,  $\frac{1}{F(D,D)}$  $\overline{\phantom{a}}$ 

Method II  
\nDecomposing the operator 
$$
\frac{1}{F(D, D')}
$$
 into partial fractions as  
\n
$$
\frac{1}{F(D, D')} = \frac{A_1}{(\alpha_1 D + \beta_1 D' + \gamma_1)} + \frac{A_2}{(\alpha_2 D + \beta_{m2} D' + \gamma_2)} + \dots + \frac{A_m}{(\alpha_m D + \beta_m D' + \gamma_m)}
$$
\n
$$
= \sum_{j=1}^m \frac{A_j}{(\alpha_j D + \beta_j D' + \gamma_j)}
$$
\n(2.20)

we then perform the inverse operation term-wise to obtain the required  $PI$  as demostrated in the following steps: we then perform the inverse operation term-<br>the following steps:<br> $\frac{A_1}{(\alpha_1 D + \beta_1 D' + \gamma_1)} f(x, y) = G(x, y)$ *PI*

we then perform the inverse operation term-wise to obtain the required PI as demos  
\nthe following steps:  
\n
$$
\frac{A_1}{(\alpha_1 D + \beta_1 D' + \gamma_1)} f(x, y) = G(x, y)
$$
\nwith the corresponding auxiliary equation  
\n
$$
\frac{dx}{\alpha} = \frac{dy}{\beta} = \frac{dG}{f - xG}
$$
\n(2.22)

with the corresponding auxiliary equation

$$
\frac{\mu_1}{\mu_1 + \beta_1 D' + \gamma_1} f(x, y) = G(x, y)
$$
\n
$$
\frac{dx}{\alpha_1} = \frac{dy}{\beta_1} = \frac{dG}{f - \gamma_1 G}
$$
\n(2.21)\n(2.22)

From the first two relation we obtain  
\n
$$
\beta_1 dx - \alpha_1 dy = 0
$$
\n*i.e.*  $\beta_1 x - \alpha_1 y = c$   
\nSimilarly, we have that  
\n
$$
\frac{dG}{f - \gamma_1 G} = \frac{dx}{\alpha_1} \Rightarrow \frac{dG}{dx} = \frac{f - \gamma_1 G}{\alpha_1}
$$
\n(2.23)

Similarly, we have that

$$
\beta_1 x - \alpha_1 y = c
$$
  
have that  

$$
\frac{dG}{f - \gamma_1 G} = \frac{dx}{\alpha_1} \Rightarrow \frac{dG}{dx} = \frac{f - \gamma_1 G}{\alpha_1}
$$

$$
\frac{dG}{dx} + \frac{\gamma_1}{\alpha_1} G = \frac{f}{\alpha_1}, \ \alpha_1 \neq 0
$$

, *ie*

$$
\frac{dG}{f - \gamma_1 G} = \frac{dx}{\alpha_1} \Rightarrow \frac{dG}{dx} = \frac{f - \gamma_1 G}{\alpha_1}
$$
  
\ni*e*,  
\n
$$
\frac{dG}{dx} + \frac{\gamma_1}{\alpha_1} G = \frac{f}{\alpha_1}, \ \alpha_1 \neq 0
$$
\n
$$
\text{This is a first order ODE with an integrating factor } (IF) \ \exp\left(\int \frac{\gamma_1}{\alpha_1} dx\right) = \exp\left(\frac{\gamma_1}{\alpha_1} x\right)
$$

 $(IF) \exp\left(\int \frac{\gamma_1}{\alpha} dx\right) = \exp\left(\frac{\gamma_1}{\alpha}\right)$  $\left(\frac{1}{\alpha_1}dx\right) = \exp\left(\frac{\gamma_1}{\alpha_1}\right)$  $\frac{dG}{dx} + \frac{\gamma_1}{\alpha_1} G = \frac{f}{\alpha_1}$ ,  $\alpha_1 \neq 0$ <br>This is a first order ODE with an integrating factor  $(IF) \exp\left(\int \frac{\gamma_1}{\alpha_1} dx\right) = \exp\left(\int \frac{\gamma_1}{\alpha_1} dx\right)$ *IF*)  $\exp\left(\int \frac{\gamma_1}{\alpha_1} dx\right) = \exp\left(\frac{\gamma_1}{\alpha_1} x\right)$ E with an integrating factor  $(IF)$  exp<br>  $\left(\frac{\gamma_1}{\alpha}x\right)G = A_1 \int \frac{f}{\alpha} \exp\left(\frac{\gamma_1}{\alpha}x\right)dx, \ \alpha_1 \neq 0$ 

This is a first order ODE with an integrating factor 
$$
(IF)
$$
  $\exp\left(\int \frac{\gamma_1}{\alpha_1} dx\right) = \exp\left(\frac{\gamma_1}{\alpha_1} x\right)$   
\ni.e,  $\exp\left(\frac{\gamma_1}{\alpha_1} x\right) G = A_1 \int \frac{f}{\alpha_1} \exp\left(\frac{\gamma_1}{\alpha_1} x\right) dx, \ \alpha_1 \neq 0$  (2.25)  
\ni.e,  $G = A_1 \exp\left(-\frac{\gamma_1}{\alpha_1} x\right) \int \frac{f}{\alpha_m} \exp\left(\frac{\gamma_1}{\alpha_1} x\right) dx$ 

*i.e,* 
$$
\exp\left(\frac{\gamma_1}{\alpha_1}x\right)G = A_1 \int \frac{f}{\alpha_1} \exp\left(\frac{\gamma_1}{\alpha_1}x\right)dx, \ \alpha_1 \neq 0
$$
 (2.25)  
\n*i.e,* 
$$
G = A_1 \exp\left(-\frac{\gamma_1}{\alpha_1}x\right) \int \frac{f}{\alpha_m} \exp\left(\frac{\gamma_1}{\alpha_1}x\right)dx
$$

$$
= \frac{A_1}{\alpha_1} \exp\left(-\frac{\gamma_1}{\alpha_1}x\right) \int \exp\left(\frac{\gamma_1}{\alpha_1}x\right) f(x, y) dx, \ \alpha_1 \neq 0
$$
(2.26)  
\nSimilarly, we have  
\n*i.e,* 
$$
G = \frac{A_1}{\beta_1} \exp\left(-\frac{\gamma_1}{\alpha_1}y\right) \int \exp\left(\frac{\gamma_1}{\alpha_1}y\right) f(x, y) dy = \psi(x, y) \text{ say, } \beta_m \neq 0
$$
(2.27)

S imilarly,we have

$$
= \frac{A_1}{\alpha_1} \exp\left(-\frac{7}{\alpha_1}x\right) \exp\left(\frac{7}{\alpha_1}x\right) f(x, y) dx, \ \alpha_1 \neq 0
$$
\nSimilarly, we have

\n
$$
i\epsilon, \qquad G = \frac{A_1}{\beta_1} \exp\left(-\frac{\gamma_1}{\alpha_1}y\right) \exp\left(\frac{\gamma_1}{\alpha_1}y\right) f(x, y) dy = \psi(x, y) \text{ say, } \beta_m \neq 0 \qquad (2.27)
$$
\nThe expression for the PI is therefore given nas

\n
$$
\frac{A_1}{\alpha} \exp\left(-\frac{\gamma_1}{\alpha}x\right) \exp\left(\frac{\gamma_1}{\alpha}x\right) f(x, y) dx + \frac{A_2}{\alpha} \exp\left(-\frac{\gamma_2}{\alpha}x\right) \exp\left(\frac{\gamma_2}{\alpha}x\right) f(x, y) dx + \dots
$$

Similarly, we have  
\ni.e, 
$$
G = \frac{A_1}{\beta_1} \exp\left(-\frac{\gamma_1}{\alpha_1} y\right) \int \exp\left(\frac{\gamma_1}{\alpha_1} y\right) f(x, y) dy = \psi(x, y)
$$
 say,  $\beta_m \neq 0$  (2.27)  
\nThe expression for the PI is therefore given nas  
\n
$$
\frac{A_1}{\alpha_1} \exp\left(-\frac{\gamma_1}{\alpha_1} x\right) \int \exp\left(\frac{\gamma_1}{\alpha_1} x\right) f(x, y) dx + \frac{A_2}{\alpha_2} \exp\left(-\frac{\gamma_2}{\alpha_2} x\right) \int \exp\left(\frac{\gamma_2}{\alpha_2} x\right) f(x, y) dx + \dots +
$$
  
\n
$$
\dots + \frac{A_m}{\alpha_m} \exp\left(-\frac{\gamma_m}{\alpha_m} x\right) \int \exp\left(\frac{\gamma_m}{\alpha_m} x\right) f(x, y) dx, \quad \alpha_j \neq 0
$$
 (2.28)  
\nor  
\n
$$
\frac{A_1}{\beta_1} \exp\left(-\frac{\gamma_1}{\beta_1} y\right) \int \exp\left(\frac{\gamma_1}{\beta_1} y\right) f(x, y) dx + \frac{A_2}{\beta_2} \exp\left(-\frac{\gamma_2}{\beta_2} x\right) \int \exp\left(\frac{\gamma_2}{\beta_2} x\right) f(x, y) dx + \dots +
$$

or

1 1 1 1 exp *A y* − ( ) ( ) ( ) ( ) 1 2 2 2 1 2 2 2 exp , exp exp , ............. .................... exp exp , , 0 2.28 *m m m j m m m A y f x y dx x x f x y dx A x x f x y dx* + − 

, *ie*

$$
\lim_{m \to \infty} + \frac{1}{\beta_m} \exp\left(-\frac{\mu}{\beta_m} x\right) \exp\left(\frac{\mu}{\beta_m} x\right) f(x, y) dx, \quad \beta_j \neq 0
$$
\n
$$
\sum_{j=1}^m \frac{A_j}{\alpha_j} \exp\left(-\frac{\gamma_j}{\alpha_j} x\right) \exp\left(\frac{\gamma_j}{\alpha_j} x\right) f(x, y) dx, \alpha_j \neq 0
$$
\n
$$
\text{or } \sum_{j=1}^m \frac{A_j}{\beta_j} \exp\left(-\frac{\gamma_j}{\beta_j} x\right) \exp\left(\frac{\gamma_j}{\beta_j} x\right) f(x, y) dx, \beta_j \neq 0
$$
\n(2.29)

*Examples*

1 Obtain the solution of the *PDE*  
\n
$$
\frac{\partial^2 u}{\partial x^2} - 6 \frac{\partial^2 u}{\partial x \partial y} + 9 \frac{\partial^2 u}{\partial y^2} = \text{Tan}(3x + y)
$$

*Solution*

In operator form the PDE is expressible as<br>  $(D^2 - 6DD' + 9D'^2)u = \text{Tan}(3x +$ 

Solution  
\nIn operator form the PDE is expressible as  
\n
$$
(D^2 - 6DD' + 9D'^2)u = \text{Tan}(3x + y)
$$
\n
$$
i\dot{e}, \qquad (D - 3D')^2 u = \text{Tan}(3x + y)
$$
\n
$$
(ii)
$$

 $(D - 3D')^{2}$ 

The corresponding homogeneous equation is

The corresponding homogeneous equation is  
\n
$$
(D-3D')^{2}u = 0
$$
\n(iii)

$$
(D-3D') u = 0
$$
  
\nwith the corresponding complementary function  
\n
$$
u_c = \phi_1(-3x - y) + x\phi_2(3x - y)
$$
 (iv)

The  $PI$  is given as *PI*

The *PI* is given as  
\n
$$
\frac{1}{(D-3D')^{2}}\text{Tan}(3x+y)
$$
\n
$$
= \frac{1}{(D-3D')}\left[\frac{1}{1}\text{exp}\left(-\frac{0}{1}x\right)\text{exp}\left(\frac{0}{1}x\right)\text{Tan}(3x+y)\,dx\right]
$$
\n
$$
= \frac{1}{(D-3D')}\text{Tan}(3x+y)\,dx = \frac{1}{(D-3D')}\text{Tan}(c)\,dx
$$
\nwhere  $c = -3x - y$ 

where  $c = -3$ , *ie*

 $(D - 3D')$  $(vi)$ 1  $x^2$ ie,<br> $PI = \frac{1}{(D-3D')} x \text{Tan } c = \frac{x^2}{2} \text{Tan } c$  $\frac{x^2}{3D'}$  x Tan  $c = \frac{x^2}{2}$ *x*  $PI = \frac{1}{(D-3D')} x \text{Tan } c = \frac{x^2}{2} \text{Tan } c$  (*vi*  $\frac{1}{D-3D}$  $=\frac{1}{(D-3D')}\pi \tan c = \frac{x^2}{2} \tan c$ 

The general solution of the *PDE* is therefore  

$$
u(x, y) = \phi_1(-3x - y) + x\phi_2(3x - y) + \frac{x^2}{2}\tan(3x + y)
$$

2 Solve the PDE  
\n
$$
(4D^2 - 4DD' + D'^2)u = 16\ln(x+2y)
$$

*Solution*

The corresponding homogeneous equation is given as<br>  $(4D^2 - 4DD' + D'^2)u = 0$  (*i*)

Solution  
The corresponding homogeneous equation is give  

$$
(4D^2 - 4DD' + D'^2)u = 0
$$
 (*i*)  
*ie*, 
$$
(2D - D')^2 u = 0
$$
 (*ii*)

 $(2D-D')^{2} u = 0$  (*ii*)

*ie,* 
$$
(2D - D') u = 0
$$
 *(ii)*  
The complementary function is given as  

$$
u_c = \phi_1(-x - 2y) + x\phi_2(-x - 2y)
$$
 *(iii)*

and the PI is given as

$$
\frac{1}{(2D - D')^{2}}.16\ln(x + 2y) \qquad (iv)
$$
  
\n
$$
= \frac{1}{(2D - D')}\left[\frac{1}{2}exp\left(-\frac{0}{2}x\right)\int exp\left(-\frac{0}{2}x\right)\ln(c)dx\right], c = -x - 2y \qquad (v)
$$
  
\n
$$
= \frac{1}{(2D - D')}.8x\ln(x + 2y) \qquad (vi)
$$
  
\n
$$
= 8\left[\frac{1}{2}exp\left(-\frac{0}{2}x\right)\int exp\left(\frac{0}{2}x\right) x\ln(c)\right] \qquad (vii)
$$
  
\n
$$
= 4\int x\ln(c) = 2x^{2}\ln(x + 2y) \qquad (viii)
$$
  
\ni.e,  
\n
$$
PI = 2x^{2}\ln(x + 2y) \qquad (ix)
$$

, *ie*

$$
= 4 \int x \ln(c) = 2x^2 \ln(x+2y)
$$
 (viii)  
ie,  

$$
PI = 2x^2 \ln(x+2y)
$$
 (ix)  
The solution to the PDE is therefore given as  

$$
u(x, y) = \phi_1(-x-2y) + x\phi_2(-x-2y) + 2x^2 \ln(x+2y)
$$

The solution to the PDE is therefore given as  
\n
$$
u(x, y) = \phi_1(-x - 2y) + x\phi_2(-x - 2y) + 2x^2 \ln(x + 2y)
$$
\n2.2.3 Some Special Cases.  
\nWe recall that the particular integral of (2.6) is given as

2.2.3 Some Special Cases.

 $(2.6)$ as

2.2.3 Some Special Cases.  
We recall that the particular integral of (2.6) is given as  

$$
u_p(x, y) = \frac{1}{F(D, D')} \cdot f(x, y)
$$
(2.30)  
This is determined almost the same way as that of *ODEs*

This is determined almost the same way as that of ODEs.

The inverse operator may be expanded using the Binomial Theorem an d thereafter performing the  $\left[ ^{1},\left( D^{\prime }\right) ^{-1}$ This is determined almost the same way as that of *ODEs*.<br>
The inverse operator may be expanded using the Binomial Theorem and thereafter performing the integration  $D^{-1}$ ,  $(D')^{-1}$  nwith respect to x and y respectively. T functions may be obtained by much shorter method than the general method.<br>
In this section we note the following pertinent rules:<br>  $\frac{1}{F(D, D')} e^{ax+by} = \frac{1}{F(a, b)} e^{ax+by}$  provided  $F(a, b) \neq 0$ integration  $D^{-1}$ ,  $(D'$ <br>functions may be ob<br>In this section we no<br>*Case I*:

functions may be obtained by much shorter method than the gen  
In this section we note the following pertinent rules:  
*Case I*:  

$$
\frac{1}{F(D,D')}e^{ax+by} = \frac{1}{F(a,b)}e^{ax+by} \text{ provided } F(a,b) \neq 0
$$
*Case II*:

 : *Case II* Case II: *e III*

$$
\frac{1}{F(D,D')}e^{ax+by} = \frac{1}{F(a,b)}e^{ax+by} \text{ provided } F(a,b) \neq 0
$$
  
Case II :  

$$
\frac{1}{F(D,D')}e^{ax+by}\phi(x,y) = e^{ax+by}\frac{1}{F(D+a,D'+b)}\phi(x,y), \phi(x,y) \text{ is arbitrary.}
$$
  
Case III :  
If  $F(a,b) = 0$  in Case I, then the PI is obtained as follow:

Case III :

 $(a,b)$ =

Case III :  
\n
$$
F(D, D')^{\text{max}} = F(D+a, D'+b)^{\text{max}(M, Y)} \times (M, Y) \text{ as a scalar.}
$$
\nIf  $F(a,b) = 0$  in Case I, then the PI is obtained as follow:  
\n
$$
\frac{1}{F(D, D')} e^{ax+by} = \frac{1}{F(D, D')} e^{ax+by} \cdot 1 = e^{ax+by} \frac{1}{F(D+a, D'+b)}.
$$

and then apply case II. *Case IV*

1

 $IV:$ 

$$
\frac{F(D, D')}{F(D, D')} = \frac{F(D, D')}{F(D, D')} = \frac{1}{F(D + a, D' + b)}.
$$
\nand then apply case II.

\n
$$
\frac{1}{F(D, D')} \cdot \frac{\cos(ax + by)}{F(D, D')} = \frac{1}{F(D^2, DD', D'^2)} \cdot \frac{\cos(ax + by)}{F(a^2, ab, b^2)} = \frac{1}{F(-a^2, -ab, -b^2)} \cdot \frac{\cos(ax + by)}{F(a^2, ab, b^2)} = 0
$$
\nIf  $F(a^2, ab, b^2) = 0$  this case fails. We then compute the PI by considering the real and imaginary parts of the following.

 $= 0$  this cas  $, -ab,$ <br>0 this *i* ails.  $F(T)$ <br>=  $\frac{1}{F(T)}$ <br> $F(a^2, ab, b)$  $\begin{pmatrix} -a & -ab & -b \\ b^2 & 0 & \text{this can} \end{pmatrix} = 0$  this control =  $(\cdot, -b^2)$ <br>is case fails. We then If  $F\left(a^2\right)$ 

In this case we apply the Binomial theorem to the inverse operator and then operate on  $x^m y^n$ . These methods are evidently shorter ways of obtaining the respective PIs. *x<sup>m</sup> y* 

: *Examples*

1 Solve the *PDE*

intetious are evidently shorter ways of obtain:

\n
$$
les:
$$
\n
$$
the PDE
$$
\n
$$
(D^2 - D'^2 - 3D + 3D')u = xy + e^{x+2y}.
$$

*Solution*

Observe that the given differential equation may be put in the form

Solution  
\n
$$
(D - D' - 3D + 3D')u = xy + e
$$
\nSolution  
\nObserve that the given differential equation may be p  
\n
$$
(D - D')(D + D' - 3)u = xy + e^{x+2y}.
$$
\nThe complimentary function is given as  
\n
$$
\phi_1(-x - y) + e^{3x}\phi_2(x - y)
$$

The complimentary function is given as  
\n
$$
\phi_1(-x - y) + e^{3x} \phi_2(x - y)
$$

The particular integral is given as

$$
\phi_1(-x-y) + e^{3x}\phi_2(x-y)
$$
  
\nThe particular integral is given as  
\n
$$
\frac{1}{(D-D')(D+D'-3)} [xy + e^{x+2y}]
$$
\n
$$
= \frac{1}{(D-D')(D+D'-3)} [xy] + \frac{1}{(D-D')(D+D'-3)} [e^{x+2y}]
$$
\n
$$
\frac{1}{(D-D')(D+D'-3)} [xy]
$$
\n
$$
= -\frac{1}{3D} \Big( 1 - \frac{D'}{D} \Big)^{-1} \Big( 1 - \frac{D+D'}{3} \Big)^{-1} [xy]
$$
\n
$$
= -\frac{1}{3D} \Big( 1 + \frac{D'}{D} + \frac{D'^2}{D^2} + \dots \Big) \Big( 1 + \frac{D+D'}{3} + \frac{(D+D')^2}{9} \Big) [xy]
$$
\n
$$
= -\frac{1}{3D} \Big( 1 + \frac{D+D'}{3} + \frac{2DD'}{9} + \frac{D'}{D} + \frac{DD'}{9} + \frac{D'}{3} \Big) [xy]
$$
\n
$$
= -\frac{1}{3D} \Big( 1 + \frac{D}{3} + \frac{DD'}{3} + \frac{D'}{D} + \frac{2D'}{3} \Big) [xy]
$$
\n
$$
= -\frac{1}{3D} \Big( xy + \frac{y}{3} + \frac{1}{3} + \frac{x^2}{2} + \frac{2x}{3} \Big)
$$
\n
$$
= -\frac{1}{3} \Big( \frac{x^2y}{2} + \frac{xy}{3} + \frac{x}{3} + \frac{x^3}{6} + \frac{x^2}{3} \Big)
$$

and

$$
= -\frac{1}{3} \left( \frac{y}{2} + \frac{y}{3} + \frac{y}{3} + \frac{y}{6} + \frac{y}{3} \right)
$$
  
and  

$$
\frac{1}{(D - D')(D + D' - 3)} \left[ e^{x+2y} \right]
$$

$$
= \frac{1}{(D + D' - 3)} \cdot \frac{1}{(1-2)} \left[ e^{x+2y} \right] = -\frac{1}{(D + D' - 3)} \cdot \left[ e^{x+2y} \right]
$$

$$
= -\left[ e^{x+2y} \right] \frac{1}{(D + 1 + D' + 2 - 3)} \cdot 1 = -\left[ e^{x+2y} \right] \frac{1}{(D + D')} \cdot 1
$$

$$
= -\left[e^{x+2y}\right] \frac{1}{D'} \left(1 + \frac{D}{D'}\right)^{-1} . 1 = -\left[e^{x+2y}\right] \frac{1}{D'} . 1
$$

$$
= -ye^{x+2y} .
$$

Thus, the general solution is

$$
= -ye^{x+2y}.
$$
  
\nThus, the general solution is  
\n
$$
u(x, y) = \phi_1(-x-y) + e^{3x}\phi_2(x-y) - ye^{x+2y} - \frac{1}{3}\left(\frac{x}{3} + \frac{xy}{3} + \frac{x^2}{3} + \frac{x^2y}{2} + \frac{x^3}{6}\right)
$$
  
\n2 Solve the *PDE*  
\n
$$
(D^2 - DD' + D' - 1)u = \cos(x + 2y) + e^y.
$$

2 Solve the *PDE*

$$
\varphi_1(-x - y) + e \varphi_2(x - y) - ye
$$
  
he *PDE*  

$$
(D^2 - DD' + D' - 1)u = \cos(x + 2y) + e^y.
$$

*Solution*

Observe that the PDE is of the form *PDE*

Solution  
Observe that the *PDE* is of the form  

$$
(D-1)(D-D+1)u = \cos(x+2y) + e^y.
$$

The reduced DE is

$$
(D-1)(D-D+1)u = \cos(x+2y) + e^{y}.
$$
  
\nThe reduced DE is  
\n
$$
(D-1)(D-D'+1)u = 0
$$
  
\n
$$
CF = e^{x} \phi_1(-y) + e^{-x} \phi_2(-x-y)
$$
  
\n
$$
PI = \frac{1}{(D^2 - DD' + D' - 1)} \Big[ \cos(x+2y) + e^{y} \Big]
$$
  
\n
$$
\frac{1}{(D^2 - DD' + D' - 1)} \Big[ \cos(x+2y) \Big]
$$
  
\n
$$
= \frac{1}{(-1^2 - (2)(-1) + D' - 1)} \Big[ \cos(x+2y) \Big] = \frac{1}{(D')} \Big[ \cos(x+2y) \Big]
$$
  
\n
$$
= \frac{1}{2} \sin(x+2y).
$$
  
\n
$$
\frac{1}{(D^2 - DD' + D' - 1)} \Big[ e^{y} \Big] = e^{y} \frac{1}{(D^2 - D(D' + 1) + (D' + 1) - 1)} [1]
$$
  
\n
$$
= e^{y} \frac{1}{(D^2 - DD' - D + D')} [1] = -e^{y} \frac{1}{D} \Big[ 1 - \Big( \frac{D'}{D} + D - D' \Big) \Big]^{-1} [1]
$$
  
\n
$$
= -e^{y} \frac{1}{D} [1] = -xe^{y}
$$

Thus,

$$
= -e^y \frac{1}{D} [1] = -xe^y
$$
  
\nThus,  
\n
$$
u(x, y) = e^x \phi_1(-y) + e^{-x} \phi_2(-x - y) + \frac{1}{2} \sin(x + 2y) - xe^y.
$$
  
\n3 Obtain the solution to the *PDE*  
\n
$$
(D^2 - D')u = xe^{ax + a^2y}.
$$

3 Obtain the solution to the *PDE*<br> $(D^2 - D')u = xe^{ax+a^2y}$ .

$$
(D2 – D')u = xeax+a2y.
$$

*Solution*

The reduced equation is

Solution  
The reduced equation is  

$$
(D^2 - D')u = 0.
$$

The operator  $D^2 - D'$  is irreducible. Hence,<br> $F(a,b) = a^2 - b = 0 \Rightarrow b = a^2$ . The operator  $D^2 - D'$  is irreducible. F<br> $F(a,b) = a^2 - b = 0 \Rightarrow b = a^2$ .  $D^2 - D$  $-D'$  $e^2 - D'$  is irreducible. Hence,<br>=  $a^2 - b = 0 \Rightarrow b = a^2$ .

erator 
$$
D^2 - D'
$$
 is irreducible. H  
\n $F(a,b) = a^2 - b = 0 \Rightarrow b = a^2$ .  
\n $u = xe^{ax+a^2y}$  is a solution of F (

<sup>2y</sup> is a solution of  $F(D^2, D')$ The operator  $D^2 - D'$  is irreducible. Hence,<br>  $F(a,b) = a^2 - b = 0 \Rightarrow b = a^2$ .<br>
Hence,  $u = xe^{ax+a^2y}$  is a solution of  $F(D^2, D') = 0$  has the complimentary function erator  $D^2 - D'$  is irreducible. Hence,<br>  $F(a,b) = a^2 - b = 0 \Rightarrow b = a^2$ .<br>  $u = xe^{ax+a^2y}$  is a solution of  $F(D^2, D)$ + ator  $D^2 - D'$  is irreducible. Hence,<br>  $(a,b) = a^2 - b = 0 \Rightarrow b = a^2$ .<br>  $= xe^{ax+a^2y}$  is a solution of  $F(D^2, D') = 0$  has the co

Hence, 
$$
u = xe^{\alpha + a^2y}
$$
 is a solution of  $F(D^2, D') = 0$  has the complementary function  
\n
$$
u_c = \sum_{r=1}^{\infty} A_r e^{\alpha_r + a_r^2y}
$$
\n
$$
PI = \frac{1}{D^2 - D'} \Big[ xe^{\alpha + a^2y} \Big] = e^{\alpha + a^2y} \cdot \frac{1}{D^2 - D'} \Big[ x \Big]
$$
\n
$$
= e^{\alpha + a^2y} \cdot \frac{1}{(D + a)^2 - (D' + a^2)} \cdot \Big[ x \Big] = e^{\alpha + a^2y} \cdot \frac{1}{(D^2 + 2aD + a^2 - D' - a^2)} \cdot \Big[ x \Big]
$$
\n
$$
= e^{\alpha + a^2y} \cdot \frac{1}{(D^2 + 2aD - D')} \cdot \Big[ x \Big] = e^{\alpha + a^2y} \cdot \frac{1}{(D^2 + 2aD)} \Big( 1 - \frac{D'}{D^2 + 2aD} \Big)^{-1} \cdot \Big[ x \Big]
$$
\n
$$
= e^{\alpha + a^2y} \cdot \frac{1}{(D^2 + 2aD)} \Big[ 1 + \frac{D'}{D^2 + 2aD} + \frac{D'^2}{(D^2 + 2aD)^2} + \dots \Big] \cdot \Big[ x \Big]
$$
\n
$$
= e^{\alpha + a^2y} \cdot \frac{1}{(D^2 + 2aD)} \cdot \Big[ x \Big] = e^{\alpha + a^2y} \cdot \frac{1}{2aD} \Big[ 1 + \frac{D}{2a} \Big)^{-1} \cdot \Big[ x \Big]
$$
\n
$$
= e^{\alpha + a^2y} \cdot \frac{1}{2aD} \Big[ 1 - \frac{D}{2a} - \frac{D^2}{4a^2} - \dots \Big] \cdot \Big[ x \Big]
$$
\n
$$
= e^{\alpha + a^2y} \cdot \frac{1}{2aD} \Big[ x - \frac{1}{2a} \Big] \cdot \Big[ x \Big]
$$
\n
$$
= e^{\alpha + a^2y} \cdot \frac{1}{2aD} \Big[ x - \frac{1}{2a} \Big] \cdot \Big[ x \Big]
$$
\n
$$
= e^{\alpha + a^2y} \cdot \frac{1}{2a
$$

Hence, the general solution of the PDE is

Hence, the general solution of the PDE is  
\n
$$
u(x, y) = \sum_{r=1}^{\infty} A_r e^{a_r x + a_r^2 y} + \frac{x}{4a^2} (ax - 1) e^{ax + a^2 y}.
$$

## CHAPTER THREE

### SECOND – ORDER DIFFERENTIAL EQUATIONS II

#### 3.1 PARTIAL DIFFERENTIAL EQUATIONS OF THE CAUCHY-EULER TYPE

Equations of the of the Cuachy-Euler type are the PDEs of the form

3.1 PARTIAL DIFFERENTIAL EQUATIONS OF THE CAUCHY-EULER TYPE  
Equations of the of the Cuachy-Euler type are the PDEs of the form  

$$
F(xD, yD')u = f(x, y)
$$
(3.1)  
where *F* is a polynomial in the indeterminate *xD* and *yD'*

where F is a polynomial in the indeterminate  $xD$  and  $yD'$ . In this case we make the following transformations: ARTIAL DIFFERENTIAL EQUATIONS OF THI<br>ons of the of the Cuachy-Euler type are the PDEs  $F(xD, yD')u = f(x, y)$ <br>*F* is a polynomial in the indeterminate *xD* and *yD*<br>case we make the following transformations:  $\overline{\phantom{a}}$ 

$$
F(xD, yD')u = f(x, y)
$$
(3.1)  
where *F* is a polynomial in the indeterminate *xD* and *yD'*.  
In this case we make the following transformations:  

$$
s = \ln x, t = \ln y, \ \mathcal{G} = \frac{\partial}{\partial s} \text{ and } \phi = \frac{\partial}{\partial t}
$$
(3.2)  
It is therefore immediate from (3.2) that  

$$
(xD)u = \mathcal{G}u, (x^2D^2)u = \mathcal{G}(\mathcal{G} - 1)u \text{ and } (x^3D^3)u = \mathcal{G}(\mathcal{G} - 1)(\mathcal{G} - 2)u
$$
(3.3)

 $(3.2)$ 

$$
\frac{\partial s}{\partial t}
$$
  
\nIt is therefore immediate from (3.2) that  
\n
$$
(xD)u = \mathcal{G}u, (x^2D^2)u = \mathcal{G}(\mathcal{G}-1)u \text{ and } (x^3D^3)u = \mathcal{G}(\mathcal{G}-1)(\mathcal{G}-2)u
$$
\n
$$
(yD')u = \phi u, (y^2D'^2)u = \phi(\phi-1)u \text{ and } (y^3D'^3)u = \phi(\phi-1)(\phi-2)u
$$
\nSubstituting (3.3) into (3.1) transforms it into linear equation with constant coefficients with  $\mathcal{G}$  and  $\phi$  as the new independent variables

 $(3.3)$  into  $(3.1)$ the new independent variables.  $\mathcal{G}$  and  $\phi$  as

Examples.

Transform the following *PDE* to linear form

The new independent variables.  
\nixamples.  
\nTransform the following *PDE* to linear form  
\n
$$
(x^2D^2 - 4xyDD' + 4y^2D'^2 + 4yD' + xD)u = x^2y.
$$
 (i)  
\n
$$
(i)
$$

Observe that the given PDE is of Cauchy-Euler type. We then define the following transformation:

Transform the following *PDE* to linear form  
\n
$$
(x^2D^2 - 4xyDD' + 4y^2D'^2 + 4yD' + xD)u = x^2y.
$$
\n(*i* Observe that the given PDE is of Cauchy-Euler type. We then det  
\n
$$
s = \ln x, t = \ln y, \ \mathcal{G} = \frac{\partial}{\partial s} \text{ and } \phi = \frac{\partial}{\partial t}
$$
\n(*ii*) Using (*ii*) in (*i*) we obtain

Using  $(ii)$  in  $(i)$  we obtain that the<br> $s =$ <br> $ii$ ) in (*i*)

$$
s = \ln x, t = \ln y, \ \mathcal{G} = \frac{\partial}{\partial s} \text{ and } \phi = \frac{\partial}{\partial t} \qquad (ii)
$$
  
Using (ii) in (i) we obtain  

$$
\left[\mathcal{G}(\mathcal{G} - 1) - 4\mathcal{G}\phi + 4\phi(\phi - 1) + 4\phi + \mathcal{G}\right]u = e^{2s}e^{t} = e^{2s+t}.
$$
  
i.e, 
$$
\left(\mathcal{G}^{2} - 4\mathcal{G}\phi + 4\phi^{2}\right)u = e^{2s+t} \qquad (iii)
$$

$$
\Rightarrow \qquad \left(\mathcal{G} - 2\phi\right)^{2}u = e^{2s+t} \qquad (iv)
$$

$$
[\mathcal{G}(\mathcal{G}-1) - 4\mathcal{G}\phi + 4\phi(\phi-1) + 4\phi + \mathcal{G}]u = e^{2s}
$$
  
\n*i.e,*  $(9^2 - 4\mathcal{G}\phi + 4\phi^2)u = e^{2s+t}.$  *(iii)*  
\n $\Rightarrow (9-2\phi)^2 u = e^{2s+t}.$  *(iv)*  
\nThis is a linear DE with constant coefficients.  
\n $CF = \phi_1(-2s-t) + s\phi_2(-2s-t)$ 

This is a linear DE with constant coefficients.

$$
\begin{aligned}\n\text{Let,} & (\mathcal{F} - 4\mathcal{H}) \mathcal{H} = e^{-\alpha} \cdot \mathcal{H} \\
& (\mathcal{F} - 2\phi)^2 u = e^{2s+t} \cdot \mathcal{H} \\
\text{This is a linear DE with constant coefficients.} \\
& CF &= \phi_1 \left( -2s - t \right) + s\phi_2 \left( -2s - t \right) \\
&= \psi_1 \left( 2s + t \right) + s\psi_1 \left( 2s + t \right) \quad \text{(v)} \\
& PI &= \frac{1}{\left( \mathcal{G} - 2\phi \right)^2} \left[ e^{2s+t} \right] = e^{2s+t} \cdot \frac{1}{\left( \mathcal{G} - 2\phi \right)^2} [1]\n\end{aligned}
$$

$$
= e^{2s+t} \cdot \frac{1}{9^2} \left(1 - \frac{2\phi}{9}\right)^{-2} [1] = e^{2s+t} \cdot \frac{1}{9^2} [1]
$$

$$
= e^{2s+t} \left(\frac{s^2}{2} + \alpha s + \beta\right)
$$
The general solution is therefore

$$
= e^{2s+t} \left( \frac{s}{2} + \alpha s + \beta \right)
$$
  
The general solution is therefore,  

$$
u = \psi_1 (2s+t) + s\psi_1 (2s+t) + e^{2s+t} \left( \frac{s^2}{2} + \alpha s + \beta \right)
$$
  
i.e, 
$$
u = \psi_1 (2 \ln x + \ln y) + \ln x \psi_1 (2 \ln x + \ln y) + e^{2 \ln x + \ln y} \left( \frac{1}{2} (\ln x)^2 + \alpha \ln x + \beta \right)
$$

$$
= \psi_1 (\ln x^2 y) + \ln x \psi_1 (2 \ln x + \ln y) + \left( \frac{1}{2} (\ln x)^2 + A \ln x \right) x^2 y
$$
  
Example  

$$
\frac{1}{x^2} \frac{\partial^2 u}{\partial x^2} - \frac{1}{x^3} \frac{\partial u}{\partial x} = \frac{1}{y^2} \frac{\partial^2 u}{\partial y^2} - \frac{1}{y^3} \frac{\partial u}{\partial y}.
$$
 (i)

*Example*

*i*

$$
= \psi_1 (\ln x \ y) + \ln x \ \psi_1 (2 \ln x + \ln y) + \left(\frac{1}{2} (\ln x) \ + A \ln x \right)
$$
  
\nExample  
\n
$$
\frac{1}{x^2} \frac{\partial^2 u}{\partial x^2} - \frac{1}{x^3} \frac{\partial u}{\partial x} = \frac{1}{y^2} \frac{\partial^2 u}{\partial y^2} - \frac{1}{y^3} \frac{\partial u}{\partial y}.
$$
\n(i)  
\nSuppose  $s = \frac{x^2}{2}$  and  $t = \frac{y^2}{2}$  (ii)  
\nThen  
\n
$$
\frac{\partial u}{\partial x} = \frac{\partial u}{\partial s} \frac{\partial s}{\partial x} = x \frac{\partial u}{\partial s} \text{ or } \frac{\partial u}{\partial s} = \frac{1}{x} \frac{\partial u}{\partial x}
$$

Then

Suppose 
$$
s = \frac{1}{2}
$$
 and  $t = \frac{1}{2}$    
\nThen  
\n
$$
\frac{\partial u}{\partial x} = \frac{\partial u}{\partial s} \frac{\partial s}{\partial x} = x \frac{\partial u}{\partial s} \text{ or } \frac{\partial u}{\partial s} = \frac{1}{x} \frac{\partial u}{\partial x}
$$
\n
$$
\frac{\partial^2 u}{\partial s^2} = \frac{\partial}{\partial s} \frac{\partial u}{\partial s} = \frac{1}{x} \frac{\partial}{\partial x} \left( \frac{1}{x} \frac{\partial}{\partial x} \right) = \frac{1}{x^2} \frac{\partial^2}{\partial x^2} - \frac{1}{x^3} \frac{\partial}{\partial x}
$$
\n
$$
\frac{1}{x^2} \frac{\partial^2}{\partial x^2} - \frac{1}{x^3} \frac{\partial}{\partial x} = \frac{\partial^2 u}{\partial s^2}
$$
\nSimilarly,  
\n
$$
\frac{1}{x^2} \frac{\partial^2}{\partial x^2} - \frac{1}{x^3} \frac{\partial}{\partial x} = \frac{\partial^2 u}{\partial t^2}
$$
 *(iv)*

Similarly,

$$
\frac{1}{x^2} \frac{\partial^2}{\partial x^2} - \frac{1}{x^3} \frac{\partial}{\partial x} = \frac{\partial^2 u}{\partial s^2}
$$
  

$$
\frac{1}{y^2} \frac{\partial^2}{\partial y^2} - \frac{1}{y^3} \frac{\partial}{\partial y} = \frac{\partial^2 u}{\partial t^2}
$$
 (*iv*)

 $(\vartheta - \phi)(\vartheta + \phi)$ 

Similarly,  
\n
$$
\frac{1}{y^2} \frac{\partial^2}{\partial y^2} - \frac{1}{y^3} \frac{\partial}{\partial y} = \frac{\partial^2 u}{\partial t^2}
$$
 (*iv*)  
\nThus the given PDE is transformed into  
\n
$$
\frac{\partial^2 u}{\partial s^2} = \frac{\partial^2 u}{\partial t^2}
$$
 or  $(9^2 - \phi^2)u = 0$  (*v*)  
\nwhere  $9 = \frac{\partial}{\partial s}$  and  $\phi = \frac{\partial}{\partial t}$   
\n $\Rightarrow$   $(9 - \phi)(9 + \phi) = 0$ 

where

Hence,

$$
(\mathcal{G} - \phi)(\mathcal{G} + \phi) = 0
$$
  

$$
u = \varphi_1(-s - t) + \varphi_2(s - t)
$$
  

$$
= \varphi_1\left(-\frac{x^2 + y^2}{2}\right) + \varphi_2\left(\frac{x^2 - y^2}{2}\right)
$$
  

$$
= \psi_1(x^2 + y^2) + \psi_2(x^2 - y^2)
$$

#### 3.2 SECOND-ORDER WITH VARIABLE COEFFICIENTS. *PDE*

#### Definition.

A partial differential equation with variable coefficients is that which contains atleast one of the partial derivative of the second order and none higher than the second. This is simplified if we consider the case<br>of two independent variables.<br>We shall define the following:<br> $p = \frac{\partial u}{\partial x}, q = \frac{\partial u}{\partial y}, r = \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial x$ of two independent variables.

We shall define the following:

derivative of the second order and none higher than the second. This is simplified in we c  
of two independent variables.  
We shall define the following:  

$$
p = \frac{\partial u}{\partial x}, q = \frac{\partial u}{\partial y}, r = \frac{\partial^2 u}{\partial x^2} = \frac{\partial p}{\partial x}, s = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x}\right)
$$

$$
= \frac{\partial p}{\partial y} = \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial y}\right) = \frac{\partial q}{\partial x}, t = \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y}\right) = \frac{\partial q}{\partial y}
$$
(3.4)  
Our discussion shall be limited to that of the variable coefficients which are of first degree  
*ie*,  $Rr + Ss + Tt = V$   
(3.5)  
in which *R*, *S*, *T* and *V* are in general functions of *Rx*, *y*, *p*, *q* and *u*.  
This will be illustrated by examples solvable by inspection

Our discussion shall be limited to that of the variable coefficients which are of first degree in  $r, s,$ *r s t*

 $(3.5)$ 

$$
ie, \t Rr + Ss + Tt = V
$$

in which *R*, *S*, *T* and *V* are in generally small be illustrated by example.<br>
1 Solve  $s = 2x + 2y$ <br>
Solution

This will be illus trated by examples solvable by inspection.

Example.

1 Solve 
$$
s = 2x + 2y
$$

*Solution*

The PDE is given by

I Solve 
$$
s = 2x + 2y
$$
  
Solution  
The PDE is given by  

$$
\frac{\partial^2 u}{\partial x \partial y} = 2x + 2y
$$
(i)  
Integrating wrt y we have

Integrating wrt y we have

$$
\frac{\partial^2 u}{\partial x \partial y} = 2x + 2y
$$
 (i)  
Integrating wrt y we have  

$$
\frac{\partial u}{\partial x} = 2xy + y^2 + h(x)
$$
 (ii)  
Finally, integrating wrt x yields  

$$
u(x, y) = x^2y + xy^2 + \int h(x)dx + g(y)
$$
 (iii)

Finally, integrating wrt  $x$  yields  $\frac{\partial x}{\partial x}$ <br>inally, integrati<br> $u(x,$ <br> $u(x,$ *x*

$$
\partial x \qquad (1)
$$
\n
$$
\partial x \qquad (2)
$$
\n
$$
\partial y \qquad (3)
$$
\n
$$
u(x, y) = x^2 y + xy^2 + \int h(x) dx + g(y) \qquad (iii)
$$
\n
$$
u(x, y) = x^2 y + xy^2 + \phi(x) + g(y) \qquad (iv)
$$
\n
$$
x + p = 9x^2 y^2.
$$
\n(given by

Finally, integrating wrt x yields  
\n
$$
u(x, y) = x^2y + xy^2 + \int h(x)dx + g(y)
$$
\n
$$
u(x, y) = x^2y + xy^2 + \phi(x) + g(y)
$$
\n
$$
u(x, y) = x^2y + xy^2 + \phi(x) + g(y)
$$
\n
$$
u(x, y) = x^2y^2 + xy^2 + \phi(x) + g(y)
$$
\n
$$
u(x, y) = x^2y^2 + xy^2 + \phi(x) + g(y)
$$
\n
$$
u(x, y) = x^2y^2 + xy^2 + \phi(x) + g(y)
$$
\n
$$
u(x, y) = x^2y^2 + xy^2 + \phi(x) + g(y)
$$
\n
$$
u(x, y) = x^2y^2 + xy^2 + \phi(x) + g(y)
$$

2 Solve  $xr + p = 9x^2y +$ <br>
2 Solve  $xr + p = 9x^2y^2$ .<br>
2 Solve  $xr + p = 9x^2y^2$ .

The PDE is given by

$$
u(x, y) = x^2 y + xy + \varphi(x) + g(y)
$$
\n
$$
2 \quad \text{Solve } xr + p = 9x^2 y^2.
$$
\n
$$
x \frac{\partial^2 u}{\partial x^2} + p = 9x^2 y^2
$$
\n
$$
i e, \qquad \frac{\partial p}{\partial x} + \frac{1}{x} p = 9xy^2
$$
\n
$$
(ii)
$$

The PDE is given by  
\n
$$
x \frac{\partial^2 u}{\partial x^2} + p = 9x^2 y^2
$$
\ni*e*,  
\n
$$
\frac{\partial p}{\partial x} + \frac{1}{x} p = 9xy^2
$$
\n(ii)  
\nThe DE in (ii) has an integrating factor (IF) x  
\ni*e*,  
\n
$$
(xp)' = 9xy^2
$$
\n(iii)  
\n
$$
[Q^2]^2 I = Q^3^2 + Q^2
$$
\n(iv)

*ie,* 
$$
(xp)' = 9xy^2
$$
 (*iii*)  
*ie,*  $xp = \int 9x^2y^2 dx = 3x^3y^2 + f(y)$  (*iv*)

The DE in (ii) has an integrating factor 
$$
(IF) x
$$
  
\ni.e,  $(xp)' = 9xy^2$  (iii)  
\ni.e,  $xp = \int 9x^2y^2 dx = 3x^3y^2 + f(y)$  (iv)  
\ni.e,  $xp = \int 9x^2y^2 dx = 3x^3y^2 + f(y)$  (v)

*i.e,* 
$$
xp = \int 9x^2 y^2 dx = 3x^3 y^2 + f(y)
$$
 (v)

*i.e.*  
\n
$$
\frac{\partial u}{\partial x} = 3x^2 y^2 + \frac{1}{x} h(y)
$$
\n
$$
u(x, y) = x^3 y^2 + \int \frac{1}{x} h(y) dx
$$

$$
\frac{d}{dx} = 3x \, y + \frac{1}{x} h(y)
$$
\n
$$
\therefore \qquad u(x, y) = x^3 y^2 + \int \frac{1}{x} h(y) dx
$$
\n
$$
i.e, \qquad u(x, y) = x^3 y^2 + h(y) \ln x + \alpha(y)
$$
\n
$$
3 \quad \text{Solve } s - t = \frac{x^2}{y}.
$$

$$
\therefore \quad u(x, y)
$$
  
*ie,* 
$$
u(x, y)
$$
  
3 Solve  $s - t = \frac{x^2}{y}$ 

*Solution*

 $(i)$ 2 The *DE* is Solve  $s - t = \frac{x^2}{y}$ .<br> *tion*<br> *DE* is  $\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x} = \frac{x^2}{y}$  (*i y*<br> $\frac{p}{y} - \frac{\partial q}{\partial x} = \frac{x^2}{y}$ <br>respect to *y*  $=\frac{x^2}{y}$ .<br>  $\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x} = \frac{x^2}{y}$  (*i*) y<br> $\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x} = \frac{x^2}{y}$  $\frac{x^2}{y}$  (*i*)<br>*y* and treating *x* 

Integrating with respect to  $y$  and treating  $x$  as a constant and conversely yields

 $\alpha$ 

$$
\frac{dP}{dy} - \frac{dq}{dx} = \frac{x}{y}
$$
 (*i*)  
g with respect to y and treating  

$$
p - q = -\frac{x}{y} + f(x)
$$
 (*ii*)

This is Lagranges linear equation with auxiliary equation

$$
p - q = -\frac{x}{y} + f(x)
$$
 (*ii*)  
ranges linear equation with auxi  

$$
\frac{dx}{1} = \frac{dy}{-1} = \frac{du}{f - x/y}
$$
 (*iii*)  
rst two ratios we obtain  

$$
-dx - dy = 0
$$
 (*iv*)

From the first two ratios we obtain

From the first two ratios we obtain  
\n
$$
-dx - dy = 0
$$
 (*i*  
\n*ie*,  $x + y = c$  (*v*)

...<br>,<br>... From the first last ratios we have

$$
u \, dx \, dy = 0 \qquad (iv)
$$
\n
$$
x + y = c \qquad (v)
$$
\nin the first last ratios we have

\n
$$
du = f(x)dx - \frac{x}{y}dx = f(x)dx - \frac{x}{c - x}dx
$$
\n
$$
= f(x)dx + \left(1 - \frac{c}{c - x}\right)dx \qquad (vi)
$$
\ngrating we have

\n
$$
u = \int f(x)dx + \int \left(1 - \frac{c}{c - x}\right)dx = \int f(x)dx + \int f(x)dx
$$

 $(iv)$ 

 $(v)$ 

Integrating we have

$$
y = c - x
$$
  
\n
$$
= f(x)dx + \left(1 - \frac{c}{c - x}\right)dx \qquad (vi)
$$
  
\nIntegrating we have  
\n
$$
u = \int f(x)dx + \int \left(1 - \frac{c}{c - x}\right)dx = \int f(x)dx + x + c\ln(c - x) + \beta(y)
$$
  
\n
$$
= \phi(x) + x + (x + y)\ln y
$$
  
\nThe general solution is therefore  
\n
$$
u = \phi(x) + (x + y)\ln y + F(x, y).
$$

The general solution is the refore

$$
u = \phi(x) + (x + y) \ln y + F(x, y).
$$

 $(3.5)$ The general solution is therefore<br>  $u = \phi(x) + (x + y) \ln y + F(x, y)$ .<br>
We note that (3.5) is a second - order quasilinear *PDE*. It is linear if it can be put in the form *PDE*

$$
u = \phi(x) + (x + y) \ln y + F(x, y).
$$
  
(3.5) is a second - order quasilinear *PDE*. It is linear if it can be put in the  $Rr + Ss + Tt + Pp + Uu = V$  (3.6)  
*T*, *P, U*, and *V*, are functions of *y* and *y*.

We note that (3.5) is a second - order quasilinear *PD*<br>  $Rr + Ss + Tt + Pp + Uu = V$ <br>
in which R, S, T, P, U and V are functions of x and y.

$$
Rr + Ss + Tt + Pp + Uu = V
$$
  
in which R, S, T, P, U and V are functions of x and y  
(a) 
$$
\frac{\partial^2 u}{\partial x^2} + x \frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = e^{xy} \sin u
$$
  
(b) 
$$
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + x \frac{\partial u}{\partial x} + xy \frac{\partial u}{\partial y} u = x + y
$$
  
Observe that (a) is a second order quasilinear PDE

$$
(b) \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + x \frac{\partial u}{\partial x} + xy \frac{\partial u}{\partial y} u = x + y
$$

 $x + y$ <br>quasilinear *PDE* while (*b*) is a linear second-order *PDE*.

#### 3.3 MONGE'S METHOD.

In this section we shall discuss the Monge's general method of solving

3.3 MONGE'S METHOD.  
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$$
Rr + Ss + Tt = V
$$
(3.7)  
in which *R S T* and *V* are functions of *x y u*, *p* and *a* with *r s* and *t* retaining t

in which R, S, T and V are functions of  $ie$ ,<br> $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial x^2}$  and  $t = \frac{\partial^2 u}{\partial x^2}$ MONGE'S METHOD.<br>
s section we shall discuss the Monge's general method of solving<br>  $Rr + Ss + Tt = V$  (3.7)<br>
ich R, S, T and V are functions of x, y, u, p and q with r, s and t retaining their ususl definitions. , *ie RRETHOD.*<br> *Rr* + *Ss* + *Tt* = *V*<br> *R, S, T* and *V* are functions of *x, y, u, p* and *q* with *r, s* and *t* 

$$
Rr + Ss + Tt = V
$$
(3.7)  
in which *R*, *S*,*T* and *V* are functions of *x*, *y*, *u*, *p* and *q* with *r*, *s* and *t* retaining  
ie,  

$$
r = \frac{\partial^2 u}{\partial x^2}, s = \frac{\partial^2 u}{\partial x \partial y} \text{ and } t = \frac{\partial^2 u}{\partial y^2}
$$
(3.8)  
From (3.7) we recall that  

$$
dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy = r dx + s dy
$$
(3.9)

 $(3.7)$ 

$$
r = \frac{\partial^2 u}{\partial x^2}, s = \frac{\partial^2 u}{\partial x \partial y} \text{ and } t = \frac{\partial^2 u}{\partial y^2}
$$
(3.8)  
From (3.7) we recall that  

$$
dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy = r dx + s dy
$$
(3.9)  

$$
dq = \frac{\partial q}{\partial x} dx + \frac{\partial q}{\partial y} dy = s dx + t dy
$$
(3.10)

From (3.7) we recall that  
\n
$$
dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy = r dx + s dy
$$
\n(3.9)  
\n
$$
dq = \frac{\partial q}{\partial x} dx + \frac{\partial q}{\partial y} dy = s dx + t dy
$$
\n(3.10)  
\nFrom (3.9) we have  
\n
$$
r = \frac{dp - sdy}{dx} \text{ and } t = \frac{dq - sdx}{dy}
$$
\n(3.11)

 $(3.9)$ 

$$
dq = \frac{cq}{\partial x} dx + \frac{\partial q}{\partial y} dy = s dx + t dy
$$
 (3.10)  
From (3.9) we have  

$$
r = \frac{dp - sdy}{dx} \text{ and } t = \frac{dq - sdx}{dy}
$$
 (3.11)  
Substituting (3.11) into (3.7) yields  

$$
R\left(\frac{dp - sdy}{dx}\right) + Ss + T\left(\frac{dq - sdx}{dy}\right) = V
$$
 (3.12)

 $(3.11)$  into  $(3.7)$ 

$$
r = \frac{dp - sdy}{dx} \text{ and } t = \frac{dq - sdx}{dy}
$$
(3.11)  
\nSubstituting (3.11) into (3.7) yields  
\n
$$
R\left(\frac{dp - sdy}{dx}\right) + S_s + T\left(\frac{dq - sdx}{dy}\right) = V
$$
(3.12)  
\nor  
\n
$$
Rdpdy - Rs(dy)^2 + Ssdxdy + Tdqdx - Ts(dx)^2 - Vdxdy = 0
$$
  
\ni*e*,  
\n
$$
(Rdpdy - Vdxdy + Tdqdx) - \left(Rs(dy)^2 - Ssdxdy + Ts(dx)^2\right) = 0
$$

or  
\n
$$
Rdpdy - Rs(dy)^2 + Ssdxdy + Tdqdx - Ts(dx)^2 - Vdxdy = 0
$$
  
\ni*e*,  
\n $(Rdpdy - Vdxdy + Tdqdx) - (Rs(dy)^2 - Ssdxdy + Ts(dx)^2)$ 

*i.e,* 
$$
(Rdpdy - Vdxdy + Tdqdx) - (Rs(dy)^2 - Ssdxdy + Ts(dx)^2) = 0
$$
  
*i.e,*  $(Rdpdy - Vdxdy + Tdqdx) - s(R(dy)^2 - Sdxdy + T(dx)^2) = 0$  (3.13)

or 
$$
Rdpdy - Rs(dy)^2 + Ssdxdy + Tdqdx - Ts(dx)^2 - Vdxdy = 0
$$
  
\n*i.e,*  $(Rdpdy - Vdxdy + Tdqdx) - (Rs(dy)^2 - Ssdxdy + Ts(dx)^2) = 0$   
\n*i.e,*  $(Rdpdy - Vdxdy + Tdqdx) - s(R(dy)^2 - Sdxdy + T(dx)^2) = 0$  (3.13)

 $(3.11)$  $(3.13)$  and  $(3.7)$ ie,  $(Rdpdy - Vdxdy + Tdqdx) - (Rs(dy)^2 - Ssdxdy + Ts(dx)^2) = 0$ <br>
ie,  $(Rdpdy - Vdxdy + Tdqdx) - s(R(dy)^2 - Sdxdy + T(dx)^2) = 0$  (3.13)<br>
If there exists a relation between x, y, u, p and q such that the terms in parenthesis in (3.11) vanish indep<br>
and<br>
and the term in pa ie,  $(Rdpdy - Vdxdy + Tdqdx) - s(R(dy)^2 - Sdxdy + T(dx)^2) =$ <br>If there exists a relation between x, y, u, p and q such that the terms in endently then it satisfies both (3.13) and (3.7). It therefore follows th tisfies both (3.13)<br> $2^2 - S dx dy + T (dx)^2$ at sts a relation between x,<br>en it satisfies both (3.13)<br> $R(dy)^2 - Sdx dy + T(dx)$ Example 12 and the same of th tion between x, y, u, p and q such the<br>sfies both (3.13) and (3.7). It there<br>–  $Sdx dy + T(dx)^2 = 0$ ation between x, y, u, p and q such that the terms in parentness in<br>tisfies both (3.13) and (3.7). It therefore follows that<br> $- S dx dy + T (dx)^2 = 0$  (3.14)<br> $- V dx dy + T dq dx = 0$  (3.15)

$$
R(dy)^{2} - Sdxdy + T(dx)^{2} = 0
$$
\n(3.14)

$$
Rdpdy - Vdxdy + Tdqdx = 0
$$
\n(3.15)

These are refered to as the Monge's subsidiary equations.

 $(3.14)$  $R(dy)^2 - Sdxdy + T(dx)^2 = 0$ <br>  $Rdpdy - Vdxdy + Tdqdx = 0$ <br>
These are refered to as the Monge's subsidiary equations.<br>
We now assume that (3.14) is resolvable into factors thus;

\n $Rdpdy - Vdxdy + Tdqdx = 0$ \n	\n        (3.15)\n	
\n        These are referred to as the Monge's subsidiary equations.\n		
\n $W = \text{row assume that } (3.14) \text{ is resolvable into factors thus;}$ \n	\n $dy - m_1dx = 0$ \n	\n        (3.16)\n
\n $dy - m_2dx = 0$ \n	\n        (3.17)\n	
\n        The first equation in (3.16) combined with (3.13) and with $du = pdx + qdy$ will y.\n		
\n $g_1 = a \text{ and } h_1 = b \text{ in which } a \text{ and } b \text{ are arbitrary constants. Then a relation of the equation of the equation of the equation:\n        h = f(a)\n$	\n        (3.17)\n	

 $(3.16)$  combined with  $(3.13)$ st equation<br> $I_1 = a$  and  $h_1$ We now assume that (3.14) is resolvable into factors thus;<br>  $dy - m_1 dx = 0$ <br>  $dy - m_2 dx = 0$  (3.16)<br>
The first equation in (3.16) combined with (3.13) and with  $du = pdx + qdy$  will yield an integral of the<br>
form  $a = a$  and  $b = b$  in whic form  $g_1 = a$  and  $h_1 = b$  in which a and b are arbitrary co a and  $h_1 = b$  in which a and b are arbitrary constants. Then a relation of the type<br>  $h_1 = f_1(g_1)$  (3.17)  $dy - m_2$ <br>equation i<br>*a* and  $h_1$ <br> $h_1 = f_1(g)$ 3) and with  $du = pdx + qdy$  will yiel<br>trary constants. Then a relation of the<br>(3.17)<br>alled an intermediate (first) integral.

$$
h_{1} = f_{1}(g_{1}) \tag{3.17}
$$

where  $f_1$  is arbitrary will be an integral. This is called an intermediate (first)

 $(3.16)$  combined with  $(3.13)$ form  $g_1 = a$  and  $h_1 = b$  in which a and b are arbitrary constants. Then a relation of the type<br>  $h_1 = f_1(g_1)$  (3.17)<br>
where  $f_1$  is arbitrary will be an integral. This is called an intermediate (first) integral.<br>
Similar type

$$
h_2 = f_2(g_2) \tag{3.18}
$$

 $h_2 = f_2(g_2)$ <br>in which  $f_2$  is also arbitrary.

 $(3.17)$  and  $(3.18)$  $h_2 = f_2(g_2)$ <br>in which  $f_2$  is also arbitrary.<br>Solving (3.17) and (3.18) we obtain<br>substituted in  $du = ndx + ddy$  which of *p* (3.18)<br>and q in terms of x, y and u. These values of p and q are then<br>integration yields the required solution substituted in  $du = pdx + qdy$  which on integration yields the required solution. in which  $f_2$  is also arbitrary.<br>Solving (3.17) and (3.18) we obtain p and q in terms of x, y<br>substituted in  $du = pdx + qdy$  which on integration yields the r<br>We however here note that if (3.16a) is a perfect square it is c<br>on (3)<br>*q* in terms of *x*, *y* and *u*. These values of *p* and *q*<br>reation vields the required solution  $f_2(g_2)$ <br>also arbitrary.<br>*add (3.18)* w<br>*du = pdx + qdy*<br>ere note that if  $(g_2)$ <br>
o arbitrary.<br>
and  $(3.18)$  we obtain *p* a<br>  $= pdx + qdy$  which on in

 $(3.16a)$  is a perfect square it is convinient in some cases to compute only one intermediate integral and integrate it with the help of Lagrange's method to get the complete solution.<br> *F x r* +  $(a + b)s + abt = xy$   $(i)$ <br> *solution* Examples.

1 Solve

$$
r + (a+b)s + abt = xy \qquad (i)
$$

. *Solution*

We recall that

$$
r + (a + b)s + abt = xy
$$
 (*i*)  
Solution.  
We recall that  

$$
dp = rdx + sdy
$$
 and  $dq = sdx + tdy$  (*ii*)

, *ie*

We recall that  
\n
$$
dp = rdx + sdy
$$
 and  $dq = sdx + tdy$  (ii)  
\ni.e,  
\n $r = \frac{dp - sdy}{dx} + \text{ and } t = \frac{dq - sdx}{dy}$  (iii)  
\nSubstituting (iii) into (i) we have

$$
r = \frac{dp - sdy}{dx} + \text{ and } t = \frac{dq - sdx}{dy} \qquad (iii)
$$
  
Substituting (iii) into (i) we have  

$$
\frac{dp - sdy}{dx} + (a + b)s + ab\left(\frac{dq - sdx}{dy}\right) = xy
$$
  
ie,  $dpdy - s(dy)^2 + (a + b)sdxdy + abdqdx - sab(dx)^2 - xydxdy = 0$   
ie,  $(dpdy - xydxdy + abdqdx) - s((dy)^2 - (a + b)dxdy + ab(dx)^2) = 0 \qquad (iv)$   
The Monge's subsidiary equation are thus;  
 $dpdy - xydxdy + abdqdx = 0 \qquad (v)$ 

The Monge's subsidiary equation are thus;

The Monge's subsidiary equation are thus;  
\n
$$
dpdy - xydxdy + abdqdx = 0
$$
 (v)  
\n $(dy)^2 - (a+b) dxdy + ab (dx)^2 = 0$  (vi)  
\nConsidering (vi) in the form  
\n
$$
[(dy)^2 - adxdy] + [ab (dx)^2 - bdxdy] = 0
$$
 (vii)

 $(vi)$ Considering  $(vi)$  in the form

$$
(dy)2 - (a+b)dxdy + ab (dx)2 = 0 \t (vi)
$$
  
Considering (vi) in the form  

$$
[(dy)2 - adxdy] + [ab (dx)2 - bdxdy] = 0 \t (vii)
$$
  
we may have  

$$
(dy)2 - adxdy = 0 \t (viii)
$$

$$
ab (dx)2 - bdxdy = 0 \t (ix)
$$

we may have

$$
(dy)2 - adxdy = 0
$$
 (viii)

$$
(dy)2 - adxdy = 0
$$
 (viii)  
\n
$$
ab (dx)2 - bdxdy = 0
$$
 (ix)  
\nwhich gives respectively  
\n
$$
dy - adx = 0 \implies y - ax = c_1
$$
 (x)

which gives respectively

$$
ab (dx)^{2} - b dx dy = 0
$$
 (*ix*)  
which gives respectively  

$$
dy - adx = 0 \implies y - ax = c_{1}
$$
 (*x*)  

$$
dy - b dx = 0 \implies y - bx = c_{2}
$$
 (*xi*)  
Substituting (*x*) into (*iv*) we obtain

$$
dy - a dx = 0 \implies y - ax = c_1 \qquad (x)
$$
  
\n
$$
dy - b dx = 0 \implies y - bx = c_2 \qquad (xi)
$$
  
\nstituting (x) into (iv) we obtain  
\n
$$
adp dx + ab dq dx - xa (c_1 + ax) (dx)^2 = 0
$$

Substituting  $(x)$  into  $(iv)$  we obtain

$$
adp dx + ab dq dx - xa (c1 + ax) (dx)2 = 0
$$

*i.e,* 
$$
dp + bdq - x(c_1 + ax)dx = 0
$$
 (x*ii*)  
Integrating (x*ii*) yields

Integrating (*xii*) yields

*ie,* 
$$
dp + bdq - x(c_1 + ax)dx = 0
$$
 (xii)  
\nIntegrating (xii) yields  
\n
$$
p + bq = \left(c_1 \frac{x^2}{2} + a \frac{x^3}{3}\right) + A
$$
\n*ie,* 
$$
p + bq = (y - ax) \frac{x^2}{2} + \frac{1}{3}ax^3 + A
$$
 (xiii)  
\nTherefore, the first integral is  
\n
$$
p + bq + \frac{1}{6}ax^3 - \frac{1}{2}x^2y = f_1(y - ax)
$$
 (xiv)

,

Therefore, the first integral is

$$
b + bq = (y - ax) \frac{1}{2} + \frac{1}{3}ax^{3} + A
$$
 (xiii)  
the first integral is  

$$
p + bq + \frac{1}{6}ax^{3} - \frac{1}{2}x^{2}y = f_{1}(y - ax)
$$
 (xiv)  
the other intermediary integral is  

$$
p + aq + \frac{1}{6}bx^{3} - \frac{1}{2}x^{2}y = f_{2}(y - bx)
$$
 (xv)

Similarly, the other intermediary integral is

$$
p + bq + \frac{1}{6}ax^3 - \frac{1}{2}x^2y = f_1(y - ax)
$$
 (xiv)  
ly, the other intermediary integral is  

$$
p + aq + \frac{1}{6}bx^3 - \frac{1}{2}x^2y = f_2(y - bx)
$$
 (xv)  
xiv) and (xv) we have  

$$
p(b - a) + \frac{1}{6}(b^2 - a^2)x^3 - \frac{1}{2}(b - a)x^2y = bf_2(y - b^2)
$$

From  $(xiv)$  and  $(xv)$  we ha ve

$$
p + aq + \frac{1}{6}bx^3 - \frac{1}{2}x^2y = f_2(y - bx)
$$
  
\n
$$
y) \text{ and } (xy) \text{ we have}
$$
  
\n
$$
p(b-a) + \frac{1}{6}(b^2 - a^2)x^3 - \frac{1}{2}(b-a)x^2y = bf_2(y - bx) - af_1(y - ax)
$$
  
\n
$$
p = \frac{1}{2}x^2y - \frac{1}{6}(b+a)x^3 + \frac{1}{b-a}(bf_2 - af_1)
$$
  
\n
$$
(xvi)
$$

, *ie*

$$
p(b-a)+\frac{1}{6}(b^2-a^2)x^3-\frac{1}{2}(b-a)x^2y=bf_2(y-bx)-af_1(y)
$$
  
\n
$$
p=\frac{1}{2}x^2y-\frac{1}{6}(b+a)x^3+\frac{1}{b-a}(bf_2-af_1)
$$
 (xvi)  
\nwe have  
\n
$$
q(b-a)-\frac{1}{6}(b-a)x^3=f_1(y-ax)-f_2(y-bx)
$$

Similarly, we have

Similarly, we have  
\n
$$
q(b-a) - \frac{1}{6}(b-a)x^3 = f_1(y - ax) - f_2(y - bx)
$$
\ni.e, 
$$
q = \frac{1}{6}x^3 + \frac{1}{b-a} [f_1(y - ax) - f_2(y - bx)]
$$
\n
$$
du = pdx + qdy
$$
\n
$$
= \left[ \frac{1}{2}x^2y - \frac{1}{6}(b+a)x^3 + \frac{1}{b-a}(bf_2 - af_1) \right]dx + \left[ \frac{1}{6}x^3 + \frac{1}{b-a}[f_1(y - ax) - f_2(y - bx)] \right]dy
$$
\ni.e, 
$$
u = \int \left[ \frac{1}{2}x^2y - \frac{1}{6}(b+a)x^3 + \frac{1}{b-a}(bf_2 - af_1) \right]dx + \int \left[ \frac{1}{6}x^3 + \frac{1}{b-a}[f_1(y - ax) - f_2(y - bx)] \right]dy
$$
\n
$$
= \frac{1}{6}x^3y - \frac{1}{24}(b+a)x^4 + \frac{1}{24}x^4 + \frac{1}{b-a}[f_1(y - af_1)dx] + \frac{1}{b-a}[f_1(y - ax) - f_2(y - bx)]dy
$$
\ni.e, 
$$
u = \frac{1}{6}x^3y - \frac{1}{24}(b+a)x^4 + \frac{1}{24}x^4 + \phi_1(y - ax) + \phi_2(y - ax).
$$
\n2 Solve  
\n
$$
t - r\sec^4 y = 2q \tan y.
$$
\n(i)  
\n
$$
dp = rdx + sdy
$$
 and  $dq = sdx + tdy$ \n(ii)

$$
ie,
$$

2 Solve

$$
a = \frac{1}{6}x^3 + \frac{24}{24}(b+a)x + \frac{24}{24}x + \frac{1}{4}(y)
$$
  
2 Solve  

$$
t - r \sec^4 y = 2q \tan y.
$$
 (i)  

$$
dp = rdx + sdy \text{ and } dq = sdx + tdy
$$
 (ii)  

$$
dp = sdy.
$$

2 Solve  
\n
$$
t - r \sec^4 y = 2q \tan y.
$$
 (i)  
\n $dp = rdx + sdy$  and  $dq = sdx + tdy$  (ii)  
\ni*e*,  $r = \frac{dp - sdy}{dx} +$  and  $t = \frac{dq - sdx}{dy}$  (iii)  
\nSubstituting (iii) into (i) we have

Substituting  $(iii)$  into  $(i)$  we have

$$
\frac{dq - sdx}{dy} - \left(\frac{dp - sdy}{dx}\right) \sec^4 y = 2q \tan y.
$$
  
\ni.e,  $\left(dqdx - dpdy \sec^4 y - 2q \tan ydx dy\right) - s\left(\left(dx\right)^2 - \left(dy\right)^2 \sec^4 y\right) = 0$  (iv)  
\nThe Monge's subsidiary equations are  
\n
$$
dqdx - dpdy \sec^4 y - 2q \tan ydx dy = 0
$$
 (v)

The Monge's subsidiary equations are

The Monge's subsidiary equations are  
\n
$$
dqdx - dpdy \sec^{4} y - 2q \tan ydxdy = 0
$$
\n
$$
(dx)^{2} - (dy)^{2} \sec^{4} y = 0
$$
\n(vi)  
\n
$$
(dx)^{2} - (dy)^{2} \sec^{4} y = 0
$$
\n(vi)  
\n
$$
(dx)^{2} + (dy)^{2} \sec^{4} y = 0
$$

$$
(dx)^{2} - (dy)^{2} \sec^{4} y = 0
$$
 (vi)

 $(vi)$ Observe that  $(vi)$  is of the form *vi*

$$
(dx)2 - (dy)2 sec4 y = 0
$$
 (vi)  
Observe that (vi) is of the form  

$$
(dx - dy sec2 y)(dx + dy sec2 y) = 0
$$
 (vii)  
ie,  

$$
dx - dy sec2 y = 0, dx + dy sec2 y = 0
$$
 (viii)  
Substituting the first of (viii) into (v) we have

, *ie*

$$
(ax - ay \sec y)(ax + ay \sec y) = 0
$$
 (vi)  
dx - dy \sec<sup>2</sup> y = 0, dx + dy \sec<sup>2</sup> y = 0 (viii)  
ing the first of (viii) into (v) we have  
dday \sec<sup>2</sup> y - dpdy \sec<sup>4</sup> y - 2q tan y \sec<sup>2</sup> y (dy)<sup>2</sup> = 0

Substituting the first of (*viii*) into (*v*) we have

$$
dx - dy \sec^2 y = 0, dx + dy \sec^2 y = 0 \qquad (viii)
$$
  
Substituting the first of  $(viii)$  into  $(v)$  we have  

$$
dqdy \sec^2 y - dpdy \sec^4 y - 2q \tan y \sec^2 y (dy)^2 = 0
$$
  
i.e, 
$$
dq - dp \sec^2 y - 2q \tan y dy = 0 \qquad (ix)
$$

*i.e,* 
$$
dq - dp \sec^2 y - 2q \tan y dy = 0
$$

\n
$$
\text{d}q\text{d}y\text{sec}^2 y - \text{d}p\text{d}y\text{sec}^2 y - 2q\tan y\text{sec}^2 y(\text{d}y) = 0
$$
\n

\n\n $\text{d}q - \text{d}p\text{sec}^2 y - 2q\tan y\text{d}y = 0$ \n

\n\n $\text{d}q\text{cos}^2 y - \text{d}p - 2q\tan y\text{sin }y\text{d}y = 0$ \n

\n\n $\text{d}q\text{cos}^2 y - \text{d}p - 2q\tan y\text{sin }y\text{d}y = 0$ \n

\n\n $\text{d}q\text{cos}^2 y = f_1(x - \tan y)$ \n

\n\n $\text{d}q\text{cos}^2 y = f_1(x - \tan y)$ \n

\n\n $\text{d}q\text{cos}^2 y = f_1(x - \tan y)$ \n

 $x^2$  y =  $f_1(x - \tan y)$  (x) 1 *viii v*

Similarly, the second of  $(viii)$  and  $(v)$  give

*i.e,* 
$$
aq\cos y - ap - 2q\tan y \sin y \, dy = 0
$$
  
\n*i.e,*  $p - q\cos^2 y = f_1(x - \tan y)$  (x)  
\nSimilarly, the second of *(viii)* and *(v)* give  
\n $p + q\cos^2 y = f_2(x + \tan y)$  (xi)  
\n*i.e,*  $p = \frac{1}{2} \Big[ f_1(x - \tan y) + f_2(x + \tan y) \Big] \qquad (xii)$ 

and

*i.e,* 
$$
p = \frac{1}{2} [f_1(x - \tan y) + f_2(x + \tan y)]
$$
 *(xii)*  
\nand  
\n*i.e,*  $q = \frac{1}{2} [f_2(x + \tan y) - f_1(x - \tan y)] \sec^2 y$  *(xiii)*  
\n $\therefore du = \frac{1}{2} [[f_1(x - \tan y) + f_2(x + \tan y)] dx + [f_2(x + \tan y) - f_1(x - \tan y)] \sec^2 y dy]$   
\n $= \frac{1}{2} [dx - dy \sec^2 y] f_2(x + \tan y) + \frac{1}{2} [dx + dy \sec^2 y] f_1(x - \tan y)$   
\n*i.e,*  $u = \phi_1(x + \tan y) + \phi_2(x - \tan y)$   
\n*Exercise*  
\nProve that the solution to the *PDE*  $q^2r - 2pqrs + p^2t = 0$  is given as the intersection between the planes  
\n $y = c$   $y + rf(c) = \phi(c)$ 

*Exercise*

 $(c) = \phi(c)$ in y j $^2r - 2pqrs + p^2$ ie,  $u = \phi_1 (x + \tan y)$ <br> *Exercise*<br>
Prove that the solution to<br>  $u = c, y + xf(c) = \phi(c)$ . *u* =  $\phi_1(x + \tan x)$ <br>*xercise*<br>*cove that the solution t*<br>*u* = *c*, *y* + *xf* (*c*) =  $\phi$  (*c*)  $u = \phi_1(x + \tan y) + \phi_2(x - \text{r} \cos \theta)$ <br>ve that the solution to the *PDE*<br>= c, y + xf (c) =  $\phi(c)$ .

# 3.4 GENERAL FORM OF SECOND-ORDER *PDE* WITH VARIABLE COEFFICIENTS ADMITTING<br> *A* FIRST INTEGRAL AND ITS SOLUTIONS.<br>
In section 3.3 we saw that a relation of the form<br>  $h = f(g)$  (3.19)<br>
in which *g* and *h* are differentiabl A FIRST INTEGRAL AND ITS SOLUTIONS.

In section 3.3 we saw that a relation of the form

$$
h = f(g) \tag{3.19}
$$

A FIRST INTEGRAL AND ITS SOLUTIONS.<br>
In section 3.3 we saw that a relation of the form<br>  $h = f(g)$  (3.19)<br>
in which g and h are differentiable functions of x, y, u, p and q and f an arbitrary differentiable function<br>
is call is called a first (intermediate)integral of a second-order PDE if the latter is obtained by eliminating  $f$  $(3.19)$  together with the relation obtained by differentiating  $(3.19)$ on In section 3.3 we saw that a relation of the form<br>  $h = f(g)$  (3.19)<br>
in which g and h are differentiable functions of x, y, u, p and q and f an arbitrary differentiable funct<br>
is called a first (intermediate)integral of a s  $h = f(g)$  (3.19)<br>in which g and h are differentiable functions of x, y, u, p and q and f an arbitrary differentiable<br>is called a first (intermediate)integral of a second-order *PDE* if the latter is obtained by elimina<br>and and  $f'$  from (3.19) together with the relation obtained by differentiating (3.19) partially wrt *x* and *y*.<br>
We now discuss the general form of second-order *PDE* if admitting first integral and its method of soltion du We now discuss the general form of second-order *PDE* if admitting first integral and its method of solution due to Monge.<br> *Differentiating* (3.19) partially wrt *x* and *y* yields<br>  $\frac{\partial h}{\partial x} + \frac{\partial h}{\partial u} \cdot p + \frac{\partial h}{\partial p} \cdot$ tion due to Monge. cuss the general form o<br>Monge.<br>ing (3.19) partially wrt<br> $\frac{\partial h}{\partial x} + \frac{\partial h}{\partial u} \cdot p + \frac{\partial h}{\partial v} \cdot r + \frac{\partial h}{\partial v} \cdot r$ 

 $(3.19)$ 

we now discuss the general form of second-order *FDE* if admitting first integral and its hence  
\ntion due to Monge.  
\nDifferentiating (3.19) partially wrt x and y yields  
\n
$$
\frac{\partial h}{\partial x} + \frac{\partial h}{\partial u} \cdot p + \frac{\partial h}{\partial p} \cdot r + \frac{\partial h}{\partial q} \cdot s = f'(g) \left( \frac{\partial g}{\partial x} + \frac{\partial g}{\partial u} \cdot p + \frac{\partial g}{\partial p} \cdot r + \frac{\partial g}{\partial q} \cdot s \right)
$$
\n(3.20)  
\n
$$
\frac{\partial h}{\partial y} + \frac{\partial h}{\partial u} \cdot q + \frac{\partial h}{\partial p} \cdot s + \frac{\partial h}{\partial q} \cdot t = f'(g) \left( \frac{\partial g}{\partial y} + \frac{\partial g}{\partial u} \cdot q + \frac{\partial g}{\partial p} \cdot s + \frac{\partial g}{\partial q} \cdot t \right)
$$
\n(3.21)  
\nEliminating  $f'(g)$  between (3.20) and (3.21) yields  
\n
$$
Rr + Ss + Tt + U(rt - s^2) = V
$$
\n(3.22)

 $(g)$  between  $(3.20)$  $'(g)$  between  $(3.20)$  and  $(3.21)$ 

$$
Rr + Ss + Tt + U\left(rt - s^2\right) = V\tag{3.22}
$$

where

Eliminating 
$$
f'(g)
$$
 between (3.20) and (3.21) yields

\n
$$
R + Ss + Tt + U(r - s^{2}) = V
$$
\nwhere

\n
$$
R = \frac{\partial(g, h)}{\partial(p, y)} + \frac{\partial(g, h)}{\partial(p, u)} \cdot q, S = \frac{\partial(g, h)}{\partial(q, y)} + \frac{\partial(g, h)}{\partial(q, u)} \cdot q + \frac{\partial(g, h)}{\partial(u, p)} \cdot p + \frac{\partial(g, h)}{\partial(x, p)}
$$
\n
$$
T = \frac{\partial(g, h)}{\partial(x, q)} + \frac{\partial(g, h)}{\partial(u, q)} \cdot p, U = \frac{\partial(g, h)}{\partial(p, q)}
$$
\n
$$
V = \frac{\partial(g, h)}{\partial(y, u)} \cdot p + \frac{\partial(g, h)}{\partial(u, x)} \cdot q + \frac{\partial(g, h)}{\partial(y, x)}
$$
\nHence, (3.22) is the most general form of second-order PDE that possesses a first (intermediate) integral.

\nWe thus proceed as in Monge's method for solving equations of this kind by determining the first integral.

Hence, (3 We thus proceed as in Monge's method for solving equations of this kind by determining the first integral.<br>
Recall that<br>  $dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy = r dx + s dy$  (3.24)<br>
and Recall that 22) is the most general form of<br>
oceed as in Monge's method for<br>  $dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy = r dx + s dy$ is the most general form of second-orde<br>ed as in Monge's method for solving equ<br>=  $\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy = rdx + sdy$ as in Monge's method for<br>  $\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy = rdx + sdy$ 

and  
\n
$$
dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy = r dx + s dy
$$
\n(3.24)\n
$$
dq = \frac{\partial q}{\partial x} dx + \frac{\partial q}{\partial y} dy = s dx + t dy
$$
\n(3.25)

and

$$
dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy = r dx + s dy
$$
 (3.24)  
and  

$$
dq = \frac{\partial q}{\partial x} dx + \frac{\partial q}{\partial y} dy = s dx + t dy
$$
 (3.25)  
i.e,  

$$
r = \frac{dp - s dy}{dx} \text{ and } t = \frac{dq - s dx}{dy}
$$
 (3.26)

, *ie*

$$
dq = \frac{1}{\partial x} dx + \frac{1}{\partial y} dy = s dx + t dy
$$
(3.25)  
ie,  

$$
r = \frac{dp - sdy}{dx} \text{ and } t = \frac{dq - sdx}{dy}
$$
(3.26)  
Putting (3.26) into (3.22) we have

 $(3.26)$  into  $(3.22)$ 

$$
= \frac{dp - sdy}{dx} \text{ and } t = \frac{dq - sdx}{dy}
$$
(3.26)  
6) into (3.22) we have  

$$
R\left(\frac{dp - sdy}{dx}\right) + Ss + T\left(\frac{dq - sdx}{dy}\right) + U\left(\frac{dp - sdy}{dx}\right)\left(\frac{dq - sdx}{dy}\right) - Us^2 = V
$$

, *ie*

i.e,  
\n*Rdpdy - Rs (dy)<sup>2</sup> + Ssdxdy + Tdqdx - Ts (dx)<sup>2</sup> + U (dpdq - sdpdx - sdqdy + s<sup>2</sup> dxdy) – Vdxdy = 0  
\ni.e,  
\n
$$
(Rdpdy + Tdqdx + Udpdq - Vdxdy) - s(R(dy)2 + Udpdx + Udqdy - Sdxdy + T (dx)2) = 0 \t(3.27)
$$*

$$
(Rdpdy + Tdqdx + Udpdq - Vdxdy) - s(R(dy)^{2} + Udpdx + Udqdy - Sdxdy + T(dx)^{2}) = 0
$$
 (3.27)  
\nMonge's subsidiary equations are:  
\n
$$
M = Rdpdy + Tdqdx + Udpdq - Vdxdy = 0
$$
 (3.27b)

Monge's subsidiary equations are:

*Kapay* + *I aqax* + *Uapaq* - *Vaxay* ) - *s* (*K*(*ay*) + *Uapax* + *Uaqay* - *Saxay* + *I* (*ax*)  
\nlonge's subsidiary equations are:  
\n
$$
M = Rdpdy + Tdqdx + Udpdq - Vdxdy = 0
$$
\n
$$
N = R(dy)^{2} + Udpdx + Udqdy - Sdxdy + T(dx)^{2} = 0
$$
\n
$$
(3.27b)
$$
\n
$$
N = R(dy)^{2} + Udpdx
$$
\nand *Udqdy N* cannot be factorized. We factorize

In view of the presence of the terms  $Udpdx$  and  $Udqdy$  N cannot be factorized. We may however try to factorize  $N = R(dy)^2 + Udpdx + Udq$ <br>sence of the terms  $Udpdx$  as<br> $N + \lambda N = 0$ <br>determined multiplier

$$
N + \lambda N = 0 \tag{3.28}
$$

where  $\lambda$  is an undetermined multiplier.

#### , *ie*

where  $\lambda$  is an undetermined multiplier.<br>
ie,<br>  $R(dy)^2 + Udpdx + Udqdy - Sdxdy + T(dx)^2 + \lambda (Rdpdy + Tdqdx + Udpdq - Vdxdy) = 0$  (3.2)  $N + \lambda N = 0$  (3.28)<br>where  $\lambda$  is an undetermined multiplier.<br> $R (dy)^2 + Udpdx + Udqdy - Sdxdy + T (dx)^2 + \lambda (Rdpdy + Tdqdx + Udpdq - Vdxdy$ <br>suppose this has factors λ  $N + \lambda N = 0$  (3.28)<br>  $\lambda$  is an undetermined multiplier.<br>  $+ Udpdx + Udqdy - Sdxdy + T(dx)^2 + \lambda (Rdpdy + Tdqdx + Udpdq - Vdxdy) = 0$  (3.29)<br>
this has feature.  $(3.29)$ Suppose this has factors ie,<br>  $R(dy)^2 + Udpdx + Udqdy - Sdxdy + T(dx)^2 + \lambda (Rdpdy + Tdqdx + Udpdq - Vdxdx$ <br>
Suppose this has factors<br>  $(Rdy + mTdx + \kappa Udp) + \lambda \left(dy + \frac{1}{m}dx + \frac{\lambda}{\kappa}dq\right) = 0$  (3.30)

Suppose this has factors  
\n
$$
(Rdy + mTdx + \kappa Udp) + \lambda \left(dy + \frac{1}{m}dx + \frac{\lambda}{\kappa}dq\right) = 0
$$
\n(3.30)  
\nComparing (3.29) and (3.30) we obtain

 $(3.29)$  and  $(3.30)$ 

$$
(Rdy + mTdx + \kappa Udp) + \lambda \left( dy + \frac{1}{m} dx + \frac{\lambda}{\kappa} dq \right) = 0 \qquad (3.30)
$$
  
Comparing (3.29) and (3.30) we obtain  

$$
\frac{R}{m} + mT = -(S + \lambda V)
$$

$$
\kappa = m \qquad (3.32)
$$

$$
\kappa = m
$$
 (3.32)  
\n
$$
\frac{R\lambda}{\kappa} = U
$$
 (3.33)  
\nEliminating  $\kappa$  and *m* from (3.31) through (3.33) we observe that  $\lambda$  satisfies the quadratic equation  
\n
$$
\lambda^2 (UV + RT) + \lambda US + U^2 = 0
$$
 (3.34)

 $(3.31)$  through  $(3.33)$ *m*

Eliminating 
$$
\kappa
$$
 and *m* from (3.31) through (3.33) we observe  
\n
$$
\lambda^2 (UV + RT) + \lambda US + U^2 = 0
$$
\n(3.34)

Recall that (3.34) has in general two roots  $\lambda_1, \lambda_2$ . Putting  $\lambda = \lambda_1$  and  $\kappa = m = \frac{K \lambda_1}{K}$  in (3.30) = 0 (3.34)<br> $\lambda_1, \lambda_2$ . Putting  $\lambda = \lambda_1$ general two roots  $\lambda_1$ ,  $\lambda_2$ . Putting<br>  ${}_{1}Tdx + \lambda_1 Udp$   $(Udx + R\lambda_1 dy + \lambda_1$ inating  $\kappa$  and  $m$  from (3.31) through (3.33) we observe that  $\lambda$  satisfies the quadratic equation<br>  $\lambda^2 (UV + RT) + \lambda US + U^2 = 0$  (3.34)<br>
Il that (3.34) has in general two roots  $\lambda_1, \lambda_2$ . Putting  $\lambda = \lambda_1$  and  $\kappa = m = \frac{R$  $m = \frac{R}{A}$ *U*  $\lambda^2 (UV + RT) + \lambda US + U^2 = 0$  (3.34)<br>
) has in general two roots  $\lambda_1$ ,  $\lambda_2$ . Putting  $\lambda = Udy + \lambda_1 Tdx + \lambda_1 Udp (Udx + R\lambda_1 dy + \lambda_1 Udq)$  $\lambda$ (3.33) we observe that  $\lambda$  satisfies the quadr<br>= 0 (3.34)<br> $\lambda_1, \lambda_2$ . Putting  $\lambda = \lambda_1$  and  $\kappa = m = \frac{R\lambda_1}{U}$  in (  $(A + RT) + \lambda US + U^2 = 0$  (3.34)<br>
a general two roots  $\lambda_1, \lambda_2$ . Putting  $\lambda = \lambda_1$  and  $\kappa = \lambda_1 T dx + \lambda_1 U dp$  (*Udx + R* $\lambda_1 dy + \lambda_1 U dq$ ) = 0 (*i*th  $\lambda$  we have as in general two roots  $\lambda_1$ .<br>  $dy + \lambda_1 T dx + \lambda_1 U dp$   $(U dx + \lambda$  with  $\lambda_2$  we have ve that  $\lambda$  satisfies the quadratic eq<br>=  $\lambda_1$  and  $\kappa = m = \frac{R\lambda_1}{U}$  in (3.30)  $(V + RT) + \lambda US + U^2 = 0$  (3.34)<br>in general two roots  $\lambda_1, \lambda_2$ . Putting  $\lambda = \lambda_1$  and  $\kappa = m = \frac{R\lambda}{U}$ <br> $+ \lambda_1 T dx + \lambda_1 U dp$  (*Udx* +  $R\lambda_1 dy + \lambda_1 U dq$ ) = 0 (3.35)

$$
(Udy + \lambda_1 Tdx + \lambda_1 Udp)(Udx + R\lambda_1 dy + \lambda_1 Udq) = 0 \qquad (3.35)
$$

Similarly, replacing  $\lambda$  with  $\lambda_2$  we have

$$
(Udy + \lambda_1 Tdx + \lambda_1 Udp)(Udx + R\lambda_1 dy + \lambda_2 Udq) = 0 \qquad (3.35)
$$
  
\n
$$
(Udy + \lambda_1 Tdx + \lambda_1 Udp)(Udx + R\lambda_1 dy + \lambda_1 Udq) = 0 \qquad (3.35)
$$
  
\n
$$
(Udy + \lambda_2 Tdx + \lambda_2 Udp)(Udx + R\lambda_2 dy + \lambda_2 Udq) = 0 \qquad (3.36)
$$

1  $dx + R\lambda_2 dy + \lambda_2 U dq$  = 0 (3.36)<br>  $\lambda_1 = a_1$  and  $h_1 = b_1$  by solving the pair  $\lambda = \lambda_1$ We now obtain two integrals of the form  $g_1 = a_1$  and  $h_1 = b_1$  by solving the pair  $\lambda = \lambda_1$  and  $\kappa = m = \frac{R}{l}$ <br>and integrals of the type  $g_2 = a_2$  and  $h_2 = b_2$  obtained from solving the pairs  $(\lambda_1, \lambda_2)$ . Hence, we g *g*<sub>1</sub> =  $a_1$  and  $h_1 = b_1$  by solving the pair  $\lambda = \lambda_1$  and  $\kappa = m = \frac{R\lambda}{U}$  $g_2 T dx + \lambda_2 U dp$   $(U d$ <br>rals of the form  $g_1$ <br> $g_2 = a_2$  and  $h_2 = b$  $\lambda$  $x + R\lambda_1 dy + \lambda_1 U dq$  = 0 (3.35)<br>  $x + R\lambda_2 dy + \lambda_2 U dq$  = 0 (3.36)<br>
=  $a_1$  and  $h_1 = b_1$  by solving the pair  $\lambda = \lambda_1$  and  $\kappa = m = \frac{R\lambda_1}{U}$  $dx + \lambda_2 U dp$   $(U dx + R \lambda_2 dy + \lambda_2 U dq) = 0$  (3.36)<br>
is of the form  $g_1 = a_1$  and  $h_1 = b_1$  by solving the pair  $\lambda = \lambda_1$  and  $\kappa = m = \frac{R \lambda_1}{U}$ <br>  $= a_2$  and  $h_2 = b_2$  obtained from solving the pairs  $(\lambda_1, \lambda_2)$ . Hence, we get the  $\lambda = \lambda_1$  and  $\kappa = m =$ <br> $\lambda_1, \lambda_2$ ). Hence, we go

and integrals of the type  $g_2 = a_2$  and  $h_2 = b_2$  obtained from solving the pairs  $(\lambda_1, \lambda_2)$ . Hence, we get the  $(g_1)$  and  $h_2 = f_2(g_2)$  $g_2 = a_2$  and  $h_2 = b_2$  obtained from solving t<br>  $h_1 = f_1(g_1)$  and  $h_2 = f_2(g_2)$  where  $f_1$  and  $f_2$ two integrals of the type  $h_1 = f_1(g_1)$  and  $h_2 = f_2(g_2)$  where  $f_1$  and  $f_2$  are arbitrary. These are solved to we now obtain two integrals of the form  $g_1 - u_1$  and  $h_1 - v_1$  by solving the pair  $\lambda = \lambda_1$  and  $\lambda = m -$ <br>and integrals of the type  $g_2 = a_2$  and  $h_2 = b_2$  obtained from solving the pairs  $(\lambda_1, \lambda_2)$ . Hence, we g<br>two int grals of the form  $g_1 = a_1$  and  $h_1 = b_1$  by solvi<br>  $g_2 = a_2$  and  $h_2 = b_2$  obtained from solving<br>  $h_1 = f_1(g_1)$  and  $h_2 = f_2(g_2)$  where  $f_1$  and  $f_2$ <br>
ctions of x, y and  $u$  thereafter substituting is Is of the form  $g_1 = a_1$  and  $h_1 = b_1$  b<br>  $= a_2$  and  $h_2 = b_2$  obtained from s<br>  $= f_1(g_1)$  and  $h_2 = f_2(g_2)$  where f determine p and q as functions of x, y and u thereafter substituting into  $du = pdx + qdy$  which when integrated gives the complete solution.

In implementing this procedure we note the following:

- (3.34) has double roots, it is only possible to obtain one integral of the form  $h_1 = f_1(g_1)$ s, it is only pos<br>  $\mathbf{a}_1 = a_1$  or  $h_1 = b_1$ 1 If (3.34) has double roots, it is only possible to obtain one integral of the form  $h_1 = f_1(g_1)$  which can<br>be obtained from either  $g = g$  or  $h = h$  to give the values of *n* and *g* to render  $du = ndx + ady$  integral be obtained from either  $g_1 = a_1$  or  $h_1 = b_1$  to give the values of p and q to render  $du = pdx + qdy$  integrable.<br>
2 Since  $\lambda_1 = \lambda_2$  we get a more general solution by taking liner relation between  $g_1$  and  $h_1$  in the form ts, it is only possible to obtain one integral of the form  $h_1 = f_1(g_1)$  whi<br> $g_1 = a_1$  or  $h_1 = b_1$  to give the values of p and q to render  $du = pdx + qdy$ = it is only possible to obtain one integral of the form  $h_1 = f_1(g_1)$  which can<br>=  $a_1$  or  $h_1 = b_1$  to give the values of p and q to render  $du = pdx + qdy$  integ rable. 4) has double roots, it is only possible to obtain one integral of the form  $h_1 = f$ <br>ained from either  $g_1 = a_1$  or  $h_1 = b_1$  to give the values of p and q to render  $du =$ <br> $\lambda_1 = \lambda_2$  we get a more general solution by takin
- $\beta_1$  =  $\lambda_2$  we get a more general solution by taking liner relation between  $g_1$  and  $h_1$ ince  $\lambda_1$ <br> $\lambda_1 = m h_1$ 2 Since  $\lambda_1 = \lambda_2$  we get a more general solution by taking liner relation between  $g_1$  and  $h_1$  in the form  $g_1 = mh_1 + n$  and integrate by Lagrange's method. a more general solution by taking liner relation betwerd<br>grate by Lagrange's method.<br> $f_1(g_1)$  and  $h_2 = f_2(g_2)$  and unsolvable for p and que combined with  $g_1 = g_2$  or  $h_2 = h_1$  to determine the v
- 3 If the first integral  $h_1 = f_1(g_1)$  and  $h_2 = f_2(g_2)$  $(g_1)$ the by Lagrange's meth<br>  $f_1^2(g_1)$  and  $h_2 = f_2(g_2)$ first integral  $h_1 = f_1(g_1)$  and  $h_2 = f_2(g_2)$  and unso<br>  $h_1 = f_1(g_1)$  may be combined with  $g_2 = a_2$  or  $h_2 = b_2$ and  $h_2 = f_2(g_2)$  and unsolvable for p and q then one of the first integrals  $h_1 = f_1(g_1)$  may be combined with  $g_2 = a_2$  or  $h_2 = b_2$  to determine the values of p and q and then *h*  $\lambda_1 = \lambda_2$  we get a more general solution by taking liner relation between  $g_1$  and  $h_1$  in the  $mh_1 + n$  and integrate by Lagrange's method.<br>
e first integral  $h_1 = f_1(g_1)$  and  $h_2 = f_2(g_2)$  and unsolvable for p and *du* and integrate<br>the praise  $h_1 = f_1$ <br> $(g_1)$  may be concept  $du = pdx + qdy$  $\lambda_1 = \lambda_2$  we get a more general solution by taking liner relation be<br>  $h_1 + n$  and integrate by Lagrange's method.<br>
irst integral  $h_1 = f_1(g_1)$  and  $h_2 = f_2(g_2)$  and unsolvable for p a<br>  $= f_1(g_1)$  may be combined with  $g$ and integrate by Lagrange's method.<br>
gral  $h_1 = f_1(g_1)$  and  $h_2 = f_2(g_2)$  and unsolvable for p<br>
(a) may be combined with  $g_2 = a_2$  or  $h_2 = b_2$  to determine<br>  $= pdx + qdy$  to obtain the complete solution (integral)

#### *Examples*

1 Solve the differential equation

integrating 
$$
du = pdx + qdy
$$
 to obtain the complete solution (integral).  
\nExamples  
\n1 Solve the differential equation  
\n
$$
u(1+q^2)r - 2pqus + u(1+p^2)t - u^2(s^2 - rt) + 1 + q^2 + p^2 = 0
$$
\n(1)  
\nSolution  
\nFrom the general PDE  
\n
$$
Rr + Ss + Tt + U\left(rt - s^2\right) = V
$$
\n(ii)

*Solution*

From the general *PDE*

$$
Rr + Ss + Tt + U\left(rt - s^2\right) = V \qquad (ii)
$$

we have

From the general *PDE*  
\n
$$
Rr + Ss + Tt + U\left(rt - s^2\right) = V \qquad (ii)
$$
\nwe have  
\n
$$
R = u\left(1 + q^2\right), S = -2pqu, T = u\left(1 + p^2\right), U = u^2, V = -\left(1 + q^2 + p^2\right) = 0 \quad (iii)
$$
\nSubstituting into the  $\lambda$ -equation  
\n
$$
\lambda^2 \left(UV - RT\right) + \lambda SU + U^2 = 0 \qquad (iv)
$$

 $(vi)$ 

Substituting into the  $\lambda$ -equation

$$
K = u(1+q^2), S = -2pqu, T = u(1+p^2),
$$
  
Substituting into the  $\lambda$ -equation  

$$
\lambda^2 (UV - RT) + \lambda SU + U^2 = 0
$$
 (iv)

w e have

Substituting into the 
$$
\lambda
$$
-equation  
\n
$$
\lambda^2 (UV - RT) + \lambda SU + U^2 = 0 \qquad (iv)
$$
\nwe have  
\n
$$
\lambda^2 \{-u^2 (1 + q^2 + p^2) - u^2 (1 + q^2) (1 + p^2) \} - 2\lambda p q u^3 + u^4 = 0
$$
\ni*e*,  
\n
$$
\lambda^2 p^2 q^2 - 2\lambda p q u + u^2 = 0 \qquad (v)
$$
\ni*e*,  
\n
$$
(\lambda pq - u)^2 = 0
$$
\n
$$
\Rightarrow \qquad \lambda_1 = \lambda_2 = \frac{u}{pq} \qquad (vi)
$$

*i.e.*  
\n
$$
(\lambda pq - u)^2 = 0
$$
\n
$$
\Rightarrow \qquad \lambda_1 = \lambda_2 = \frac{u}{pq}
$$
\n
$$
(vi)
$$

The intermediate integral is thus given as

*u*

*pq*

$$
\Rightarrow \qquad \lambda_1 = \lambda_2 = \frac{u}{pq} \qquad (vi)
$$
\nThe intermediate integral is thus given as\n
$$
Udy + \lambda_1 T dx + \lambda_1 Udp = 0
$$
\n
$$
Udx + \lambda_2 R dy + \lambda_2 Udq = 0
$$
\ni.e,\n
$$
u^2 dy + \frac{u^2}{pq} (1 + p^2) dx + \frac{u^3}{pq} dp = 0
$$
\n
$$
u^2 dx + \frac{u^2}{pq} (1 + q^2) dy + \frac{u^3}{pq} dq = 0
$$
\ni.e,\n
$$
pq dy + (1 + p^2) dx + udp = 0
$$
\ni.e,\n
$$
pq dx + (1 + q^2) dy + udq = 0
$$
\n
$$
du = pdx + qdy = 0 \qquad (vii)
$$
\n
$$
(viii)
$$

Also, we have

, *ie*

$$
du = pdx + qdy = 0
$$
 (ix)

From the  $(viiia)$  and  $(ix)$  we have From the (*viiia*) and (*ix*) we have<br> $dx + udp + pdu = 0$ (*ie*, *viiia*) and (*ix* 

*u*) and (*ix*) we have  

$$
dx + udp + pdu = 0
$$
 (*ie*, *viiia* – *p* × *ix*)

From the *(viiia)* and *(ix)* we have  
\n
$$
dx + udp + pdu = 0 (ie, viii)
$$
\n*ie,*\n
$$
dx + d (up) = 0
$$
\nIntegrating gives  
\n
$$
x + up = a
$$

Integrating gives

$$
dx + d (up) = 0
$$
  
es  

$$
x + up = a
$$
 (x)  
the (viiib) and (ix) we have

Similarly, from the (*viiib*) and  $(ix)$  we have

$$
x + up = a
$$
 (x)  
\nSimilarly, from the *(viiib)* and *(ix)* we have  
\n
$$
dy + udq + qdu = 0
$$
 (*ie, viiib* – *q* × *ix*)  
\n*ie,* 
$$
dy + d( uq ) = 0
$$
  
\nIntegrating gives  
\n
$$
y + uq = b
$$
 (*xi*)

Integrating gives

$$
y + uq = b \tag{xi}
$$

From  $(x)$ 

$$
y + uq = b \qquad (xi)
$$
  
From (x)  

$$
p = \frac{a - x}{u}
$$
  
and from (xi)  

$$
q = \frac{b - y}{u}
$$
  
∴  

$$
du = \frac{a - x}{u} dx + \frac{b - y}{u} dy
$$
 (xii)

*y dy*

*u*

, *ie*

$$
\therefore \quad du = \frac{a - x}{u} dx + \frac{b - y}{u} dy
$$
  
\n*i.e.*  
\n
$$
u du = (a - x) dx + (b - y) dy
$$
  
\n
$$
\frac{u^2}{2} = ax - \frac{x^2}{2} + by - \frac{y^2}{2}
$$
  
\n*i.e.*  
\n
$$
u^2 + (x - a)^2 + (y - b)^2 = A
$$

 $\left( qu+y\right)$ 

 $\ddot{\phantom{0}}$ is the required solution.

By note 2 we can find a more general solution of the given *PDE*. Hence, we assume solution.<br>
an find a more general solution of t<br>  $pu + x = m(qu + y) + n$  (*xiii*) *p* m find a more general<br>  $pu + x = m(qu + y) + n$ <br>  $pp - mq)u = my - x + n$ ution.<br>
find a more general solution of<br>  $+ x = m (qu + y) + n$  (*xiii*) find a more general solution of<br>  $(- + x = m(qu + y) + n)$ <br>  $-mq$ ) $u = my - x + n$  (*xiii* 

or

which is a Lagrange's linear equationwith corresponding auxiliary equation given as

$$
(p-mq)u = my - x + n
$$
  
\n
$$
\left(\frac{xuu}{u}\right)
$$
  
\n
$$
range's linear equation with corresponding auxiliary of \n
$$
\frac{dx}{u} = \frac{dy}{-mu} = \frac{du}{my - x + n} = \frac{\frac{x}{u}dx + \frac{y}{u}dy + du}{n}
$$
$$

From the first two we have

 $(p - mq)$ 

$$
\frac{du}{u} = \frac{dy}{-mu} = \frac{du}{my - x + n} = \frac{u}{n}
$$
  
From the first two we have  

$$
\frac{dx}{u} = \frac{dy}{-mu} \Rightarrow mdx + dy = 0 \Rightarrow y + my = c_1
$$
  
From first and last we have  

$$
\frac{dx}{du} = \frac{\frac{x}{u}dx + \frac{y}{u}dy + du}{u}
$$

From first and last we have

$$
\frac{dx}{u} = \frac{dy}{-mu} \Rightarrow mdx + dy
$$
  
last we have  

$$
\frac{dx}{u} = \frac{\frac{x}{u}dx + \frac{y}{u}dy + du}{n}
$$

$$
ndx = xdx + ydy + udu = \frac{1}{2}d(x^2 + y^2 + u^2)
$$
  
we have  

$$
x^2 + y^2 + u^2 - 2nx = c_2
$$
  
solution is thus

Integrating we have

Integrating we have  
\n
$$
x^2 + y^2 + u^2 - 2nx = c_2
$$

The general solution is thus

Integrating we have  
\n
$$
x^2 + y^2 + u^2 - 2nx = c_2
$$
\nThe general solution is thus  
\n
$$
x^2 + y^2 + u^2 - 2nx = f(y + mx).
$$
\n2 Determine the general solution of the differential equs

Integrating we have<br>  $x^2 + y^2 + u^2 - 2nx = c_2$ <br>
The general solution is thus<br>  $x^2 + y^2 + u^2 - 2nx = f(y + mx)$ .<br>
2 Determine the general solution of the differential equation 2 Determine the general solut<br> $ar + bs + ct + e(i)$ <br>where  $a, b, c, e, h$  are constants. *x*<br>termine th<br>*a*<br>*a*,*b*,*c*,*e*,*h* 

lution is thus  
\n
$$
x^2 + y^2 + u^2 - 2nx = f(y + mx)
$$
.  
\nthe general solution of the differential equation  
\n $ar + bs + ct + e(rt - s^2) = h$  (i)  
\n*b* are constants.

Solution

We consider the equation

where *a*, *b*, *c*, *e*, *h* are constants.  
\nSolution  
\nWe consider the equation  
\n
$$
Rr + Ss + Tt + U\left(rt - s^2\right) = V \qquad (ii)
$$
\nComparing (i) and (ii) we have  
\n
$$
R = a, S = b, T = c, U = e \text{ and } V = h \qquad (iii)
$$
\nBut the  $\lambda$ -equation is in general given as

Comparing  $(i)$  and  $(ii)$  we have

$$
R = a, S = b, T = c, U = e \text{ and } V = h \tag{iii}
$$

But the  $\lambda$ -equation is in general given as

ring (*i*) and (*ii*) we have  
\n
$$
R = a, S = b, T = c, U = e
$$
 and  $V = h$  (iii)  
\nλ-equation is in general given as  
\n
$$
λ2(UV + RT) + λSU + U2 = 0
$$
 (iv)

, *ie*

But the 
$$
\lambda
$$
-equation is in general given as  
\n
$$
\lambda^2 (UV + RT) + \lambda SU + U^2 = 0 \qquad (iv)
$$
\ni.e,  
\n
$$
\lambda^2 (ac + eh) + \lambda be + e^2 = 0 \qquad (v)
$$
\nFor convenience we set  $\lambda m + e = 0$  in (v) to obtain  
\n
$$
m^2 - bm + ac + eh = 0 \qquad (vi)
$$
\nNow

Foe convinience we set  $\lambda m + e = 0$  in  $(v)$  to obtain  $\lambda$ i

$$
\lambda^2 (ac + eh) + \lambda be + e^2 = 0 \qquad (v)
$$
  
vinience we set  $\lambda m + e = 0$  in (v) to obtain  

$$
m^2 - bm + ac + eh = 0 \qquad (vi)
$$
  
me further that (vi) admits roots  $m_1$  and  $m_2$ .

 $(vi)$ We assume further that  $(vi)$  admitts roots  $m_1$  and  $m_2$ .

The first system of integrals is

$$
m^{2} - bm + ac + eh = 0
$$
 (vi)  
We assume further that (vi) admits roots  $m_{1}$  and  $m_{2}$ .  
The first system of integrals is  

$$
cdx + edp - m_{1}dy = 0
$$
 (vii)  

$$
ady + edq - m_{2}dx = 0
$$
 (vii)  
An intermediate integral is  

$$
cx + ep - m_{1}y = f_{1}(ay + eq - m_{2}x)
$$
 (viii)  
The second system of integral is given by

An intermediate integral is

e integral is  
\n
$$
cx + ep - m_1 y = f_1 (ay + eq - m_2 x)
$$
\n
$$
cdy + edq - m_2 dx = 0
$$
\n
$$
ay + eq - m_2 x = constant
$$
\n
$$
(viii)
$$

The seco nd system of integral is given by

$$
ady + edq - m_2 dx = 0
$$

ie,  $ay + eq - m_2x = constant$ 

and

 $ady + edq - m_2 dx = 0$ <br>
ie,  $ay + eq - m_2x = \text{constant}$ <br>
and  $cdx + edp - m_2dy = 0 \Rightarrow cx + ep - m_2y = \text{constant}$ <br>
Therefore the other intermediate integral is

Therefore the other intermediate integral is

$$
cdx + edp - m_2dy = 0 \Rightarrow cx + ep - m_2y = \text{constant}
$$
  
other intermediate integral is  

$$
cx + ep - m_2y = f_2(ay + eq - m_2x)
$$
 (*ix*)  
and the easily solved from the above intermediate into

Clearly,  $p$  and  $q$  can not be easily solved from the above intermediate integrals. Therefore we comb  $\frac{c}{p}$  and  $\frac{p}{q}$ <br>**p** and  $\frac{q}{q}$ ine any particular integral of the second with the general integral of the firstsystem.

or

$$
cx + ep - m_2 y = A
$$
 (x)

From (*viii*) we obtain

$$
cx + ep - m_2 y = A
$$
\n
$$
(x)
$$
\n
$$
f_1(ay + eq - m_2 x) cx + ep = (m_2 - m_1)y + A
$$
\n
$$
ay + eq = -m_2x + \psi \{(m_2 - m_1)y + A\}
$$
\n
$$
\psi
$$
\nis an inverse function of  $\phi$ . Using the values of  $p$  and  $q$  from  $(x)$  and  $(xi + ady)$  we thus have

where  $\psi$  is an inverse functionn of  $\phi$ . Using the values of p and q from  $(x)$  and  $(xi)$  in the general relation  $du = pdx + qdy$  we thus have where  $\psi$  is an  $du = pdx + qdy$  $f_1(ay + eq - b)$ <br>  $ay + eq = -$ <br>
ere  $\psi$  is an inverse funct<br>  $= pdx + qdy$  we thus have  $\begin{aligned} xy + eq &= -m_2 x + \psi \left\{ (m_2 - m_1) y + A \right\} \end{aligned}$ <br> *edu*  $= (A + m_2 y - cx) dx + \left[ m_2 x + \psi \left\{ (m_2 - m_1) y + A \right\} - ay \right] dy$  $q = -m_2x + \psi \{(m_2 - m_1)y + A\}$ <br>
Exercision of  $\phi$ . Using the values of p and q from  $(x)$  and  $(xi)$  in the<br>
nus have<br>  $= (A + m_2y - cx)dx + [m_2x + \psi \{(m_2 - m_1)y + A\} - ay]dy$ <br>  $= Adx - cxdx + m(xdy + ydx) - gydx + \psi \{(m_2 - m_1)y + A\}dy$ 

where 
$$
\psi
$$
 is an inverse function of  $\phi$ . Using the values of p and q from (x) and (xi) in the g  
\n $du = pdx + qdy$  we thus have  
\n
$$
edu = (A + m_2y - cx)dx + \left[m_2x + \psi\left\{(m_2 - m_1)y + A\right\} - ay\right]dy
$$
\n
$$
= Adx - cxdx + m_2\left(xdy + ydx\right) - aydy + \psi\left\{(m_2 - m_1)y + A\right\}dy
$$
\nIntegration gives  
\n
$$
eu = Ax - \frac{1}{2}cx^2 + 2m_2xy - \frac{1}{2}ay^2 + F\left\{(m_2 - m_1)y + A\right\} + B
$$

Integration gives

whe

$$
= Adx - cxdx + m_2 (xdy + ydx) - aydy + \psi \{(m_2 - m_1)y + A
$$
  
gration gives  

$$
eu = Ax - \frac{1}{2}cx^2 + 2m_2xy - \frac{1}{2}ay^2 + F \{(m_2 - m_1)y + A\} + B
$$
  
re  $F(m_2 - m_1)y + A = \int \psi \{(m_2 - m_1)y + A\} dy$ 

, *ie*

### CHAPTER FOUR BOUNDARY VALUE PROBLEMS.

#### 4.1 BOUNDARY CONDITIONS AND BOUNDARY VALUE PROBLEMS.

If a second-order differential equation

4.1 BOUNDARY CONDITIONS AND BOUNDARY VALUE PROBLEMS.  
If a second-order differential equation  

$$
F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0
$$
4.1

is to be solved within a specified region  $R$  of space in which the values of the dependent variables  $u$  are specified at the boundary  $\partial R$  then the resulting problem is refered to as a *R* D BOUNDARY VALUE PROBLEMS.<br>
FORMALLY ALLY AND ALLY AND ALLY AND ALLY AND AND AND ALLY AND AND AND UNITAL SURVIVIDE AND UNITAL SURVIVIDE TO UNITAL THE TO AS A *houndary value problem R*  $(\mu_x, \mu_{xy}, \mu_{yy}) = 0$ <br> *R* of space in which the values of the dependent variables *u* are<br> *R* then the resulting problem is refered to as a *boundary value problem*. These<br> *R* then the resulting problem is refered to boundaries need not enclose a finite volume. In this case one of the boundaries may be at infinity. A PDE in which one of the independent variables is time, the value of the depe ndent variable and often 4.1<br> **endent variables**<br> *y value problem*<br> *y* **be at infinity.**  $t = 0$  (say) specified at the boundary  $\partial R$  then the resulting problem is refered to as a *boundary value problem*. These<br>boundaries need not enclose a finite volume. In this case one of the boundaries may be at infinity.<br>A PDE in wh conditions. Hence, the term boundary and initial condtions will be used as appropriate. boundaries need not enclose a finite volume. In this case<br>*A* PDE in which one of the independent variables is tim<br>its time derivatives at some instant of time,  $t = 0$  (say)<br>*initial conditions*. Hence, the term *boundary* We shall concern ourselves here primarily with two ntypes of boundary conditions that arise frequently in the description of physical phenomena and which we encounter frequently in many applications:<br>
(*a*) Dirichlet Conditions; where the dependent variable *u* is specified at each point of a boundary<br>
ion. For example at the (a) Dirichlet Conditions; where the dependent variable u is specified at each pointof a boundary in a region. For example at the end of a rectangular region. *a u*

 $R: a \leq x \leq b, c \leq y \leq d.$ 

(b) Cauchy Condition; if one of the independent variables is time (t) and the values of both u and  $\frac{\partial u}{\partial x}$  are (b) Cauchy Condition; if one of the independent variables is time (t) and the values of both  $u$  as specified on the boundary at time  $t = 0$  (at some initial time) then this condition is refered to as type *u b*) Cauchy Condition; if one of the independent variables is time (*t*) and the values of both *u* and  $\frac{\partial u}{\partial t}$ the independent variables is time  $(t)$  and the values of both  $u$  and  $t = 0$  (at some initial time) then this condition is refered to as *ca*  $\hat{o}$  $\hat{c}$ type.

on the boundary at time  $t = 0$  (at some initial time) then this condition is refered to as *cauchy*<br> *Mathematics*, Physics and Engineering, *PDEs* generally arise from the mathematical formula<br> *real – life* physical pro In applied Mathematics, Physics and Engineering, PDEs generally arise from the mathematical formulation of the real – life physical problems. Often, boundary conditions are imposed on the dependent variacondition is solving a boundary value proble (*BVP*). It is initial value problem if initial conditions are bles and certain of its derivatives. The process of determining a PDE subject to the imposed boundary imposed on the differential equation. *METHOD OF SEPERATION OF VARIABLE*<br>*METHOD OF SEPERATION OF VARIABLE*<br>*METHOD OF SEPERATION OF VARIABLE* 

#### 3.2 METHOD OF SEPERATION OF VARIABLE.

This is perhaphs the oldest and commonest method of solving a partial differential equation. Given the unknown function wn function<br>  $(x_1, x_2, x_3, x_4, \dots, x_{m-1}, x_m)$ <br>  $(4.2)$ 3.2 *METHOD OF SEPERATION OF VARIABLE*.<br>This is perhaphs the oldest and commonest method of solving a partial differential equation.<br>Given the unknown function<br> $u = u(x_1, x_2, x_3, x_4, \dots, x_{m-1}, x_m)$  (4.2)<br>we shall on the onset  $_{m-1}$ ,  $x_m$ 

$$
u = u(x_1, x_2, x_3, x_4, \cdots x_{m-1}, x_m)
$$
\n(4.2)

we shall on the onset make some fundamental assumptions thus: that

$$
u = u(x_1, x_2, x_3, x_4, \dots, x_{m-1}, x_m)
$$
\nshall on the onset make some fundamental assumptions thus:

\n
$$
u(x_1, x_2, \dots, x_{m-1}, x_m) = X_1(x_1) \cdot X_2(x_2) \cdot X_3(x_3) \cdot \dots \cdot X_{m-1}(x_{m-1}) \cdot X_m(x_m)
$$
\nwhich

\n
$$
X_k = X_k(x_k)
$$
\nfunction of a single independent variable

\n
$$
(4.4)
$$

in which

$$
X_k = X_k(x_k)
$$
 (4.4)

a function of a single independent variable.

 $(4.3)$  into  $(4.1)$  and simplifying we obtain ordinary differential equations  $(ODEs)$  $(k = 1(1) m).$ in which<br>  $X_k = X_k(x_k)$  (4.4)<br>
a function of a single independent variable.<br>
On substituting (4.3) into (4.1) and simplifying we obtain ordinary differential equations (*ODEs*) in the<br>
unknown functions  $X_k(x_k, 1(1), y_k)$  Some  $X_k = X_k(x_k)$ <br>a function of a single independent variable.<br>On substituting (4 · 3) into (4 · 1) and simplifying we obtain<br>unknown functions  $X_k (k = 1(1)m)$ . Some of the boundary *ODEs*  $(x_k)$ <br>(le independent variable.<br>(3) into (4.1) and simplifyin  $=1(1)$ *m*). Some of the boundary conditions of the original *PDE* will give rise  $_{k}(k=1(1) m)$ a function of a single independent variable.<br>
On substituting  $(4.3)$  into  $(4.1)$  and simplifying we obtain ordinary differential equations  $(ODEs)$  in<br>
unknown functions  $X_k$   $(k = 1(1)m)$ . Some of the boundary conditions of therefore have to solve *m* uncoupled ordinary differential equations some of which may be *BVPs* or *IVPs*. *PDE* fial equations<br> *X<sub>k</sub>* ( $k = 1(1)m$ <br> *X<sub>k</sub>* ( $k = 1(1)m$ These particular solutions  $X_k$  are then used to constitute the most general solution of the original *PDE*. Consider the  $PDE$  in two independent variables x and y in the form *B*(*ODES*) in the use is set that the use of the use of *BVPs* or *IVPs*<br>in *BVPs* or *IVPs*  $Z = 1(1)m$ . Some of the boundary conditions of the original *PDE* will give ris<br>*X* conditions to be satisfied by some of the functions  $X_k (k = 1(1)m)$ . We will<br>uncoupled ordinary differential equations some of which may be ling boundary conditions to be satisfied by<br> *e* to solve *m* uncoupled ordinary differential<br> *PDE* in two independent variables *x* and *y*<br>  $+ S_S + Tt + P_D + Oa + Uu - V$ therefore have to solve *m* uncoupled ordinary differential equations some of which may be *BV*<br>These particular solutions  $X_k$  are then used to constitute the most general solution of the origi<br>Consider the *PDE* in two These particular solutions  $X_k$  are then use<br>Consider the *PDE* in two independent van<br> $Rr + Ss + Tt + Pp + Qq + Uu = 1$ <br>Suppose the solution of  $(4.5)$  is given as *Rr* +  $S_s$  +  $Tt$  +  $Pp$  +  $Qq$  +  $Uu$  =  $V$  and  $Rr$  and  $Rr$  and  $Rr$  is a single  $Rr + S_s$  +  $Tt$  +  $Pp$  +  $Qq$  +  $Uu$  =  $V$  (4.5)

$$
Rr + Ss + Tt + Pp + Qq + Uu = V \tag{4.5}
$$

 $(4.5)$ 

the *PDE* in two independent variables *x* and *y* in the form  
\n
$$
Rr + Ss + Tt + Pp + Qq + Uu = V
$$
\nthe solution of (4.5) is given as  
\n
$$
u = X(x) \cdot Y(y)
$$
\n
$$
V = \frac{X}{Y}
$$
\n(4.6)

ppose the solution of  $(4.5)$  is<br>  $u = X(x) \cdot Y(y)$ <br>
which X and Y are functions<br>
(6) into  $(4.5)$  and simplifyin  $(4.6)$ in which X and Y are functions of x and y respectively and  $u$  is the dependent variable. Substituting Suppose the solution of  $(4.5)$  is given as<br>  $u = X(x) \cdot Y(y)$ <br>
n which X and Y are functions of x and y<br>  $4.6$ ) into  $(4.5)$  and simplifying we obtain  $Rr + Ss + Tt + Pp + Qq + Uu = V$ <br>
he solution of (4.5) is given as<br>  $u = X(x) \cdot Y(y)$ <br> *X* and *Y* are functions of *x* and *y* respectively and *u*<br>
2. (4.5) and simplifying we obtain  $u = X(x) \cdot Y(y)$ <br>in which X and Y are functions of x and y respectively and u is the depende<br>(4.6) into (4.5) and simplifying we obtain<br> $\frac{1}{X} f(D) \cdot X(x) = \frac{1}{Y} \phi(D') \cdot Y(y)$  (4.7)

and Y are functions of x and y respectively and u  
\n(4.5) and simplifying we obtain  
\n
$$
\frac{1}{X} f(D) \cdot X(x) = \frac{1}{Y} \phi(D') \cdot Y(y) \qquad (4.7)
$$

where  $f(D)$  and  $\phi(D')$  are quadratic functions of  $D = \frac{b}{\epsilon}$  and  $D' = \frac{c}{\epsilon}$  respectively. We observe that  $\frac{1}{X} f(D) \cdot X ($ <br>where  $f(D)$  and  $\phi(D')$ <br>the lhs of  $(4 \cdot 7)$  is a fun nto  $(4.5)$  and simplifying we obtain<br>  $\frac{1}{X} f(D) \cdot X(x) = \frac{1}{Y} \phi(D') \cdot Y(y)$  (4.7)<br> *f (D)* and  $\phi(D')$  are quadratic functions of  $D = \frac{\partial}{\partial x}$  and *D* 4.7)<br> $\frac{\partial}{\partial x}$  and  $D' = \frac{\partial}{\partial y}$  $\phi$ simplifying we obtain<br>  $I(x) = \frac{1}{Y} \phi(D') \cdot Y(y)$  (4.7)<br>
(4.7)<br>
(4.7)<br>
(4.7)<br>
(4.7)<br>
(4.7)<br>
(4.7)<br>
(4.7)  $(4.7)$ <br>  $\frac{\partial}{\partial x}$  and  $D' = \frac{\partial}{\partial y}$  respectively. We function of y only and the two  $(D') \cdot Y(y)$   $(4 \cdot 7)$ <br> *x* only while the rhs is a function of *y*<br>  $\frac{\partial}{\partial x}$  and *D'* =  $\frac{\partial}{\partial y}$ 

 $(4\cdot 7)$  is a function of x only while the rhs is a function of y only and the two can not be equal  $-\lambda$ (say) where  $f(D)$  and  $\phi(D')$  are quadratic functi<br>the lhs of  $(4.7)$  is a function of x only whil<br>except each is equal to a constant  $-\lambda$  (say).

We thus have

$$
f(D) \cdot X(x) = \lambda X
$$
  
\n
$$
\phi(D') \cdot Y(y) = \lambda Y
$$
\nThe solution of (4.5) therefore reduces to the solution of (4.8).  
\nThe usefulness of the solutions of *PDE* is quite limited because of the difficulty in choosing the

 $(4.5)$ 

The usefulness of the solutions of PDE is quite limited because of the difficulty in choosing the appropriate arbitrary functions that will satisfy the imposed boundary conditions. Thi s is however eliminated for of  $(4.5)$  therefore reduces to the solution of  $(4.5)$  s of the solutions of *PDE* is quite limited becaus functions that will satisfy the imposed boundary *PDEs (linear)* by certain techniques one of whis This states some class of PDEs (linear) by certain techniques one of which is based on the principle of superposition of solutions. This states that riate arbitrary functions that will satisfy the some class of *PDEs* (*linear*) by certain tech<br>ion of solutions. This states that<br>"If each of the m functions  $z_k$  ( $k = 1(1)m$ ) s

 $(k = 1(1) m)$ 

From this state, the initial distribution of solutions. This states that

\nIf each of the m functions 
$$
z_k
$$
 ( $k = 1(1)m$ ) satisfies a linear PDE, then an arbitrary linear combination

\n
$$
Z = \alpha_1 z_1 + \alpha_2 z_2 + \alpha_1 z_1 + \alpha_2 z_2 + \cdots + \alpha_1 z_1 + \alpha_2 z_2 = \sum_{j=1}^{m} \alpha_j z_j
$$
 (4.9)

\nwhere  $\alpha_k$  ( $k = 1(1)m$ ) are constants also satisfies the differential equation". The combination of the method of separation of variables and the superposition of solution is usually known as Fourier method.

 $(k = 1(1) m)$ "If each of the m functions  $z_k (k = 1(1)m$ <br>  $Z = \alpha_1 z_1 + \alpha_2 z_2 + \alpha_1 z_1 + \alpha_2 z_2 + \cdots$ <br>
where  $\alpha_k (k = 1(1)m)$  are constants also sof separation of variables and the superpose *k*  $\alpha_k$  ( $k = 1(1)$ m) are constants also satisfies the differential equation". The combination of the method of seperation of variables and the superposition of solution is usually known as Fourier method. *Example* of seperation of variables and t<br> *Example*<br>
1 Solve by the method of seper<br>  $\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$ on of variables and the superposition<br>
the method of seperation of variable  $\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$ the method of seperation of va<br>  $\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$ 

1 Solve by the method of seper ation of variables the differential equation

Example  
\n1 Solve by the method of separation of variable  
\n
$$
\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0
$$
\nSolution  
\nSetting  $u(x, y) = X(x) \cdot Y(y) \neq 0$   
\ninto the differential equation we have

*Solution*

$$
\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0
$$
  
\nSetting  $u(x, y) = X(x) \cdot Y(y) \neq 0$  (*i*)  
\nferential equation we have  
\n
$$
X'' \cdot Y - 2X' \cdot Y + Y'X = 0
$$
 (*ii*)  
\ntrough by  $u(x, y)$  by virtue of (*i*) yields  
\n
$$
X'' = X' - Y'
$$

into the differential equation we have

$$
X'' \cdot Y - 2X' \cdot Y + Y'X = 0 \tag{ii}
$$

 $(x, y)$  by vitue of  $(i)$ Setting  $u(x, y) = X(x) \cdot Y(y) \neq 0$ <br>into the differential equation we have<br> $X'' \cdot Y - 2X' \cdot Y + Y'X = 0$ <br>Dividing through by  $u(x, y)$  by vitue of (*i*) yields<br> $X'' = X' - Y'$ 

into the differential equation we have

\n
$$
X'' \cdot Y - 2X' \cdot Y + Y'X = 0
$$
\nDividing through by  $u(x, y)$  by virtue of (i) yields

\n
$$
\frac{X''}{X} - 2\frac{X'}{X} + \frac{Y'}{Y} = 0
$$
\niie,

\n
$$
\frac{1}{X}(X'' - 2X') = -\frac{Y'}{Y}
$$
\n(iv)

, *ie*

$$
\frac{X''}{X} - 2\frac{X'}{X} + \frac{Y'}{Y} = 0
$$
 (*iii*)  
*ie*,  

$$
\frac{1}{X}(X'' - 2X') = -\frac{Y'}{Y}
$$
 (*iv*)  
We observe here that the lhs and rhs of (*iv*) are functions of *x* and *y* respectively. For this equation to be

valid each side must be independently equal to a constant  $(iv)$ <br>*iv*) are functions of *x* and *y*<br>*y* yel to a constant  $\frac{3}{2}$  (sev). The  $(f \times g)$  and y respectively. For this equation to be (say). The implication of this yields the following uncoupled ordinary differential equation: We observe here that the lhs and<br>valid each side must be indeper<br>wing uncoupled ordinary differ<br> $X'' - 2X' - \lambda X = 0$ <br> $Y' + \lambda Y = 0$ valid each side must be<br>wing uncoupled ordina<br> $X'' - 2X' -$ <br> $Y' + \lambda Y = 0$ Exercibent the lift<br> *X*  $\alpha$  *X*  $\alpha$ here that the lhs and rhs of  $(iv)$  and rhs of  $(iv)$  and the must be independently equal to lead ordinary differential equation  $x - 2X' - \lambda X = 0$ 

valid each side must be independently equal to a constant 
$$
\lambda
$$
  
wing uncoupled ordinary differential equation:  

$$
X'' - 2X' - \lambda X = 0
$$

$$
Y' + \lambda Y = 0
$$

$$
i\dot{e},
$$

$$
(D' \cdot 2)Y = 0
$$

$$
(D' \cdot 2)Y = 0
$$

, *ie*

$$
Y' + \lambda Y = 0
$$
  
*i.e.*  

$$
(D^2 - 2D - \lambda)X = 0
$$
  

$$
(D' + \lambda)Y = 0
$$
 $\left\{\n \begin{array}{c}\n (vi)\n \end{array}\n \right.$ 

The solution of the ordinary differential equations in  $(vi)$  above are given as *vi*

the solution of the ordinary differential equations in 
$$
(vi)
$$
 above are given by:  
\n
$$
X(x) = A \exp(1 + \sqrt{1 + \lambda})x + B \exp(1 - \sqrt{1 + \lambda})x
$$
\nand  
\n
$$
Y(y) = C \exp(-\lambda y)
$$
\n
$$
y \text{ virtue of (i) and (vii) therefore we have}
$$

By virtue of  $(i)$  and  $(vii)$  therefore we have

and 
$$
Y(y) = C \exp(-\lambda y)
$$
  
\nBy virtue of (i) and (vii) therefore we have  
\n $u(x, y) = (D \exp(1 + \sqrt{1 + \lambda})x + E \exp(1 - \sqrt{1 + \lambda})x) \exp(-\lambda y)$   
\nwhere  $D = AC$  and  $E = BC$  are arbitrary constants of integration.  
\n2 Determine the solution to the 3 – *D* wave equation

where  $D = AC$  and  $E = BC$  are arbitrary constants of integration.

ne solution to the  $3-D$ <br> $2\nabla^2 u = \frac{\partial^2 u}{\partial x^2}$ −

C and E = BC are arl  
the solution to the 3  

$$
c^2 \nabla^2 u = \frac{\partial^2 u}{\partial t^2}
$$

b y method of seperation of variables.

#### . *Solution*

Assuming the unknown function  $t$  is seperable and of the form *t*

Solution.  
\nAssuming the unknown function *t* is separable and of the form  
\n
$$
u(x, y, z, t) = X(x) \cdot Y(y) \cdot Z(z) \cdot T(t) \neq 0
$$
\nthen the partial differential equation yields  
\n
$$
c^{2}(X''YZT + Y''XZT + Z''XYT) = \ddot{T} XYZ
$$
\n(ii)

then the partial di fferential equation yields

$$
c^{2}\left(X''YZT+Y''XZT+Z''XYT\right)=\ddot{T}XYZ\qquad \qquad (ii)
$$

, *ie*

$$
c^{2}\left(X''YZT+Y''XZT+Z''XYT\right)=\ddot{T} XYZ
$$
\n
$$
c^{2}\left(\frac{X''}{X}+\frac{Y''}{Y}+\frac{Z''}{Z}\right)=\frac{\dddot{T}}{T}
$$
\n(iii)

 $\Rightarrow$ 

$$
c^{2}\left(\frac{x}{X} + \frac{y}{Y} + \frac{z}{Z}\right) = \frac{1}{T}
$$
\n
$$
\Rightarrow \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = \frac{1}{c^{2}}\frac{\ddot{T}}{T}
$$
\n
$$
(iv)
$$

⇒<br>  $\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = \frac{1}{c^2} \frac{\ddot{T}}{T}$  (*iv*)<br>
This equation is true only if each of the component parts is equal to a constant.<br>  $\frac{X''}{X} = -p^2, \frac{Y''}{Y} = -q^2, \frac{Z''}{Z} = -r^2, \frac{1}{c^2} \frac{\ddot{T}}{T} = -s^2$  (*v* , *ie* true only if each of the componen<br> $\frac{r}{r} = -p^2$ ,  $\frac{Y''}{Y} = -q^2$ ,  $\frac{Z''}{Z} = -r^2$ ,  $\frac{1}{2} \frac{T}{Z}$ 

This equation is true only if each of the component parts is equal to a constant.  
\n*ie*,  
\n
$$
\frac{X''}{X} = -p^2, \frac{Y''}{Y} = -q^2, \frac{Z''}{Z} = -r^2, \frac{1}{c^2} \frac{\ddot{T}}{T} = -s^2
$$
\n(v)  
\nThis yields the following uncoupled ordinary differential equations:

$$
\frac{}{X} = -p^2, \frac{}{Y} = -q^2, \frac{}{Z} = -r^2, \frac{}{c^2} = \frac{}{T}
$$
\nThis yields the following uncoupled ordinary difference

\n
$$
X'' + p^2 X = 0
$$
\n
$$
Y'' + q^2 Y = 0
$$
\n
$$
Z'' + r^2 Z = 0
$$
\n
$$
\ddot{T} + c^2 s^2 T = 0
$$
\nif  $d = 1, 2, \ldots$ 

with solutions

$$
\ddot{T} + c^2 s^2 T = 0
$$
\nwith solutions\n
$$
X_p(x) = A_p \cos px + B_p \sin px
$$
\n
$$
Y_q(y) = C_q \cos qy + D_q \sin qy
$$
\n
$$
Z_r(z) = E_r \cos rz + F_r \sin rz
$$
\n
$$
T_s(t) = P_s \cos (cs)t + Q_s \sin (cs)t
$$
\n
$$
(vii)
$$

 $(iv)$  we may express  $T(t)$ Since the parameters  $p, q, r$  and s are dependent by virture of  $(iv)$  we may express  $T(t)$  as *p*, *q*, *r* and *s* are dependent by virture of  $(iv)$  we may express  $T(t)$ 

Since the parameters 
$$
p, q, r
$$
 and  $s$  are dependent by virtue of  $(iv)$  we may express  $T(t)$ :  
\n
$$
T_{pqr}(t) = G_{pqr} \cos\left(\sqrt{p^2 + q^2 + r^2}\right) t + Q_s \sin\left(\sqrt{p^2 + q^2 + r^2}\right) t \qquad (viii)
$$
\nHence by virtue of  $(i)$  and  $(vii)$  we thus have that\n
$$
u_{pqr}(x, y, t, t) = X_p(x) Y_q(y) Z_r(z) T_{pqr}(t) \qquad (ix)
$$
\nThe most general solution is thus given as

Hence by vitue of  $(i)$  and  $(vii)$  we thus have that

$$
u_{pqr}(x, y, t, t) = X_p(x)Y_q(y)Z_r(z)T_{pqr}(t)
$$
 (ix)

The most general solution is thus given as

$$
u_{pqr}(x, y, t, t) = X_p(x)Y_q(y)Z_r(z)T_{pqr}(t)
$$
 (*ix*  
The most general solution is thus given as  

$$
u_{pqr}(x, y, t, t) = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \sum_{r=1}^{\infty} u_{pqr}(x, y, t, t)
$$

$$
(x)
$$
in which the function  $u_{pqr}(x, y, t, t)$  are as defined in *(vii)* and *(ix)*.

 $(x, y, t, t)$  are as defined in (*vii*) and (*ix*) *pqr*

 $(I)$  Cylindrical;  $(r, \vartheta, z)$  $u_{pqr}(x, y, t, t) = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \sum_{r=1}^{\infty} u_{pqr}(x, y, t, t)$  (x)<br>in which the function  $u_{pqr}(x, y, t, t)$  are as defined in (vii) and (ix).<br>4.3 SOLUTION OF 3 - D LAPLACE's EQUATION IN CURVILINEAR COORDINATE SYSTEM.<br>(1)  $OF 3-D LAPLACE 81$ <br> $(r, 9, z)$ <br> $\frac{2u}{u} + \frac{1}{2}\frac{\partial u}{\partial u} + \frac{1}{2}\frac{\partial^2 u}{\partial v^2} + \frac{\partial^2 u}{\partial v^2}$ ich the function  $u_{pq}$ <br> *pdf* (*r*, *y*, *z*)<br> *plindrical;*  $(r, \theta, z)$ *<br>*  $\frac{\partial^2 u}{\partial t^2}$  *1.*  $\frac{\partial^2 u}{\partial t^2}$ in which the function  $u_{pqr}(x, y, t, t)$  are as defi<br>
4.3 *SOLUTION OF* 3 - *D LAPLACE's EQUA*<br>
(*I*) Cylindrical;  $(r, \vartheta, z)$ <br>  $\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \vartheta^2} + \frac{\partial^2 u}{\partial z^2} = 0$ *r <i>r <i>r s* - *p LAPLACE* '*s EQUATION IN*<br> *r*  $(r, \vartheta, z)$ <br>  $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \vartheta^2} + \frac{\partial^2 u}{\partial z^2} = 0$ <br>  $(r, \vartheta, \phi)$ i the function  $u_{pqr}(x, y, t, t)$  are as defined in (*vii*) an<br>
UTION OF 3- D LAPLACE's EQUATION IN CUI<br>
ndrical;  $(r, 9, z)$ <br>  $\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial 9^2} + \frac{\partial^2 u}{\partial z^2} = 0$ 

*I*) Cylindrical; 
$$
(r, \vartheta, z)
$$
  
\n
$$
\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \vartheta^2} + \frac{\partial^2 u}{\partial z^2} = 0
$$
\n*II*) Spherical;  $(r, \vartheta, \phi)$ 

 $(H)$  Spherical;  $(r, \vartheta, \phi)$ 

$$
\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0
$$
  
(*H*) Spherical;  $(r, \theta, \phi)$   

$$
\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{Cot \theta}{r^2} \frac{\partial u}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta \phi^2} = 0
$$
  
In this section we will solve the problem for the spherical coordinate system. The s

(*II*) Spherical;  $(r, \theta, \phi)$ <br>  $\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{Cot \theta}{r^2} \frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} = 0$ <br>
In this section we will solve the problem for the shperical coord

In this section we will solve the problem for the spherical coordinate system  
drical coordinate follows the same procedure.  
The corresponding differential equation is given by  

$$
\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{Cot \theta}{r^2} \frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} = 0
$$
 (*i*)  
Assume the unknown function *u* is separable in the form  

$$
u(r, \theta, \phi) = R(r) \cdot \Theta(\theta) \cdot \Phi(\phi) \neq 0
$$
 (*ii*)  
Substitution of (*ii*) into (*i*) and dividing through the result by  $u(r, \theta, \phi)$  by

Assume the unknown function  $u$  is seperable in the form *u*

$$
\frac{1}{r^2} + \frac{1}{r} \frac{\partial r}{\partial r} + \frac{1}{r^2} \frac{\partial \theta^2}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial \theta^2}{\partial \theta^2} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial \phi^2}{\partial \phi^2} = 0
$$
 (i)  
ne unknown function *u* is separable in the form  

$$
u(r, \theta, \phi) = R(r) \cdot \Theta(\theta) \cdot \Phi(\phi) \neq 0
$$
 (ii)  
on of (ii) into (i) and dividing through the result by  $u(r, \theta, \phi)$  yields

Substitution of  $(ii)$  into  $(i)$  and

Assume the unknown function *u* is separable in the form  
\n
$$
u(r, \theta, \phi) = R(r) \cdot \Theta(\theta) \cdot \Phi(\phi) \neq 0
$$
\n(ii)  
\nSubstitution of (ii) into (i) and dividing through the result by  $u(r, \theta, \phi)$  y  
\n
$$
\frac{R''}{R} + \frac{2}{r} \frac{R'}{R} + \frac{1}{r^2} \frac{\Theta''}{\Theta} + \frac{Cot \theta}{r^2} \frac{\Theta'}{\Theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\Phi''}{\Phi} = 0
$$
\n(iii)  
\ni*e*,  
\n
$$
\left(\frac{R''}{R} + \frac{2}{r} \frac{R'}{R} + \frac{1}{r^2} \frac{\Theta''}{\Theta} + \frac{Cot \theta}{r^2} \frac{\Theta'}{\Theta}\right) r^2 \sin^2 \theta = -\frac{\Phi''}{\Phi}
$$
\n(iv)

, *ie*

$$
\frac{R''}{R} + \frac{2}{r} \frac{R'}{R} + \frac{1}{r^2} \frac{\Theta''}{\Theta} + \frac{Cot \theta}{r^2} \frac{\Theta'}{\Theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\Phi''}{\Phi} = 0
$$
 (*iii*)  
*ie*,  

$$
\left(\frac{R''}{R} + \frac{2}{r} \frac{R'}{R} + \frac{1}{r^2} \frac{\Theta''}{\Theta} + \frac{Cot \theta}{r^2} \frac{\Theta'}{\Theta}\right) r^2 \sin^2 \theta = -\frac{\Phi''}{\Phi}
$$
 (*iv*)  
Observe that the lhs of (*iv*) are functions of *r* and *9* while the rhs is a function of  $\phi$  only. This can only be

valid if each side is a constant  $m^2$ , say. Therefore, we have that  $\frac{\Theta''}{\Theta} + \frac{Cot \theta}{r^2} \frac{\Theta'}{\Theta} \Bigg| r^2 \sin^2 \theta = -\frac{\Phi''}{\Phi}$  (*iv*)<br>*iv*) are functions of *r* and *9* while the rhs is a function of  $\phi$  only. This can constant  $m^2$ , soy. Therefore, we have that  $(R + r R + r^2 \Theta + r^2 \Theta)^r$  on  $V = \Phi$  (*iv*)<br>that the lhs of (*iv*) are functions of *r* and *9* while the rhs is a functions of *r* and *9* while the rhs is a function side is a constant  $m^2$ , say. Therefore, we have that  $\Phi''$ Observe that the lhs of (*iv*) are functions of *r* and *9* while the rhs is a furtial<br>valid if each side is a constant  $m^2$ , say. Therefore, we have that<br> $\Phi'' + m^2 \Phi = 0$  (*v*)<br> $\frac{1}{R} (r^2 R'' + 2rR') + \frac{1}{\Theta} (\Theta'' + Cot \Theta'') = \frac{m$ 

$$
\Phi'' + m^2 \Phi = 0 \tag{v}
$$

$$
\frac{1}{R}(r^2 R'' + 2rR') + \frac{1}{\Theta}(\Theta'' + \cot 9\Theta') = \frac{m^2}{\sin^2 9}
$$
 (vi)  
ie,  

$$
\frac{1}{\Theta}(\Theta'' + \cot 9\Theta') - \frac{m^2}{\sin^2 9} = -\frac{1}{R}(r^2 R'' + 2rR')
$$
 (vi)

, *ie*

*ie,*  
\n
$$
\frac{1}{\Theta}(\Theta'' + \text{Cot}\mathcal{P}\Theta') - \frac{m^2}{\text{Sin}^2 \mathcal{G}} = -\frac{1}{R}(r^2 R'' + 2rR')
$$
 (*vii*)  
\nEqn (*vii*) is true if only each side is a constant  $-l(l+1)$ . This condition gives rise to the following uncoup-

(*vii*) is true if only each side is a constant  $-l(l+1)$ 

uncoupled ordinary differential equations:

uncoupled ordinary differential equations:  
\n
$$
r^2 R'' + 2rR' - l(l+1) R = 0
$$
 (viii)

uncoupled ordinary differential equations:  
\n
$$
r^2 R'' + 2rR' - l(l+1) R = 0
$$
 (viii)  
\n $\Theta'' + \text{Cot } \mathcal{S}\Theta' + \left\{ l(l+1) - \frac{m^2}{\sin^2 \theta} \right\} \Theta = 0$  (ix)  
\nSubstituting  $\text{Cos } \theta = \mu$  in (ix) yields

 $(ix)$ *ix* =

$$
\Theta'' + \cot \theta \Theta' + \left\{ l(l+1) - \frac{m^2}{\sin^2 \theta} \right\} \Theta = 0 \qquad (ix)
$$
  
Substituting  $\cos \theta = \mu$  in  $(ix)$  yields  

$$
\left(1 - \mu^2\right) \frac{d^2 \Theta}{d \mu^2} - 2\mu \frac{d\Theta}{d \mu} + \left\{ l(l+1) - \frac{m^2}{1 - \mu^2} \right\} \Theta = 0 \qquad (x)
$$
  
Eqn  $(x)$  is associated Legendre differential equation.  
Solving Eqns  $(v)$ ,  $(viii)$  and  $(x)$  in standard form we obtain  

$$
\Phi_m(\phi) = A_m \cos m\phi + B_m \sin m\phi \qquad (xi)
$$

Eqn  $(x)$  is associated Legendre differential equation.

Solving Eqns  $(v)$ ,  $(viii)$  and  $(x)$ Eqn (x) is associated Legendre differential equals Solving Eqns (v), (viii) and (x) in standard for  $\Phi_m(\phi) = A_m \cos m\phi + B_m \sin m\phi$ 

$$
\Phi_m(\phi) = A_m \cos m\phi + B_m \sin m\phi \qquad (xi)
$$

Eqn *(x)* is associated Legendre differential equation.  
\nSolving Eqns *(v)*,*(viii)* and *(x)* in standard form we obtain  
\n
$$
\Phi_m(\phi) = A_m \cos m\phi + B_m \sin m\phi
$$
\n*(xi)*\n
$$
R_l(r) = C_l r^l + \frac{D_l}{r^{l+1}}
$$
\n(xi)  
\nand  
\n
$$
\Theta_{ml}(\theta) = E_{ml} P_l^m (\cos \theta) + F_{ml} Q_l^m (\cos \theta)
$$
\n
$$
(xii)
$$
\nThe general solution of the *PDF* is therefore

and

$$
P_{nl}^{(1)}(x) = P_{ml} P_l^m \left( \cos \theta \right) + F_{ml} Q_l^m \left( \cos \theta \right) \qquad (xiii)
$$
\nSolution of the *PDE* is therefore

\n
$$
\sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \left( A_n \cos m\phi + B_n \sin m\phi \right) \left( C_n x^l + D_l \right) \left( E_n - D_m^m \left( C_n \sin \theta \right) + E_n \right)
$$

The general solution of the PDE is therefore

$$
R_{i}(r) = C_{i}r^{i} + \frac{D_{i}}{r^{i+1}}
$$
 (*xii*)  
and  

$$
\Theta_{ml}(9) = E_{ml}P_{i}^{m}(\cos 9) + F_{ml}Q_{i}^{m}(\cos 9)
$$
 (*xiii*)  
The general solution of the *PDE* is therefore  

$$
u(r, 9, \phi) = \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} (A_{m} \cos m\phi + B_{m} \sin m\phi) \left(C_{i}r^{i} + \frac{D_{i}}{r^{i+1}}\right) \left(E_{ml}P_{i}^{m}(\cos 9) + F_{ml}Q_{i}^{m}(\cos 9)\right)
$$
 (*xiv*)  
The arbitrary constants are chosen in a manner that the solution is bounded. This implies that  $F_{ml} = 0$   

$$
\therefore Q_{i}^{m}(\cos 9) \rightarrow \infty \text{ as } 9 \rightarrow 0. \text{ Consequently the general solution is}
$$

 $(\cos \theta)$ The arbitrary constants are chosen in a manner that the solution is bounded. This implies that  $F_{ml} = 0$ *m l Q* Constants are<br>  $\phi \rightarrow \infty$  as  $\theta -$ <br>  $\sum_{m=0}^{\infty} \sum_{m=0}^{\infty} E_{m} p_{m}$ ution is bounded. This implies<br>
ution is<br>  $\left(C_l r^l + \frac{D_l}{r^{l+1}}\right)$  (*x* 

$$
u(r, \theta, \phi) = \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} (A_m \cos m\phi + B_m \sin m\phi) \left( C_l r^l + \frac{D_l}{r^{l+1}} \right) \left( E_{ml} P_l^m \left( \cos \theta \right) + F_{ml} Q_l^m \left( \cos \theta \right) \right) \qquad (xiv)
$$
  
The arbitrary constants are chosen in a manner that the solution is bounded. This implies that  $F_{ml} = 0$   

$$
\therefore Q_l^m \left( \cos \theta \right) \to \infty \text{ as } \theta \to 0. \text{ Consequently the general solution is}
$$

$$
u(r, \theta, \phi) = \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} E_{ml} P_l^m \left( \cos \theta \right) \left( A_m \cos m\phi + B_m \sin m\phi \right) \left( C_l r^l + \frac{D_l}{r^{l+1}} \right) \qquad (xv)
$$
  
A solution of the problem in the form  $(xi)$ ,  $(xii)$  and  $(xiii)$  are called *sperical harmonics* while the solution  $(xi)$  and  $(xiii)$  called *plane harmonics*.  
3 Determine the potential outside and inside a spherical surface kent at a fixed distribution of electrical

 $(xi), (xii)$  and  $(xiii)$ tion  $(x_i)$  and  $(xiii)$  called *plane harmonics*.

potential of the form  $u = f(g)$  assuming that the space inside and outside th 3 Determine the potential outside and inside a spherical surface kept at a fixed distribution of electrical blem in the called planet planet  $u = f(g)$ e sphere is free of charge. . *Solution* 3 Determine the potential outside and inside a spherical surface kept at a fixed distribution of electric<br>potential of the form  $u = f(\theta)$  assuming that the space inside and outside the sphere is free of charg<br>*Solution*.<br>I potential of the form  $u = f(\mathcal{G})$  assuming that the spa<br>
Solution.<br>
In potential theory it is known that the potential u sat:<br>
ie,  $\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \mathcal{G}^2} + \frac{Cot \mathcal{G}}{r^2} \frac{\partial u}{\partial \mathcal$ wherical surface kept at a fixed distribution of  $\alpha$ <br>
e space inside and outside the sphere is free of<br> *u* satisfies the Laplace equation  $\nabla^2 u = 0$  in (*r* of the form  $u = f(\theta)$  assuming that the space inside<br>ial theory it is known that the potential u satisfies the<br>ie,  $\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{Cot \theta}{r^2} \frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\$ lectrical<br>charge.<br> $\mathcal{G}, \phi$ . istribution of electrical<br>bhere is free of charge.<br> $\nabla^2 u = 0$  in  $(r, \theta, \phi)$ .

 $^{2}u = 0$  in  $(r, \vartheta, \phi)$ 

l of the form 
$$
u = f(\theta)
$$
 assuming that the space inside and outside the  
\n.  
\ntial theory it is known that the potential u satisfies the Laplace equation  
\n
$$
ie, \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{Cot \theta}{r^2} \frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} = 0.
$$
 (i)  
\nof spherical symmetry, u is independent of  $\phi$ .  
\ni.e,  $\frac{\partial u}{\partial \phi} = \frac{\partial^2 u}{\partial \phi^2} = 0.$  (ii)

In veiw of spherical symmetry,  $u$  is independent of  $\phi$ . *u*  $\phi$ .

*ie,* 
$$
\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{Cot \theta}{r^2} \frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial u}{\partial \theta}
$$
  
In view of spherical symmetry, *u* is independent of  $\phi$ .  
*ie,*  $\frac{\partial u}{\partial \phi} = \frac{\partial^2 u}{\partial \phi^2} = 0.$  (*ii*)  
By virtue of (*ii*) the governing equation (*i*) reduces to

By vitue of  $(ii)$  the governing equation  $(i)$  reduces to

*ie,* 
$$
\frac{\partial u}{\partial \phi} = \frac{\partial^2 u}{\partial \phi^2} = 0.
$$
 (*ii*)  
\nBy virtue of (*ii*) the governing equation (*i*) reduces to  
\n*ie,*  $\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{Cot \theta}{r^2} \frac{\partial u}{\partial \theta} = 0.$  (*iii*)

Assume the unknown function  $u$  is seperable in the form *u*

Assume the unknown function *u* is separable in the form  
\n
$$
u(r, \theta, \phi) = R(r) \cdot \Theta(\theta) \neq 0
$$
\n
$$
(iv)
$$
\nSubstitution of (ii) into (i) and dividing through the result

(*ii*) into (*i*) and dividing through the resuly by  $u(r, \theta, \phi)$ Assume the unknown function *u* is seperable in the form<br>  $u(r, \theta, \phi) = R(r) \cdot \Theta(\theta) \neq 0$  (*iv*)<br>
Substitution of (*ii*) into (*i*) and dividing through the resuly by  $u(r, \theta, \phi)$  yields<br>  $R'' = 2 R' = 1 \Theta''$   $Cot \Theta'$ 

the unknown function *u* is separable in the form  
\n
$$
u(r, \theta, \phi) = R(r) \cdot \Theta(\theta) \neq 0
$$
 (3)  
\nion of *(ii)* into *(i)* and dividing through the re  
\n
$$
\frac{R''}{R} + \frac{2}{r} \frac{R'}{R} + \frac{1}{r^2} \frac{\Theta''}{\Theta} + \frac{Cot \theta}{r^2} \frac{\Theta'}{\Theta} = 0
$$
 (v)

, *ie*

$$
\frac{R''}{R} + \frac{2}{r} \frac{R'}{R} + \frac{1}{r^2} \frac{\Theta''}{\Theta} + \frac{Cot \theta}{r^2} \frac{\Theta'}{\Theta} = 0 \qquad (v)
$$
  
ie,  

$$
\frac{1}{\Theta} (\Theta'' + Cot \theta \Theta') + \frac{1}{R} (r^2 R'' + 2rR') = 0 \qquad (vi)
$$

$$
\frac{1}{\Theta} (\Theta'' + Cot \theta \Theta') = -\frac{1}{R} (r^2 R'' + 2rR') \qquad (vii)
$$
Observe that the lhs of  $(vii)$  is a function of  $\theta$  while the rhs  
valid if each side is a constant  $-l(l+1)$ , say. Therefore, we

Observe that the lhs of  $(vii)$  is a function of  $\theta$  while the rhs is a function of r only. This can only be  $(l+1)$  $\frac{1}{\Theta}(\Theta'' + \text{Cot}\mathcal{\theta}\Theta') = -\frac{1}{R}(r^2R'' + 2rR')$  (vii)<br>Observe that the lhs of (vii) is a function of  $\theta$  while the rhs is a function of  $r$  only. This can only<br>valid if each side is a constant  $-l(l+1)$ , say. Therefore, we Observe that the lhs of  $(vii)$  is a function of<br>valid if each side is a constant  $-l(l+1)$ , say<br> $r^2R'' + 2rR' - l(l+1)R = 0$  (viii t the lhs of  $(vii)$  is a function of  $\theta$ <br>h side is a constant  $-l(l+1)$ , say. The  $r + 2rR' - l(l+1)R = 0$  (viii) between that the lhs of  $(vu)$  is a function of  $\theta$  will be if each side is a constant  $-l(l+1)$ , say. The  $r^2 R'' + 2rR' - l(l+1)R = 0$  (viii)<br>  $\Theta'' + Cot \theta \Theta' + l(l+1) \Theta = 0$  (ix)

Find the equation 
$$
u = k
$$
 and  $u = k$  and  $u = k$ .

\nFind the equation  $u = k$  and  $u = k$ .

\nUsing (viii) and (ix).

\nUsing (viii) and (ix).

\nWe set  $u = \text{Cose}(0, \text{in.}(i))$ .

 $(viii)$  and  $(ix)$ 

( ) 2 2 2 2 2 We set Cos in Sin 1 1 But 1 *ix d d d d d d d d d d d d d d d d d d d* = = − = − − = − − = = − − = ( ) ( ) 2 2 2 2 2 1 1 1 1 2 1 0 This is the Lagendre with solution *d d DE* ( ) ( ) . Similarly, the solution of is obtained by assuming *d d d d x d d d d d d l l xi* = − − − − = − − − − + + =

Hence,  $(ix)$  transforms to

$$
\left(1 - \mu^2\right) \frac{d^2 \Theta}{d\mu^2} - 2\mu \frac{d\Theta}{d\mu} + l(l+1)\Theta = 0 \qquad (xi)
$$
\nLagendre *DE* with solution  $P_l(\mu)$ . Similarly, the solution of *(viii)*  $R(r) = r^{\alpha}$  (xii)

 $\mu$ 

$$
R(r) = r^{\alpha} \tag{xii}
$$

giving the solutions

Equation 2.2 with standard 
$$
P_I(\mu)
$$
. Similarly, the solution of  $(\nu \mu)$ 

\n $R(r) = r^{\alpha}$  (xii)

\n $R(r) = A_I r^I + \frac{B_I}{r^{I+1}}$  (xiii)

\n $\Theta(\mu) = C_I P_I(\mu) + D_I Q_I(\mu)$  (xiv)

\n $\Theta(\theta) = C_I P_I(\cos \theta) + D_I Q_I(\cos \theta)$  (xiv)

\nwe share is of radius  $a$ . Then, we have

, *ie*

*i.e,*  
\n
$$
\Theta(\mathcal{G}) = C_l P_l (\cos \mathcal{G}) + D_l Q_l (\cos \mathcal{G}) \qquad (xv)
$$
\nSuppose the sphere is of radius *a*. Then, we have  
\n
$$
u(a, \mathcal{G}) = f(\mathcal{G}) \qquad (xvi)
$$
\nAlso, the potential *u* remains bounded everywhere  $\Rightarrow u \prec \infty$  as  $\mathcal{G} \rightarrow 0$ .

Suppose the sphere is of radius  $a$ . Then, we have *a*

$$
u(a, \theta) = f(\theta) \qquad (xvi)
$$

Also, the potential  $u$  rema

$$
\therefore Q_l (\cos \theta) \to \infty \text{ as } \theta \to 0 \Rightarrow D_l = 0
$$
  

$$
\therefore \Theta(\theta) = C_l P_l (\cos \theta)
$$
  
Hence,  

$$
u_m (r, \theta) = \left( A_l r^l + \frac{B_l}{r^{l+1}} \right) C_l P_l (\cos \theta)
$$
 (xvii)

Hence,

$$
\therefore \Theta(\mathcal{G}) = C_l P_l (\cos \mathcal{G})
$$
  
Hence,  

$$
u_m(r, \mathcal{G}) = \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) C_l P_l (\cos \mathcal{G})
$$
  
The most general solution is therefore given as

The most general solution is therefore given as

$$
u_m(r, \theta) = \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) C_l P_l \left(\cos \theta\right)
$$
  
The most general solution is therefore given as  

$$
u(r, \theta) = \sum_{l=0}^{\infty} \left(E_l r^l + \frac{F_l}{r^{l+1}}\right) P_l \left(\cos \theta\right)
$$

$$
Potential Outside the Sphere.
$$
  
We also recall from potential theory that  $u = 0$  as  $r \to \infty \Rightarrow E_l = 0$ .  
Therefore, solution for  $r \succ 0$  (outside the sphere) is given as

Potential Outside the Sphere.

We also recall from potential theory that  $u = 0$  as  $r \to \infty \Rightarrow E_1 = 0$ . *l*

We also recall from potential theory that 
$$
u = 0
$$
 as  $r \to \infty \Rightarrow E_l = 0$ .  
\nTherefore, solution for  $r > 0$  (outside the sphere) is given as  
\n
$$
u(r, \theta) = \sum_{l=0}^{\infty} \frac{F_l}{r^{l+1}} P_l(\cos \theta)
$$
\n
$$
\text{Setting } r = a \text{ and applying } u(a, \theta) = f(\theta) \text{ we have}
$$
\n
$$
\int_{a}^{b} F \, dt
$$

Therefore, solution for 
$$
r > 0
$$
 (outside the sphere) is given as  
\n
$$
u(r, \theta) = \sum_{l=0}^{\infty} \frac{F_l}{r^{l+1}} P_l(\cos \theta)
$$
\n
$$
\text{Setting } r = a \text{ and applying } u(a, \theta) = f(\theta) \text{ we have}
$$
\n
$$
\sum_{l=0}^{\infty} \frac{F_l}{a^{l+1}} P_l(\cos \theta) = f(\theta) = \sum_{l=0}^{\infty} \frac{F_l}{a^{l+1}} P_l(\mu) \qquad (xx)
$$
\nThis is the Legendre series. To determine the coefficients  $F$  therefore we

This is the Legendre series. To determine the coefficients  $F_t$  therefore we multiply  $(xx)$  by  $P_m(\mu)$  and Example series. To determine<br>result in  $-1 \le \mu \le 1$  to obtain<br> $\int_{0}^{1} \sum_{i=1}^{\infty} F_{i} P_{i}(u) P_{i}(u) du = 0$  $\sum_{l=0}^{\infty} \frac{F_l}{a^{l+1}} P_l(\cos \theta) = f(\theta) = \sum_{l=0}^{\infty}$ <br>This is the Legendre series. To determine integrate the result in  $-1 \le \mu \le 1$  to obtain *m xx*) by *P*,  $\mu$ es. To determine the coefficients  $F_t$  therefore we<br>  $\leq \mu \leq 1$  to obtain<br>  $\mu$ )  $P_m(\mu) d\mu = \int_{-1}^{1} f(\vartheta) P_m(\mu) d\mu$ 

$$
\sum_{l=0}^{\infty} \frac{F_l}{a^{l+1}} P_l(\cos \theta) = f(\theta) = \sum_{l=0}^{\infty} \frac{F_l}{a^{l+1}} P_l(\mu)
$$
  
This is the Legendre series. To determine the coefficients  $F_l$  there  
integrate the result in  $-1 \le \mu \le 1$  to obtain  

$$
\int_{-1}^{1} \sum_{l=0}^{\infty} \frac{F_l}{a^{l+1}} P_l(\mu) P_m(\mu) d\mu = \int_{-1}^{1} f(\theta) P_m(\mu) d\mu
$$
  
i.e, 
$$
\frac{F_m}{a^{m+1}} \int_{-1}^{1} P_m^2(\mu) d\mu = \int_{-1}^{1} f(\theta) P_m(\mu) d\mu
$$
  
i.e, 
$$
\frac{F_m}{a^{m+1}} \cdot \frac{2}{2m+1} = \int_{-1}^{1} f(\theta) P_m(\mu) d\mu
$$

,<br>,<br>,<br>,

*i.e,*  
\n
$$
\frac{F_m}{a^{m+1}} \int_{-1}^{1} P_m^2(\mu) d\mu = \int_{-1}^{1} f(\vartheta) P_m(\mu) d\mu
$$
\n*i.e,*  
\n
$$
\frac{F_m}{a^{m+1}} \cdot \frac{2}{2m+1} = \int_{-1}^{1} f(\vartheta) P_m(\mu) d\mu
$$
\n
$$
\therefore \qquad F_m = \frac{2m+1}{2} \cdot a^{m+1} \int_{0}^{\pi} f(\vartheta) P_m(\cos \vartheta) \sin \vartheta d\vartheta \qquad (xxi)
$$
\nTherefore, the required potential is given by  $(xix)$  with coefficient as given by  $(xxi)$ .

Therefore, the required potential is given by  $(xix)$  with cosfficient as given by  $(xxi)$ . inside the sphere.  $F_m = \frac{2m+1}{2}$ <br>Therefore, the require<br>*Potential inside the s*<br>From potential theory Therefore, the required potential is given by  $(xix)$  with cosff<br> *Potential inside the sphere*.<br>
From potential theory  $u \prec \infty$  as  $r \to 0 \Rightarrow F_1 = 0$ . Potential inside the sphere.<br>
From potential theory  $u \prec \infty$  as  $r \to 0 \Rightarrow F_l = 0$ .<br>  $\therefore$   $u(r, \theta) = \sum_{l=0}^{\infty} E_l r^l P_l (\cos \theta)$ 

<i>Potential inside the sphere.</i>		
From potential theory $u \prec \infty$ as $r \to 0 \Rightarrow F_l = 0$ .		
$\therefore$	$u(r, \theta) = \sum_{l=0}^{\infty} E_l r^l P_l (\cos \theta)$	$(x x ii)$
By virtue of the condition on the surface of the sphere $(u(a, \theta) = f(\theta))$ we thus have		

 $(u(a, \vartheta) = f(\vartheta))$  $= f(g)$  we thus have

$$
u(r, \theta) = \sum_{l=0}^{\infty} E_l r^l P_l (\cos \theta)
$$
 (xxii)  
By virtue of the condition on the surface of the sphere  $(u(a, \theta) = f(\theta))$  we thus have  

$$
u(a, \theta) = \sum_{l=0}^{\infty} E_l r^l P_l (\cos \theta) = f(\theta)
$$
 (xxiii)  

$$
\int_{-1}^{1} \sum_{l=0}^{\infty} E_l a^l P_l (\cos \theta) P_m (\cos \theta) \sin \theta d\theta = \int_{-1}^{1} f(\theta) P_m (\cos \theta) \sin \theta d\theta
$$

*i.e,* 
$$
E_m a^m \cdot \frac{2}{2m+1} = \int_{-1}^{1} f(\vartheta) P_m(\cos \vartheta) \sin \vartheta d\vartheta
$$

$$
\therefore \qquad E_m = \frac{2m+1}{2a^m} \int_{-1}^{1} f(\vartheta) P_m(\cos \vartheta) \sin \vartheta d\vartheta \qquad (xxiv)
$$

Therefore, inside the sphere the potential is given as

$$
E_m = \frac{2m+1}{2a^m} \int_{-1}^{1} f(\mathcal{G}) P_m(\cos \mathcal{G}) \sin \mathcal{G} d\mathcal{G}
$$
 (xxiv  
efore, inside the sphere the potential is given as  

$$
u(a, \mathcal{G}) = \sum_{l=0}^{\infty} E_l r^l P_l(\cos \mathcal{G}) = f(\mathcal{G})
$$
 (xxv)  
be the coefficients  $E_l$  are as given in (xxiv).

where the coefficients  $E_i$  are as given in  $(xxiv)$ .

#### *Exercise*

Determine the steady-state temperature of a semi-circulr plate of radius a whose circumference is maintained at temperature  $T_0$  and the base at  $T = 0$ . *erca*<br>etern<br>ed at<br>int : Determine the steady-state temperature of a semi-circulr plate of radius *a* whos<br>ined at temperature  $T_0$  and the base at  $T = 0$ .<br>*H* int:<br>This is a Laplace equation in polar coordinate  $(r, \theta)$  with boundary conditions: The *T<sub>l</sub>* are as given in<br>ty-state temperature<br> $T_0$  and the base at *T*  $T = 0.$ <br>
rdinate  $(r, \theta)$  with bo<br>  $\le a \le r$ 

#### $H$  int:

 $(r, \vartheta)$ *r*

Find the base at 
$$
T = 0
$$
.

\nH int:

\nThis is a Laplace equation in polar coordinate  $(r, \theta)$  with

\n
$$
T(r, 0) = 0 = T(r, \pi); 0 \le a \le r
$$
\n
$$
T(a, \theta) = T_0, T \prec \infty \text{ as } r \to 0
$$
\nSolution

*Solution*

$$
T(r,0) = 0 = T(r,\pi); 0 \le a \le r
$$
  
\n
$$
T(a, \theta) = T_0, T \prec \infty \text{ as } r \to 0
$$
  
\nSolution  
\n
$$
T(r, \theta) = \frac{4T_0}{\pi} \sum_{m=1}^{\infty} \frac{1}{2m-1} \left(\frac{r}{2}\right)^{2m-1} \sin(2m-1)\theta.
$$
  
\n4.4 *SOLUTION OF THE 3-D Wave EQUATIONS IN CURVILINEAR*  
\n(I) Cylindrical;  $(r, \theta, z)$   
\n
$$
\nabla^2 u = \frac{\partial^2 u}{\partial t} + \frac{1}{2} \frac{\partial u}{\partial t} + \frac{1}{2} \frac{\partial^2 u}{\partial t} + \frac{\partial^2 u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial t}
$$

 $(I)$  Cylindrical;  $(r, \vartheta, z)$ Solution<br>  $T(r, \theta) = \frac{4T_0}{\pi} \sum_{m=1}^{\infty} \frac{1}{2m-1} \left(\frac{r}{2}\right)^{2m-1} \sin(2m-1)\theta.$ <br>
4.4 SOLUTION OF THE 3-D WAVE EQUATIONS IN CURVILINEAR COORDINATE SYSTEM.<br>
(1) Cylindrical: (r. 9 z)  $T(r, \theta) = \frac{1}{r^2}$ <br> *UTION OF TH*<br>
drical;  $(r, \theta, z)$  $\pi$   $\frac{1}{m=1}$  2m - 1 (2)<br> *OF THE 3-D WAVE EQUATI*<br>  $(r, \theta, z)$ <br>  $\frac{u}{z^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$ *v OF THE 3-D WAVE EQUAT*<br>  $(r, \vartheta, z)$ <br>  $\frac{e^2u}{r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \vartheta^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t}$ <br>  $(r, \vartheta, \varphi)$ 9. LUTION OF THE 3-D WAVE EQUATIONS IN CUR<br>
ndrical;  $(r, \theta, z)$ <br>  $\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$ . *NN OF THE 3-D WAVE EQUATIONS IN CUR*<br>  $(\vec{r}, \theta, z)$ <br>  $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}.$ <br>  $(\vec{r}, \theta, \phi)$ 

$$
(I) \text{ Cylindrical; } (r, \vartheta, z)
$$
\n
$$
\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \vartheta^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}.
$$
\n
$$
(II) \text{ Spherical; } (r, \vartheta, \phi)
$$
\n
$$
\frac{\partial^2 u}{\partial \vartheta^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}.
$$

 $(H)$  Spherical;  $(r, \vartheta, \phi)$ 

$$
\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial g^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}.
$$
  
(*H*) Spherical;  $(r, \theta, \phi)$   

$$
\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial g^2} + \frac{Cot \theta}{r^2} \frac{\partial u}{\partial g} + \frac{1}{r^2 S} \frac{\partial^2 u}{\partial g^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}.
$$
  
In this section we will solve the problem for the cylindrical coordinate system, the the spherical case

follows the same procedure. In this section we will solve the problem for the cylindrical coordinate system, the follows the same procedure.<br>
Solution.<br>
We recall that the governing eqution in the coordinate system  $(r, \theta, z)$  is given as  $\theta^2$ <br>  $r^2$ 

Solution.

 $(r, \vartheta, z)$ 

follows the same procedure.  
\nSolution.  
\nWe recall that the governing equation in the coordinate system 
$$
(r, \theta)
$$
  
\n
$$
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}.
$$
\n
$$
\text{Assuming a separable solution of the form}
$$
\n
$$
u(r, \theta, z, t) = R(r) \Theta(\theta) Z(z) T(t) \neq 0 \qquad (ii)
$$
\nand dividing through by  $u$  we have

Assuming a seperable solution of the form

$$
u(r, \vartheta, z, t) = R(r) \Theta(\vartheta) Z(z) T(t) \neq 0 \qquad (ii)
$$

and dividing through by *u* we have

separable solution of the form

\n
$$
u(r, \vartheta, z, t) = R(r) \Theta(\vartheta) Z(z) T(t) \neq 0 \qquad (ii)
$$
\nthrough by  $u$  we have

\n
$$
\frac{1}{R} \left( R'' + \frac{1}{r} R' \right) + \frac{\Theta''}{r^2 \Theta} + \frac{Z''}{Z} = \frac{1}{c^2} \frac{\ddot{T}}{T} \qquad (iii)
$$

of (*iii*) is a function of r and  $\theta$  while the rhs is a function of t. The equation is only true if they are both constant say  $-p^2$ .<br> *ie*,<br>  $\ddot{T} = c^2 p^2 T$ *lhs* of *(iii)* is a function of *r* and *9* while the *rhs* is a function of *t* beth constant sex. a function<br>say  $-p^2$ .<br> $\ddot{T} = c^2 p^2 T$ <br>1 ( 9 both constant say  $-p^2$ .<br> *ie*,<br>  $\ddot{T} = c^2 p^2 T$ <br>
and<br>  $\frac{1}{R} \left( R'' + \frac{1}{r} \right)$ 

, *ie*

say 
$$
-p^2
$$
.  
\n
$$
\ddot{T} = c^2 p^2 T \qquad (iv)
$$
\n
$$
\frac{1}{R} \left( R'' + \frac{1}{r} R' \right) + \frac{\Theta''}{r^2 \Theta} + \frac{Z''}{Z} = -p^2 \qquad (v)
$$
\n
$$
\frac{1}{R} \left( 1 - \frac{1}{r^2} \right) + \frac{1}{r^2 \Theta} = \frac{Z''}{r^2} \qquad (v)
$$

າ<br>,<br>,

$$
\ddot{T} = c^2 p^2 T \qquad (iv)
$$
  
and  

$$
\frac{1}{R} \left( R'' + \frac{1}{r} R' \right) + \frac{\Theta''}{r^2 \Theta} + \frac{Z''}{Z} = -p^2 \qquad (v)
$$
  
*ie*,  

$$
\frac{1}{R} \left( R'' + \frac{1}{r} R' \right) + \frac{1}{r^2 \Theta} \Theta'' + p^2 = -\frac{Z''}{Z} = s^2 \qquad (vi)
$$
  

$$
\Rightarrow Z'' + s^2 Z = 0 \qquad (vii)
$$

$$
\frac{1}{R}\left(R'' + \frac{1}{r}R'\right) + \frac{1}{r^2\Theta}\Theta'' + p^2 = -\frac{Z''}{Z} = s^2 \quad (vi)
$$
  

$$
Z'' + s^2Z = 0 \qquad (vii)
$$
  

$$
\frac{1}{R}\left(r^2R'' + rR'\right) + \left(p^2 - s^2\right)r^2 = -\frac{\Theta''}{\Theta} = \alpha^2 \quad (viii)
$$
  
n (viii) results in the following uncoupled *ODEs*:

$$
\frac{1}{R}(r^2R'' + rR') + (p^2 - s^2)r^2 = -\frac{\Theta''}{\Theta} = \alpha^2 \quad (viii)
$$
  
sults in the following uncoupled *ODEs*:  

$$
\Theta'' + \alpha^2 \Theta = 0 \qquad (ix)
$$

Eqn  $(viii)$  results in the following uncoupled *ODEs*:

$$
\Theta'' + \alpha^2 \Theta = 0 \tag{ix}
$$

$$
R^{(1,1,1,1,1)}(P^{(2,2)})^T \Theta^{(1,2,1,1)}
$$
\n
$$
(viii) \text{ results in the following uncoupled } ODEs:
$$
\n
$$
\Theta'' + \alpha^2 \Theta = 0 \qquad (ix)
$$
\n
$$
r^2 R'' + rR' + (\beta^2 r^2 - \alpha^2) R = 0 \qquad (x)
$$
\n
$$
\text{re } \beta^2 = p^2 - s^2.
$$
\n
$$
(x) \text{ is the Bessel's differential equation}
$$

 $r^2 R'' +$ <br>where  $\beta^2 = p^2 - s^2$ .  $\beta$ 

Eqn  $(x)$  is the Bessel's differential equation. where  $\beta^2 = p^2 - s^2$ .<br>
Eqn (x) is the Bessel's differential equation.<br>
We thus have the following solutions:<br>  $T(t) = A_p \cos(cpt) + B_p \sin(cpt)$ <br>  $T(s) = C \cos(s\pi) + D \sin(s\pi)$ *x*

We thus have the following solutions:

$$
^{2}-s^{2}
$$
.  
\nthe following solutions:  
\n $T(t) = A_{p} \cos(cpt) + B_{p} \sin(cpt)$  (xi)  
\n $Z(s) = C \cos(st) + D \sin(st)$  (vii)

Eqn 
$$
(x)
$$
 is the Bessel's differential equation.  
\nWe thus have the following solutions:  
\n
$$
T(t) = A_p \cos(cpt) + B_p \sin(cpt)
$$
\n
$$
Z(z) = C_s \cos(sz) + D_s \sin(sz)
$$
\n
$$
\Theta(\theta) = E_{\alpha} \cos(\alpha \theta) + F_{\alpha} \sin(\alpha \theta)
$$
\n
$$
R(r) = G_{ps\alpha} J(\beta r) + H_{ps\alpha} Y(\beta r)
$$
\n
$$
(xii)
$$
\nThe general solution is therefore given by

$$
\Theta(\mathcal{G}) = E_{\alpha} \cos(\alpha \mathcal{G}) + F_{\alpha} \sin(\alpha \mathcal{G}) \qquad (xiii)
$$
  

$$
R(r) = G_{\beta \alpha} J(\beta r) + H_{\beta \alpha} Y(\beta r) \qquad (xiv)
$$

$$
R(r) = G_{\rho s\alpha}J(\beta r) + H_{\rho s\alpha}Y(\beta r) \qquad (xiv)
$$

T he general solution is therefore given by

$$
B(r) = B_{\rho s\alpha}J(\beta r) + H_{\rho s\alpha}Y(\beta r)
$$
\n(*xtu*)

\nThe general solution is therefore given by

\n
$$
u(r, \theta, z, t) = \sum_{p=0}^{\infty} \sum_{s=0}^{\infty} \sum_{\alpha=0}^{\infty} u_{\rho s\alpha}(r, \theta, z, t)
$$
\n(*xv*)

\nin which  $u_{\rho s\alpha}$  is as defined in (*xi*) through (*xiv*).

\nIn practical application  $u_r$  is a converguchon ineludine  $r = 0 \Rightarrow H$ .

in which  $u_{ps\alpha}$  is as defined in (*xi*) through (*xiv*). 'a

In practical application  $u \prec \infty$  everywhere including  $r = 0 \Rightarrow H_{_{ps\alpha}} = 0 \therefore Y(\beta r)$ 9, z, t)  $(xv)$ <br>
here including  $r = 0 \Rightarrow H_{ps\alpha} = 0 \therefore Y(\beta r) \rightarrow \infty$  as  $r \rightarrow 0$ . Therefore, the finite solution is given by In practical application  $u \prec \infty$  everywhere including  $r = 0 \Rightarrow H_{ps\alpha} = 0 \therefore Y(\beta r) \rightarrow \infty$  as r<br>Therefore, the finite solution is given by<br> $u(r, \theta, z, t) = \sum_{p=0}^{\infty} \sum_{s=0}^{\infty} \sum_{\alpha=0}^{\infty} G_{ps\alpha} J(\beta r) \Big\{ A_p \cos(cpt) + B_p \sin(cpt) \Big\} \Big\{$ *ps*  $(xv)$ <br> $r = 0 \Rightarrow H_{ps\alpha} = 0 \therefore Y(\beta r) \rightarrow \infty$  as r  $_{\alpha} = 0$  :  $Y(\beta)$ application  $u \prec \infty$  everywhere including r:<br>
the finite solution is given by<br>  $\mathcal{G}(x,t) = \sum_{p=0}^{\infty} \sum_{s=0}^{\infty} \sum_{\alpha=0}^{\infty} G_{ps\alpha} J(\beta r) \Big\{ A_p \cos(cpt) \Big\}$  $u \prec \infty$  everyw.<br>solution is given<br> $\sum_{n=1}^{\infty} \sum_{n=1}^{\infty} G_n = I(t)$  $(xv)$ <br>= 0.  $\Rightarrow$   $H_{ps\alpha} = 0$  :  $Y(\beta r) \rightarrow \infty$  as  $r \rightarrow 0$ . ation  $u \prec \infty$  everywhere including  $r = 0 \Rightarrow H_{ps\alpha} = 0 \therefore Y(\beta r) \rightarrow \infty$  as  $r \rightarrow 0$ .<br>
te solution is given by<br>  $= \sum_{p=0}^{\infty} \sum_{s=0}^{\infty} \sum_{\alpha=0}^{\infty} G_{ps\alpha} J(\beta r) \Big\{ A_p \cos(cpt) + B_p \sin(cpt) \Big\} \Big\{ C_s \cos(sz) + D_s \sin(sz) \Big\} \times$ 

trivial application 
$$
u \prec \infty
$$
 everywhere including  $r = 0 \Rightarrow H_{ps\alpha} = 0 \therefore Y(\beta r) \rightarrow \infty$  as  $r \rightarrow 0$ .

\nore, the finite solution is given by

\n
$$
u(r, \theta, z, t) = \sum_{p=0}^{\infty} \sum_{s=0}^{\infty} \sum_{\alpha=0}^{\infty} G_{ps\alpha} J(\beta r) \{ A_p \cos(cpt) + B_p \sin(cpt) \} \{ C_s \cos(sz) + D_s \sin(sz) \} \times
$$
\n
$$
\{ E_{\alpha} \cos(\alpha \theta) + F_{\alpha} \sin(\alpha \theta) \}
$$
\nin the solution of the transverse vibration of a thin membrane bounded by a circle of radius.

 ${E_{\alpha} \cos(\alpha \beta) + F_{\alpha} \sin(\alpha \beta)}$ <br>f the transverse vibration of a thin membrane b<br> $(r, \beta, t)$  satisfying the wave equation  $\nabla^2 u = c^{-2}$ 4 Obtain the solution of the transverse vibration of a thin membrane bounded by a circle of radius a desc-Figure  $\{E_{\alpha}\cos(\alpha\vartheta) + F_{\alpha}\sin(\alpha\vartheta)\}$ <br>4 Obtain the solution of the transverse vibration of a thin membrane bound<br>ribed by the function  $u(r, \vartheta, t)$  satisfying the wave equation  $\nabla^2 u = c^{-2}u_{xx}$  s<br> $u(\alpha, \vartheta, t) = 0$ ,  $u(r, \vartheta,$  $(\theta, t)$  satisfying the wave equation  $\nabla^2 u = c^{-1}$ where  $\left\{\nabla^2 u = c^{-2} u_{xx}\n\right\}$  satisfying  $u(a, \theta, t) = 0, u(r, \theta, 0) = f(r, \theta), u_r(r, \theta, 0) = \phi(r, \theta).$ atisfying the conditions:  $\{E_{\alpha}\cos(\alpha\vartheta) + F_{\alpha}\sin(\alpha\vartheta)\}\$ <br>4 Obtain the solution of the transverse vibration of a thin membrane<br>ribed by the function  $u(r, \vartheta, t)$  satisfying the wave equation  $\nabla^2 u = c$ <br> $u(a, \vartheta, t) = 0, u(r, \vartheta, 0) = f(r, \vartheta), u_t(r, \vartheta, 0) = \phi(r$ 

. *Solution*

Solution.  
\nThe initial boundary value problem is represented by  
\n
$$
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial g^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}.
$$
\n
$$
u(a, \theta, t) = 0, \quad -\pi \le \theta \le \pi, t \ge 0
$$
\n
$$
u(r, \theta, 0) = f(r, \theta), u_r(r, \theta, 0) = \phi(r, \theta). \quad 0 \le r \le a, -\pi \le \theta \le \pi
$$
\nAssuming a separable solution of the form  
\n
$$
u(r, \theta, t) = R(r) \Theta(\theta) T(t) \ne 0
$$
\n(ii)

Assuming a seperable solution of the form

$$
u(r, \vartheta, t) = R(r) \Theta(\vartheta) T(t) \neq 0 \qquad (ii)
$$

and dividing through by  $u$  we have *u*

Assuming a separable solution of the form  
\n
$$
u(r, \theta, t) = R(r)\Theta(\theta)T(t) \neq 0 \qquad (ii)
$$
\nand dividing through by *u* we have\n
$$
\frac{1}{R}\left(R'' + \frac{1}{r}R'\right) + \frac{\Theta''}{r^2\Theta} = \frac{1}{c^2}\frac{\ddot{T}}{T} = -\lambda^2 \qquad (iii)
$$
\n
$$
\ddot{T} + c^2\lambda^2T = 0 \qquad (iv)
$$
\n
$$
\frac{1}{R}\left(R'' + \frac{1}{r}R'\right) + \frac{\Theta''}{r^2\Theta} + \lambda^2 = 0 \qquad (v)
$$

$$
R\left(\begin{array}{cc}r\end{array}\right) r^2\Theta c^2T
$$
  
\n
$$
\ddot{r}+c^2\lambda^2T=0
$$
\n
$$
\frac{1}{R}\left(R''+\frac{1}{r}R'\right)+\frac{\Theta''}{r^2\Theta}+\lambda^2=0
$$
\n
$$
\frac{1}{R}\left(R''+\frac{1}{r}R'\right)+\lambda^2=-\frac{\Theta''}{r^2\Theta}
$$
\n
$$
(vi)
$$

$$
\frac{1}{R}\left(R'' + \frac{1}{r}R'\right) + \frac{\Theta''}{r^2\Theta} + \lambda^2 = 0 \qquad (v)
$$
\n
$$
\frac{1}{R}\left(R'' + \frac{1}{r}R'\right) + \lambda^2 = -\frac{\Theta''}{r^2\Theta} \qquad (vi)
$$

, *ie*

$$
\frac{1}{R}\left(R'' + \frac{1}{r}R'\right) + \lambda^2 = -\frac{\Theta''}{r^2\Theta} \qquad (vi)
$$
\n
$$
\frac{1}{R}\left(r^2R'' + rR'\right) + r^2\lambda^2 = -\frac{\Theta''}{\Theta} = m^2 \qquad (vii)
$$
\nwe\n
$$
\Theta'' + m^2\Theta = 0 \qquad (viii)
$$

Hence, we have

$$
\Theta'' + m^2 \Theta = 0 \qquad (viii)
$$

Hence, we have  
\n
$$
\Theta^{n} + m^2 \Theta = 0 \qquad (viii)
$$
\n
$$
R^n + \frac{1}{r}R' + \left(\lambda^2 - \frac{m^2}{r^2}\right)R = 0 \qquad (ix)
$$
\nThe solutions of *(iv)* and *(viii)* are respectively\n
$$
T(t) = A_{\lambda} \cos(c\lambda t) + B_{\lambda} \sin(c\lambda t) \qquad (x)
$$

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$$
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$$
 and  $(viii)$  are respectively  
\n
$$
T(t) = A_{\lambda} \cos(c\lambda t) + B_{\lambda} \sin(c\lambda t) \qquad (x)
$$
\n
$$
\Theta(\theta) = C_{\lambda} \cos(m\theta) + D_{\lambda} \sin(m\theta) \qquad (xi)
$$
\nEqn  $(ix)$  is the standard Bessel's differential equation with  $\Theta$  so  $R(r) = E_{\lambda} J_m(r\lambda) + F_{\lambda} Y_m(r\lambda) \qquad (xii)$ 

Eqn (*ix*) is the standard Bessel's differential equation with solution  
\n
$$
R(r) = E_{\lambda}J_{m}(r\lambda) + F_{\lambda}Y_{m}(r\lambda)
$$
\n
$$
(xii)
$$
\nSince solution must remain finite everywhere, we observe that  $Y_{m}(r\lambda)$ .  
\n
$$
\therefore R(r) = E_{\lambda}J_{m}(r\lambda)
$$
\n
$$
(xii)
$$

Since solution must remain finite everywhere, we observe that  $Y_m(r\lambda)$ the solution<br>  $(xii)$ <br>
e that  $Y_m(r\lambda) \to \infty$  as  $r \to 0 \Rightarrow F_{\lambda} = 0$ *m* lution<br> *Y<sub>m</sub>* (*r*λ) → ∞as *r* → 0 ⇒ *F*<sub>*i*</sub>:  $\lambda$ )  $\rightarrow \infty$ as  $r \rightarrow 0 \Rightarrow F_{\lambda}$  $\rightarrow \infty$  as  $r \rightarrow 0 \Rightarrow F_{\lambda} = 0$ 

Eqn (*tx*) is the standard Bessers differential equation with so that  
\n
$$
R(r) = E_{\lambda}J_{m}(r\lambda) + F_{\lambda}Y_{m}(r\lambda)
$$
\n
$$
(xii)
$$
\nSince solution must remain finite everywhere, we observe that  $Y_{m}$  ( $\lambda$ ).

Thus,

 $\ddot{\cdot}$ 

Since solution must remain finite everywhere, we observe that 
$$
Y_m(r\lambda) \rightarrow \infty
$$
 as  $r \rightarrow 0 \Rightarrow F_{\lambda} = 0$ .  $\therefore$   $R(r) = E_{\lambda} J_m(r\lambda)$   $(xii)$   $u(r, \theta, t) = J_m(r\lambda) \left\{ A_{\lambda}^{\prime} \cos(c\lambda t) + B_{\lambda}^{\prime} \sin(c\lambda t) \right\} \left\{ C_{\lambda} \cos(m\theta) + D_{\lambda} \sin(m\theta) \right\}$   $(xiii)$  in which  $A_{\lambda}^{\prime} = A_{\lambda} E_{\lambda}$  and  $B_{\lambda}^{\prime} = B_{\lambda} E_{\lambda}$ .

in which  $A_{\lambda}^{\prime} = A_{\lambda} E_{\lambda}$  and  $' =$  Recall that

Recall that  
\n
$$
u(a, \theta, t) = 0; -\pi \le \theta \le \pi, t \ge 0
$$
\n
$$
\Rightarrow R(a)\Theta(\theta)T(t) = 0 \quad ie, R(a) = 0: \Theta(\theta)T(t) = 0 \Rightarrow u(r, \theta, t) = 0 \text{ trivially}
$$
\n
$$
\Rightarrow J_m(\lambda a) = 0 \qquad \text{(xiv)}
$$
\nThis is an eigenvalue problem with infinite solutions.  
\nThus, suppose  $\lambda_k$  ( $k = 1, 2, 3 \cdots$ ) are the positive roots of (*xiv*) then the general solution becomes

This is an eigenvalue problem with i nfinite solutions.

 $\lambda_{\scriptscriptstyle\! L}$ 

$$
\Rightarrow R(a)\Theta(\vartheta)T(t) = 0 \quad ie, R(a) = 0 \quad \because \Theta(\vartheta)T(t) = 0 \Rightarrow u(r, \vartheta, t) = 0 \text{ trivially}
$$
\n
$$
\Rightarrow J_m(\lambda a) = 0 \quad \text{(xiv)}
$$
\nThis is an eigenvalue problem with infinite solutions.  
\nThus, suppose  $\lambda_k$  ( $k = 1, 2, 3 \cdots$ ) are the positive roots of (*xiv*) then the general solution becomes\n
$$
u(r, \vartheta, t) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} J_m(\lambda_k r) \Big\{ A_{\lambda} \Big\{ \cos(c\lambda_k t) + B_{\lambda} \Big\} \sin(c\lambda_k t) \Big\} \Big\{ C_{\lambda} \cos(m\vartheta) + D_{\lambda} \sin(m\vartheta) \Big\} \quad (xv)
$$
\nAxiisymmetric solutions.

Axisymmet ric solutions.

This is the case where *u* is independent of *9*.<br> *u*(*r*  $9t$ ) –  $\sum_{n=1}^{\infty} \sum_{n=1}^{\infty} I_n (3r) \frac{1}{2}$   $\sum_{n=1}^{\infty} C_0$ , *ie*

$$
u(r, \theta, t) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} J_m(\lambda_k r) \Big\{ A_{\lambda}^{\prime} \cos(c\lambda_k t) + B_{\lambda}^{\prime} \sin(c\lambda_k t) \Big\} \Big\{ C_{\lambda} \cos(m\theta) + D_{\lambda}
$$
  
Axisymmetric solutions.  
This is the case where *u* is independent of  $\theta$ .  
*ie*,  

$$
u(r, \theta, t) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} J_0(\lambda_k r) \Big\{ A_{\lambda}^{\prime} \cos(c\lambda_k t) + B_{\lambda}^{\prime} \sin(c\lambda_k t) \Big\} \qquad (xvi)
$$
  
in which  $\lambda_k$  are the positive roots of  $J_0(\lambda_k r) = 0$ . In view of the boundary condition we  

$$
u(r, \theta, 0) = \sum_{k=0}^{\infty} A_{\lambda}^{\prime} J_0(\lambda_k r) = f(r) \qquad (xvii)
$$

 $(\lambda_k r)$ in which  $\lambda_k$  are the positive roots of  $J_0(\lambda_k r) = 0$ . In  $= 0$ . In view of the boundary condition we have

$$
u(r, \vartheta, 0) = \sum_{k=0}^{\infty} A_{\lambda}^{\prime} J_0(\lambda_k r) = f(r)
$$
 (xvii)

This is Fourier-Bessel series. To obtain the coefficients  $A_{\lambda}^{\prime}$  we have

are the positive roots of 
$$
J_0(\lambda_k r) = 0
$$
. In view of the bound  
\n $u(r, \theta, 0) = \sum_{k=0}^{\infty} A_{\lambda}^{\dagger} J_0(\lambda_k r) = f(r)$  (xvii)  
\nrier-Bessel series. To obtain the coefficients  $A_{\lambda}^{\dagger}$  we have  
\n
$$
\int_0^a r J_0(\lambda_j r) f(r) dr = \int_0^a \sum_{k=0}^{\infty} A_{\lambda}^{\dagger} r J_0(\lambda_j r) J_0(\lambda_k r) dr
$$

, *ie*

This is Fourier-Bessel series. To obtain the coefficients 
$$
A_{\lambda}
$$
' we have  
\n
$$
\int_{0}^{a} rJ_{0}(\lambda_{j}r)f(r)dr = \int_{0}^{a} \sum_{k=0}^{\infty} A_{\lambda}^{'}rJ_{0}(\lambda_{j}r)J_{0}(\lambda_{k}r)dr
$$
\n*i.e.*\n
$$
\int_{0}^{a} \sum_{k=0}^{\infty} A_{\lambda}^{'}rJ_{0}(\lambda_{j}r)J_{0}(\lambda_{k}r)dr = \int_{0}^{a} rJ_{0}(\lambda_{j}r)f(r)dr
$$
\n
$$
\Rightarrow A_{j}^{'} \int_{0}^{a} rJ_{0}^{2}(\lambda_{j}r)dr = \int_{0}^{a} rJ_{0}(\lambda_{j}r)f(r)dr
$$
\nBut\n
$$
\int_{0}^{a} rJ_{p}^{2}(\lambda_{j}r)dr = \frac{a^{2}}{2} \Big[ J_{p}^{'}^{2}(\lambda_{j}a) + \Big(1 - \frac{p^{2}}{a^{2} \lambda^{2}} \Big) J_{p}^{2}(\lambda_{j}a) \Big]
$$

$$
\Rightarrow A_{j} \int_{0}^{a} r J_{0}^{2} (\lambda_{j} r) dr = \int_{0}^{a} r J_{0} (\lambda_{j} r) f (r) dr
$$
  
\nBut 
$$
\int_{0}^{a} r J_{p}^{2} (\lambda_{j} r) dr = \frac{a^{2}}{2} \left[ J_{p}^{'2} (\lambda_{j} a) + \left( 1 - \frac{p^{2}}{a^{2} \lambda_{j}^{2}} \right) J_{p}^{2} (\lambda_{j} a) \right]
$$
 (xviii)  
\nRecall also that  $J_{p}^{'2} (\lambda_{j} a) = J_{p+1}^{2} (\lambda_{j} a)$   
\ni.e, 
$$
\int_{0}^{a} r J_{0}^{2} (\lambda_{j} r) dr = \frac{a^{2}}{2} J_{0}^{'2} (\lambda_{j} a) = \frac{a^{2}}{2} J_{1}^{2} (\lambda_{j} a)
$$

 $\int^2 \left( \lambda_j a \right) = J_{p+1}$ 

But 
$$
\int_0^r r J_p^2(\lambda_j r) dr = \frac{a}{2} \left[ J_p^{\prime 2}(\lambda_j a) + \left[ 1 - \frac{P}{a^2 \lambda_j^2} \right] J_p
$$
  
Recall also that  $J_p^{\prime 2}(\lambda_j a) = J_{p+1}^2(\lambda_j a)$   
*i.e.* 
$$
\int_0^a r J_0^2(\lambda_j r) dr = \frac{a^2}{2} J_0^{\prime 2}(\lambda_j a) = \frac{a^2}{2} J_1^2(\lambda_j a)
$$

$$
\Rightarrow A_j^{\prime} \int_0^a r J_0^2(\lambda_j r) dr = \frac{a^2}{2} J_1^2(\lambda_j a) A_j^{\prime} = \int_0^a r J_0(\lambda_j r).
$$

Recall also that 
$$
J_p^{'2}(\lambda_j a) = J_{p+1}^2(\lambda_j a)
$$
  
\ni.e,  $\int_0^a r J_0^2(\lambda_j r) dr = \frac{a^2}{2} J_0^{'2}(\lambda_j a) = \frac{a^2}{2} J_1^2(\lambda_j a)$   
\n $\Rightarrow$   $A_j' \int_0^a r J_0^2(\lambda_j r) dr = \frac{a^2}{2} J_1^2(\lambda_j a) A_j' = \int_0^a r J_0(\lambda_j r) f(r) dr$   
\n $\therefore$   $A_j' = \frac{2}{a^2 J_1^2(\lambda_j a)} \int_0^a r J_0(\lambda_j a) f(r) dr$  (xix)

$$
\Rightarrow A_{j}^{\prime} \int_{0}^{a} r J_{0}^{2} (\lambda_{j} r) dr = \frac{a^{2}}{2} J_{1}^{2} (\lambda_{j} a) A_{j}^{\prime} = \int_{0}^{a} r J_{0} (\lambda_{j} r) f(r)
$$
  
\n
$$
\therefore A_{j}^{\prime} = \frac{2}{a^{2} J_{1}^{2} (\lambda_{j} a)} \int_{0}^{a} r J_{0} (\lambda_{j} a) f(r) dr \qquad (xix)
$$
  
\nFrom the initial condition we have  
\n
$$
\frac{\partial u}{\partial t}\Big|_{t=0} = g(r) = \sum_{k=0}^{\infty} c \lambda_{k} B_{k}^{\prime} J_{0} (\lambda_{k} r) \qquad (xx)
$$

From the initial condition we have

$$
= \frac{2}{a^2 J_1^2 (\lambda_j a)} \int_0^r r J_0(\lambda_j a) f(r) dr \qquad (xix)
$$
  
ial condition we have  

$$
\frac{\partial u}{\partial t}\Big|_{t=0} = g(r) = \sum_{k=0}^\infty c \lambda_k B_k' J_0(\lambda_k r) \qquad (xx)
$$

As in the above, we therefore have

above, we therefore have

\n
$$
c \int_{0}^{a} \sum_{k=0}^{\infty} \lambda_{k} B_{k}^{\prime} J_{0}(\lambda_{j} r) J_{0}(\lambda_{k} r) dr = \int_{0}^{a} J_{0}(\lambda_{j} r) g(r) dr
$$
\n
$$
c \lambda_{j} B_{j}^{\prime} \int_{0}^{a} J_{0}^{2}(\lambda_{j} r) dr = \int_{0}^{a} J_{0}(\lambda_{j} r) g(r) dr
$$

, *ie*

$$
c\lambda_j B_j' \int_0^a J_0^2(\lambda_j r) dr = \int_0^a J_0(\lambda_j r) g(r) dr
$$

, *ie*

$$
c\lambda_j B_j' \int_0^1 J_0^2 (\lambda_j r) dr = \int_0^a J_0 (\lambda_j r) g(r) dr
$$
  
\n
$$
B_j' = \frac{\int_0^a J_0 (\lambda_j r) g(r) dr}{c\lambda_j \int_0^a J_0^2 (\lambda_j r) dr}
$$
  
\n
$$
B_j' = \frac{2}{c\lambda_j a^2 J_1^2 (\lambda_j a)} \int_0^a J_0 (\lambda_j r) g(r) dr
$$
 (xxi)  
\n*xyi*) is the solution for radially symmetric wave with coefficients defined in (*xix*) and (*xxi*)

Therefore,  $(xvi)$  is the solution for radially symmetric wave with coefficients defined in  $(xix)$  and  $(xxi)$ .