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A 4-STAGE RUNGE-KUTTA TYPE METHOD FOR SOLUTION OF STIFF ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, a 2 step implicit block hybrid linear multistep method was reformulated into a 4-stage block hybrid Runge-Kutta Type Method via the butcher analysis. The method can be used to solve first order stiff ordinary differential equations. A numerical example solved with the proposed method showed a better result in comparison with an existing method.

1. INTRODUCTION

Ordinary Differential Equations (ODEs) arise frequently in the study of physical problems. Unfortunately, many of these equations cannot be solved exactly. This is why solving these equations numerically is important. Traditionally, mathematicians have used one of two classes of methods for solving numerically ordinary differential equations. These are one step methods and Linear Multistep Methods (LMMs) [7].

Runge-Kutta methods are very popular because of their simple coefficients, efficiency, and numerical stability [2]. The methods are fairly simple to program, easy to implement, and their truncation error can be controlled in a more straightforward manner than multistep methods [5]. The application of Runge-Kutta methods has provided many satisfactory solutions to problems that were previously considered unsolvable. The popularity and growth of these methods, coupled with the amount of research effort being undertaken, are further evidence that the applications are still a leading source of inspiration for mathematical creativity [1]. With the advancement in computer technology, numerical methods are now an increasingly attractive and efficient way to obtain approximate or nearly accurate solutions to differential equations which have hitherto proved difficult or even impossible to solve analytically.

2. LITERATURE REVIEW

Yahaya and Adegboye [9] reformulated the block hybrid Quade's method into Runge-Kutta type method of order 6. The method was extended to the case in which the approximate to a second order (special or general) as well as first order initial value problem can be calculated. The method was A-stable, possessed the Runge-Kutta stability property, however, it was limited to the step number $k = 4$. Chollom et al. [4] constructed a class of A-stable Block Adams Bashforth Explicit Method (BABE) including their hybrid forms. The method tested on non-linear initial value problems performed well and competed favourably with the block hybrid Adams Moulton of higher order, but the step number was restricted to 2. Yahaya and

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Ajibade [10] reformulated the two-step hybrid linear multistep method into a 3-Stage Runge-Kutta Type method through the idea of general linear method. The method was used to solve only first order initial value problems.

Sofoluwe et al. [8] derived some Backward Differentiation Formulae (BDF) capable of generating solutions to stiff initial value problems using Lagrangian interpolation technique. The BDF derived were implemented on some standard initial value problems. The region of absolute stability was constructed and the nature obtained established some facts about the choice of BDF for numerical treatment of stiff problems. Non-stiff problems were not considered. Okunuga et al. [6] presented a direct integration of second order ordinary differential equations using only Explicit Runge-Kutta Nystrom (RKN) method with higher derivative. They derived and tested various numerical schemes on standard problems. Due to the limitations of Explicit Runge-Kutta (ERK) in handling stiff problems, the extension to higher order Explicit Runge-Kutta Nystrom (RKN) was considered and results obtained showed an improvement over conventional Explicit Runge-Kutta schemes. The Implicit Runge-Kutta scheme was however not considered.

3. Methodology

Butcher defined an S-stage Runge-Kutta method for the first order differential equation in the form:

(1)
$$y_{n+1} = y_n + h \sum_{i,j=1}^{s} a_{ij} k_i$$

where for $i = 1, 2, \ldots, s$,

(2)
$$k_i = f(x_i + \alpha_j h, y_n + h \sum_{i,j=1}^{s} a_{ij} k_j)$$

The real parameters α_j, k_i, a_{ij} define the method. The method in Butcher array form can be written as:

$$\begin{array}{c|c} \alpha & \beta \\ \hline & b^T \end{array}$$

Where $a_{ij} = \beta$.

Consider the approximate solution to $y' = f(x, y), y(x_0) = y_0$ in the form of power series given as:

(3)
$$y(x) = \sum_{j=0}^{t+m-1} \alpha_j x^j = y_{n+j}$$

(4)
$$\alpha \in R, j = 0, 1, ..., t + m - 1, y \in C^m(a, b) \subset P(x)$$

(5)
$$y'(x) = \sum_{j=1}^{t+m-1} j\alpha_j x^{j-1} = f(x, y)$$

Where α_j are the parameters to be determined, t and m are the points of interpolation and collocation respectively. For $K = 2$, we choose $t = 3$ and $m = 1$ at (3). Also, interpolate (3) at $x = x_{n+i}, i = 0, \frac{1}{2}, 1$ and collocate (4) at $x = x_{n+i}, i = 2$ to have the following system of linear equations of the form:

(6)
$$y(x) = \sum_{j=0}^{t+m-1} \alpha_j x^j = y_{n+i}, \quad i = 0, \frac{1}{2}, 1$$

(7)
$$y'(x) = \sum_{j=1}^{t+m-1} j\alpha_j x^{j-1} = f_{n+i}, \quad i = 2$$

The general form of the method upon addition of one off grid point is expressed as:

$$y(x) = \alpha_1(x)y_n + \alpha_2(x)y_{n+1} + \alpha_3 y_{n+\frac{1}{2}} + h\beta_0(x)f_{n+2} \tag{8}$$

The matrix D of dimension $(t+m)\times(t+m)$ of the proposed method is expressed as:

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & x_3^3 \\ 1 & x_n + h & (x_n+h)^2 & (x_n+h)^3 \\ 1 & x_n + \frac{1}{2}h & (x_n+\frac{1}{2}h)^2 & (x_n+\frac{1}{2}h)^3 \\ 0 & 1 & 2x_n + 4h & 3(x_n+2h)^2 \end{bmatrix}$$

We invert the matrix D, to obtain columns which form the matrix C i.e. $C = D^{-1}$. The elements of C are used to generate the continuous coefficients of the method equations:

$$\begin{aligned} \alpha_1(x) &= C_{11} + C_{21}x + C_{31}x^2 + C_{41}x^3 \\ \alpha_2(x) &= C_{12} + C_{22}x + C_{32}x^2 + C_{42}x^3 \\ \alpha_3(x) &= C_{13} + C_{23}x + C_{33}x^2 + C_{43}x^3 \\ \beta_0(x) &= C_{14} + C_{24}x + C_{34}x^2 + C_{44}x^3 \end{aligned} \tag{9}$$

The values of the continuous coefficients (9) are substituted into (8) to give the continuous form of the two-step block hybrid BDF with one off step interpolation point.

$$\begin{aligned} y(x) = &\left[1 - \frac{44}{13h}(x-x_n) + \frac{41}{13h^2}(x-x_n)^2 - \frac{10}{13h^3}(x-x_n)^3\right]y_n + \left[-\frac{20}{13h}(x-x_n) + \frac{47}{13h^2}(x-x_n)^2 \right. \\ &\left. -\frac{14}{13h^3}(x-x_n)^3\right]y_{n+1} + \left[\frac{64}{13h}(x-x_n) - \frac{88}{13h^2}(x-x_n)^2 + \frac{10}{13h^3}(x-x_n)^3\right]y_{n+\frac{1}{2}} + \left[\frac{1}{13h}(x-x_n) - \right. \\ &\left. \frac{3}{13h}(x-x_n)^2 + \frac{2}{13h^2}(x-x_n)^3\right]f_{n+2} \end{aligned} \tag{10}$$

Evaluating (10) at point $x = x_{n+2}$ and its derivative at $x = x_{n+\frac{1}{2}}, x = x_{n+1}$ yields the following three discrete hybrid schemes which are used as block integrator:

$$\begin{aligned} -\frac{36}{13}y_{n+1} + y_{n+2} + \frac{32}{13}y_{n+\frac{1}{2}} &= \frac{9}{13}y_n + \frac{6}{13}hf_{n+2} \\ \frac{33}{12}y_{n+1} + y_{n+\frac{1}{2}} &= \frac{21}{12}y_n + \frac{26}{12}hf_{n+\frac{1}{2}} + \frac{1}{12}hf_{n+2} \\ y_{n+1} - \frac{40}{32}y_{n+\frac{1}{2}} &= -\frac{8}{32}y_n + \frac{13}{32}hf_{n+1} - \frac{1}{32}hf_{n+2} \end{aligned} \tag{11}$$

Equation (11) is of order $[3,3,3]^T$ with error constant $\left[-\frac{3}{52}, \frac{17}{832}, -\frac{19}{624}\right]^T$ respectively. Equation (11) is transformed as ;

$$\begin{aligned} y_{n+\frac{1}{2}} &= y_n + \frac{h}{72}\left(0f_n + 64f_{n+\frac{1}{2}} - 33f_{n+1} + 5f_{n+2}\right) \\ y_{n+1} &= y_n + \frac{h}{72}\left(0f_n + 80f_{n+\frac{1}{2}} - 12f_{n+1} + 4f_{n+2}\right) \\ y_{n+2} &= y_n + \frac{h}{9}\left(0f_n + 8f_{n+\frac{1}{2}} + 6f_{n+1} + 4f_{n+2}\right) \end{aligned} \tag{12}$$

Reformulating the block hybrid method with the coefficient as characterized by the Butcher array form as:

$$\begin{array}{c|c} \alpha & \beta \\ \hline & b^T \end{array}$$

where $a_{ij} = \beta$.

Table 3.1: The Butcher Table for Mehtod (12)

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{8}{9} & \frac{-11}{24} & \frac{5}{72} \\ 2 & 0 & \frac{8}{9} & \frac{2}{3} & \frac{4}{9} \\ 1 & 0 & \frac{10}{9} & \frac{-1}{6} & \frac{1}{18} \\ \hline & 0 & \frac{10}{9} & \frac{-1}{6} & \frac{1}{18} \end{array}$$

NOTE: The Butcher table is being arranged with the off grid points appearing first, followed by the $c_i's$ in descending order. This is done in order to satisfy the consistency condition. Using (1) we obtained an implicit 4-stage block Runge-Kutta type method of uniform order 3.

$$y_{n+\frac{1}{2}} = y_n + h\left(0k_1 + \frac{8}{9}k_2 - \frac{11}{24}k_3 + \frac{5}{72}k_4\right)$$

$$y_{n+2} = y_n + h\left(0k_1 + \frac{8}{9}k_2 + \frac{2}{3}k_3 + \frac{4}{9}k_4\right)$$

$$y_{n+1} = y_n + h\left(0k_1 + \frac{10}{9}k_2 - \frac{1}{6}k_3 + \frac{1}{18}k_4\right) \tag{13}$$

where

$$k_1 = f(x_n, y_n)$$

$$k_2 = f\left(x_n + \frac{1}{2}h, y_n + h\left[\frac{8}{9}k_2 - \frac{11}{24}k_3 + \frac{5}{72}k_4\right]\right)$$

$$k_3 = f\left(x_n + h, y_n + h\left[\frac{10}{9}k_2 - \frac{1}{6}k_3 + \frac{1}{8}k_4\right]\right)$$

$$k_4 = f\left(x_n + 2h, y_n + h\left[\frac{8}{9}k_2 + \frac{2}{3}k_3 + \frac{4}{9}k_4\right]\right) \tag{14}$$

4. Results and Discussion

The newly derived block integrators were used to solve the below problem within the interval $0 \le x \le 0.1$. Okunuga et al. (2013) solved this stiff problem by adopting a new $3-$ point block method of order five. Consider the highly stiff Ordinary Differential Equation (ODE)

$$y' = -10(y-1)^2$$

$$y(0) = 2$$

$$h = 0.01$$

Exact Solution:

$$y(x) = 1 + \frac{1}{1+10x}$$

We applied the proposed Runge-Kutta Type Method (RKTM) to the above problem and we obtained the following tabulated results:

Table 4.1: Absolute Error and Comparison of Result with Okunuga et al. (2013) for Problem 1 Using method (13)

t	$Exact\,Solution$	$Computed\,Solution$	$Error\,(RKTM)$	$Error\,Okunuga(2013)$
0.01	1.909090909	1.909125964	$3.51E-05$	$1.07E-04$
0.02	1.833333333	1.833397888	$6.46E-05$	$2.38E-04$
0.03	1.769230769	1.769301368	$7.06E-05$	$4.51E-04$
0.04	1.714285714	1.714362172	$7.65E-05$	$6.20E-04$
0.05	1.666666667	1.666741036	$7.44E-05$	$8.84E-04$
0.06	1.625000000	1.625073119	$7.31E-05$	$1.03E-03$
0.07	1.588235294	1.588304277	$6.90E-05$	$1.27E-03$
0.08	1.555555556	1.555621291	$6.57E-05$	$1.53E-03$
0.09	1.526315789	1.526377230	$6.14E-05$	$1.75E-03$
0.1	1.500000000	1.500057888	$5.79E-05$	$1.81E-03$

Results of the proposed method of Order # with fewer function evaluations displayed in the table showed a better accuracy than results obtained by Okunuga et al. (2013) of order five. Also there is an increase in accuracy as the computed solution moves closer to the exact solution.

5. Conclusion

This research work shows the link between a k-step linear multistep methods and Runge-Kutta methods which leads to a more accurate Block implicit Runge-Kutta Type Method (RKTM) for solving first order stiff ordinary differential equation (ODE). The method can also be extended to second and higher order as derivation is done only once.

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