A12: Development of Falkner-Type method for Numerical Solution of Second Order Initial Value Problems (IVPs) in ODEs

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ABSTRACT

In this research work, Falkner type method for *k=2* with four off-step point were derived for numerical solution of second order initial value problems. The idea of collocation and interpolation techniques was adopted in the derivation of the schemes. The basic properties of numerical methods were analyzed and the methods were found to be consistent, zero stable and therefore, convergent. Numerical experiments were carried out on five (5) problems of second order initial value problems (IVPs). The results obtained for the proposed methods, in comparison with the exact solutions and some existing methods from the literatures show the efficiency and reliability of the proposed schemes.

INTRODUCTION

Differential equation of the form

$$y''(x) = f\left(x, y(x), y'(x)\right),\ y(a) = y_0,\ y'(a) = y'_0 \qquad (1)$$

where $x \in [a,b]$, $y:[a,b] \to \mathbb{R}$ and $f:[a,b] \times \mathbb{R} \to \mathbb{R}$ are sufficiently differentiable functions; is usually used to model numerous problems such as chemical kinetics, orbital dynamics, circuit and control theory and Newton's second law of motion. However, in most cases, the differential equations so formed for these real life problems often do not have analytical solution. Therefore one of the possible ways to tackle this problem is to consider a discrete domain rather a continuous one. Hence for practical purposes such as engineering, a numerical approximation to the solution is often sufficient. Although it is possible to integrate (1) by reducing it to a first-order system and applying one of the methods available for such systems, it however, seems natural to employ numerical methods to integrate the problem directly as this result to more efficiency of the method (Ramos *et al.,* 2016, Mohammed *et al.,* 2010, Mohammed *et al.,* 2019, Badmus and Yahaya, 2009, Awoyemi, 2001). Scholars have proposed numerous numerical methods for approximating initial value problems such as (1); these methods range from discrete schemes (Lambert, 1973; Butcher, 2008; Fatunla, 1988) to predictor corrector methods (Onuman;yi*et al.,* 1994; Fatunla 1994; Awoyemi, and Idowu, 2005; Areo and Adeniyi, 2013; Omar and Kuboye, 2015; Ndanusa and Tafida, 2016) and then block methods ((Badmus and Yahaya, 2009; Jator and Li, 2012; Mohammed, 2011; Mohammed and Adeniyi, 2014; Badmus, *et al.,* 2015; Akinfenwa, *et al.,* 2013; Omar and Adeyeye, 2016; Akinfenwa*et al.,* 2017).

In this paper, we present the hybrid-block form of the Falkner formulas where generalized 6 off-step points are considered within $0 \le x \le 2$ in order to increase the number of function evaluation.

Derivation of the Methods

In this section, we derive some linear multi-step methods in the form

$$y_{n+1} = y_n + hy'_n + h^2 \sum_{j=0}^{k-1} \beta_j \nabla^j f_n \tag{2}$$

$$y'_{n+1} = y'_n + h^2 \sum_{j=0}^{k-1} \gamma_j \nabla^j f_n \tag{3}$$

where h is the step-size, y_n and y'_n are numerical approximations to the theoretical solution and its derivative at the grid point $x_n = a + nh;\ n = 0,1,2,3,...,N$, $h = \frac{(b-a)}{N}$, $f_n = f\left(x_n, y_n, y'_n\right)$ and $\nabla^j f_n$ is the standard notation for the backward differences.

We then construct the continuous approximation by imposing the following conditions

$$\left.\begin{aligned} y_{n+k-r} &= Y(x_{n+k-r}) \\ y'_{n+k-r} &= Y'(x_{n+k-r}) \\ y''_{n+j} &= Y''(x_{n+j}) = f(x_{n+j}) \end{aligned}\right\} \tag{4}$$

Equation (4) leads to a system of equations and unknowns written in the form $AX = B$

$$X = \begin{bmatrix} 1 & x_{n+k-r} & x_{n+k-r}^2 & \cdots & x_{n+k}^{k+2} \\ 0 & 1 & 2x_{n+k-r} & \cdots & (k+2)x^{k+1} \\ 0 & 0 & 2 & \cdots & (k+2)(k+1)x_n^k \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 2 & \cdots & (k+2)(k+1)x_{n+k}^k \end{bmatrix}$$

$$A = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{k+2} \end{bmatrix} B = \begin{bmatrix} y_{n+k-r} \\ y'_{n+k-r} \\ f_n \\ \vdots \\ f_{n+2} \end{bmatrix} \tag{5}$$

solving (5) by Gaussian elimination method, the coefficients α_j can be obtained. Substituting the coefficients α_j into (2) yields the continuous scheme:

$$Y(x) = \alpha_0(x)y_n + \alpha_0'(x)hy'_{n+j} + h^2\left[\sum_{j=0}^{k}\beta_j(x)f_{n+j} + \beta_v(x)f_{n+v}\right] \tag{6}$$

where $\alpha_j(x), \beta_j(x) and \beta_v(x)$are continuous coefficients. We note that (4) involves first derivative, which can be obtained by substituting the coefficients of α_j into the first derivative of (3) to yield

$$Y'(x) = \alpha_0'(x) y'_{n+j} + h\left[\sum_{j=0}^{k} \beta_j'(x) f_{n+j} + \beta_v'(x) f_{n+v}\right] \qquad (7)$$

The main and the additional methods can be obtained from (6) and (7). Both methods are called Hybrid Falkner-type Block methods (HFBM);

Two step method with $\frac{1}{2}, \frac{3}{4}, \frac{5}{4}$ and $\frac{7}{4}$ off-step points

To derive a continuous method by considering four off-step points $\frac{1}{2}, \frac{3}{4}, \frac{5}{4}$ and $\frac{7}{4}$, the following specifications are considered $r = 2, k = 2, s = 7$and the continuous form is given as:

$$y(x) = \alpha_0 y_n + \alpha_0' y_n' + h^2\left[\beta_0 f_n + \beta_1 f_{n+\frac{1}{2}} + \beta_2 f_{n+\frac{3}{4}} + \beta_3 f_{n+1} + \beta_4 f_{n+\frac{5}{4}} + \beta_5 f_{n+\frac{7}{4}} + \beta_6 f_{n+2}\right] \qquad (8)$$

Evaluating (8) above at point $x = x_{n+\frac{1}{2}}, x_{n+\frac{3}{4}} x_{n+1}, x_{n+\frac{5}{4}}, x_{n+\frac{7}{4}}$ and x_{n+2} gives the following six discrete scheme that form the block method

$$y_{n+\frac{1}{2}} = y_n + \frac{1}{2} h y_n' + \frac{36179}{705600} h^2 f_n + \frac{20263}{75600} h^2 f_{n+\frac{1}{2}} - \frac{781}{1800} h^2 f_{n+\frac{3}{4}} + \frac{1901}{5040} h^2 f_{n+1}$$
$$- \frac{2921}{18900} h^2 f_{n+\frac{5}{4}} + \frac{1999}{88200} h^2 f_{n+\frac{7}{4}} - \frac{1727}{302400} h^2 f_{n+2} \qquad (9)$$

$$y_{n+\frac{3}{4}} = y_n + \frac{3}{4} h y_n' + \frac{843009}{10035200} h^2 f_n + \frac{98013}{179200} h^2 f_{n+\frac{1}{2}} - \frac{140457}{179200} h^2 f_{n+\frac{3}{4}} + \frac{13977}{20480} h^2 f_{n+1}$$
$$- \frac{24987}{89600} h^2 f_{n+\frac{5}{4}} + \frac{51129}{125440} h^2 f_{n+\frac{7}{4}} - \frac{14709}{1433600} h^2 f_{n+2} \qquad (10)$$

$$y_{n+1} = y_n + hy'_n + \frac{3431}{29400}h^2 f_n + \frac{3928}{4725}h^2 f_{n+\frac{1}{2}} - \frac{568}{525}h^2 f_{n+\frac{3}{4}} + \frac{139}{140}h^2 f_{n+1} - \frac{272}{675}h^2 f_{n+\frac{5}{4}} + \frac{72}{1225}h^2 f_{n+\frac{7}{4}} - \frac{559}{37800}h^2 f_{n+2} \quad (11)$$

$$y_{n+\frac{5}{4}} = y_n + \frac{5}{4}hy'_n + \frac{539725}{3612672}h^2 f_n + \frac{215875}{193536}h^2 f_{n+\frac{1}{2}} - \frac{88625}{64512}h^2 f_{n+\frac{3}{4}} + \frac{349375}{258048}h^2 f_{n+1} - \frac{50425}{96768}h^2 f_{n+\frac{5}{4}} + \frac{34625}{451584}h^2 f_{n+\frac{7}{4}} - \frac{29875}{1548288}h^2 f_{n+2} \quad (12)$$

$$y_{n+\frac{7}{4}} = y_n + \frac{7}{4}hy'_n + \frac{132251}{614400}h^2 f_n + \frac{1154881}{69200}h^2 f_{n+\frac{1}{2}} - \frac{146461}{76800}h^2 f_{n+\frac{3}{4}} + \frac{40817}{20480}h^2 f_{n+1} - \frac{189679}{345600}h^2 f_{n+\frac{5}{4}} + \frac{3577}{25600}h^2 f_{n+\frac{7}{4}} - \frac{175273}{5529600}h^2 f_{n+2} \quad (13)$$

$$y_{n+2} = y_n + 2hy'_n + \frac{2738}{11025}h^2 f_n + \frac{1312}{675}h^2 f_{n+\frac{1}{2}} - \frac{3392}{1575}h^2 f_{n+\frac{3}{4}} + \frac{716}{315}h^2 f_{n+1} - \frac{2432}{4725}h^2 f_{n+\frac{5}{4}} + \frac{2624}{11025}h^2 f_{n+\frac{7}{4}} - \frac{164}{4725}h^2 f_{n+2} \quad (14)$$

The following schemes are obtained by differentiating equation (8) and evaluating at point$x = x_{n+\frac{1}{2}}, x_{n+\frac{3}{4}} x_{n+1},\ x_{n+\frac{5}{4}}, x_{n+\frac{7}{4}}$ and x_{n+2} gives the following

$$y'_{n+\frac{1}{2}} = y'_n + \frac{92621}{705600}hf_n + \frac{39581}{37800}hf_{n+\frac{1}{2}} - \frac{9329}{6300}hf_{n+\frac{3}{4}} + \frac{12659}{10080}hf_{n+1} - \frac{4807}{9450}hf_{n+\frac{5}{4}} + \frac{3253}{44100}hf_{n+\frac{7}{4}} - \frac{5603}{302400}hf_{n+2} \quad (15)$$

$$y'_{n+\frac{3}{4}} = y'_n + \frac{163911}{1254400}hf_n + \frac{25597}{22400}hf_{n+\frac{1}{2}} - \frac{14319}{11200}hf_{n+\frac{3}{4}} + \frac{21309}{17920}hf_{n+1} - \frac{2729}{5600}hf_{n+\frac{5}{4}} + \frac{5583}{78400}hf_{n+\frac{7}{4}} - \frac{3211}{179200}hf_{n+2} \quad (16)$$

$$y'_{n+1} = y'_n + \frac{481}{3675}hf_n + \frac{5354}{4725}hf_{n+\frac{1}{2}} - \frac{604}{525}hf_{n+\frac{3}{4}} + \frac{281}{210}hf_{n+1} - \frac{2392}{4725}hf_{n+\frac{5}{4}} + \frac{268}{3675}hf_{n+\frac{7}{4}} - \frac{173}{9450}hf_{n+2} \quad (17)$$

$$y'_{n+\frac{5}{4}} = y'_n + \frac{59015}{451584}hf_n + \frac{27575}{24192}hf_{n+\frac{1}{2}} - \frac{4775}{4032}hf_{n+\frac{3}{4}} + \frac{48625}{32256}hf_{n+1} - \frac{2392}{6048}hf_{n+\frac{5}{4}} + \frac{1975}{28224}hf_{n+\frac{7}{4}} - \frac{3425}{193536}hf_{n+2} \quad (18)$$

$$y'_{n+\frac{7}{4}} = y'_n + \frac{10213}{76800}hf_n + \frac{92953}{86400}hf_{n+\frac{1}{2}} - \frac{4459}{4800}hf_{n+\frac{3}{4}} + \frac{7889}{7680}hf_{n+1} + \frac{4459}{21600}hf_{n+\frac{5}{4}} + \frac{1309}{4800}hf_{n+\frac{7}{4}} - \frac{25039}{691200}hf_{n+2} \quad (19)$$

$$y'_{n+2} = y'_n + \frac{1451}{11025}hf_n + \frac{5248}{4725}hf_{n+\frac{1}{2}} - \frac{1664}{1575}hf_{n+\frac{3}{4}} + \frac{388}{315}hf_{n+1} + \frac{256}{4725}hf_{n+\frac{5}{4}} + \frac{5248}{11025}hf_{n+\frac{7}{4}} + \frac{247}{4725}hf_{n+2} \quad (2$$

Analysis of the method

In this section, we discuss in general the order and error constants, consistency, zero-stability and convergence of the proposed method

Order and error constants

Let the linear difference operator L associated with k-step method be defined as

$$L\left[y\left(x_n\right);h\right]=\sum_{j=0}^{k}\left(\alpha_j y\left(x_n+jh\right)-h\beta y'\left(x_n\right)-h^2\gamma_{vj}f\left(x_n+jvh\right)\right) \tag{21}$$

and

$$L\left[y'\left(x_n\right);h\right]=\sum_{j=0}^{k}\left(h\overline{\beta}_j y'\left(x_n+jvh\right)-h^2\overline{\gamma}_{vj}hf\left(x_n+jvh\right)\right) \tag{22}$$

respectively. Assuming that $y\left(x_n\right)$ and $y'\left(x_n\right)$ are sufficiently differentiable, we can expand the terms in (21) and (22) as Taylor series about the point x_n to obtain the expression

$$L\left[y\left(x_n\right);h\right]=C_0 y\left(x_n\right)+C_1 hy'\left(x_n\right)+\ldots+C_q h^q y^{(q)}\left(x_n\right)+\ldots \tag{23}$$

and

$$L\left[y'\left(x_n\right);h\right]=\overline{C}_0 y'\left(x_n\right)+\overline{C}_1 hy''\left(x_n\right)+\ldots+\overline{C}_q h^q y^{(q+1)}\left(x_n\right)+\ldots \tag{24}$$

respectively;

where the constants C_q and $\overline{C}_q$ $q=0,1,\ldots$ are given as follows

$$\left.\begin{aligned}
&C_0 = \sum_{j=0}^{k} \alpha_j \\
&C_1 = \sum_{j=1}^{k} j\alpha_j \\
&C_2 = \frac{1}{2!}\sum_{j=1}^{k} (j)^2 \alpha_j - \sum_{j=0}^{k} \beta_j \\
&\vdots \\
&C_q = \frac{1}{q!}\sum_{j=1}^{k} (j)^q \alpha_j - \frac{1}{(q-2)!}\sum_{j=1}^{k} j^{q-1}\beta_j
\end{aligned}\right\} \qquad (25)$$

$q = 2, 3, \ldots$

$$\left.\begin{aligned}
&\bar{C}_0 = \sum_{j=0}^{k} \bar{\alpha}_j \\
&\bar{C}_1 = \sum_{j=1}^{k} j\bar{\alpha}_j - \sum_{j=0}^{k} \bar{\beta}_j \\
&\bar{C}_2 = \frac{1}{2!}\sum_{j=1}^{k} (j)^2 \bar{\alpha}_j - \sum_{j=1}^{k} j\bar{\beta}_j \\
&\vdots \\
&\bar{C}_q = \frac{1}{q!}\sum_{j=1}^{k} (j)^q \bar{\alpha}_j - \frac{1}{(q-1)!}\sum_{j=1}^{k} j^{q-1}\bar{\beta}_j
\end{aligned}\right\} \qquad (26)$$

The methods (25) and (26) are of order p if $C_0 = C_1 = \ldots C_p = C_{p+1} = 0,\ C_{p+2} \neq 0$ and C_{p+2} is the error constant and $C_{p+2} h^{p+2} y^{(p+2)}(x_n)$ the principal truncation error at the point x_n.

Zero Stability

This is the concept concerning the behavior of a numerical method as $h \to 0$, the system of equation (7)becomes

$$\left.\begin{aligned} y_{n+1} &= y_{n+k-r} \\ y_{n+2} &= y_{n+k-r} \\ &\vdots \\ y_{n+k-2} &= y_{n+k-r} \\ y_{n+k-1} &= y_n \\ y_{n+k} &= y_{n+k-r} \end{aligned}\right\} \tag{27}$$

which can be written in matrix form as

$$A^0 \bar{Y}_\mu - A^1 \bar{Y}_{\mu-1} = 0 \tag{28}$$

Where $\bar{Y}_\mu = (y_{n+1}, y_{n+2}, \ldots, y_{n+k})^T$, $\bar{Y}_{\mu-1} = (y_n, y_{n+1}, y_{n+k-r})^T$, A^0 is the identity matrix of dimension k and A^1 is a matrix of dimension K.

Consistency

Each of the methods is consistent as they all have order > 1.

Convergence

The convergence of the proposed methods, are considered in the light of the basic properties in conjunction with the fundamental theorem of Dahlquist (Henrichi 1962) for linear multistep methods. We state here the Dahlquist theorem without proof.

Theorem

The necessary and sufficient condition for a multistep method to be convergent is for it to be consistent and zero-stable.

RESULTS AND DISCUSSION

Numerical Experiments

In this section, we solve some standard second order initial value problems of ordinary differential equations using the proposed Falkner type method in order to demonstrate its efficacy. However the implementation is carried as a block (self-starting) method whereby the continuous forms of the methods generates the main and additional discrete Falkner formulas to produce approximation simultaneously at each step of implementation within the interval of integration. Comparisons were made with the exact solutions of the problems considered and absolute errors were compared with some other existing methods found in the literature and presented in tables.

For the purpose of comparative analysis, the following notations are adopted.

FTM: The proposed Falkner Type Method with $\left\{\frac{1}{2},\frac{3}{4},\frac{5}{4},\frac{7}{4}\right\}$ as off-grid points

$HFBM_{2,1}$: 2-step, one off-step hybrid block Falkner-type method by Nicholas (2019)

$HFBM_{2,2}$: 2-step, 2 off-step hybrid block Falkner-type method by Nicholas (2019)

$HFBM_{2,4.}$: 2-step, one off-grid hybrid block Falkner-type method by Nicholas (2019)

BFM_6: Block Falkner method for k=6 by Ramos *et al.,* (2016)

Problem 1. (*Source:* Ramos *et al.* (2016))

Consider the non-linear homogeneous problem given by:

$$y'' = x(y')^2, \quad y(0) = 1, \ y'(0) = 0.5$$
$$0 \le x \le 1, \quad h = 0.1$$

Exact solution: $y(x) = 1 + \frac{1}{2}\ln\left(\frac{2+x}{2-x}\right)$

Table 1: comparison of absolute errors for problem 1

x	James *et al.* (2013) h=0.1	BFM_6 h=0.05	Mohammad and Zurni (2017), h=0.05	$HFBM_{22}$ h=0.1	**FTM h=0.1**
0.1	$1.110*10^{-15}$	$3.114*10^{-12}$	$2.220*10^{-16}$	$2.000*10^{-12}$	$\mathbf{1.40*10^{-22}}$
0.2	$5.995*10^{-15}$	$6.660*10^{-12}$	$2.220*10^{-16}$	$3.000*10^{-12}$	$\mathbf{6.30*10^{-22}}$
0.3	$2.554*10^{-14}$	$9.833*10^{-12}$	$6.661*10^{-16}$	$6.000*10^{-12}$	$\mathbf{1.36*10^{-21}}$
0.4	$7.105*10^{-14}$	$2.173*10^{-11}$	$1.110*10^{-15}$	$9.000*10^{-11}$	$\mathbf{2.66*10^{-21}}$
0.5	$1.157*10^{-13}$	$3.570*10^{-11}$	$4.440*10^{-16}$	$1.400*10^{-11}$	$\mathbf{4.38*10^{-21}}$
0.6	$1.199*10^{-13}$	$4.859*10^{-11}$	$8.881*10^{-16}$	$2.200*10^{-11}$	$\mathbf{6.67*10^{-21}}$
0.7	$6.857*10^{-13}$	$1.310*10^{-10}$	$1.554*10^{-15}$	$3.500*10^{-12}$	$\mathbf{9.60*10^{-21}}$
0.8	$3.475*10^{-12}$	$2.313*10^{-10}$	$4.440*10^{-15}$	$5.900*10^{-11}$	$\mathbf{1.35*10^{-20}}$
0.9	$1.222*10^{-11}$	$3.286*10^{-10}$	$8.660*10^{-16}$	$1.010*10^{-10}$	$\mathbf{1.83*10^{-20}}$
1.0	$7.728*10^{-11}$	$1.335*10^{-09}$	$1.266*10^{-14}$	-	$\mathbf{2.41*10^{-20}}$

Table 1 shows the comparison of performance of the proposed method FTM with some existing methods for problem 1. It is shown that the FTM yield higher accurate results than the existing methods.

Problem 2. (*Source:* Ramos *et al.* (2016))

Consider a linear homogeneous problem given by

$y'' = y', \quad y(0) = 0, \; y'(0) = -1$
$0 \le x \le 1, \quad h = 0.01$

Exact solution: $y(x) = 1 - e^x$

Table 2: comparison of absolute errors for problem 2

X	Kayode and Adeyeye. (2013), h=0.1	BFM_6 h=0.1	$HFBM_{2,4}$ h=0.1	**FTM** h=0.1
0.2	$8.171*10^{-07}$	$2.427*10^{-11}$	$2.000*10^{-12}$	$\mathbf{1.063*10^{-14}}$
0.3	$3.103*10^{-06}$	$4.001*10^{-11}$	$1.000*10^{-12}$	$\mathbf{2.272*10^{-14}}$
0.4	$6.569*10^{-06}$	$5.746*10^{-11}$	$1.010*10^{-12}$	$\mathbf{3.786*10^{-14}}$
0.5	$1.143*10^{-05}$	$7.741*10^{-11}$	$1.400*10^{-11}$	$\mathbf{6.090*10^{-14}}$
0.6	$1.796*10^{-05}$	$9.517*10^{-11}$	$2.100*10^{-11}$	$\mathbf{8.853*10^{-14}}$
0.7	$2.644*10^{-05}$	$1.221*10^{-10}$	$3.000*10^{-12}$	$\mathbf{1.268*10^{-13}}$
0.8	$3.722*10^{-05}$	$1.604*10^{-10}$	$4.000*10^{-11}$	$\mathbf{1.717*10^{-13}}$
0.9	$5.067*10^{-05}$	$2.013*10^{-10}$	$5.000*10^{-11}$	$\mathbf{2.307*10^{-13}}$
1.0	$5.255*10^{-05}$	$2.466*10^{-10}$	-	$\mathbf{2.992*10^{-13}}$

Table 2 shows the comparison of performance of the proposed method FTM with some existing methods for problem 2. It is shown that the FTM yield higher accurate results than the existing methods

Problem 3. (*Source:* Adediran and Ogundare,(2015))

Consider a highly stiff initial value problem given by

$$y'' = -1001y' - 1000y, \quad y(0) = 1, \;\; y'(0) = -1$$
$$0 \le x \le 1, \quad h = 0.05$$

Exact solution: $y(x) = e^{-x}$

Table 3: comparison of absolute errors for problem 3

X	Adediran and Ogundare. (2015)	Mohammad and Zurni (2017)	**FTM h=0.1**
0.1	$2.050*10^{-11}$	$1.055*10^{-14}$	$\mathbf{1.005*10^{-16}}$
0.2	$4.390*10^{-11}$	$1.776*10^{-14}$	$\mathbf{9.642*10^{-17}}$
0.3	$6.550*10^{-11}$	$2.342*10^{-14}$	$\mathbf{4.795*10^{-16}}$
0.4	$8.380*10^{-11}$	$2.798*10^{-14}$	$\mathbf{4.530*10^{-16}}$
0.5	$9.860*10^{-10}$	$3.131*10^{-14}$	$\mathbf{8.329*10^{-16}}$
0.6	$1.100*10^{-10}$	$3.397*10^{-14}$	$\mathbf{7.743*10^{-16}}$
0.7	$1.190*10^{-10}$	$3.564*10^{-14}$	$\mathbf{1.080*10^{-15}}$
0.8	$1.240*10^{-10}$	$3.675*10^{-14}$	$\mathbf{9.960*10^{-16}}$
0.9	$1.280*10^{-10}$	$3.730*10^{-14}$	$\mathbf{1.223*10^{-15}}$
1.0	$1.300*10^{-10}$	$3.741*10^{-14}$	$\mathbf{1.122*10^{-15}}$

Table 3 shows the comparison of performance of the proposed method FTM with some existing methods for problem 3. It is shown that the FTM yield higher accurate results than the existing methods

Problem 4. Dynamic Problem (*Source:* Nicholas,(2019))

A 10kg mass is attached to a spring having a constant of 140N/m. The mass is started in motion from the equilibrium position with an initial value of 1m/sec in upward direction and with an applied external force $F(t)=0.5\sin(t)$. The resulting equation due to air resistance $9y'N$ is given as

$$y''=-9y'-14y+\frac{1}{2}\sin x,\quad y(0)=1,\ y'(0)=-1$$
$$0\le x\le 0.1,\quad h=0.001$$

Exact solution: $y(x)=-\frac{9}{50}e^{-2x}+\frac{99}{500}e^{-7x}-\frac{9}{500}\cos x$

Table 4: comparison of absolute errors for problem 4

X	$HFBM_{2,1}$	$HFBM_{2,2}$	$HFBM_{2,4}$	**FTM**
0.01	$1.304*10^{-10}$	$4.500*10^{-13}$	$1.700*10^{-13}$	$1.01*10^{-24}$
0.02	$3.323*10^{-10}$	$1.000*10^{-13}$	$4.000*10^{-13}$	$3.75*10^{-24}$
0.03	$6.448*10^{-10}$	$6.000*10^{-13}$	$2.000*10^{-15}$	$7.86*10^{-24}$
0.04	$1.003*10^{-09}$	$1.500*10^{-12}$	$7.130*10^{-13}$	$1.31*10^{-23}$
0.05	$1.438*10^{-09}$	$9.000*10^{-12}$	$1.000*10^{-15}$	$1.93*10^{-23}$
0.06	$1.899*10^{-09}$	$1.400*10^{-12}$	$4.000*10^{-13}$	$2.62*10^{-23}$
0.07	$2.412*10^{-09}$	$2.001*10^{-12}$	$1.010*10^{-12}$	$3.39*10^{-23}$
0.08	$2.933*10^{-09}$	$1.500*10^{-12}$	$4.000*10^{-13}$	$4.19*10^{-23}$
0.09	$3.489*10^{-09}$	$1.600*10^{-12}$	$5.000*10^{-13}$	$5.02*10^{-23}$
0.10	$4.041*10^{-09}$	$1.400*10^{-12}$	$3.000*10^{-13}$	$5.88*10^{-23}$

Table 4 shows the comparison of performance of the proposed method FTM with some existing methods for problem 4. It is shown that the FTM yield higher accurate results than the existing methods

Problem 5. Van Der Pol Oscillator (*Source:* Mohammed *et al.*,(2019))

$$y'' - 2\xi\left(1 - y^2\right)y' - y = 0,\ \ y(0) = 0,\ y'(0) = 0.5,\ \ \xi = 0.025,\ \ 0 \le x \le 1$$

This problem has no exact solution, our result is however validated using Runge-Kutta (RK45) and compared with Mohammed *et al.,* (2019).

Table 5: Results for the Van Der Pol Oscillator Problem with h=0.1

X	RK(5)	FTM	Mohammed *et al.*(2019)
1.0	0.431051	0.431431	0.431051
2.0	0.47631	0.478239	0.476309
3.0	0.076077	0.0765766	0.076076
4.0	-0.41546	-0.417868	-0.41546
5.0	-0.53857	-0.543708	-0.53857
6.0	-0.16135	-0.163413	-0.16134
7.0	0.386024	0.390437	0.386025
8.0	0.595231	0.604590	0.59523
9.0	0.254655	0.259731	0.254653
10.0	-0.34157	-0.347672	-0.34157

Table 5 presents the numerical solutions obtained using the proposed methods for problem 5. It is evident from the table that the numerical solutions are in agreement with the Runge-Kutta (R-K5) solution and Mohammed *et al.* (2019).

Conclusion

In this research work, we solved some standard second order initial value problems of ordinary differential equations using the proposed Falkner type method involving four off-step point using

Block hybrid method. The order of the developed methods is 7. It is zero stable and convergent.The developed methods were used to solve five test problems in Ramos *et al.* (2016). The exact results were compared with result from the source as well as the result from the proposed methods. The desirable property of a numerical solution is to behave like the exact solution of the problem which can be seen in the tables of the results represented.

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