

SENSITIVITY ANALYSIS FOR THE MATHEMATICAL MODELING OF MEASLES DISEASE INCORPORATING TEMPORARY PASSIVE IMMUNITY

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Abstract

Measles is an airborne disease which spreads easily through the coughs and sneezes of those infected. Measles antibodies are transferred from mothers who have been vaccinated against measles or have been previously infected with measles to their newborn children. These antibodies are transferred in low amounts and usually last six months or less. In this paper a mathematical model of measles disease was formulated incorporating temporary passive immunity. There exist two equilibria in the model; Disease Free Equilibrium (DFE) and Endemic Equilibrium (EE). The Disease Free Equilibrium (DFE) state was analyzed for local and global stability. The Basic Reproduction Number R_0 was computed and used to carry out the sensitivity analysis with some parameters of the model. The analysis shows that as contact rate α increases the R_0 increases and as the vaccination rate v increases the R_0 decreases. Sensitive parameters with the R_0 were presented graphically. The disease will die out of the population if the attention is given to high level immunization.

Keywords: Basic Reproduction Number; equilibrium state; sensitivity; stability.

1. Introduction

Measles is an airborne disease which spreads easily through the coughs and sneezes of those infected. It may also be spread through contact with saliva or nasal secretions. Nine out of ten people who are not immune and share living space with an infected person will likely catch it. People are infectious to others from four days before to four days after the start of the rash. People usually do not get the disease more than once in a life time; indicating that once recovered from the disease, the person becomes permanently immune (Atkinson, 2011).

According to WHO, measles is one of the leading causes of death among young children even though a safe and cost-effective vaccine is available. In 2015, there were 134 200 measles deaths globally – about 367 deaths every day or 15 deaths every hour. Measles vaccination resulted in a 79% drop in measles deaths between 2000 and 2015 worldwide. In 2015, about 85% of the world's children receive 'one dose of measles vaccine by their first birthday through routine health services – up from 73% in 2000. During 2000-2015, measles vaccination prevented an estimated 20.3 million deaths making measles vaccine one of the best buys in public health.

According to, Leuridan *et al.* (2012), in developed countries, children are immunized against measles at 12 months, generally as part of a three-part measles, mumps, and rubella (MMR) vaccine. The vaccination is generally not given before this age because such infants respond inadequately to the vaccine due to an

immature immune system. Measles antibodies are transferred from mothers who have been vaccinated against measles or have been previously infected with measles to their newborn children. However, such antibodies are transferred in low amounts and usually last six months or less. Infants under one year of age whose maternal anti-measles antibodies have disappeared become susceptible to infection with the measles virus. A second dose of the vaccine is usually given to children between the ages of four and five, to increase rates of immunity.

Sensitivity analysis tells us how important each parameter is to disease transmission. Such information is crucial not only for experimental design, but also to data assimilation and reduction of complex nonlinear models, (Powell *et al.*, 2005). Sensitivity analysis is commonly used to determine the robustness of model predictions to parameter values, since there are usually errors in data collection and presumed parameter values. It is used to discover parameters that have a high impact on R_0 and should be targeted for intervention strategies.

In this paper, we formulated a mathematical model of disease incorporating temporary passive immunity. The existence of equilibrium point was verified and the local and global stability of Disease Free Equilibrium (DFE) were analyzed using basic reproduction number. We also carried out the sensitivity analysis of the basic reproduction number with some parameters of the model.

2. Literature Review

Abubakar *et al.*, (2012), formulated a mathematical model for measles disease dynamics. They divided the total population into three compartments of: Susceptible $S(t)$, Infected $I(t)$ and Recovered $R(t)$. In their model they did not consider vaccination rate, they did not compute basic reproduction number.

Somma *et al.*, (2015), modified the Maternally-Derived-Immunity Susceptible Infectious Recovered (MSIR) Model by adding the vaccination rate and death rate due to the disease to the existing MSIR model. They determined the basic reproduction number using next generation Matrix and Jacobian matrix methods. In their model they did not considered the immunity rate.

Okyere-Siabouh and Adetunde (2013), they formulated a mathematical model of Measles with respect to Cape-Coast Metropolis. They consider Susceptible-Exposed-Infected-Recovered (SEIR) epidemiological model. Their model assumes that individuals are equally likely to be infected by the infectious individuals in a case of an outbreak, except those who are immune.

Fred *et al.* (2014), used a population with variable size to provide a framework. Their model relied on a compartmental model expressed by a set of ordinary differential equations (O.D.E) and partial differential equations (P.D.E) based on the dynamics of measles infections. The mathematical model equations, the mathematical analysis and the numerical simulations that followed served to reveal quantitatively as well as qualitatively the consequences of the mathematical modeling on measles vaccination. They performed the numerical and qualitative analyses of the model at different state variables.

Derdei *et al.*, (2014), formulated a MSIR model, they did not incorporate vaccination rate into their model. They did not analyze the stability of the model.

Yano *et al.* (2016), investigate the transmission dynamics of a Childhood disease outbreak in a community with direct inflow of susceptible and vaccinated new-born. In their model they did not considered the maternally-derive- immunity and immunity rate.

3. Methodology

3.1 Model Formulation

The total population $N(t)$ is divided into four compartment based on the epidemiological status of individuals: Maternally-Derive-Immunity $M(t)$, Susceptible $S(t)$, Infected $I(t)$ and Recovered/Immuned $R(t)$, $t = \text{time}$. In this model it is assume that the new babies are born into M class and S at constant rate Λ . The proportion of the new born with immunity is θ while the proportion of the new born without immunity is $(1 - \theta)$. The new babies loss their immunity after some time at a rate ω and move to susceptible class. The susceptible individuals become infected with measles at a contact rate α . The susceptible class is vaccinated at a rate v and thereby move to recovered/immuned class. The treated infected individuals recover at a rate γ and move to recovered/immuned class. The death rate due to disease δ while the natural death rate of the entire population is μ . The schematic diagram and model equations for the measles transmission as discuss in this paper is presented below:

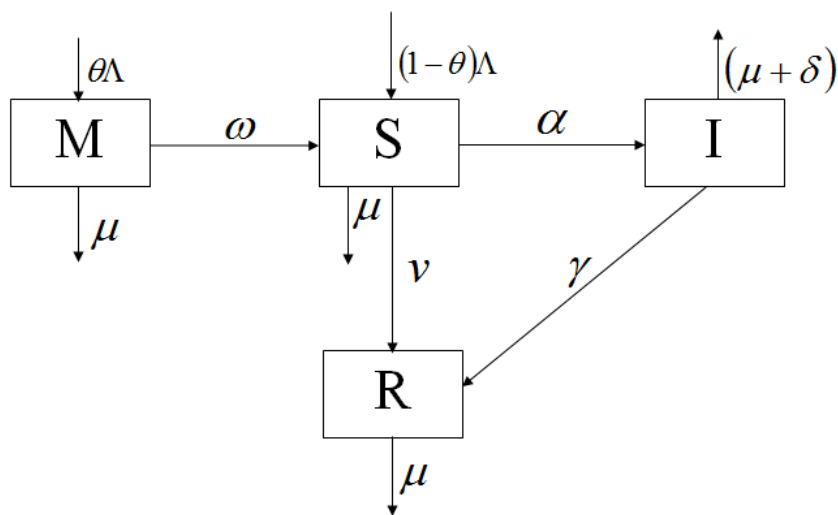


Figure 3.1: Schematic Diagram of the Model

$$\frac{dM}{dt} = \theta\Lambda - (\omega + \mu)M \quad (3.1)$$

$$\frac{dS}{dt} = (1 - \theta)\Lambda - \frac{\alpha SI}{N} + \omega M - (v + \mu)S \quad (3.2)$$

$$\frac{dI}{dt} = \frac{\alpha SI}{N} - (\mu + \delta + \gamma)I \quad (3.3)$$

$$\frac{dR}{dt} = \gamma I + vS - \mu R \quad (3.4)$$

Where

$$N = M + S + I + R \quad (3.5)$$

Table 3.1: Variables and Parameters of the Model

Variables/Parameter	Description
N	Total Population
M	Maternally-Derived –Immunity
S	Susceptible
I	Infected
R	Recovered/Immune
Λ	Recruitment rate
θ	Immunity Rate
α	Contact Rate
δ	Death Rate due to Disease
γ	Recovery Rate
μ	Natural Death Rate
ν	Vaccination Rate
ω	Loss of Immunity Rate

3.2 Existence of Equilibrium Points of the Model

At equilibrium point

$$\frac{dM}{dt} = \frac{dS}{dt} = \frac{dI}{dt} = \frac{dR}{dt} = 0 \quad (3.6)$$

Let

$$(M, S, I, R) = (M^*, S^*, I^*, R^*) \quad (3.7)$$

be arbitrarily equilibrium point

$$\theta\Lambda - A_1M^* = 0 \quad (3.8)$$

$$(1 - \theta)\Lambda - \frac{\alpha S^* I^*}{N^*} + \omega M^* - A_2 S^* = 0 \quad (3.9)$$

$$\frac{\alpha S^* I^*}{N^*} - A_3 I^* = 0 \quad (3.10)$$

$$\gamma I^* + \nu S^* - \mu R^* = 0 \quad (3.11)$$

Where

$$A_1 = (\omega + \mu), A_2 = (\nu + \mu) \text{ and } A_3 = (\mu + \delta + \gamma) \quad (3.12)$$

From (3.10) we have

$$\left(\frac{\alpha S^*}{N^*} - A_3 \right) I^* = 0 \quad (3.13)$$

$$I^* = 0 \quad (3.14)$$

Or

$$\left(\frac{\alpha S^*}{N^*} - A_3 \right) = 0 \quad (3.15)$$

Equations (3.14) and (3.15) shows the existence of two equilibria; Disease Free Equilibrium (DEF) and Endemic Equilibrium (EE) respectively.

3.3 Disease Free Equilibrium (DEF) Point

The Disease Free Equilibrium (DEF) is the absence of the disease in a population and equation (3.14) show the implies the disease.

Let

$$(M, S, I, R) = (M^0, S^0, I^0, R^0) = E^0 \quad (3.16)$$

Disease Free Equilibrium (DEF) point.

Substituting (3.14) into equations (3.9) and (3.11) gives

$$M^0 = \frac{\theta\Lambda}{A_1} \quad (3.17)$$

$$S^0 = \frac{A_1(1-\theta)\Lambda + \omega\theta\Lambda}{A_1A_2} \quad (3.18)$$

$$R^0 = \frac{v[A_1(1-\theta)\Lambda + \omega\theta\Lambda]}{\mu A_1A_2} \quad (3.19)$$

The Disease Free Equilibrium (DFE) point is given as

$$(M^0, S^0, I^0, R^0) = \left(\frac{\theta\Lambda}{A_1}, \frac{A_1(1-\theta)\Lambda + \omega\theta\Lambda}{A_1A_2}, 0, \frac{v[A_1(1-\theta)\Lambda + \omega\theta\Lambda]}{\mu A_1A_2} \right) \quad (3.20)$$

3.4 Basic Reproduction Number R_0

Basic reproduction number R_0 , is average number of secondary cases produced by a single infection in a completely susceptible population.

Applying next generation matrix operator to compute the Basic Reproduction Number of the model as used by Diekmann, *et al.*, (1990), and improved by (Driessche, *et al.*, 2002).

$$FV^{-1} = \left[\frac{\partial F_i(E^0)}{\partial x_i} \right] \left[\frac{\partial V_i(E^0)}{\partial x_i} \right]^{-1} \quad (3.21)$$

Where

F_i are the new infections, while the V_i are transfers of infections from one compartment to another. E^0 is the disease-free equilibrium point. The basic reproduction number, R_0 is the largest eigenvalue or spectral radius of FV^{-1} .

$$F = \frac{\alpha S^0}{N^0} \quad (3.21)$$

$$V = A_3 \quad (3.22)$$

$$V^{-1} = \frac{1}{A_3} \quad (3.23)$$

$$FV^{-1} = \frac{\alpha S^0}{A_3 N^0} \quad (3.34)$$

At DFE

$$FV^{-1} = \frac{\alpha\mu[A_1(1-\theta) + \omega\theta]}{A_1 A_2 A_3} \quad (3.25)$$

$$R_0 = \frac{\alpha\mu[A_1(1-\theta) + \omega\theta]}{A_1 A_2 A_3} \quad (3.26)$$

3.5 Local Stability of Disease Free Equilibrium (DFE) E^0

Theorem 3.1: The Disease Free Equilibrium of the model system (3.1)-(3.4) is locally asymptotically stable if $R_0 < 1$.

Proof

In order to prove the theorem above, we are going to use the Jacobian Matrix stability techniques.

The all eigenvalues of the matrix are expected to be less than zero (i.e. $\lambda_i < 0$).

$$J(E^0) = \begin{bmatrix} -A_1 & 0 & 0 & 0 \\ \omega & -A_2 & -\frac{\alpha\mu[A_1(1-\theta) + \omega\theta]}{A_1 A_2} & 0 \\ 0 & 0 & \frac{\alpha\mu[A_1(1-\theta) + \omega\theta] - A_1 A_2 A_3}{A_1 A_2} & 0 \\ 0 & \nu & \gamma & -\mu \end{bmatrix} \quad (3.27)$$

$$|J(E^0) - \lambda I| = 0 \quad (3.28)$$

$$\begin{vmatrix} -A_1 - \lambda & 0 & 0 & 0 \\ \omega & -A_2 - \lambda & -\frac{\alpha\mu[A_1(1-\theta) + \omega\theta]}{A_1A_2} & 0 \\ 0 & 0 & \frac{\alpha\mu[A_1(1-\theta) + \omega\theta]}{A_1A_2} - A_1A_2A_3 - \lambda & 0 \\ 0 & \nu & \gamma & -\mu - \lambda \end{vmatrix} = 0 \quad (3.29)$$

The characteristic equation of (3.29) is given as

$$(-A_1 - \lambda)(-A_2 - \lambda)[\alpha\mu[A_1(1-\theta) + \omega\theta] - A_1A_2A_3] - \lambda(-\mu - \lambda) = 0 \quad (3.30)$$

$$\lambda_1 = -A_1, \lambda_2 = -A_2, \lambda_3 = -\mu \quad (3.31)$$

It is observed that, $\lambda_1, \lambda_2, \lambda_3 < 0$

But, $\lambda_4 < 0$ if

$$\alpha\mu[A_1(1-\theta) + \omega\theta] - A_1A_2A_3 < 0 \quad (3.32)$$

$$\frac{\alpha\mu[A_1(1-\theta) + \omega\theta]}{A_1A_2A_3} < 1 \quad (3.33)$$

The Left Hand Side (LHS) of equation (3.33) is equivalent to the Right Hand Side of (3.26), therefore,

$$R_0 < 1 \quad (3.34)$$

Equation (3.34) proved the theorem 3.1. Equation (3.34), implies that, the disease will not persist in the population.

3.7 Global Stability of Disease Free Equilibrium (DFE), E^0

Theorem 3.2: The DFE, E_0 of the model system is globally asymptotically stable if $R_0 \leq 1$.

Proof

To establish the global stability of the disease-free equilibrium, we construct the following Lyapunov function:

$$V(M, S, I, R) = A_3 I \tag{3.35}$$

Differentiating (3.35) with respect to t gives

$$\frac{dV}{dt} = A_3 \frac{dI}{dt} \tag{3.36}$$

$$\frac{dV}{dt} = A_3 \left(\frac{\alpha S}{N} - A_3 \right) I \tag{3.37}$$

Since $S \leq S^0$ and $N \leq N^0$

$$\frac{dV}{dt} \leq A_3 \left(\frac{\alpha S^0}{N^0} - A_3 \right) I \tag{3.38}$$

$$\frac{dV}{dt} \leq A_3 \left[\frac{\alpha \mu [A_1 (1 - \theta) + \omega \theta]}{A_1 A_2} - A_3 \right] I \tag{3.39}$$

$$\frac{dV}{dt} \leq A_3^2 (R_0 - 1)I \quad (3.40)$$

When $R_0 < 1$, the derivative $\frac{dV}{dt} < 0$ and $R_0 = 1$, the derivative $\frac{dV}{dt} = 0$. Consequently, the largest

compact invariant set in $\left\{ (M, S, I, R) \in \Omega, \frac{dV}{dt} = 0 \right\}$, when $R_0 \leq 1$, is the singleton E^0 . Hence,

LaSalle's invariance principle, LaSalle (1976) implies that E^0 is globally asymptotically stable in Ω . This completes the proof.

3.8 Endemic Equilibrium (EE) Point

The Endemic Equilibrium (EE) is the persistence of the disease in a population and equation.

Let

$$(M, S, I, R) = (M^{**}, S^{**}, I^{**}, R^{**}) = E^1 \quad (3.41)$$

$$\left. \begin{aligned} \theta\Lambda - A_1 M^{**} &= 0 \\ (1-\theta)\Lambda - \frac{\alpha S^{**} I^{**}}{N^{**}} + \omega M^{**} - A_2 S^{**} &= 0 \\ \frac{\alpha S^{**} I^{**}}{N^{**}} - A_3 I^{**} &= 0 \\ \gamma I^{**} + \nu S^{**} - \mu R^{**} &= 0 \end{aligned} \right\} \quad (3.42)$$

$$\left. \begin{aligned} \theta\Lambda - A_1 M^{**} &= 0 \\ (1-\theta)\Lambda - \lambda^{**} S^{**} + \omega M^{**} - A_2 S^{**} &= 0 \\ \lambda^{**} S^{**} - A_3 I^{**} &= 0 \\ \gamma I^{**} + \nu S^{**} - \mu R^{**} &= 0 \end{aligned} \right\} \quad (3.43)$$

Where,

$$\lambda^{**} = \frac{\alpha I^{**}}{N^{**}} \quad (3.44)$$

Is the force of infection

From (3.43)

$$\left. \begin{aligned}
 M^{**} &= \frac{\theta\Lambda}{A_1} \\
 S^{**} &= \frac{A_1\Lambda(1-\theta) + \theta\omega\Lambda}{A_1(\lambda^{**} + A_2)} \\
 I^{**} &= \frac{\lambda^{**}[A_1\Lambda(1-\theta) + \theta\omega\Lambda]}{A_1A_3(\lambda^{**} + A_2)} \\
 R^{**} &= \frac{\nu A_3[A_1\Lambda(1-\theta) + \theta\omega\Lambda] + \gamma[A_1\Lambda(1-\theta) + \theta\omega\Lambda]\lambda^{**}}{A_1A_3\mu(\lambda^{**} + A_2)}
 \end{aligned} \right\} \quad (3.45)$$

$$N^{**} = M^{**} + S^{**} + I^{**} + R^{**} \quad (3.46)$$

$$N^{**} = \frac{A_3\theta\Lambda\mu(\lambda^{**} + A_2) + [A_1\Lambda(1-\theta) + \theta\omega\Lambda][(\mu + \gamma)\lambda^{**} + A_2A_3]}{A_1A_3\mu(\lambda^{**} + A_2)} \quad (3.47)$$

Substituting I^{**} and N^{**} into (3.44) gives

$$\begin{aligned}
 [A_3\theta\Lambda\mu + (\mu + \gamma)[A_1\Lambda(1-\theta) + \theta\omega\Lambda]]\lambda^{**} + A_2A_3\theta\Lambda\mu + A_2A_3[A_1\Lambda(1-\theta) + \theta\omega\Lambda] \\
 - \alpha\mu[A_1\Lambda(1-\theta) + \theta\omega\Lambda] = 0
 \end{aligned} \quad (3.48)$$

$$B_1\lambda^{**} + B_2 = 0 \quad (3.49)$$

Where,

$$\left. \begin{aligned}
 B_1 &= A_3\theta\Lambda\mu + (\mu + \gamma)[A_1\Lambda(1-\theta) + \theta\omega\Lambda] \\
 B_2 &= A_2A_3[\theta\Lambda\mu + [A_1\Lambda(1-\theta) + \theta\omega\Lambda]] - \alpha\mu[A_1\Lambda(1-\theta) + \theta\omega\Lambda]
 \end{aligned} \right\} \quad (3.50)$$

$$B_2 = A_1 A_2 A_3 \Lambda (1 - R_0) \tag{3.51}$$

Therefore, equation (3.49) becomes

$$B_1 \lambda^{**} + A_1 A_2 A_3 \Lambda (1 - R_0) = 0 \tag{3.52}$$

3.9 Bifurcation Analysis

We illustrate the phenomenon of Bifurcation by considering the equation (3.52) resulting from the endemic equilibrium. The estimated parameter values in table 4.1 are used to plot the diagram.

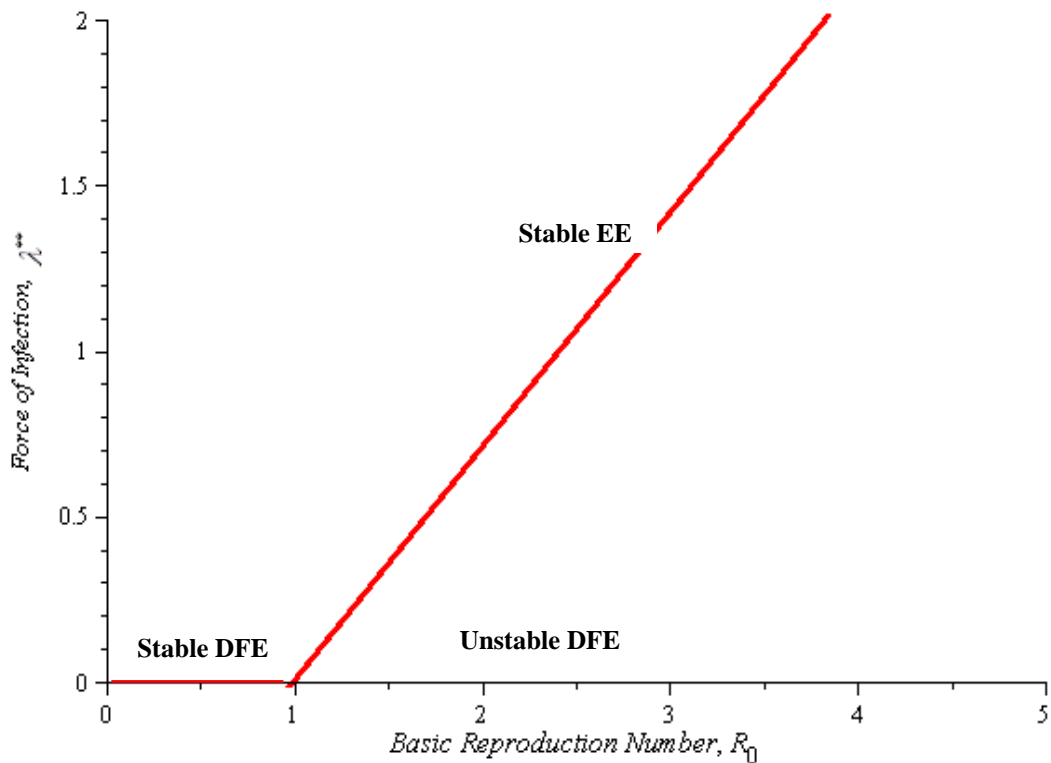


Figure 3.2: Forward Bifurcation Diagram for the Model

In figure 3.2 above, the two equilibrium points exchange stabilities depending on the value of R_0 . A transcritical/forward bifurcation in the equilibrium points occur at $R_0 = 1$. If, $R_0 < 1$ the disease free equilibrium (DFE) is stable. But if $R_0 > 1$, the endemic equilibrium exists and it is stable while the disease free equilibrium is a saddle point. Thus there is a forward bifurcation because in the neighbourhood of the bifurcation point, the force of infection, λ^{**} is an increasing function of R_0 .

4 Results and Discussion

4.1 Sensitivity Analysis of the Basic Reproduction Number, R_0 with Some Parameter of the Model

Sensitivity indices allow us to measure the relative change in a variable when a parameter changes. The normalized forward sensitivity index of a variable with respect to a parameter is the ratio of the relative change in the variable to the relative change in the parameter. When the variable is a differentiable function of the parameter, the sensitivity index may be alternatively defined using partial derivatives.

In determining how best to reduce human mortality and morbidity due to measles, the sensitivity indices of the basic reproduction number to the parameters of the model was calculated following similar approaches as in Arriola and Hyman (2005), Chitnis *et al.* (2008), Mikuchi *et al.* (2012) and Abdulrahman *et al.* (2013). The normalized forward sensitivity indices with respect to a parameter value, P is defined as

$$S_P^{R_0} = \frac{\partial R_0}{\partial P} \times \frac{P}{R_0} \tag{4.1}$$

Where,

$$P = \{\alpha, \gamma, \nu, \theta\} \tag{4.2}$$

The sensitivity indices of the parameters of the basic reproduction number R_0 were calculated using Maple 13 software. See appendix B, for the estimation of variables and parameter values used in sensitivity analysis as shown on Table 4.1 below.

Table 4.1: Values for Parameters used for Sensitivity Analysis

Variables	Values per year	Source
$M(0)$	82,010,000	B9
$S(0)$	7,099,464,364	B10
$I(0)$	254,918	B3
$R(0)$	118,270,718	B4
N	7,300,000,000	B1
Λ	139,000,000	B2
α	0.9	B12
δ	0.53	B6
γ	0.47	B5
μ	0.008	B7
ω	0.39	B11
ν	0.85	B8
θ	0.61	B13

Table 4.2: Sensitivity Indices of R_0 to Parameters of the model, evaluated at the parameter values given in Table 4.1

Parameter	Low transmission	High transmission
	Sensitivity Index	Sensitivity Index
α	1.000000000	1.000000000
γ	-0.466269843	-0.3172588832
ν	-0.9906759907	-0.9878419453
θ	-0.01241351242	-0.006270226538

Table 4.2 shows that all the parameters have either positive or negative effects on the basic reproduction number, R_0 . The positive parameters will increase the basic reproduction number while the negative parameters will decrease the basic reproduction number. The contact rate, α has the highest sensitivity index follow by vaccination rate, ν and recovery rate, γ and immunity rate θ has the lowest sensitivity analysis.

4.2 Graphical Representation of Basic Reproduction Number with Sensitive Parameter

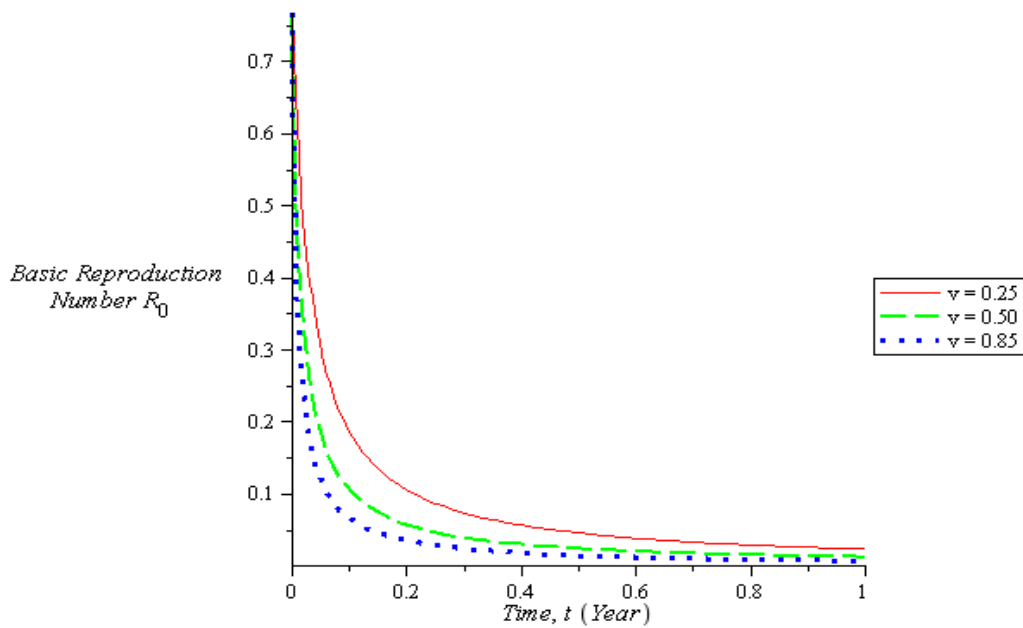


Figure 4.1: The Graph of Basic Reproduction Number against different values of Vaccination Rate
 Figure 4.1 shows that as vaccination rate increases with time the Basic reproduction number decreases. It is observe that, with increase in vaccination rate, the basic reproduction number decrease to almost zero.

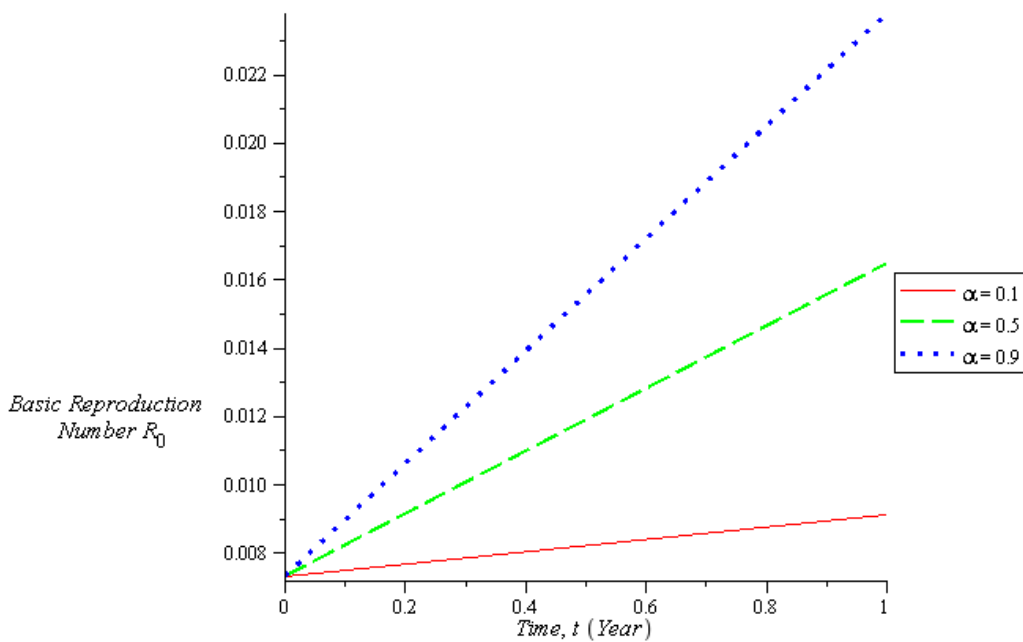


Figure 4.2: The Graph of Basic Reproduction Number against different values of Contact Rate

Figure 4.2 shows that as contact rate increases with time the Basic reproduction number increases. It also show that low contact rate gives low basic reproduction number. The children infected with measles should be separated from those that are not infected.

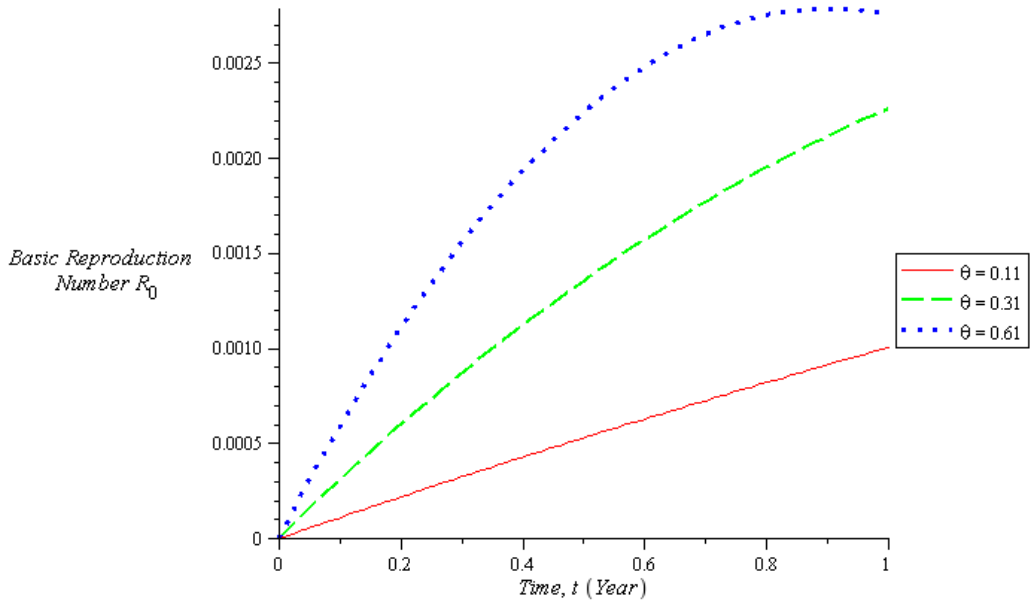


Figure 4.3: The Graph of Basic Reproduction Number against different values of Immunity Rate
 Figure 4.3 shows that as immunity rate increases with time the Basic reproduction number decreases. The immunity depends on vaccination and treatment.

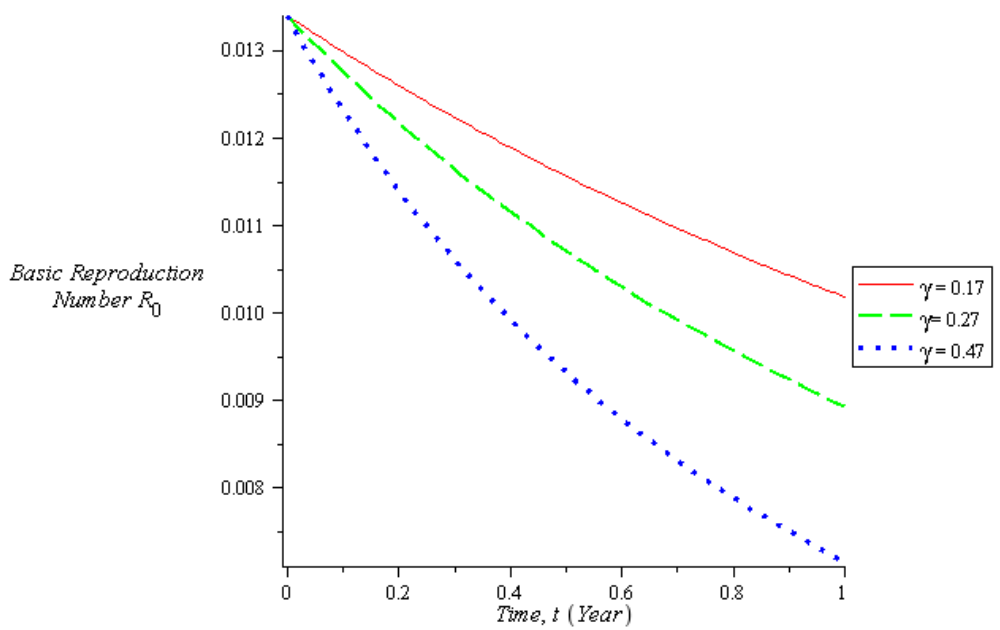


Figure 4.4: The Graph of Basic Reproduction Number against different values of Recovery Rate

Figure 4.4 shows that as recovery rate increases with time the basic reproduction number decreases. It is observe that, with increase in recovery rate, the basic reproduction number decrease to almost zero.

5 Conclusion

The model equations are formulated using first order ordinary differential equation. The existence of Disease Free Equilibrium (DFE) and Endemic Equilibrium (EE) was proved. The Disease Free Equilibrium (DFE) is locally asymptotically stable if $R_0 < 1$ and globally asymptotically stable if $R_0 \leq 1$. The bifurcation analysis reveal that the model exhibit forward bifurcation if $R_0 = 1$.

Four parameters of the model were used to carried out the sensitivity analysis with the basic reproduction number R_0 . The contact rate α , is the most sensitive parameter that will increase the basic reproduction number R_0 while the vaccination rate ν is the most sensitive parameter that will decrease the basic reproduction number R_0 . Other parameter that were used in sensitivity analysis are immunity rate θ and recovery rate γ .

The graphical representation of the basic reproduction number R_0 with these sensitive parameters give the better understanding on how the parameters affect the basic reproduction number R_0 negatively or positively. Measles will die out of the population if attention is given to high level immunization of children. The rural dweller should be sensitizing on the risk of contracting the disease. The susceptible individuals should not share the same living space with the infected individuals

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Appendices

Appendix A: Reported Measles Cases by WHO Region 2015 and 2016, as of November 2016

Table A1: Reported Measles Cases by WHO Region 2015, as of November 2016

WHO Region	Member States Reported (Expected)	Total Suspected	Total Measles	Clinical Confirmed	Epidemiological Link	Laboratory Confirmed	Data Received
Africa	41(47)	86984	55263	21111	26163	7989	Nov-16
America	34(35)	18670	210	0	0	210	Nov-16
Eastern Mediterranean	21(21)	34654	14053	639	4559	8855	Nov-16
Europe	50(53)	28025	26776	19835	1014	5926	Nov-16
South-East Asia	11(11)	114726	90860	64484	22353	4023	Nov-16
Western Pacific	27(27)	143289	67756	22337	611	44808	Nov-16
Total	184(194)	426348	254918	128406	54700	71811	

Source: WHO (2016)

Table A2: Reported Measles Cases by WHO Region 2016, as of November 2016

WHO Region	Member States Reported (Expected)	Total Suspected	Total Measles	Clinical Confirmed	Epidemiological Link	Laboratory Confirmed	Data Received
Africa	42(47)	46474	28126	12459	11085	4582	Nov-16
America	34(35)	9564	65	0	0	65	Nov-16
Eastern Mediterranean	20(21)	19763	4518	153	947	3418	Nov-16
Europe	50(53)	3849	2537	241	385	1910	Nov-16
South-East Asia	11(11)	86302	63169	51015	11004	1150	Nov-16
Western Pacific	27(27)	100517	55620	27594	638	27388	Nov-16
Total	184(194)	266469	154035	91462	24059	38513	

Source: WHO (2016)

Appendix B: Estimation of Variables and Parameter Values

It is difficult to get a reliable data, we estimated the parameter values based on the available data from the World Health Organization (WHO), Population Reference Bureau and reliable related literature. The estimates are clearly explained in the following sub-sections.

B1: The Total Population, N

According to Population Reference Bureau, the world total population at 2015, is 7.3 billion.

$$N = 7,300,000,000$$

B2: Recruitment Number, Λ

According to Population Reference Bureau the birth rate per year is $\frac{19}{1,000}$

The number of new birth in 2015 is 139,000,000.

Therefore,

$$\Lambda = 139,000,000$$

B3: Number of Infected, I

The WHO estimate that, there are 254,918 cases of measles worldwide each year, resulting in 134,200 deaths. (See Table A1)

$$I = 254,918$$

B4: Number of Recovered/Immune, R

Recovered/Immune Human population, $R = \text{recovered} + \text{immune}$

From B3 the number of cases is 254,918 and number of death is 134,200.

Recovered= 254,918 -134,200 = 120,718 the number of surviving infants in 2015 is 139,000,000

and the percentage of vaccinated is 85%. Therefore,

Vaccinated = 85% of 139,000,000 =118,150,000.

Hence,

Recovered/Immune Human population, $R = 120,718 + 118,150,000$

$$R = 118,270,718$$

B5: Recovery Rate, γ

From B3 and B4

$$\gamma = \frac{\text{Recovered}}{\text{Number of cases}}$$
$$\gamma = \frac{120,718}{254,918} = 0.47$$

B6: Disease Induce death rate, δ

From B3 the number of cases of measles is 254,918 and the number of death from measles is 134,200

$$\delta = \frac{\text{Number of Death from measles}}{\text{Number of cases}}$$
$$\delta = \frac{134,200}{254,918} = 0.53$$

B7: Natural Death Rate, μ

According to WHO, the death rate is 8 deaths per 1,000. Therefore,

$$\mu = \frac{8}{1000} = 0.008$$

B8: Vaccination rate, v

According to, WHO in 2015, about 85% of the world's children received one dose of measles vaccine. Therefore,

$$v = 0.85$$

B9: Maternally-Derived-Immunity, M

According to Millennium Development Goal (MDG4), every year nearly 41% of all under-five child deaths are among newborn infants, babies in their first 28 days of life or the neonatal

period.

$$M = 59\% \text{ of } 139,000,000$$

$$M = 82,010,000$$

B10: Number of Susceptible, S

Recall $N = M + S + I + R$ therefore,

$$S = N - (M + I + R)$$

$$S = 7,300,000,000 - (82,010,000 + 254,918 + 118,270,718)$$

$$S = 7,300,000,000 - 200,535,636$$

$$S = 7,099,464,364$$

B11: Loss of immunity, ω

According to WHO Immunization coverage fact sheet, national immunization schedule reported that, only 61% of children received 2 doses of measles. Therefore,

$$\omega = 39\% = 0.39$$

B12: Contact Rate, α

According to, Atkinson, (2011), nine out of ten people who are not immune and share living space with an infected person will catch it. Therefore

$$\alpha = \frac{9}{10} = 0.9$$

B13: Immunity Rate, θ

WHO doctors recommend that two doses of the vaccine be given at six and nine months of age to ensure immunity and prevent outbreaks, as about 15% of vaccinated children fail to develop immunity from the first dose. According to WHO Immunization coverage fact sheet, national immunization schedule reported that, only 61% of children received 2 doses of measles. Therefore, the Immunity Rate, θ is assumed to be

$$\theta = 0.61$$